# HOMOCLINIC BIFURCATIONS AND HYPERBOLIC DYNAMICS

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## Foreword

Perhaps one of the most striking (and still somewhat puzzling) ways of performing substantial change in the dynamical structure (bifurcation) of a system is through the creation and unfolding of a cycle, in particular a homoclinic cycle. Poincaré first noticed the existence of homoclinic orbits in his prize essay on the 3-body problem. Subsequently, in "Les méthods nouvelles de la Mécanique Céleste", he expressed amazement about the complexity of the orbit structure of a diffeomorphisms in the presence of a transverse homoclinic orbit. More than forty years latter, Birkhoff showed that any such homoclinic orbit is accumulated by periodic ones and in the sixties Smale put this fact into the framework of (persistent) hyperbolic sets with dense subsets of periodic orbits.

In the last twenty years or so several results were obtained concerning the dynamics of a parametrized family of diffeomorphisms going through a homoclinic bifurcation. (Some authors relate homoclinic bifurcations to chaotic behavior and strange attractors). Similarly for vector fields and endomorphisms (but this is not discussed here). We intend in these notes to present part of such development, specially in two dimensions, and to indicate how fractional dimensions of Cantor sets play a relevant role in this context. In particular, they are instrumental in measuring how frequently a value of the parameter yields a hyperbolic diffeomorphism (i.e., with a hyperbolic limit set). A number of related topics are also treated, like cascades of bifurcations (homoclinic tangencies, period doubling, saddle-nodes), comparison of unfoldings of homoclinic tangencies and those of quadratic maps of the interval, diffeomorphisms which persistently exhibit homoclinic tangencies (most of them with infinitely many sinks), Markov partitions of basic sets and a result on the shape of some strange attractors. Several open questions and conjectures as well as some generalizations to higher dimensions are mentioned, specially at the and of Chapter VI. We present proofs of the results or at least the concepts and main ideas involved.

We should end by saying that our interest here is to treat general (or generic) families of diffeomorphisms and so, as significant as they may be, we do not discuss specific equations (the or Melnikov method). Nor do we discuss results on homoclinic orbits associated to singularities of vector fields, such as Silnikov's. Finally, we only mention references we consider necessary for our text without intention of being in any way complete.

# INDEX

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# CHAPTER I

## INTRODUCTION - EXAMPLES

In this introduction we discuss a number of dynamical systems with (transverse) homoclinic orbits just to have some examples which motivate the following chapters. First we need some definitions. We deal with diffeomorphisms $\varphi: M \to M$ of a compact manifold to itself. In this chapter it will be enough to assume $\varphi$ to be $C^1$, for some of the later results we need $\varphi$ to be $C^2$ or $C^3$. Also the compactness of $M$ is not always needed - some of the examples in this chapter will be on $\mathbb{R}^2$.

We say that $p \in M$ is a hyperbolic fixed point of $\varphi$ if $\varphi(p) = p$ and if $(d\varphi)_p$ has no eigenvalue of norm one. For such a hyperbolic fixed point, one defines the stable and the unstable manifold as

$$W^s(p) = \{x \in M \mid \varphi^i(x) \to p \quad \text{for} \quad i \to +\infty\}$$

and

$$W^u(p) = \{x \in M \mid \varphi^i(x) \to p \quad \text{for} \quad i \to -\infty\}.$$

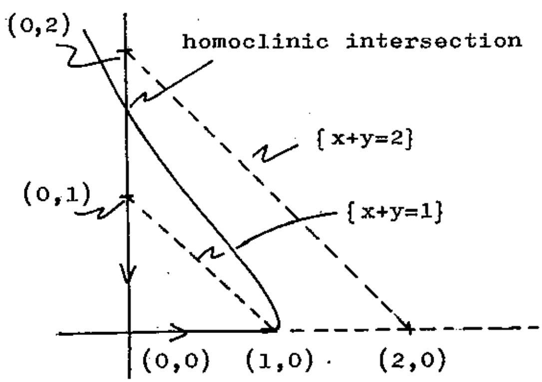
According to the invariant manifold theorem [9] (see also the Appendix) both $W^s(p)$ and $W^u(p)$ are injectively immersed submanifolds of $M$, are as differentiable as $\varphi$, and have dimensions equal to the number of eigenvalues of $(d\varphi)_p$ with norm smaller than one, respectively bigger than one. One can give the corresponding definitions for periodic points, i.e. fixed points of some power of $\varphi$.

If $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with no eigenvalues of norm one, then the origin 0 is a hyperbolic fixed point and $W^s(0)$, $W^u(0)$ are complementary linear subspaces: $\mathbb{R}^n = W^s(0) \oplus \oplus W^u(0)$.

