20º COLÓQUIO BRASILEIRO DE MATEMÁTICA

UMA INTRODUÇÃO AOS PROCESSOS ESTOCÁSTICOS ESPACIAIS

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INTRODUÇÃO

Este minicurso é uma introdução elementar aos processos estocásticos espaciais. A leitura dessas notas requer apenas familiaridade com cálculo (em particular séries) e com um primeiro curso em probabilidade. Eu tentei escolher o caminho mais curto para chegar a dois exemplos simples de sistemas de partículas (esse é o outro nome para processos estocásticos espaciais): passeios aleatórios com ramificação e processo de contato na árvore. Esses dois exemplos já são suficientes para modelizar varios fenomenos físicos e biológicos. A análise matemática desses sistemas de partículas requer o uso de técnicas importantes como acoplamento e renormalisação. Parte deste material é baseado em artigos recentes (posteriores à 1990) e é um raro exemplo de pesquisa matemática atual que pode ser exposta ao nivel de graduação.

O capítulo I é uma rápida introdução à probabilidade. O capítulo II considera cadeias de Markov, em particular passeios aleatórios e processos de ramificação em tempo discreto. O capítulo III trata do primeiro exemplo de sistema de partículas: passeio aleatório com ramificação. Mostramos que é possivel que esse sistema morra localmente e sobreviva globalmente e damos uma condição necessária e suficiente para que isto aconteça. Calculamos os valores críticos para dois exemplos e discutimos a continuidade das transições de fase.

O capítulo IV considera o processo de contato na árvore. O processo de contato foi introduzido por Harris (1974) no reticulado Z^d e só recentemente foi considerado na árvore por Pemantle (1992). O estudo do processo de contato na árvore usa os processos vistos nos capítulos anteriores: ramificação e passeio aleatório com ramificação. Isso faz com que a análise do processo de contato na árvore seja muito mais simples do que em Z^d. Mas mesmo na árvore o estudo do processo de contato é longe de ser trivial e já se encontram as dificuldades usuais no estudo de sistemas de partículas, em particular não se consegue fazer contas exatas. No final do capítulo IV menciono problemas abertos atuais.

Agradeço a Pablo Ferrari por sugerir a minha participação no Colóquio Brasileiro de Matemática (1995) e a Ellen Saada por ter lido partes dessas notas e por várias sugestões.

Chapter I

A SHORT REVIEW OF PROBABILITIES ON COUNTABLE SPACES

1. Probability space

Consider Ω a countable space. A probability P is a function from the set of subsets of Ω to [0,1] with the two following properties.

$$P(\Omega) = 1$$

and if $A_n \subset \Omega$ for $n \geq 1$, and $A_i \cap A_j = \emptyset$ for $i \neq j$ then

$$P(\cup_{n\geq 1} A_n) = \sum_{n\geq 1} P(A_n)$$

Problem 1.1 Prove that $P(\emptyset) = 0$ and that $P(A^c) = 1 - P(A)$ for any subset A of Ω.

We now give an example of a probability model. Assume a coin is tossed 2 times then $\Omega = \{HH, HT, TH, TT\}$ is all possible outcomes. It seems reasonable (in the case of a fair coin) to define $P(s) = 1/4$ for each $s \in \Omega$.

The subsets of Ω are called events. We say that the sequence of events A_n is increasing if $A_n \subset A_{n+1}$ for each $n \geq 1$.

Proposition 1.1 *Let (Ω, P) be a probability space, A, B, and A_n be events. We have the following properties*

(i) If $B \subset A$ then $P(A \cap B^c) = P(A/B) = P(A) - P(B)$.

(ii) For any sequence of events A_n we have $P(\cup_{n\geq 1} A_n) \leq \sum_{n\geq 1} P(A_n)$.

(iii) If A_n is a sequence of increasing events then $\lim_{n\to\infty} P(A_n) = P(\cup_{n\geq 1} A_n)$.

(iv) If A_n is a sequence of decreasing events then $\lim_{n\to\infty} P(A_n) = P(\cap_{n\geq 1} A_n)$.

Proof of Proposition 1.1

Observe that B and $A \cap B^c = A/B$ are disjoint and their union is A. Hence

$$P(A) = P(A/B) + P(B)$$

and this proves (i).

We now prove (iii). Assume A_n is an increasing sequence of events, define

$$A = \bigcup_{n\geq 1} A_n \text{ and } B_1 = A_1, B_n = A_n/A_{n-1} \text{ for } n \geq 2$$

Now the B_n are disjoint and their union is still A therefore

$$P(A) = \sum_{n\geq 1} P(B_n) = P(A_1) + \lim_{n\to\infty} \sum_{p=2}^{n} (P(A_p) - P(A_{p-1}))$$

So we get

$$P(A) = P(A_1) + \lim_{n\to\infty} (P(A_n) - P(A_1))$$

and this proves (iii).

We now use (iii) to prove (ii). For any sequence of events A_n we may define

$$C_n = \bigcup_{p=1}^{n} A_p$$

and C_n is increasing. We also have that for any two events A and B

$$P(A \cup B) = P((A/(A \cap B)) \cup B) = P(A/(A \cap B)) + P(B)$$

$$= P(A) + P(B) - P(A \cap B)$$

Hence

$$P(A \cup B) \leq P(A) + P(B)$$

and by induction we get for any finite union

$$P(C_n) = P(\bigcup_{p=1}^{n} A_p) \leq \sum_{p=1}^{n} P(A_p) \tag{1.1}$$

Now using (iii) we know that $P(C_n)$ converges to the probability of the union of C_n which is the same as the union of the A_n. Making n go to infinity in (1.1) yields

$$\lim_{n\to\infty} P(C_n) = P(\bigcup_{p=1}^{\infty} A_p) \leq \sum_{p=1}^{\infty} P(A_p)$$

and this concludes the proof of (ii). For (iv) it is enough to observe that if A_n is a decreasing sequence then A_n^c is an increasing sequence and by (iii)

$$\lim_{n\to\infty} P(A_n^c) = P(\bigcup_{n\geq 1} A_n^c) = P((\bigcap_{n\geq 1} A_n)^c) = 1 - P(\bigcap_{n\geq 1} A_n)$$

and this concludes the proof of Proposition 1.1.

Problem 1.2 Prove that for any events A and B we have

$$|P(A) - P(B)| \leq P(A \cap B^c) + P(A^c \cap B)$$

For any sequence of events A_n we define the events

$$\limsup A_n = \bigcap_{n\geq 1} \bigcup_{p\geq n} A_p \text{ and } \liminf A_n = \bigcup_{n\geq 1} \bigcap_{p\geq n} A_p$$

Problem 1.3 Check that ω is in $\limsup A_n$ if and only if ω is in A_n for infinitely many distinct n. And check that ω is in $\liminf A_n$ if and only if ω is in A_n for all n except possibly for a finite number of n.

Proposition 1.2 *We have the following inequalities for any sequence of events* A_n

$$P(\liminf A_n) \leq \liminf_{n\to\infty} P(A_n)$$

$$P(\limsup A_n) \geq \limsup_{n\to\infty} P(A_n)$$

Proof of Proposition 1.2

Define $B_n = \cap_{p\geq n} A_p$ and observe that B_n is an increasing sequence of events. Hence by Proposition 1.1

$$\lim_{n\to\infty} P(B_n) = P(\bigcup_{n\geq 1} B_n) = P(\liminf A_n)$$

Since $P(A_n) \geq P(B_n)$ we get

$$\liminf_{n\to\infty} P(A_n) \geq \lim_{n\to\infty} P(B_n) = P(\liminf A_n)$$

and this proves the first inequality in Proposition 1.2

Problem 1.4 Prove the second inequality in Proposition 1.2.

2. Independence

Given an event B such that $P(B) > 0$ we define the conditional probability

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Problem 2.1 Prove that the function $A \to P(A|B) = P_B(A)$ defines a new probability Ω. Prove that $P_B(A|C) = P(A|B\cap C)$ if $P(B\cap C) > 0$.

We say that the events A and B are independent if $P(A\cap B) = P(A)P(B)$.

Problem 2.2 Prove that if A and B are independent so are A and B^c, and A^c and B^c.

More generally we say that the events $A_1, A_2, \ldots, A_n$ are independent if for all integers $i_1, i_2, \ldots, i_p$ in $\{1, \ldots, n\}$ we have

$$P(\bigcap_{j=1}^{p} A_{i_j}) = \Pi_{j=1}^{p} P(A_{i_j})$$

We now state a property that we will use several times in the sequel.

Borel-Cantelli Lemma *If A_n is a sequence of events such that*

$$\sum_{n \geq 1} P(A_n) < \infty$$

then $P(\limsup A_n) = 0$.

Conversely, if the A_n are independent events and

$$\sum_{n \geq 1} P(A_n) = \infty$$

then $P(\limsup A_n) = 1$.

Observe that the independence assumption is only needed for the converse.

Proof of Borel-Cantelli Lemma

We first assume that $\sum_{n \geq 1} P(A_n) < \infty$.

Define $B_n = \cup_{p \geq n} A_p$ and since B_n is a decreasing sequence we have

$$\lim_{n \to \infty} P(B_n) = P(\cap_{n \geq 1} B_n) = P(\limsup A_n)$$

On the other hand

$$P(B_n) \leq \sum_{p \geq n} P(A_p)$$

but the last sum is the tail of a convergent series therefore

$$\lim_{n \to \infty} P(B_n) = P(\limsup A_n) \leq \lim_{n \to \infty} \sum_{p \geq n} P(A_p) = 0$$

For the converse we need the two assumptions: independence of the A_n and that the series is infinite. Using that the A_n^c are independent, for any integers $m < n$ we have that

$$P(\bigcap_{p=m}^{n} A_p^c) = \Pi_{p=m}^{n} P(A_p^c) = \Pi_{p=m}^{n}(1 - P(A_p))$$

Since $1 - u \leq e^{-u}$ we get

$$P(\bigcap_{p=m}^{n} A_p^c) \leq e^{-\sum_{p=m}^{n} P(A_p)} \tag{2.1}$$

Fix m and define $C_n = \bigcap_{p=m}^{n} A_p^c$. The sequence C_n is decreasing and

$$\bigcap_{n \geq m} C_n = \bigcap_{p=m}^{\infty} A_p^c$$

We now make n go to infinity in (2.1) to get

$$\lim_{n\to\infty} P(C_n) = P(\bigcap_{n\geq m} C_n) = P(\bigcap_{p=m}^{\infty} A_p^c) \leq \lim_{n\to\infty} e^{-\sum_{p=m}^{n} P(A_p)} = 0$$

where we are using that for any m, $\lim_{n\to\infty} \sum_{p=m}^{n} P(A_p) = \infty$. So we have that

$$P((\bigcap_{p=m}^{\infty} A_p^c)^c) = P(\bigcup_{p=m}^{\infty} A_p) = 1$$

and now we make m go to infinity to get

$$P(\bigcap_{m\geq 1} \bigcup_{p=m}^{\infty} A_p) = P(\limsup A_n) = 1$$

And this completes the proof of the Borel-Cantelli Lemma.

3. Discrete random variables

A random variable is a function from Ω to some set S. If the set S is countable we say that the random variable is discrete.

If $X_1, \ldots, X_n$ are random variables with values in $S_1, \ldots, S_n$ countable sets, we say that these random variables are independent if for any $A_i \subset S_i$, $1 \leq i \leq n$, we have

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \times \ldots \times P(X_n \in A_n)$$

If S is a subset of the integers and if $\sum_{x \in S} |x| P(X = x)$ is finite we define the expected value of the random variable X by

$$E(X) = \sum_{x \in S} x P(X = x)$$

Problem 3.1 Prove that if X is a random variable whose values are positive integers then

$$E(X) = \sum_{k \geq 1} P(X \geq k)$$

We define the conditional expected value of X by

$$E(X|A) = \sum_{x \in S} x P(X = x|A)$$

We conclude this quick overview with the notion of generating function. For a random variable whose values are positive integers we define the generating function

$$g_X(s) = E(s^X) = \sum_{n \geq 0} s^n P(X = n) \text{ for } s \in [-1, 1]$$

Problem 3.2 Prove that g_X is well defined on [-1,1].