We say that if $p$ is a hyperbolic fixed point of $\varphi$, $q$ is <u>homoclinic</u> to $p$ if $p \neq q \in W^s(p) \cap W^u(p)$, i.e. if $q \neq p$ and if $\lim_{i \to \pm\infty} \varphi^i(q) = p$ (this last form of the definition makes clear why Poincaré called such points "<u>bi-assymptotique</u>"). We say that $q$ is a <u>transverse homoclinic</u> point if $W^s(p)$ and $W^u(p)$ intersect transversally at $q$, i.e. if

$$T_q(M) = T_q(W^s(p)) \oplus T_q(W^u(p)).$$

It is clear that <u>linear diffeomorphisms</u> have <u>no homoclinic points</u>. We proceed now to show how to deform a linear map in such a way as to get a homoclinic point. We start with the linear map $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x,y) = (2x, 1/2\ y)$. The stable manifold is the y-axis, the unstable manifold is the x-axis. Next consider the




composition $\Psi \circ \varphi$, where $\Psi: \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism of the form

$$\Psi(x,y) = (x-f(x+y),y+f(x+y)),$$

where $f$ is some smooth function. This means that $\Psi$ is pushing points along lines of the form $\{x+y=c\} = \ell_c$ over a distance which only depends on $c$. We take $f$ a smooth function which is zero on $(-\infty,1]$ and such that $f(2) > 2$. In this case the stable and unstable manifold $W^s(0)$ and $W^u(0)$ for the diffeomorphism $\Psi \circ \varphi$ intersect outside the origin.

In fact, due to the construction, $\{(x,y) \mid x=0, y \leq 2\}$ belongs to $W^s(0)$ and $\{(x,y) \mid x \leq 1, y=0\}$ belongs to $W^u(0)$. Also

$$\Psi(\{(x,y) \mid 1 \leq x \leq 2, \quad y = 0\})$$

belongs to $W^u(0)$. From this and the description of $\Psi$ we obtain the homoclinic intersection, see the above figure. By choosing $f$ appropriately we can produce a transverse homoclinic point. Not much can be said about the global configuration of $W^s(0)$ and $W^u(0)$, but this configuration will be very complicated - see later in this introduction.

Our next example, the pendulum, contains a line of non-transverse homoclinic points. Consider the differential equation

$$\ddot{\theta} = -\sin\theta, \qquad \theta \in \mathbb{R}/2\pi,$$

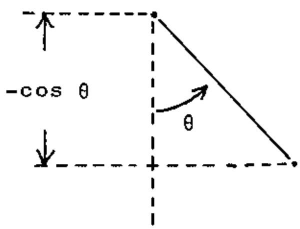
which defines a system of ordinary differential equations on the anulus:

$$\left.\begin{aligned} \dot{\theta} &= y \\ \dot{y} &= -\sin\theta \\ &\text{with } \theta \in \mathbb{R}/2\pi, \quad y \in \mathbb{R}. \end{aligned}\right\} \qquad (1)$$

We take the time one map of this system, i.e. the diffeomorphism $\varphi$ such that $\varphi(\theta,y) = (\tilde{\theta},\tilde{y})$ whenever there is a solution $(\theta(t),y(t))$ of (1) with $(\theta(0),y(0)) = (\theta,y)$ and $(\theta(1),y(1)) = (\tilde{\theta},\tilde{y})$. Then the fixed points of $\varphi$ are $(\theta=0,\ y=0)$ and $(\theta=\pi,\ y=0)$. The first is not hyperbolic (the eigenvalues of $(d\varphi)_{(0,0)}$ have norm one) but the second is: it has a one dimensional stable and a one dimensional unstable manifold. In order to determine the positions of these stable and unstable manifold it is important to note that the function

$$E(\theta,y) = -\cos\theta + \frac{1}{2}y^2$$

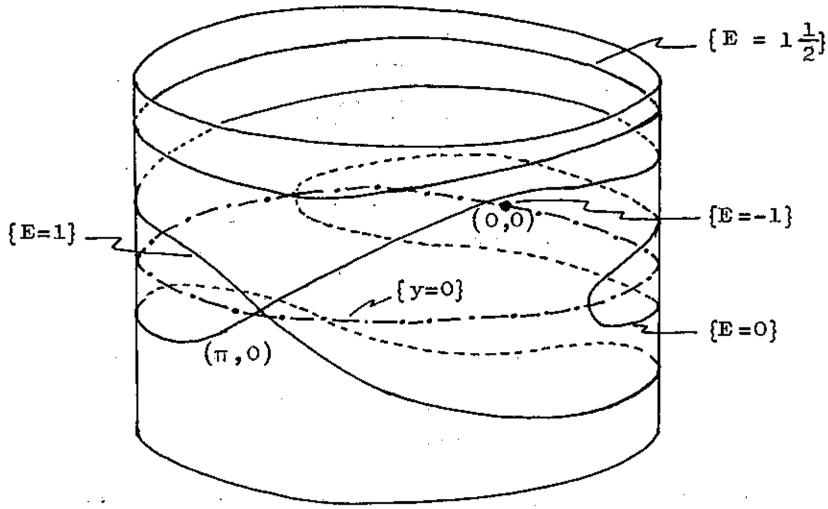
is constant along solutions of (1) - it is the energy, $-\cos\theta$