Proposition 3.1 *If X and Y are two independent random variables with values on the positive integers we have that*

$$g_{X+Y}(s) = g_X(s)g_Y(s)$$

Proof of Proposition 3.1

$$g_{X+Y}(s) = \sum_{n\geq 0} s^n P(X+Y=n)$$

but

$$P(X+Y=n) = \sum_{k=0}^{n} P(X=k; Y=n-k) = \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

where the last equality comes from the independence of X and Y.

$$g_{X+Y}(s) = \sum_{n\geq 0} s^n \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

$$= \sum_{n\geq 0} s^n P(X=n) \sum_{n\geq 0} s^n P(Y=n)$$

where the last equality comes from results about the product of two absolute convergent series. This completes the proof of Proposition 3.1.

Chapter II

DISCRETE TIME MARKOV CHAINS

1. Definitions

A discrete time stochastic process is a sequence of random variables $(X_n)_{n\geq 0}$ defined on the same probability space and having values on the same countable space $\mathcal{S}$. We will take $\mathcal{S}$ to be the set of positive integers in most of what we will do in this chapter.

A Markov process is a stochastic process with the property that for $n > k$ the conditional distribution of X_n given X_k is independent of $(X_l)_{l\leq k}$. More precisely, a stochastic process X_n is said to be Markovian if for any states $x_1, x_2, \ldots, x_k, x_n$ in $\mathcal{S}$, any integers $n_1 < n_2 < \ldots < n_k < n$ we have

$$P(X_n = x_n|X_{n_1} = x_1, X_{n_2} = x_2, \ldots, X_{n_k} = x_k) = P(X_n = x_n|X_{n_k} = x_k)$$

We define the one-step transition probability by

$$p(i,j) = P(X_{n+1} = j|X_n = i) \text{ for all } i,j \in \mathcal{S} \text{ and all } n \geq 0$$

Observe that we are assuming that the transition probabilities do not depend on the time variable n, that is we consider Markov chains with stationary transition probabilities. The $p(i,j)$ clearly have the following (probability) properties

$$p(i,j) \geq 0 \text{ and } \sum_{j\in\mathcal{S}} p(i,j) = 1$$

Problem 1.1. Prove that if the distribution of X_0 and the $p(i,j)$ are given then the law of the Markov chain is completely determined. More precisely, prove that for any $n \geq 0$ and any $i_0, i_1, \ldots, i_n$ in $\mathcal{S}$ we have

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_0 = i_0)p(i_0, i_1)\ldots p(i_{n-1}, i_n)$$

Define the n-steps transition probability by

$$p_n(i,j) = P(X_{n+m} = j | X_m = i) \text{ for all } i,j \in \mathcal{S} \text{ and all } m \geq 0$$

We have the following matrix multiplication like rule:

Proposition 1.1 *For any positive integers $r \leq n$*

$$p_n(i,j) = \sum_{k \in \mathcal{S}} p_r(i,k) p_{n-r}(k,j)$$

where we define $p_0(i,j) = 1$ if $i = j$ and $p_0(i,j) = 0$ if $i \neq j$.

Proof of Proposition 1.1

We have the following decomposition for $r \leq n$

$$\{X_n = j\} = \bigcup_{k \in \mathcal{S}} \{X_n = j; X_r = k\}$$

and the events in the union are mutually exclusive. Hence

$$p_n(i,j) = P(X_n = j | X_0 = i) = \sum_{k \in \mathcal{S}} P(X_n = j; X_r = k | X_0 = i)$$

The last term is equal to

$$\sum_{k \in \mathcal{S}} P(X_n = j | X_r = k; X_0 = i) P(X_r = k | X_0 = i)$$

but by the Markov property we get that this sum is equal to

$$\sum_{k \in \mathcal{S}} P(X_n = j | X_r = k) P(X_r = k | X_0 = i) = \sum_{k \in \mathcal{S}} p_r(i,k) p_{n-r}(k,j)$$

this completes the proof of Proposition 1.1.

We will now turn to the classification of the states of a Markov chain. We say that two states i and j are in the same class if there are integers

$n_1 \geq 0$ and $n_2 \geq 0$ such that $p_{n_1}(i,j) > 0$ and $p_{n_2}(j,i) > 0$. In words, i and j are in the same class if the Markov chain can go from i to j and j to i in a finite number of steps with positive probability.

Problem 1.2 Prove that the relation "to be in the same class than" is an equivalence relation on $\mathcal{S}$.

For each state i in $\mathcal{S}$ we define the random variable

$$T_i = \inf\{n \geq 1 : X_n = i\}$$

and if the inf does not exist we take $T_i = \infty$. We define the probability that the first return to i occurs at time $n \geq 1$ by

$$f_n(i) = P(T_i = n | X_0 = i)$$

Observe that

$$\sum_{n=0}^{\infty} f_n(i) = P(T_i < \infty | X_0 = i) \leq 1$$

We say that state i is recurrent if $\sum_{n=0}^{\infty} f_n(i) = 1$, a nonrecurrent state is said to be transient. In words, a recurrent state is one for which the probability that the Markov chain will return to it after a finite (random) time is one. The next theorem gives us a recurrence criterion in terms of the $p_n(i,i)$ which are usually much easier to estimate than the $f_n(i)$.

Theorem 1.1 *A state i in $\mathcal{S}$ is recurrent if and only if*

$$\sum_{n=0}^{\infty} p_n(i,i) = \infty$$

To prove Theorem 1.1 we will need several observations and one lemma.

Observe that the Markov chain reaches i at time $n \geq 1$ only if T_i is less than n. Hence

$$\{X_n = i\} = \bigcup_{k=1}^{n} \{X_n = i; T_i = k\}$$

where the events in the union are pairwise disjoint. Therefore

$$P(X_n = i | X_0 = i) = p_n(i,i) = \sum_{k=1}^{n} P(X_n = i; T_i = k | X_0 = i)$$

But $P(X_n = i; T_i = k | X_0 = i) = P(X_n = i | T_i = k; X_0 = i) P(T_i = k | X_0 = i) = p_{n-k}(i,i) f_k(i)$ and this implies that

(1.1) $$p_n(i,i) = \sum_{k=0}^{n} p_{n-k}(i,i) f_k(i) \text{ for } n \geq 1$$

where we define $f_0(i) = 0$. We define the generating functions by

$$P_{ii}(s) = \sum_{n=0}^{\infty} p_n(i,i) s^n \text{ and } F_i(s) = \sum_{n=0}^{\infty} f_n(i) s^n \text{ for } |s| < 1$$

There is a very simple and useful relation between these two generating functions. First observe that by the multiplication rule for absolute convergent series we have

$$F_i(s) P_{ii}(s) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_k(i) p_{n-k}(i,i) s^n$$

but using (1.1) for $n \geq 1$ and $f_0(i) = 0$ we get

$$F_i(s) P_{ii}(s) = \sum_{n=1}^{\infty} p_n(i,i) s^n = P_{ii}(s) - p_0(i,i) = P_{ii}(s) - 1$$

and so

(1.2) $$P_{ii}(s) = \frac{1}{1 - F_i(s)} \text{ for all } |s| < 1$$

We will now need the following well known power series result

Lemma 1.1 *Assume that* $a_k \geq 0$ *for all* $k \geq 0$ *then*
(a) if $\sum_{k=0}^{\infty} a_k$ *converges then the following limit from the left exists*

$$\lim_{s \to 1^-} \sum_{k=0}^{\infty} a_k s^k = \sum_{k=0}^{\infty} a_k$$

(b) If

$$\lim_{s \to 1^-} \sum_{k=0}^{\infty} a_k s^k = L \leq \infty$$

then $\lim_{N \to \infty} \sum_{k=0}^{N} a_k$ *is equal to* L*.*

Problem 1.3 Prove Lemma 1.1.

We are now ready for the

Proof of Theorem 1.1

Assume that state i is recurrent then by Lemma 1.1 (a) we have

$$\lim_{s \to 1^-} F_i(s) = \sum_{n=0}^{\infty} f_n(i) = 1$$

So (1.2) implies that

$$\lim_{s \to 1^-} P_{ii}(s) = \infty$$

Using Lemma (1.1) (b) we get

$$\lim_{s \to 1^-} P_{ii}(s) = \sum_{n=0}^{\infty} p_n(i,i) = \infty$$

And this proves that recurrence implies that the preceding series is infinite.

For the converse, assume i is transient then by Lemma 1.1 (a) we get

$$\lim_{s \to 1^-} F_i(s) = \sum_{n=0}^{\infty} f_n(i) < 1$$

and so by (1.2)

$$\lim_{s \to 1^-} P_{ii}(s) < \infty$$

but the preceding limit is by Lemma 1.1 (b) equal to $\sum_{n=0}^{\infty} p_n(i,i) < \infty$. This completes the proof of Theorem 1.1.

We now show that recurrence is a class property.

Corollary 1.1 *All states that are in the same class are either all recurrent or all transient.*

Proof of Corollary 1.1

Assume that states i and j are in the same class, by definition this means that there are positive integers m and n such that

$$p_n(i,j) > 0 \text{ and } p_m(j,i) > 0$$

Observe that by Proposition 1.1 we have for any states i, j, ℓ

$$p_{m+n}(i,j) = \sum_{k \in S} p_m(i,k) p_n(k,j) \geq p_m(i,\ell) p_n(\ell,j)$$

We iterate twice the preceding the inequality to get

$$p_{m+n+r}(j,j) \geq p_m(j,i) p_r(i,i) p_n(i,j)$$

for any positive integer r. We now sum on all r to get

$$\sum_{r=0}^{\infty} p_{m+n+r}(j,j) \geq \sum_{r=0}^{\infty} p_m(j,i) p_r(i,i) p_n(i,j) = p_m(j,i) p_n(i,j) \sum_{r=0}^{\infty} p_r(i,i)$$

Therefore if $\sum_{r=0}^{\infty} p_r(i,i)$ diverges so does $\sum_{r=0}^{\infty} p_r(j,j)$. Since i and j can be interchanged this completes the proof of Corollary 1.1.

We now give an interesting characterization of the recurrence property.

Theorem 1.2 *A state i is recurrent if and only if the Markov chain returns to i infinitely many times with probability 1.*

Proof of Theorem 1.2

We define the event of at least n returns to i by

$$E_n(i) = \{X_k = i \text{ for at least } n \text{ distinct } k\}$$

and let $g_n(i) = P(E_n(i))$. The Markov chain will visit state i at least n times if and only if after the first return it visits i at least $n-1$ times. Hence

$$g_n(i) = \sum_{k=1}^{\infty} f_k(i) g_{n-1}(i) = g_{n-1}(i) f(i) \text{ for } n \geq 2$$

where $f(i) = \sum_{k=1}^{\infty} f_k(i)$. Proceeding by induction and using that $g_1(i) = f(i)$ we get that

$$g_n(i) = f(i)^n$$

Observe that $\lim_{n\to\infty} g_n(i)$ is 1 or 0 according whether $f(i) = 1$ (i is recurrent) or $f(i) < 1$ (i is transient), respectively. Since the events $(E_n(i))_{n\geq 1}$ are decreasing we get

$$\lim_{n\to\infty} g_n(i) = \lim_{n\to\infty} P(E_n(i)) = P(\bigcap_{n\geq 1} E_n(i))$$

$$= P(X_k = i \text{ for infinitely many } k)$$

and this completes the proof of Theorem 1.2.

2. Random walks

This is our first example of Markov chain. We begin by considering the one dimensional random walk on the integers. The states of this random walk are on $S = Z$. The one-step transition probabilities are $p(i, i+1) = p$ and $p(i, i-1) = 1 - p = q$ and all other entries of $p(i, j)$ are 0. We use

Theorem 1.1 to determine for which values of $p \in [0,1]$ the one-dimensional random walk is recurrent.

Problem 2.1 Prove that if p is 0 or 1 then all states are transient.

In the sequel we take p to be in (0,1). In this case it is clear that the random walk has a single class: all states communicate. So by corollary 1.1 it is enough to consider one state, take the origin 0 for instance. If n is odd then $p_n(0,0) = 0$ so we only need to consider even times. It is easy to see that

$$p_{2n}(0,0) = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!n!} p^n q^n$$

In order to estimate the preceding probability we use Stirling's formula:

$$\lim_{n \to \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$$

see for instance Feller (1968) for a proof. Stirling's formula yields

$$\lim_{n \to \infty} \frac{p_{2n}(0,0)}{(4pq)^n / \sqrt{\pi n}} = 1$$

It is easy to check that $4pq < 1$ if $p \neq 1/2$ and $4pq = 1$ if $p = 1/2$. Therefore $\sum_{n=0}^{\infty} p_n(0,0) = \infty$ if and only if $p = 1/2$. The one-dimensional simple random walk is recurrent if and only if it is symmetric ($p = 1/2$).

We now consider the two dimensional simple symmetric random walk. That is, $\mathcal{S} = Z^2$ and

$$p((i,j),(i+1,j)) = p((i,j),(i,j+1))$$

$$= p((i,j),(i-1,j)) = p((i,j),(i,j-1)) = 1/4$$

all the other entries of p are 0. We see that there is only one class and we consider the origin $O = (0,0)$. In order to analyse the two dimensional random walk we will need the following combinatorial identity.

Problem 2.2 Prove that

$$\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i} = \binom{2n}{n}$$

One way of proving this is to write Newton's formula for $(1+t)^n$ and $(1+t)^{2n}$ in the identity

$$(1+t)^n(1+t)^n = (1+t)^{2n}$$

and identify the coefficients of the polynomials on both sides of the equality.

For the two dimensional random walk to return to the origin in $2n$ steps it must move i units to the right, i units to the left, j units up and j units down where $2i + 2j = 2n$. Hence

$$p_{2n}(O,O) = \sum_{i=0}^{n} \frac{(2n)!}{i!i!(n-i)!(n-i)!}(1/4)^{2n}$$

Dividing and multiplying by $(n!)^2$ yields

$$p_{2n}(O,O) = \frac{(2n)!}{n!n!} \sum_{i=0}^{n} \frac{n!n!}{i!i!(n-i)!(n-i)!}(1/4)^{2n}$$

hence

$$p_{2n}(O,O) = \binom{2n}{n} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i} (1/4)^{2n}$$

By Problem 2.2 we get

$$p_{2n}(O,O) = \binom{2n}{n}^2 (1/4)^{2n}$$

Using Stirling's formula we have

$$\lim_{n\to\infty} \frac{p_{2n}(O,O)}{\frac{1}{\pi n}} = 1$$

and this proves that the two dimensional random walk is recurrent.

We now turn to the analysis of the three dimensional simple symmetric random walk. This time $\mathcal{S} = Z^3$ and the only transitions that are allowed are plus or minus one unit for one coordinate at the time, the probability of each of these six transitions is $1/6$. By a reasoning similar to the one we did in two dimensions we get

$$p_{2n}(O,O) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(2n)!}{i!i!j!j!(n-i-j)!(n-i-j)!}(1/6)^{2n}$$

We multiply and divide by $(n!)^2$ to get

$$p_{2n}(O,O) = (1/2)^{2n} \binom{2n}{n} \sum_{i=0}^{n} \sum_{j=0}^{n-i} (\frac{n!}{i!j!(n-i-j)!})^2 (1/3)^{2n}$$

Let

$$c(i,j) = \frac{n!}{i!j!(n-i-j)!} \text{ for } 0 \leq i+j \leq n$$

and

$$m_n = \max_{i,j,0 \leq i+j \leq n} c(i,j)$$

We have

$$p_{2n}(O,O) \leq (1/2)^{2n} \binom{2n}{n} m_n (1/3)^n \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{n!}{i!j!(n-i-j)!}(1/3)^n$$

but

$$(1/3 + 1/3 + 1/3)^n = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{n!}{i!j!(n-i-j)!}(1/3)^n = 1$$

Hence

$$p_{2n}(O,O) \leq (1/2)^{2n} \binom{2n}{n} m_n (1/3)^n \tag{2.1}$$

We now need to estimate m_n. Suppose that the maximum of the $c(i,j)$ occurs at (i_0, j_0) then the following inequalities must hold

$$c(i_0, j_0) \geq c(i_0 - 1, j_0)$$

$$c(i_0, j_0) \geq c(i_0 + 1, j_0)$$

$$c(i_0, j_0) \geq c(i_0, j_0 - 1)$$

$$c(i_0, j_0) \geq c(i_0, j_0 + 1)$$

These inequalities imply that

$$n - i_0 - 1 \leq 2j_0 \leq n - i_0 + 1$$

$$n - j_0 - 1 \leq 2i_0 \leq n - j_0 + 1$$

and it shows that the approximate values for i_0 and j_0 with one unit error are $n/3$. We use this in (2.1) to get

$$p_{2n}(O,O) \leq (1/2)^{2n} \binom{2n}{n} c(n/3, n/3)(1/3)^n$$

We use again Stirling's formula to get that the right hand side of the last inequality is asymptotic to $\frac{3\sqrt{3}}{2\pi^{3/2}n^{3/2}}$. This proves that $\sum_{n\geq 0} p_n(O,O)$ is convergent and therefore the three dimensional random walk is transient. In other words there is a positive probability that the three dimensional random walk will never return to the origin. This is in sharp contrast with what happens for the random walk in dimensions one and two.

3. The Bienaymé-Galton-Watson branching process

This process was introduced independently by Bienaymé and by Galton and Watson to model the survival of family names. An initial set of men which we call the zero-th generation have male children that are called the first generation; their children are called the second generation and so on. We denote the size of the n-th generation by Z_n, $n \geq 0$. This model has also been used in several biological problems (survival of genes, for instance) and physical problems (electron multipliers, for instance). We now give the mathematical definition of the Bienaymé-Galton-Watson (BGW) process Z_n. We call the objects that are generated particles. The state space $\mathcal{S}$ of Z_n is the set of positive (including zero) integers. We suppose that each particle gives rise to Y particles in the next generation where Y is a positive integer-valued random variable with distribution $(p_k)_{k \geq 0}$. In other words

$$P(Y = k) = p_k, \text{ for } k = 0, 1, \ldots$$

Moreover we assume that the number of offspring of the various particles in the various generations are chosen independently according to the distribution (p_k). With these assumptions it is clear that Z_n is a Markov process: in order to have the distribution of Z_{n+1} the information we need in $(Z_k)_{k \leq n}$ is Z_n. In particular the one-step transition probabilities are given by

$$p(i,j) = P(Z_{n+1} = j | Z_n = i) = P(\sum_{k=1}^{i} Y_k = j) \text{ for } i \geq 1, j \geq 0$$

where $(Y_k)_{1 \leq k \leq i}$ is a sequence of independent identically distributed random variables with distribution (p_k). We assume that state 0 (no particles) is an absorbing state (or trap) for Z_n in the sense that

$$p(0,i) = 0 \text{ if } i \geq 1 \text{ and } p(0,0) = 1$$

We also assume that the first moment exists for the offspring distribution p_k: there is a finite number m such that

$$\sum_{k=0}^{\infty} k p_k = m$$

We define the generating function of the offspring distribution by

$$f(s) = \sum_{k=0}^{\infty} p_k s^k \text{ for } |s| \leq 1$$

Theorem 3.1 *Assume that $p_0 + p_1 < 1$ then the BGW branching process exhibits a phase transition in the sense that*

$$\textit{if } m \leq 1 \textit{ then } P(Z_n \geq 1, \textit{ for all } n \geq 0 | Z_0 = 1) = 0$$

$$\textit{if } m > 1 \textit{ then } P(Z_n \geq 1, \textit{ for all } n \geq 0 | Z_0 = 1) = 1 - q > 0$$

Moreover q is the unique solution stricly less than 1 of the equation $f(s) = s$ when $m > 1$.

The process BGW is said to be subcritical, critical and supercritical according wether $m < 1$, $m = 1$ or $m > 1$. Observe that the BGW process may survive forever if and only if $m > 1$. So the only relevant parameter of the offspring distribution for survival is m. Of course, the probability $1 - q$ of surviving forever depends on the whole distribution (p_k) through its generating function.

Before proving Theorem 3.1 we will need to prove a couple of properties of the BGW process. We now introduce the iterates of the generating function f: let $f_1 = f$ and $f_{n+1} = f \circ f_n$ for $n \geq 1$.

Proposition 3.1 *For $n \geq 1$, the generating function of Z_n conditioned on $Z_0 = 1$ is f_n.*

Proof of Proposition 3.1

We prove this by induction. Let g_n be the generating function of Z_n given that $Z_0 = 1$. We have

$$g_1(s) = E(s^{Z_1} | Z_0 = 1) = E(s^{Y_1}) = f(s) = f_1(s)$$

so the property holds for $n = 1$. Assume that $g_n = f_n$. First observe that given $Z_n = k$, the distribution of Z_{n+1} is the same as the distribution of $\sum_{i=1}^{k} Y_i$ where the Y_i are i.i.d. random variables with distribution p_k. Hence

$$E(s^{Z_{n+1}}|Z_n = k) = (E(s^{Y_1}))^k = f(s)^k$$

where the first equality comes from the independence of the Y_i. Therefore

$$g_{n+1}(s) = \sum_{k=0}^{\infty} P(Z_n = k|Z_0 = 1)f(s)^k = g_n(f(s))$$

and by our induction hypothesis we get $g_{n+1} = g_n \circ f = f_n \circ f = f_{n+1}$. This completes the proof of Proposition 3.1

Proposition 3.2 *We have that*

$$E(Z_n|Z_0 = 1) = m^n \text{ for } n \geq 0$$

Proof of Proposition 3.2

$$E(Z_{n+1}|Z_0 = 1) = \sum_{k=0}^{\infty} kP(Z_{n+1} = k|Z_0 = 1) = \sum_{k=0}^{\infty} kp_{n+1}(1,k)$$

By Proposition 1.1 we get

$$E(Z_{n+1}|Z_0 = 1) = \sum_{k=0}^{\infty} k \sum_{j=0}^{\infty} p_n(1,j)p_1(j,k)$$

We interchange the two sums (we can do this since all the terms are positive) and use that

$$\sum_{k=0}^{\infty} kp_1(j,k) = E(Z_1|Z_0 = j)$$

to get

$$E(Z_{n+1}|Z_0 = 1) = \sum_{j=0}^{\infty} p_n(1,j)E(Z_1|Z_0 = j)$$

But

$$E(Z_1|Z_0 = j) = jE(Z_1|Z_0 = 1)$$

and so

$$E(Z_{n+1}|Z_0 = 1) = \sum_{j=0}^{\infty} p_n(1,j)jE(Z_1|Z_0 = 1)$$

We finally get that

$$E(Z_{n+1}|Z_0 = 1) = m\sum_{j=0}^{\infty} jP(Z_n = j|Z_0 = 1) = mE(Z_n|Z_0 = 1)$$

We iterate this equality and use that $E(Z_1|Z_0 = 1) = m$ to complete the proof of Proposition 3.2.

We are now ready for the

Proof of Theorem 3.1

We start by dealing with the easiest case: $m < 1$. By Proposition 3.2

$$P(Z_n \geq 1|Z_0 = 1) \leq E(Z_n|Z_0 = 1) = m^n$$

Hence

$$\lim_{n\to\infty} P(Z_n \geq 1|Z_0 = 1) = 0$$

and the convergence occurs exponentially fast. Observe that since 0 is a trap for Z_n the sequence of events $\{Z_n \geq 1\}$ is decreasing. We have

$$\lim_{n\to\infty} P(Z_n \geq 1|Z_0 = 1) = P(\bigcap_{n\geq 0}\{Z_n \geq 1\}|Z_0 = 1)$$

$$= P(Z_n \geq 1 \text{ for all } n \geq 0|Z_0 = 1) = 0$$

and this takes care of the case $m < 1$.

For the cases $m = 1$ and $m > 1$ we will need the following observations.

$$P(Z_n = 0|Z_0 = 1) = f_n(0)$$

and since the events $\{Z_n = 0\}$ are increasing (in n) we have

$$\lim_{n\to\infty} f_n(0) = P(\bigcup_{n\geq 0}\{Z_n = 0\}|Z_0 = 1)$$

Define q to be the probability of extinction

$$q = \lim_{n\to\infty} f_n(0) = P(\bigcup_{n\geq 0}\{Z_n = 0\}|Z_0 = 1)$$

$$= P(Z_n = 0 \text{ for some } n \geq 1|Z_0 = 1)$$

Since $f_{n+1}(0) = f(f_n(0))$ by the continuity of f we must have $f(q) = q$. Our task now will be to show that depending on the value of m we will have $q = 1$ (extinction is certain) or $q < 1$ (survival has positive probability).