being the potential energy and $\frac{1}{2}y^2 = \frac{1}{2}\dot{\theta}^2$ being the kinetic energy. This means that both $W^u(\pi,0)$ and $W^s(\pi,0)$ are given by

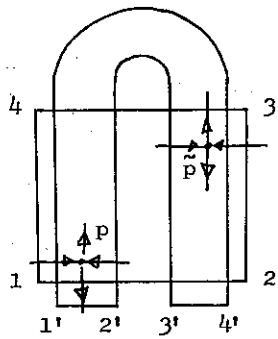
$$-\cos\theta + y^2 = 1.$$

In the next figure this homoclinic line is indicated together with some other energy levels.

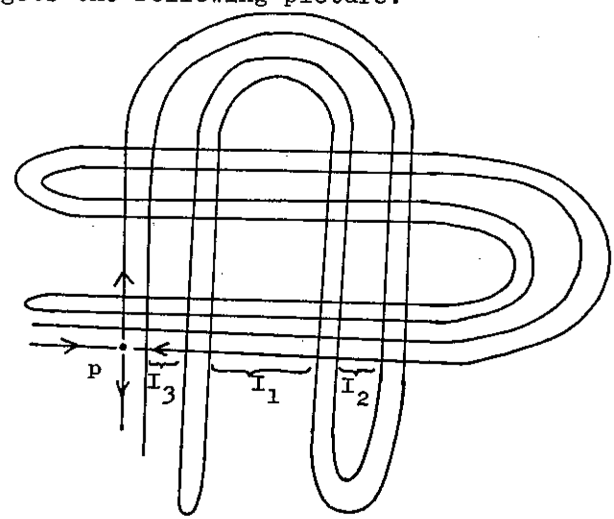



By a small perturbation of $\varphi$ one can make $W^u(\pi,0)$ and $W^s(\pi,0)$ to intersect transversally (using a perturbation as in the first example, or refering to the Kupka-Smale theorem [29]).

In the following example, the horseshoe, see Smale [29,30-A], we have transverse homoclinic points and still are fairly well able to describe globally the stable and the unstable manifold. In order to describe the diffeomorphism, let $Q$ be a square in $\mathbb{R}^2$ and let $\varphi$ map $Q$ as indicated below, such that on both components of $Q \cap \varphi^{-1}(Q)$, $\varphi$ is affine and preserves both horizontal and vertical directions, and such that 1, 2, 3, and 4




are mapped to 1′, 2′, 3′, and 4′. In Q, $\varphi$ has two fixed points p and $\tilde{p}$ as indicated; we restrict our attention to p. Since $\varphi$ is affine on $Q \cap \varphi^{-1}(Q)$, the stable and unstable manifolds $W^s(p)$ and $W^u(p)$, near p, are straight lines. In order to find the continuation one has to iterate $\varphi^{-1}$ (for $W^s(p)$) and $\varphi$ (for $W^u(p)$). Inside Q this gives pieces of straight lines, horizontal for $W^s(p)$ and vertical for $W^u(p)$. With a few iterations one gets the following picture.

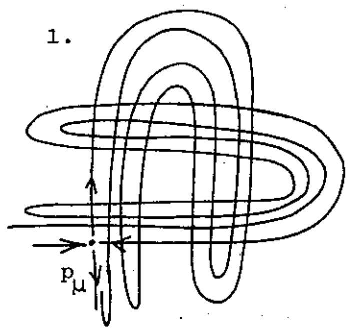


As a last remark on this example, observe that, however far we iterate, the intervals $I_1, I_2, I_3 \subset W^s(p)$ will never be intersected by $W^u(p)$. There are in fact a countably infinite number of such intervals, and $W^s(p) \cap W^u(p)$ consists of the boundary points of a Cantor set in $W^s(p)$.

As a last example we discuss a <u>homoclinic bifurcation</u> which appears when modifying the previous example. We speak of a homoclinic bifurcation if in a one-parameter family of diffeomorphisms, a pair of homoclinic intersections colides, forms a tangency and then disappears, or, reversing the direction, if a pair of homoclinic points is generated after a tangency.