We first consider $m = 1$.

$$f'(s) = \sum_{k\geq 1} kp_k s^{k-1} < \sum_{k\geq 1} kp_k = m = 1 \text{ for } s < 1$$

So by the mean value Theorem (recall that a power series is infinitely differentiable on any open interval where it is defined)

$$f(1) - f(s) = f'(c)(1-s) < 1 - s \text{ for } s < 1$$

and so

$$f(s) > s \text{ for } s < 1$$

Therefore there is no solution to the equation $f(s) = s$ other than $s = 1$ in the interval [0,1]. Hence $q = 1$.

We consider now $m > 1$. By continuity of f' there is $\eta > 0$ such that if $s > 1-\eta$ then $1 < f'(s) < f'(1) = m$. So by the mean value Theorem, for $s > 1-\eta$ we have $f(s) < s$. But at $s = 0$ we know that $f(0) \geq 0$ so by the intermediate value Theorem we have at least one solution to the equation $f(s) = s$ in [0,1), we denote this solution by s_1. We now show that the solution is unique in [0,1). By contradiction assume there is another solution t_1 in [0,1). But since $f(1) = 1$ we would have by Rolle's Theorem $\xi_1 \neq \xi_2$ in (0,1) such that $f'(\xi_1) = f'(\xi_2) = 1$. Since $p_0 + p_1 < 1$ we have that $f''(s) > 0$ for s in (0,1) and therefore f' is strictly increasing, we have a contradiction. At this point we know that $q = s_1$ or $q = 1$ since these are the two only solutions of $f(s) = s$ in [0,1]. By contradiction assume that $q = \lim_{n\to\infty} f_n(0) = 1$. For n large enough, $f_n(0) > 1-\eta$. As observed above this implies that $f(f_n(0)) < f_n(0)$. But this contradicts the fact that $f_n(0)$ is increasing so q must be the unique solution of $f(s) = s$ which is strictly less than 1. This completes the proof of Theorem 3.1.

Problem 3.1 Find a necessary and sufficient condition on the p_k in order to have $q = 0$ (survival is certain).

Problem 3.2 Find q in function of r if $p_k = (1-r)r^k$.

We now show that the BGW process is not stable in the sense that it either goes to zero or to infinity: it does not remain bounded and positive.

Proposition 3.3 *Assume that $p_0 + p_1 < 1$ then all the states $k \geq 1$ of the BGW Z_n are transient. Moreover*

$$P(\lim_{n\to\infty} Z_n = 0) = q$$

$$P(\lim_{n\to\infty} Z_n = \infty) = 1-q$$

Proof of Proposition 3.3

If $p_0 = 0$ then the number of particles can only increase from one generation to the following and we get for each $k \geq 1$

$$p_n(k,k) = (p_1^k)^n$$

and since $p_1 < 1$ by Theorem 1.1 each state $k \geq 1$ is transient.

If $p_0 > 0$ then we define for a fixed integer $k \geq 1$ the first time of return to k by

$$T_k = \inf\{n \geq 1 : X_n = k\}$$

Observe that if the first generation has no particles then there will be no return to k. So

$$P(T_k < \infty | Z_0 = k) \leq 1 - p_0^k < 1$$

This proves that k is transient.

By Theorem 1.2 we know that Z_n returns to each $k \geq 1$ only a finite number of times with probability 1. This is only possible if Z_n goes to zero or if Z_n goes to infinity as n goes to infinity. But we know that Z_n goes to 0 with probability q so Z_n goes to infinity with probability $1 - q$. This completes the proof of Proposition 3.3.

Problem 3.3 Using that $\limsup_n P(Z_n = k) \leq P(\limsup_n \{Z_n = k\})$ prove that $\lim_{n\to\infty} P(Z_n = k) = 0$ for each $k \geq 1$.

4. Notes and references

There are many good books on Markov chains. At an elementary level see for instance Hoel, Port and Stone (1972) or Karlin (1966). A more advanced book is Bhattacharya and Waymire (1990). On random walks on a countable space there is the beautiful book of Spitzer (1976) based on measure theory and Fourier analysis. On branching processes the reader can consult Harris (1963) or Athreya and Ney (1972).

Chapter III

CONTINUOUS TIME BRANCHING MARKOV CHAINS

1. The critical values

We will now consider a continuous time Markov process. In order to get the Markov property we need to use exponential distributions to describe the evolution of the process. Recall that a (continuous) random variable T is said to be exponentially distributed with rate k if for each real number t we have

$$P(T > t) = e^{-kt}$$

and the expected value of T is then $1/k$.

Consider a system of particles which undergo branching and random motion on S in continuous time. Let $p(x, y)$ be the probability transitions of a given Markov chain on a countable set S. The evolution of a continuous time branching Markov chain on S is governed by the two following rules.

(i) If $p(x, y) > 0$, and if there is a particle at x then this particle waits a random exponential time with rate $\lambda p(x, y)$ and then gives birth to a new particle at y, $\lambda > 0$ is a parameter.

(ii) A particle (anywhere on S) waits an exponential time with rate 1 and then dies.

Let $b_t^{x,\lambda}$ denote the branching Markov chain starting from a single particle at x and let $b_t^{x,\lambda}(y)$ be the number of particles at site y at time t. If the process starts with a finite number of particles it is not difficult to prove that it has a finite number of particles at all finite times. So the states of the process $b_t^{x,\lambda}$ are in

$$\mathcal{S} = \{\eta \in N^S : \sum_{y \in S} \eta(y) < \infty\}$$

where N denotes the set of positive integers (including 0). Observe that S is a countable space. The continuous time stochastic process $b_t^{x,\lambda}$ has the Markov property in the following sense. Take a sequence of times $0 \leq s_0 < s_1 < \ldots < s_k < t$ and take a sequence of states (or configurations) in S $\eta_0, \eta_1, \ldots, \eta_k, \eta$ then

$$P(b_t^{x,\lambda} = \eta | b_{s_0}^{x,\lambda} = \eta_0; b_{s_1}^{x,\lambda} = \eta_1; \ldots b_{s_k}^{x,\lambda} = \eta_k) = P(b_t^{x,\lambda} = \eta | b_{s_k}^{x,\lambda} = \eta_k)$$

The Markov property of the branching Markov chain is a direct consequence of the lack of memory property of the exponential distributions appearing in (i) and (ii).

We denote the total number of particles of $b_t^{x,\lambda}$ by

$$|b_t^{x,\lambda}| = \sum_{y \in S} b_t^{x,\lambda}(y)$$

Let O be a fixed site of S. We define the following critical parameters.

$$\lambda_1 = \inf\{\lambda : P(|b_t^{O,\lambda}| \geq 1, \forall t > 0) > 0\}$$

$$\lambda_2 = \inf\{\lambda : P(\limsup_{t \to \infty} b_t^{O,\lambda}(O) \geq 1) > 0\}.$$

In words, λ_1 is the critical value corresponding to the global survival of the branching Markov chain while λ_2 is the critical value corresponding to the local survival of the branching Markov chain. It is clear that $\lambda_1 \leq \lambda_2$, we are interested in necessary and sufficient conditions to have two distinct phase transitions: $\lambda_1 < \lambda_2$.

When there will be no ambiguity about the value of λ that we are considering we will drop λ from our notation.

Let $X(t)$ denote the continuous time Markov chain corresponding to $p(x, y)$. More precisely, consider a particle which undergoes random motion only (no branching), waits a mean 1 exponential time and jumps, going from x to y with probability $p(x,y)$. We denote by $X(t)$ the position in S at time t of this particle. We define

$$P_t(x,y) = P(X(t) = y | X(0) = x)$$

We have

$$P_{t+s}(O,O) = \sum_{y \in S} P(X(t+s) = O; X(t) = y | X(0) = O)$$

and since

$$P(X(t+s) = O; X(t) = y | X(0) = O) =$$

$$P(X(t+s) = O | X(t) = y; X(0) = O) P(X(t) = y | X(0) = O)$$

and using that $X(t)$ has the Markov property we get that the last term is $P_s(y,O)P_t(O,y)$. Therefore

$$P_{t+s}(O,O) = \sum_{y \in S} P_t(O,y)P_s(y,O) \geq P_t(O,O)P_s(O,O)$$

Lemma 1.1 *If a continuous function f has the property that*

$$f(t+s) \geq f(t) + f(s)$$

then

$$\lim_{t \to \infty} \frac{1}{t} f(t) = \sup_{t>0} \frac{1}{t} f(t) \in (-\infty, \infty]$$

We will prove this lemma below. Lemma 1.1 applied to the function

$$f(t) = \log P_t(O,O)$$

gives the existence of the following limit,

$$\lim_{t \to \infty} \frac{1}{t} \log P_t(O,O) = -\gamma = \sup_{t>0} \frac{1}{t} \log P_t(O,O). \tag{1.1}$$

Moreover it is clear that $\gamma \geq 0$ and since $P_t(O,O) \geq e^{-t}$ (if there are no jumps up to time t) we get that γ is in [0,1].

We are now ready to state the main result of this section.

Theorem 1.1. *The first critical value for a branching Markov chain is*

$$\lambda_1 = 1$$

The second critical value is

$$\lambda_2 = \frac{1}{1-\gamma} \text{ for } \gamma \text{ in } [0,1)$$

$$\lambda_2 = \infty \text{ for } \gamma = 1.$$

In particular there are two phase transitions for this model if and only if $\gamma \neq 0$.

Problem 1.1 Show that $\gamma = 1$ if and only if for all $k \geq 1$ we have $p_k(O,O) = 0$. Use the fact that

$$P_t(O,O) = \sum_{k \geq 0} e^{-t} \frac{t^k}{k!} p_k(O,O)$$

We start with

Proof of Lemma 1.1

Fix $s > 0$ then for $t > s$ we can find an integer $k(t,s)$ such that

$$0 \leq t - k(t,s)s < s$$

and iterating the hypothesis of Lemma 1.1 yields

$$f(t) \geq k(t,s)f(s) + f(t - k(t,s)s)$$

Let $m(s) = \inf_{r<s} f(r)$ and we get

$$\frac{1}{t} f(t) \geq \frac{1}{t} k(t,s) f(s) + \frac{1}{t} m(s)$$

We now let t go to infinity and use that $k(t,s)/t$ converges to $1/s$ to get

$$\liminf_{t\to\infty} \frac{1}{t} f(t) \geq \frac{1}{s} f(s)$$

Hence

$$\liminf_{t\to\infty} \frac{1}{t} f(t) \geq \sup_{s>0} \frac{1}{s} f(s)$$

On the other hand we have

$$\limsup_{t\to\infty} \frac{1}{t} f(t) \leq \sup_{s>0} \frac{1}{s} f(s)$$

and this completes the proof of Lemma 1.1.

To prove Theorem 1.1 it will be useful to have a differential equation whose solution is $P_t(x,y)$. Conditioning on what happens in the time interval $[0,h]$ we have that

$$P_{t+h}(x,O) = \sum_{y\in S} hp(x,y)P_t(y,O) + (1-h)P_t(x,O)$$

where we are are neglecting terms of order higher than h. As $h \to 0$ we get

$$P_t'(x,O) = \sum_y p(x,y)P_t(y,O) - P_t(x,O) \tag{1.2}$$

The differential equation (1.2) is called the Kolmogorov's backward equation corresponding to the process $X(t)$. It has a unique solution with the initial conditions $P_0(x,O) = 0$ for $x \neq O$ and $P_0(O,O) = 1$ (see Bhattacharya and Waymire (1990), chapter IV, section 3). We will use this to prove

Proposition 1.1. *For all x in S and times t we have the representation formula*

$$E(b_t^x(O)) = e^{(\lambda-1)t} P_{\lambda t}(x,O)$$

Proof of Proposition 1.1.

Define $m(t,x) = E(b_t^x(O))$. Conditioning on what happens in the interval $[0,h]$ and using that b_t^O is a Markov process gives

$$(1.3)\ \ m(t+h,x) = \sum_{y \in S} \lambda h p(x,y)(m(t,x)+m(t,y)) + (1-(\lambda+1)h)m(t,x)$$

Again we are neglecting terms of order higher than h. As $h \to 0$ in (1.3) we get

$$(1.4) \qquad m'(t,x) = \sum_{y \in S} \lambda p(x,y) m(t,y) - m(t,x)$$

where the derivative is taken with respect to t. Using (1.2) it is easy to check that

$$t \to e^{(\lambda-1)t} P_{\lambda t}(x,O)$$

is a solution to (1.4) with the initial value $m(0,x) = 0$ for $x \neq O$ and $m(0,O) = 1$.

This completes the proof of Proposition 1.1.

Problem 1.2 Show that

$$t \to e^{(\lambda-1)t} P_{\lambda t}(x,O)$$

is the unique solution to (1.4) with the initial conditions $m(0,x) = 0$ for $x \neq O$ and $m(0,O) = 1$.

To prove Theorem 1.1 we also need the following

Lemma 1.2. *If there is a time T such that $E(b_T^O(O)) > 1$ then* $\limsup_{t\to\infty} P(b_t^O(O) \geq 1) > 0$.

Proof of Lemma 1.2.

We will construct a super-critical Bienaymé-Galton-Watson process which is dominated (in a certain sense) by the branching Markov chain.

To do so we will first consider the following Markov process $\tilde{b}_t$ whose evolution is coupled with the evolution of b_t^O in the following way. Up to time T $\tilde{b}_t$ and b_t^O are identical. At time T we suppress all the particles of $\tilde{b}_t$ which are not at site O and we keep the particles which are at O. Between times T and $2T$ the particles of $\tilde{b}_t$ which were at O at time T evolve like the particles of b_t^O which were at O at time T. At time $2T$ we suppress again all the particles of $\tilde{b}_t$ which are not at O. And so on, at times kT $(k \geq 1)$ we suppress all the particles of $\tilde{b}_t$ which are not at O and between kT and $(k+1)T$ the particles of $\tilde{b}_t$ evolve like the corresponding particles of b_t^O.

Now we can define the following discrete time process Z_k. Let $Z_0 = 1$ and $Z_k = \tilde{b}_{kT}(O)$. We may write

$$Z_k = \sum_{i=1}^{Z_{k-1}} Y_i \text{ for } k \geq 1$$

where Y_i is the number of particles located at O that a single particle initially located at O generates in T units time. In other words each Y_i has the same law as $b_T^O(O)$. Moreover the Y_i are independent one of the other and of the ones appearing in previous generations, therefore Z_n is a Bienaymé Galton Watson process. We also have that $E(Z_1) > 1$, so Z_k is a supercritical BGW process. On the other hand by our construction we have coupled the processes b_t^O and $\tilde{b}_t$ in such a way that $\tilde{b}_t(x) \leq b_t^O(x)$ for all x in S and all $t \geq 0$. And so

$$P(Z_k \geq 1) \leq P(b_{kT}^O(O) \geq 1)$$

But $P(Z_k \geq 1, \forall k \geq 0) > 0$ so making k go to infinity in the last inequality concludes the proof of Lemma 1.2.

We are now ready for the

Proof of Theorem 1.1

We first compute the first critical value λ_1. By Proposition 1.1 we have

$$E(|b_t^O|) = \sum_{x \in S} E(b_t^O(x)) = \sum_{x \in S} P_{\lambda t}(O,x)e^{(\lambda-1)t} = e^{(\lambda-1)t}$$

Fix any time $T > 0$ then $E(|b_T^O|) = e^{(\lambda-1)T}$, define $Z_k = |b_{kT}^O|$ then Z_k is a BGW process. Observe that Z_k is supercritical if and only if $\lambda > 1$. We also have that

$$P(Z_k \geq 1, \text{ for all } k \geq 0) = P(|b_{kT}^O| \geq 1, \text{ for all } k \geq 0) = \rho$$

Since 0 is an absorbing state we have for $s < t$ that $\{|b_t^O| \geq 1\} \subset \{|b_s^O| \geq 1\}$ and so the function $t \to P(|b_t^O| \geq 1)$ is decreasing and therefore it has a limit as t goes to infinity. This limit must be equal to ρ. But $\rho > 0$ if and only if $\lambda > 1$. This proves that $\lambda_1 = 1$.

We now turn to the computation of λ_2. Using (1.1) and Proposition 1.1 we get that for all $k \geq 0$

$$E(b_k^O(O)) \leq e^{Ck}$$

for $C = \lambda - 1 - \lambda\gamma$. Consider first the case $\gamma < 1$, observe that if $\lambda < \frac{1}{1-\gamma}$ then $C < 0$. Let

$$A_k = \{b_k^O(O) \geq 1\}$$

we have

$$P(A_k) \leq E(b_k^O(O)) \leq e^{Ck}$$

By Borel-Cantelli Lemma,

$$P(\limsup_k A_k) = 0 \tag{1.5}$$

In other words $P(b_k^O(O) \geq 1 \text{ for infinitely many } k) = 0$. So the process is dying out locally along integer times. We now take care of the non-integer times. If we had particles at O for arbitrarily large continuous times it would mean that the following event happens for infinitely many integers k: all particles at O disappear between times k and $k+1$. But for distinct k's these are independent events which are bounded below by a positive probability uniform in k. So the probability that this event happens for infinitely many k's is zero and (1.5) implies that

$$P(\limsup_{t\to\infty} b_t^O(O) \geq 1) = 0$$

This shows that $\lambda_2 \geq \frac{1}{1-\gamma}$ for $\gamma < 1$. Consider now $\gamma = 1$,

$$P(A_k) \leq e^{Ck}$$

where $C = -1$, so again by Borel Cantelli $\lambda < \lambda_2$ for any λ. This shows that $\lambda_2 = \infty$.

For $\gamma < 1$ we will now prove the opposite inequality for λ_2. Suppose that $\lambda > \frac{1}{1-\gamma}$. For $\epsilon > 0$ small enough we have that $\lambda > \frac{1}{1-\gamma-\epsilon}$. By (1.1) there is T large enough so that

$$\frac{1}{T}\log P_T(O,O) > -\gamma - \epsilon$$

and therefore by proposition 1

$$E(b_T^O(O)) \geq e^{DT}$$

with $D = \lambda(-\gamma-\epsilon+1)-1 > 0$. Since $E(b_T^O(O)) > 1$ we can apply Lemma 1.2 and we get

$$\limsup_{t\to\infty} P(b_t^O(O) \geq 1) > 0 \qquad (1.6)$$

We also have that

$$P(b_t^O(O) \geq 1) \leq P(\exists s \geq t : b_s^O(O) \geq 1)$$

we make $t \to \infty$ and get

$$\limsup_{t\to\infty} P(b_t^O(O) \geq 1) \leq P(\limsup_{t\to\infty} b_t^O(O) \geq 1)$$

this together with (1.6) show that

$$P(\limsup_{t\to\infty} b_t^O(O) \geq 1) > 0$$

hence $\lambda_2 \leq \frac{1}{1-\gamma}$. This finishes the proof that $\lambda_2 = \frac{1}{1-\gamma}$ and the proof of Theorem 1.1.

2. Two examples.

We first consider the simple random walk. In this example, $S = Z$ and

$$p(x, x+1) = p \qquad p(x, x-1) = q = 1 - p.$$

Consider the asymmetric branching random walk on Z. A particle at x gives birth to a particle at $x+1$ at rate λp. A particle at x gives birth to a particle at $x-1$ at rate λq. A particle dies at rate 1.

For this example γ is easily computed. Let $p_{2n}(O, O)$ be the probability that the discrete time random walk is in O at time $2n$. An elementary computation shows that

$$\lim_{n\to\infty} \frac{p_{2n}(O,O)}{(4pq)^n(\pi n)^{-1/2}} = 1 \tag{2.1}$$

It is then easy to get the

Lemma 2.1.

$$\lim_{t\to\infty} \frac{1}{t} \log P_t(O, O) = 2\sqrt{pq} - 1 = -\gamma.$$

Proof of Lemma 2.1

Observe that the continuous time simple random walk on Z $P_t(x,y)$ waits a mean 1 exponential time between two jumps. Therefore the number of jumps up to time t is a Poisson process with parameter t. This implies that

$$P_t(x,y) = \sum_{n\geq 0} e^{-t}\frac{t^n}{n!}p_n(x,y)$$

where $p_n(x,y)$ is the probability that the discrete time random walk goes from x to y in n steps. Observe that $p_n(O,O) = 0$ if and only if n is odd, therefore

$$P_t(O,O) = \sum_{n\geq 0} e^{-t}\frac{t^{2n}}{(2n)!}p_{2n}(O,O)$$

Using (2.1) we see that there are strictly positive constants C_1 and C_2 such that for all $n \geq 1$

$$C_1(4pq)^n(\pi n)^{-1/2} \leq p_{2n}(O,O) \leq C_2(4pq)^n(\pi n)^{-1/2} \tag{2.2}$$

Therefore

$$P_t(O,O) \leq 1 + \sum_{n\geq 1} e^{-t}\frac{t^{2n}}{(2n)!}C_2(4pq)^n(\pi n)^{-1/2}$$

and we get that

$$P_t(O,O) \leq \sum_{n\geq 0} e^{-t}\frac{t^n}{n!}C_2\sqrt{4pq}^n = C_2 e^{t(-1+2\sqrt{pq})}$$

this last inequality implies

$$\limsup_{t\to\infty}\frac{1}{t}\log P_t(O,O) \leq 2\sqrt{pq} - 1$$

We now turn to the other inequality.

$$P_t(O,O) \geq 1 + \sum_{n\geq 1} e^{-t}\frac{t^{2n}}{(2n)!}C_1(4pq)^n(\pi n)^{-1/2}$$

Set $C_3 = C_1/\pi$ and observe that $\sqrt{n} \leq 2n+1$ to get

$$P_t(O,O) \geq \frac{1}{t}\sum_{n\geq 0} e^{-t}\frac{t^{2n+1}}{(2n+1)!}C_3\sqrt{4pq}^{2n+1}$$

A simple computation shows that there is $C_4 > 0$ such that for any $t \geq 0$

$$\sum_{n\geq 0}\frac{t^{2n+1}}{(2n+1)!} \geq C_4 e^t$$

We get

$$P_t(O,O) \geq \frac{1}{t}C_3C_4e^{t(-1+2\sqrt{pq})}$$

and so

$$\liminf_{t\to\infty}\frac{1}{t}\log P_t(O,O) \geq 2\sqrt{pq}-1$$

this completes the proof of Lemma 2.1.

Problem 2.1 Show that there is $C > 0$ such that for all $t > 0$ we have

$$\sum_{n\geq 0}\frac{t^{2n+1}}{(2n+1)!} \geq Ce^t$$

Now that γ has been computed we may apply Theorem 1.1 to get

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{1}{2\sqrt{pq}}$$

Observe that $pq \leq 1/4$ and that the equality holds if and only if $p = q = 1/2$. Therefore the simple branching random walk on Z has two phase transitions if and only if $p \neq q$. In other words, any asymmetry in this model provokes the appearance of two phase transitions.

The other example we will consider is the branching random walk on a homogeneous tree. Here S is a homogeneous tree (also called Bethe lattice) which is an infinite connected graph without cycles, in which every vertex has the same number of neighbors that we denote by $d \geq 3$. We assume here that $p(x, y) = 1/d$ for each of the d neighbors y of x. So in this sense this is a symmetric model but we will see that the behavior is very similar to the one of the asymmetric branching random walk on Z.

Sawyer (1978) has computed asymptotic estimates of $p_n(O, O)$ for a large class of random walks on a homogeneous tree. In our particular case (nearest neighbor symmetric random walk) his computation gives

$$\lim_{n\to\infty} \frac{p_{2n}(O,O)}{n^{-3/2}R^n} = C$$

where $C > 0$ is a constant and $R = \frac{2\sqrt{d-1}}{d}$. Doing exactly the same type of computation than in Lemma 2.1 gives

$$\lim_{t\to\infty} \frac{1}{t} \log P_t(O,O) = R - 1 = \frac{2\sqrt{d-1}}{d} - 1 = -\gamma$$

and therefore by Theorem 1.1

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{d}{2\sqrt{d-1}}$$

It is easy to check that for any $d \geq 3$, γ is strictly positive and therefore there are two distinct phase transitions for the simple symmetric branching random walk on a tree.

3. The first phase transition is continuous

Define $\rho(\lambda) = P(|b_t^{O,\lambda}| \geq 1, \forall t > 0)$. Recall that λ_1 has been defined as

$$\lambda_1 = \inf\{\lambda > 0 : \rho(\lambda) > 0\}$$

Theorem 3.1 *The first phase transition is continuous in the sense that the function*

$$\lambda \to \rho(\lambda)$$

is continuous at λ_1.

Proof of Theorem 3.1

By definition of λ_1 we have that if $\lambda < \lambda_1$ then $\rho(\lambda) = 0$. So the limit from the left at λ_1 is

$$\lim_{\lambda \to \lambda_1^-} \rho(\lambda) = 0$$

We also know what the value of ρ at λ_1 is. For positive integers k define $Z_k = |b_k^O|$. Then Z_k is a BGW and

$$P(Z_k \geq 1, \text{ for all } k \geq 0) = P(|b_k^O| \geq 1, \text{ for all } k \geq 0) = \rho(\lambda)$$

where the last equality comes from the fact that $\lim_{t\to\infty} P(|b_t^O| \geq 1)$ exists and is equal to $\rho(\lambda)$. By proposition 1.1 we have that $E(Z_1) = e^{(\lambda-1)}$. So at $\lambda_1 = 1$ the BGW process Z_k is critical and therefore $\rho(\lambda_1) = 0$. This together with the previous computation of the left limit shows that ρ is left continuous at λ_1.

We now turn to the right continuity. We may construct simultaneously two branching Markov chains with birth rates respectively equal to $\lambda_1 p(x,y)$ and $\lambda_2 p(x,y)$ where $\lambda_1 < \lambda_2$. Denote the two processes by b_t^{O,λ_1} and b_t^{O,λ_2}. To do our simultaneous construction we construct b_t^{O,λ_2} in the usual way. That is, for all sites x, y in S each particle at x waits an exponential time with rate $\lambda_2 p(x,y)$ and gives birth to a new particle located at y. After each birth we consider a Bernoulli random variable independent of everything else which has a success probability equal to λ_1/λ_2. If we have a success then a particle is also created at y for the process b_t^{O,λ_1} provided that the particle at x which gives birth in the process b_t^{O,λ_2} also exists in the process b_t^{O,λ_1}. Observe that this construction shows that the process with higher birth rates has more particles on each site of S at any time. In particular this implies that $\lambda \to \rho(\lambda)$ is an increasing function.

Consider now for a fixed time t the following function

$$f_t(\lambda) = P(|b_t^{O,\lambda}| \geq 1)$$

We will show that $\lambda \to f_t(\lambda)$ is a continuous function. As explained above we may construct simultaneously the branching Markov chains with parameters λ and $\lambda + h$ where $h > 0$. We get

$$(3.1) \qquad 0 \le f_t(\lambda + h) - f_t(\lambda) = P(|b_t^{O,\lambda+h}| \ge 1; |b_t^{O,\lambda}| = 0)$$

Consider $N(t)$ the total number of particles born up to time t for the process $b_t^{O,\lambda+h}$. From (3.1) we get for any positive integer n

(3.2)

$$0 \le f_t(\lambda+h) - f_t(\lambda) \le P(N(t) \le n; |b_t^{O,\lambda+h}| \ge 1; |b_t^{O,\lambda}| = 0) + P(N(t) > n)$$

In order to have $|b_t^{O,\lambda+h}| > |b_t^{O,\lambda}|$ at least one of the Bernoulli random variables involved in the simultaneous construction must have failed, therefore from (3.2) we get

$$0 \le f_t(\lambda + h) - f_t(\lambda) \le 1 - (\frac{\lambda}{\lambda + h})^n + P(N(t) > n)$$

We now make $h \to 0$ to get

$$(3.3) \qquad 0 \le \limsup_{h \to 0} f_t(\lambda + h) - f_t(\lambda) \le P(N(t) > n)$$

Since t is fixed, the number of births up to time t is finite and

$$\lim_{n \to \infty} P(N(t) > n) = 0$$

Using this observation in (3.3) shows that $\lambda \to f_t(\lambda)$ is right continuous. The proof of left continuity is similar and this proves that f_t is continuous.

Observe that

$$(3.4) \qquad \rho(\lambda) = \inf_{t>0} f_t(\lambda)$$

A function g is said to be upper semicontinuous at λ if a constant c is such that $c > g(\lambda)$ then there is a $\delta > 0$ such that if $|h| < \delta$ then $c \ge g(\lambda+h)$.

We now prove that ρ is upper semicontinuous. Assume $c > \rho(\lambda)$ then by (3.4) there is a t_0 such that $c > f_{t_0}(\lambda) \geq \rho(\lambda)$. But f_{t_0} is continuous so for any $\epsilon > 0$ there is $\delta > 0$ such that if $|h| < \delta$ then

$$f_{t_0}(\lambda + h) \leq f_{t_0}(\lambda) + \epsilon$$

Pick $\epsilon = (c - f_{t_0}(\lambda))/2$ to get

$$f_{t_0}(\lambda + h) \leq c$$

By (3.4) again this implies that

$$\rho(\lambda + h) \leq c$$

This concludes the proof that ρ is upper semicontinuous. The final step is to prove that since ρ is semicontinuous and increasing it is right continuous. Fix $\lambda > 0$, let $c = \rho(\lambda) + \epsilon$, by semicontinuity there is $\delta > 0$ such that if $|h| < \delta$ then

$$\rho(\lambda + h) \leq c = \rho(\lambda) + \epsilon$$

But if h is also strictly positive we use that ρ is an increasing function to get

$$0 \leq \rho(\lambda + h) - \rho(\lambda) \leq \epsilon$$

and this shows that ρ is right continuous and therefore completes the proof of Theorem 3.1.

Problem 3.1 A function g is said to be lower semicontinuous if $-g$ is upper semicontinuous. Prove that a function g is continuous at t if and only if it is upper and lower semicontinuous at t.

Remark. Observe that the proof that ρ is right continuous is fairly general and can be applied to a wide class of processes. That the limit from the left is zero at λ_1 is always true. So the main problem in showing that the phase transition is continuous is to prove that $\rho(\lambda_1) = 0$. This is in general a difficult problem, here we take advantage of the branching structure of the process and there is no difficulty.

4. The second phase transition is discontinuous

We say that the second phase transition is discontinuous in the following sense.

Theorem 4.1. *Assume that $S = Z^d$ or a homogeneous tree and assume that $p(x,y)$ is translation invariant. If the branching Markov chain has two distinct phase transitions i.e. $\lambda_1 < \lambda_2$ then the function σ defined by*

$$\sigma(\lambda) = P(\limsup_{t\to\infty} b_t^{O,\lambda}(O) \geq 1)$$

is not continuous at λ_2.

Proof of Theorem 4.1

We will prove that if $\lambda > \lambda_2$ then

$$\sigma(\lambda) = P(\limsup_{t\to\infty} b_t^{O,\lambda}(O) \geq 1) = P(|b_t^{O,\lambda}| \geq 1, \text{ for all } t > 0) = \rho(\lambda) \tag{4.1}$$

In words, above λ_2 the process must survive locally if it survives globally. We now show that Theorem 4.1 follows directly from (4.1). Make λ approach λ_2 from the right in (4.1). Since ρ is right continuous we have that $\rho(\lambda)$ approaches $\rho(\lambda_2)$. But this last quantity is strictly positive if $\lambda_1 < \lambda_2$. This shows that σ has a limit from the right at λ_2 which is strictly positive. But the limit from the left at λ_2 is zero. So there is a discontinuity at λ_2.

We now turn to the proof of (4.1). For x, y in S we define B_x^y the event that the site x is visited infinitely often by the offspring of a single particle started at the site y. More precisely

$$B_x^y = \{\limsup_{t\to\infty} b_t^y(x) \geq 1\}$$

Define

$$C^y = \cap_{x \in S} B_x^y$$

Under the translation invariance assumptions for S and $p(x, y)$ we get $P(B_y^y) = P(B_O^O) = \sigma(\lambda) > 0$ if $\lambda > \lambda_2$. Observe that

$$|P(B_y^y) - P(B_x^y)| \leq P(B_y^y \cap (B_x^y)^c) + P(B_x^y \cap (B_y^y)^c)$$

In order to have the event $B_y^y \cap (B_x^y)^c$ it is necessary that y gets occupied at arbitrarily large times while after a finite time x is empty. But y is occupied at arbitrarily large times only if there are infinitely many distinct particles that occupy y after any finite time. Each of these particles has the same positive probability of occupying x and since distinct particles are independent one of the other x will get occupied with probability one at arbitrarily large times. Therefore $P(B_y^y \cap (B_x^y)^c) = 0$ and

$$P(B_y^y) = P(B_x^y) = \sigma(\lambda)$$

We now consider

$$P(B_x^y) - P(C^y) = P(B_x^y \cap (C^y)^c) \leq \sum_{z \in S} P(B_x^y \cap (B_z^y)^c)$$

For the same reason as above each term in this last sum is zero so

$$P(C^y) = P(B_x^y) = \sigma(\lambda) \tag{4.2}$$

for any $x, y \in S$. We have for any integer time k and integer n that

$$P(C^O) \geq P(C^O | \{|b_k^O| \geq n\}) P(|b_k^O| \geq n) \tag{4.3}$$

By the Markov property we have that

$$P(C^O | \{|b_k^O| \geq n\}) \geq \inf_{x_1, \ldots, x_n} P(\cup_{i=1}^n C^{x_i})$$

But since offspring generated by different particles are independent we have

$$P(C^O | \{|b_k^O| \geq n\}) \geq 1 - (1 - P(C^O))^n$$

Using this lower bound in (4.3) gives

$$P(C^O) \geq (1 - (1 - P(C^O))^n) P(|b_k^O| \geq n) \tag{4.4}$$

Now we consider

$$P(|b_k^O| \geq 1) - P(|b_k^O| \geq n) = P(1 \leq |b_k^O| < n)$$

Recall that $Z_k = |b_k^O|$ is a BGW and we know that a BGW either gets extinct or goes to ∞ as time goes to infinity. Let

$$A_k = \{1 \leq Z_k < n\}$$

We know that

$$P(\limsup_k A_k) \geq \limsup_k P(A_k)$$

Since the event $\limsup_k A_k$ excludes the possibility that Z_k goes to zero or to infinity as k goes to infinity, this event must have probability zero. So

$$\lim_{k \to \infty} P(A_k) = 0$$

We use this observation and make k go to infinity in (4.4)

$$P(C^O) \geq (1 - (1 - P(C^O))^n) \rho(\lambda)$$

By (4.2) $P(C^O) = \sigma(\lambda) > 0$ for $\lambda > \lambda_2$, using this and making n go to infinity yields

$$\sigma(\lambda) \geq \rho(\lambda) \text{ for } \lambda > \lambda_2$$

Since the reverse inequality is always true this concludes the proof of (4.1) and of Theorem 4.1.

5. Notes and references

We have given an informal description of a branching Markov chain. For a formal construction of continuous time Markov chains, see for instance Bhattacharya and Waymire (1990) or Harris (1963).

Theorem 1.1 was first proved in the particular case of trees by Madras and Schinazi (1992). The general case was proved by Schinazi (1993). Theorem 4.1 is due to Madras and Schinazi (1992).

Chapter IV

THE CONTACT PROCESS ON A HOMOGENEOUS TREE

1. The critical values

Let S be a homogeneous tree in which d branches emanate from each vertex of S. Thus S is an infinite connected graph without cycles in which each site has d neighbors for some integer $d \geq 3$.

We consider the contact process on S whose state at time t is denoted by η_t and which evolves according to the following rules.

(i) If there is a particle at site $x \in S$ then for each of the d neighbors y of x it waits a mean $\frac{1}{\lambda}$ exponential time and then gives birth to a particle on y.

(ii) A particle waits a mean 1 exponential time and then dies.

(iii) There is at most one particle per site: births on occupied sites are suppressed.

Observe that (i) and (ii) are particular cases of (i) and (ii) in chapter III. Here λd plays the role of λ in chapter III and $p(x,y) = 1/d$ if x and y are nearest neighbors on the tree. So the contact process follows the same rules as a branching Markov chain with the additional restriction (iii) that there is at most one particle per site for the contact process. This additional rule breaks the independence property between offspring of distinct particles that holds for branching Markov chains. Without independence we will not be able to make exact computations for the contact process. Instead, we will have to proceed by comparisons to simpler processes.

Let O be a distinguished vertex of the tree that we call the root. Let η_t^x be the contact process with only one particle at time 0 located at site $x \in S$. Let $\eta_t^x(y)$ be the number of particles at site y and let $|\eta_t^x| = \sum_{y \in S} \eta_t^x(y)$ be the total number of particles. We define the following critical values

$$\lambda_1 = \inf\{\lambda : P(|\eta_t^{O,\lambda}| \geq 1, \forall t > 0) > 0\}$$

$$\lambda_2 = \inf\{\lambda : P(\limsup_{t\to\infty} \eta_t^{O,\lambda}(O) = 1) > 0\}.$$

We include λ in the notation only when there may be an ambiguity about which value we are considering.

Theorem 1.1

$$\frac{1}{d} \le \lambda_1 \le \frac{1}{d-2}.$$

and

$$\lambda_2 \ge \frac{1}{2\sqrt{d-1}}$$

In particular we have two phase transitions for the contact process ($\lambda_1 < \lambda_2$) on trees if $d \ge 7$.

Proof of Theorem 1.1

To get lower bounds for λ_1 and λ_2 we will consider a branching Markov chain that has more particles than the contact process. Define the branching Markov chain b_t^O where a particle at x gives birth to a particle at y with rate $\lambda d p(x,y)$, where $p(x,y) = 1/d$ if y is one of the d neighbors of x. A particle dies at rate 1. Since there is no restriction on the number of particles per site for b_t^O we may construct η_t^O and b_t^O simultaneously in such a way that $\eta_t^O(x) \le b_t^O(x)$ for each x in S. We denote the two critical values of b_t by $\lambda_1(b)$ and $\lambda_2(b)$. Since b_t has more particles than η_t we have

$$\lambda_1 \ge \lambda_1(b) \text{ and } \lambda_2 \ge \lambda_2(b)$$

We have computed the critical values for this branching Markov chain in III.2, observe that the parametrisation is slightly different here and $d\lambda$ plays the role here that λ plays in III.2. So we get

$$d\lambda_1(b) = 1 \text{ and } d\lambda_2(b) = \frac{d}{2\sqrt{d-1}}$$

This gives the lower bounds for λ_1 and λ_2 in Theorem 1.1.

To get an upper bound for λ_1, consider a process $\tilde{\eta}_t$ with the following rules. Start the process with a single particle at the root, pick $d-1$ sites among the d nearest neighbors. The particle at the root gives birth to a new particle at rate λ on each of the $d-1$ sites previously picked. Each new particle can give birth on all neighboring sites but the parent site. Once a site has been occupied by a particle and this particle dies, the site remains empty forever. The death rate for each particle is 1 and there is at most one particle per site.

Define the distance between sites x and y in the homogeneous tree to be the length of the shortest path between x and y. Define $Z_0 = 1$ and Z_k to be the number of sites at distance k of O that will ever be occupied by a particle of $\tilde{\eta}_t$. Observe that each particle in $\tilde{\eta}_t$ gives birth (before dying) with probability $\frac{\lambda}{\lambda+1}$ on each of the $d-1$ sites it is allowed to give birth on. So the the size of the offspring of each particle has a binomial distribution with parameters $d-1$ and $\frac{\lambda}{\lambda+1}$. Since a tree has no cycles two distinct particles of $\tilde{\eta}_t$ have independent offspring. Hence Z_k is a BGW and it is supercritical if and only if

$$(d-1)\frac{\lambda}{\lambda+1} > 1$$

and the last inequality is equivalent to $\lambda > \frac{1}{d-2}$. On the other hand it is clear that

$$\{Z_k \geq 1, \text{ for all } k \geq 0\} = \{\tilde{\eta}_t \geq 1, \text{ for all } t \geq 0\}$$

So the first critical value of $\tilde{\eta}_t$ is $\frac{1}{d-2}$. Since the birth rules for $\tilde{\eta}_t$ are more restrictive than the one for η_t, we may construct η_t and $\tilde{\eta}_t$ simultaneously in such a way that $\tilde{\eta}_t(x) \leq \eta_t(x)$ for each x in S. This implies that $\frac{1}{d-2}$ is an upper bound for λ_1 and this concludes the proof of Theorem 1.1.

Problem 1.1. Show that $\lambda_1 < \lambda_2$ if $d \geq 7$.

Problem 1.2. Show that the second critical value of the process $\tilde{\eta}_t$ (in the proof of Theorem 1.1) is infinite.

2. Characterization of the first phase transition

While it is possible to improve the bounds in Theorem 1.1 (see the notes and references at the end of the chapter), the exact computation of the critical values seems hopeless. In order to analyse the phase transitions we need to characterize the critical values in a way that is amenable to analysis. We will achieve a very useful characterization of the first phase transition. Some of the proofs in this section are more technical than in the previous sections, we will give these proofs at the end of the section.

We start with

Theorem 2.1 *For the contact process on a homogeneous tree with degree $d \geq 3$, there exist constants $c(\lambda, d)$ and $C(d)$ such that*

$$e^{c(\lambda,d)t} \leq E(|\eta_t^O|) \leq C(d)e^{c(\lambda,d)t}.$$

Moreover $c(\lambda, d)$ is a continuous function of λ.

Theorem 2.1 tells us that the expected value of the number of particles is "almost" an exponential function of time. Observe that for a branching process this expected value is exactly an exponential.

It is easy to prove that

Theorem 2.2 *If $\lambda > \lambda_1$ we have that $c(\lambda, d) > 0$.*

The following converse of Theorem 2.2 is much harder to prove.

Theorem 2.3 *If $c(\lambda, d) > 0$ then $\lambda \geq \lambda_1$.*

We now can state the characterization of the first phase transition.

Corollary 2.1 *We have that*

$$\lambda_1 = \sup\{\lambda : c(\lambda, d) \leq 0\}$$

Moreover $c(\lambda_1, d) = 0$.

So λ_1 is the largest possible value for which we have $c(\lambda, d) = 0$.

Proof of Corollary 2.1

From Theorem 2.2 and Theorem 2.3 we get the following

$$\lim_{\lambda \to \lambda_1^-} c(\lambda, d) \leq 0 \text{ and } \lim_{\lambda \to \lambda_1^+} c(\lambda, d) \geq 0$$

Now using that $\lambda \to c(\lambda, d)$ is a continuous function we get $c(\lambda_1, d) = 0$. From Theorem 2.2 we know that λ_1 is an upper bound of the set $\{\lambda : c(\lambda, d) \leq 0\}$. We just saw that λ_1 is also in this set therefore

$$\lambda_1 = \sup\{\lambda : c(\lambda, d) \leq 0\}$$

and this completes the proof of Corollary 2.1.

An immediate consequence of Corollary 2.1 and Theorem 2.1 is

Corollary 2.2 *We have that at* $\lambda = \lambda_1$

$$1 \leq E(|\eta_t^{O,\lambda_1}|) \leq C(d)$$

where $C(d)$ *is a constant depending on* d *only.*

So the expected value of the number of particles of the critical contact process remains bounded at all times. This is similar to the critical branching process for which this expected value is a constant equal to one.

Corollary 2.3 *The survival probability for the contact process on a homogeneous tree with* $d \geq 3$

$$\lambda \to P(|\eta_t^{O,\lambda}| \geq 1, \forall t > 0)$$

is continuous at λ_1, i.e., the first phase transition is continuous.

As for the second phase transition, the same type of argument that we used for Branching Markov chains works here too. The proof is complicated by the lack of independence in the contact process. See Madras and Schinazi (1992) for a proof of the following

Theorem 2.4. *If $\lambda_1 < \lambda_2$ then the function*

$$\lambda \to P(\limsup_{t\to\infty} \eta_t^{O,\lambda}(O) = 1)$$

is not continuous at λ_2.

We will now prove the results of this section in the following order: first a lemma then Corollary 2.3, Theorem 2.1, Theorem 2.2 and finally Theorem 2.3. The following lemma will be useful in two proofs and also tells us that the contact process is not stable: it either gets extinct or it goes to infinity.

Lemma 2.1 *On the event*

$$\Omega_\infty = \{|\eta_t^O| \geq 1, \text{ for all } t \geq 0\}$$

we have that

$$\lim_{t\to\infty} |\eta_t^O| = \infty$$

And this implies that if $P(\Omega_\infty) > 0$ then

$$\lim_{t\to\infty} E(|\eta_t^O|) = \infty$$

Proof of Lemma 2.1

For a fixed $k \geq 1$, define a sequence of random times T_n by

$$T_1 = \inf\{t \geq 1 : |\eta_t^O| = k\}$$

and for $n \geq 2$

$$T_n = \inf\{t \geq T_{n-1} + 1 : |\eta_t^O| = k\}$$

If $T_n = \infty$ for some $n \geq 1$ then we take $T_k = \infty$ for $k > n$.

Observe that if there are k particles or less the contact process will get extinct in less than one unit time with probability at least $p(k)$ where

$$p(k) = (\int_0^1 e^{-s} ds)^k (1 - \int_0^1 \lambda e^{-\lambda s} ds)^{kd}$$

To see the preceding equality observe that the first term in the product is the probability that k particles die in one unit time while the second term in the product is the probability that none of the k particles attempt to give birth during the first unit time. Therefore, by the strong Markov property for $n \geq 2$

$$P(T_n < \infty | T_{n-1} < \infty) \leq 1 - p(k)$$

Since the sequence $\{T_n < \infty\}$ is decreasing

$$P(T_n < \infty) = P(T_n < \infty | T_{n-1} < \infty) P(T_{n-1} < \infty)$$

$$\leq P(T_{n-1} < \infty)(1 - p(k))$$

We iterate the preceding inequality n times to get

$$P(T_n < \infty) \leq (1 - p(k))^{n-1}(1 - p(1))$$

where the last term comes from the fact that we start with one particle. We now make n go to infinity

$$\lim_{n \to \infty} P(T_n < \infty) = P(\bigcap_{n \geq 1} \{T_n < \infty\}) = 0$$

Therefore after a finite random time there no return to $k \geq 1$. On the other hand, on Ω_∞ we know that there is no return to 0 so $|\eta_t^O|$ must go to infinity as t goes to infinity. And this completes the proof of the first part of Lemma 2.1.

For any $k \geq 1$ we have

$$E(|\eta_t^O|) \geq E(|\eta_t^O|\{|\eta_t^O| \geq k\})P(|\eta_t^O| \geq k) \geq kP(|\eta_t^O| \geq k)$$

But we know that

$$\liminf_{t\to\infty} P(|\eta_t^O| \geq k) \geq P(\liminf_{t\to\infty} |\eta_t^O| \geq k)$$

The event $\{\liminf_{t\to\infty} |\eta_t^O| \geq k\}$ happens if and only if there is a time after which the number of particles is always larger than k. But this may happen if and only if the process does not get extinct. Hence

$$\liminf_{t\to\infty} E(|\eta_t^O|) \geq k \liminf_{t\to\infty} P(|\eta_t^O| \geq k) \geq kP(\liminf_{t\to\infty} |\eta_t^O| \geq k) = kP(\Omega_\infty)$$

Assume that $P(\Omega_\infty) > 0$ then since k is arbitrarily large we get

$$\lim_{t\to\infty} E(|\eta_t^O|) = \infty \text{ if } P(\Omega_\infty) > 0$$

and this completes the proof of Lemma 2.1.

Proof of Corollary 2.3

Define

$$\rho(\lambda) = P(|\eta_t^{O,\lambda}| \geq 1, \text{ for all } t \geq 0)$$

By definition of λ_1, the left limit of ρ at λ_1 is zero. The same arguments we used to prove that ρ is right continuous for a branching Markov chain work here (see Theorem 3.1 in chapter III). So ρ is continuous at λ_1 if and only if $\rho(\lambda_1) = 0$.

By Lemma 2.1 we have that if $\rho(\lambda_1) > 0$ then $E(|\eta_t^{O,\lambda_1}|)$ is not a bounded function of t and this contradicts Corollary 2.2. Therefore $\rho(\lambda_1) = 0$ and this completes the proof of Corollary 2.3.

Proof of Theorem 2.1

Let A be a subset of S, we denote by η_t^A the contact process that started with one particle on every site of A. The contact process is said to be additive in the sense that if A and B are subsets of S and $C = A \cup B$ then we may construct the three processes η_t^A, η_t^B and η_t^C in such a way that for any x in S $\eta_t^C(x) = 1$ if and only if $\eta_t^A(x) = 1$ or $\eta_t^B(x) = 1$. In particular we get that

$$|\eta_t^C| \leq |\eta_t^A| + |\eta_t^B|$$

Define $m(t) = E(|\eta_t^O|)$. Using the Markov property, the additivity and translation invariance of the contact process gives

$$E(|\eta_{t+s}^O| \,|\, |\eta_t^O| = k) \leq kE(|\eta_s^O|)$$

and so

$$m(t+s) \leq m(t)m(s)$$

Subadditivity (see Lemma 1.1 in III.1) implies the existence of the following limit:

$$\lim_{t\to\infty} \frac{1}{t} \log m(t) = \inf_{t>0} \frac{1}{t} \log m(t) = c(\lambda, d) \tag{2.1}$$

In a way similar to what was done in section 3 of chapter III it is possible to show that for fixed t the function

$$\lambda \to E(|\eta_t^{O,\lambda}|)$$

is continuous and therefore the function

$$\lambda \to c(\lambda, d)$$

is upper semicontinuous.

We now turn to the upper bound in Theorem 2.1. For a given site x if we remove $d-1$ of the d edges incident on x we are left with d disjoint subtrees, only one of which contains x. This particular subtree is called a branch adjacent to x, each site has d such adjacent branches. For a finite subset A of S we say that a site x of A is in the boundary of A if at least one branch adjacent to x has an intersection with A exactly equal to $\{x\}$.

Lemma 2.2 *Let A be a finite subset of the tree S. There are at least $(1-\frac{1}{d-1})|A|$ sites of A that are on the boundary of A.*

What Lemma 2.2 says is that the number of the boundary points is of the same order as the total number of points for a subset of the tree. For a proof of Lemma 2.2 see Pemantle (1992) (Lemma 6.2).

We will now prove that

$$m(t+s) \geq (1-\frac{1}{d-1})m(t)B(s) \tag{2.2}$$

where $B(s)$ is the expected number of particles of η_s^O that are located on a given adjacent branch of O. To see (2.2), we keep at time t only the particles of η_t^O that are located on the boundary of the set of occupied sites. For each of these particles we consider at time $t+s$ only its offspring located on the branch adjacent to the particle and which contained only this particle at time t. Then (2.2) is a consequence of Lemma 2.2, the preceding remarks, the additivity and the Markov properties of the contact process. Again by additivity we have

$$B(s) \geq m(s)/d$$

This last inequality together with (2.2) gives

$$m(t+s) \geq (1-\frac{1}{d-1})\frac{1}{d}m(t)m(s)$$

Let $K(d) = (1-\frac{1}{d-1})\frac{1}{d}$, we have that

$$K(d)m(t+s) \geq K(d)m(t)K(d)m(s)$$

By superadditivity we get

$$\lim_{t\to\infty}\frac{1}{t}\log K(d)m(t)=\sup_{t>0}\frac{1}{t}\log K(d)m(t) \tag{2.3}$$

But $K(d)$ is just a constant therefore the limit in (2.3) must be equal to the limit in (2.1): $c(\lambda,d)$. From (2.3) we see that $c(\lambda,d)$ is a lower semicontinuous function of λ. This together with (2.1) proves that

$$\lambda\to c(\lambda,d) \text{ is continuous}$$

Putting together (2.1) and (2.3) gives

$$e^{c(\lambda,d)t}\le m(t)\le\frac{(d-1)d}{d-2}e^{c(\lambda,d)t}$$

and this completes the proof of Theorem 2.1.

Proof of Theorem 2.2

By Lemma 2.1 if $\rho(\lambda)>0$ then $E(|\eta_t^O|)$ goes to infinity with t. So by Theorem 2.1 we must have $c(\lambda,d)>0$. This completes the proof of Theorem 2.2.

Proof of Theorem 2.3

We begin by recalling Harris' graphical construction of the contact process (for more details see Durrett (1988)). We associate each site of S with $d+1$ independent Poisson processes, one with rate 1 and the d others with rate λ. Make these Poisson processes independent from site to site. For each x, let $\{T_n^{x,k}:n\ge1\}$, $k=0,1,2,..,d$ be the arrival times of these $d+1$ processes, respectively; the process $\{T_n^{x,0}:n\ge1\}$ has rate 1, the others rate λ. For each x and $n\ge1$ we write a δ mark at the point $(x,T_n^{x,0})$ while if $k\ge1$ we draw arrows from $(x,T_n^{x,k})$ to $(x_k,T_n^{x,k})$ where x_k, $k=1,\ldots,d$ are the neighbors of x. We say that there is a path from (x,s) to (y,t) if there is a sequence of times $s_0=s<s_1<s_2<..<s_n<s_{n+1}=t$ and

spatial locations $x_0 = x, x_1, .., x_n = y$ so that for $i = 1, 2, .., n$ there is an arrow from x_{i-1} to x_i at time s_i and the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ for $i = 0, 1, .., n$ do not contain any δ. We use the notation $\{(x, s) \longrightarrow (y, t)\}$ to denote the event that there is path from (x, s) to (y, t). To construct the contact process from the initial configuration A (i.e. there is one particle at each site of A) we let $\eta_t^A(y) = 1$ if there is a path from $(x, 0)$ to (y, t) for some x in A. We will also need to construct the contact process restricted to a branch $\mathcal{B}$ adjacent to O. We will use the same construction as above with the additional restriction that the paths must be inside $\mathcal{B}$. We denote by $\{(x, s) \xrightarrow{\mathcal{B}} (y, t)\}$ the event that there is a path from (x, s) to (y, t) inside $\mathcal{B}$. We denote by ξ_t^x the contact process restricted to $\mathcal{B}$, starting with one particle at the site $x \in \mathcal{B}$.

Using the graphical construction we construct η_t^O and ξ_t^O simultaneously but for the restricted contact process we only use the arrows that are located in $\mathcal{B}$.

We now prove Theorem 2.3. Fix a λ such that $c(\lambda, d) > 0$.

Let $x \neq O$ in $\mathcal{B}$, let O' be the neighboring site of O in $\mathcal{B}$ and let $\{T_n^{O,1} : n \geq 1\}$ be the arrival times corresponding to the arrows from O to O'. We have $\eta_t^O(x) = 1$ if and only if there is a path from $(O, 0)$ to (x, t) and there is a time smaller than t such that the spatial location of the path from this time on is in $\mathcal{B}$. In particular there must be an arrow from O to O' at a time $T_n^{O,1}$ and a path from $(O', T_n^{O,1})$ to (x, t) inside $\mathcal{B}$. Therefore

$$P(\eta_t^O(x) = 1) \leq \int_0^t P(\exists n \geq 1 : T_n^{O,1} \in ds \text{ and } (O', T_n^{O,1}) \xrightarrow{\mathcal{B}} (x, t)).$$

Using that $\{T_n^{O,1} : n \geq 1\}$ has rate λ and the Markov property we get that

$$P(\eta_t^O(x) = 1) \leq \int_0^t P((O', s) \xrightarrow{\mathcal{B}} (x, t)) \lambda ds \quad .$$

We sum over all $x \neq O$ in $\mathcal{B}$ to get

$$\sum_{x \in \mathcal{B}, x \neq O} P(\eta_t^O(x) = 1) \leq \lambda \int_0^t \sum_{x \in \mathcal{B}, x \neq O} P((O', s) \xrightarrow{\mathcal{B}} (x, t)) ds$$

So we have

$$\sum_{x\in\mathcal{B}} P(\eta_t^O(x)=1) \le \lambda \int_0^t \sum_{x\in\mathcal{B}} P((O',s)\xrightarrow{\mathcal{B}}(x,t))ds + 1 \tag{2.4}$$

But $P((O',s)\xrightarrow{\mathcal{B}}(x,t)) = P(\xi_t^{O',s}(x)=1)$ where $\xi_t^{O',s}$ is the contact process restricted to $\mathcal{B}$ starting at time s with one particle at O'. Observe also that by symmetry the l.h.s. of (2.4) is larger than $\frac{1}{d}E(|\eta_t^O|)$ and therefore we get from (2.4) that

$$\frac{1}{d}E(|\eta_t^O|) - 1 \le \int_0^t \lambda E(|\xi_s^{O'}|)ds.$$

We use the lower bound in Theorem 2.1 to get

$$\frac{1}{d}e^{c(\lambda,d)t} - 1 \le \lambda t \sup_{s\le t} E(|\xi_s^{O'}|).$$

Since $c(\lambda,d) > 0$, the last inequality proves that for any constant K we can find a time $T > 1$ such that

$$E(|\xi_{T-1}^{O'}|) \ge K. \tag{2.5}$$

Let $p > 0$ be the probability that at time $t=1$ ξ_t^O has exactly one particle and that this particle is located at O'. By the Markov property we get from (2.5) that

$$E(|\xi_T^O|) \ge pE(|\xi_{T-1}^{O'}|) \ge pK. \tag{2.6}$$

We are now going to show that (2.6) is enough to prove that the contact process restricted to $\mathcal{B}$ survives with positive probability. Recall that we say that a site $x \in A$ is in the boundary of A if at least one of the d branches emanating from x has no other site in A than x. Denote the boundary of a subset A of the tree by $\delta(A)$. Using the graphical construction we define a new process $\tilde{\xi}_t$ as follows. $\tilde{\xi}_t$ evolves like ξ_t^O up to time T. At time T we suppress all the particles of $\tilde{\xi}_T$ which are not in its boundary, and we restrict the spatial evolution of the remaining particles in the following way. Each particle in the boundary at time T generates a process for which births are allowed only on the branch, adjacent to the initial particle, and which was empty at time T except for the initial particle. Moreover we only keep boundary particles whose adjacent empty branch is contained in $\mathcal{B}$; this last condition eliminates at most one particle. We create like this at time T, $|\delta(\xi_T^O)| - 1$ restricted contact processes which are independent one of the other. We repeat the preceding step at all times kT: we only keep the particles in the boundary of $\tilde{\xi}_{kT}$ and we only allow births in the corresponding empty branch for each process generated by the particle at the border of $\tilde{\xi}_{kT}$. Between times kT and $(k+1)T$ the process evolves using the graphical construction. If we define the discrete time process

$$Z_k = |\tilde{\xi}_{kT}|$$

we get that

$$Z_k = \sum_{i=1}^{Z_{k-1}} Y_i$$

where Y_i is the number of particles generated by a single particle in the contact process restricted to a branch in T units time. All the branches are disjoint and therefore the Y_i are independent and they all have the same law as $|\xi_T^O|$. Hence Z_k is a BGW process. By (2.6) and Lemma 2.2 we get

$$E(Z_1) \geq pK(1 - \frac{1}{d-1}) - 1.$$

We can pick K large enough so that Z_k is supercritical and therefore

$$P(Z_k \geq 1, \forall k \geq 0) > 0.$$

But if Z_k survives with positive probability so does ξ_t^O and therefore η_t^O. We have shown that if $c(\lambda, d) > 0$ then $\lambda \geq \lambda_1$. This completes the proof of Theorem 2.3.

3. Notes and references

The contact process on Z^d was introduced by Harris (1974). See Durrett (1991) for a recent account of the contact process on Z^d. Pemantle (1992) started the study of the contact process on trees. He has better bounds than the ones in Theorem 1.1. His bounds prove that there are two phase transitions for all $d \geq 4$. But the methods used by Pemantle are much more involved than ours. Observe that our comparisons with branching Markov processes do not allow us to get an upper bound for λ_2. See problem 4.2 below.

Theorem 2.1 is due to Madras and Schinazi (1992), Theorems 2.2 and 2.3 are due to Morrow, Schinazi and Zhang (1994).

4. Open problems

4.1 Is $\lambda_1 < \lambda_2$ when $d = 3$? People usually conjecture that the answer is yes and this is supported by what is known for branching Markov chains.

4.2 Theorem 1.1 proves that $d\lambda_1$ converges to 1 as d goes to infinity. Pemantle has proved that $\limsup_{d\to\infty} \lambda_2\sqrt{d-1} \leq e$, this together with the lower bound for λ_2 in Theorem 1.1 make us conjecture that $\lim_{d\to\infty} \lambda_2\sqrt{d-1}$ exists, is this true? Is it possible to compute the limit? See Griffeath (1983) for a related problem on Z^d.

4.3 Is is true that $c(\lambda, d) < 0$ if $\lambda < \lambda_1$? This is related to the uniqueness of the critical point in percolation theory, see Grimmett (1989).

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