In the previous example such bifurcations are obtained when composing $\varphi$ with a map $(x,y) \mapsto (x,y-\mu)$, which slides the image, in particular the image of Q, down. In the next figures we show the effect on the geometry of the stable and unstable manifold $W^s(p_\mu)$ and $W^u(p_\mu)$ for increasing values of $\mu$; in the first figure we have just the previous example.

1.



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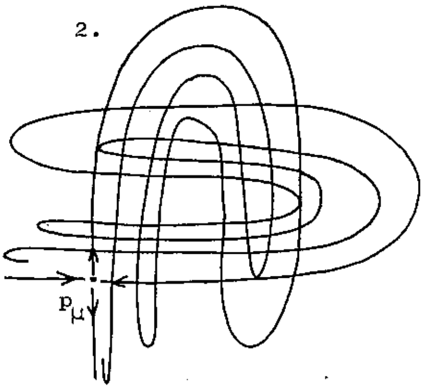


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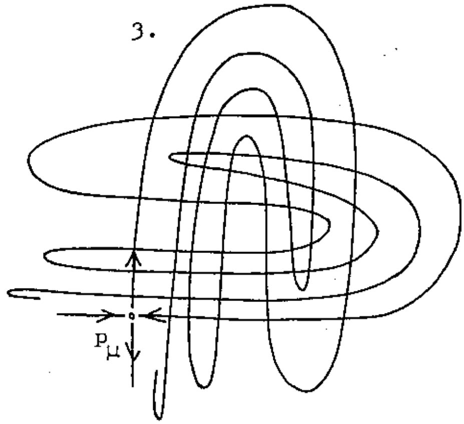
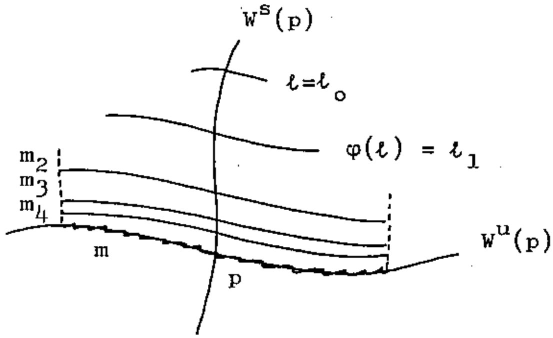



In the second figure one sees the first non-transverse homoclinic orbit (four iterations are indicated); from the third figure it is clear that near one such homoclinic bifurcation there are many others - see also Chapter III.

The complexity of the configuration of stable and unstable manifold in the last examples is typical for the case where one has at least one transverse homoclinic point. One can convince oneself of this complexify by trying to draw examples, keeping in mind that:

- $W^s(p)$ and $W^u(p)$ are $\varphi$-invariant, i.e. $\varphi(W^s(p)) = W^s(p)$ and $\varphi(W^u(p)) = W^u(p)$;
- $W^s(p)$ and $W^u(p)$ have no self intersections;
- near $p$, $\varphi$ is well approximated by the linear map $(d\varphi)_p$ - this has the following consequence ($\lambda$.lemma, [22], see also the Appendix): if $\ell$ is a smooth curve intersecting $W^s(p)$ transversally then the forward images $\ell_i = \varphi^i(\ell)$ contain




compact arcs $m_i \subset \ell_i$ which approach differentiably a compact arc $m$ in $W^u(p)$.

For n-dimensional diffeomorphisms the situation is basically the same except that if the codimension of $W^s(p)$ or $W^u(p)$ is bigger than one, they don't "separate" the ambient manifold any more.

The first time that these homoclinic points were constructed was by Poincaré in his prize essay [27]. The existence of these homoclinic points implied the non-convergence of certain power series expressions for solutions of a Hamiltonian system, comparable with the Hamiltonian system describing the restricted 3-body problem. This indicated that certain qualitative information, like "stability", was unobtainable by these analytic power series methods.

Later it was realized that the dynamics of a diffeomorphism $\varphi$, or the topology of its orbits, shows a great complexity if and only if $\varphi$ has some hyperbolic periodic point with a homoclinic intersection of its stable and unstable manifold. Although this last statement is in no way a theorem (and can be expected to be true only in a generic sense) many results in these notes can be interpreted as partial results in this direction. On the other hand we also deal with "stability results" concerning the dynamics in the presence of a homoclinic point. They deal with the not infrequent situation that the dynamics of a diffeomorphism $\varphi$ (with homoclinic point), although extremely complicated, remains unchanged in a topological sense when $\varphi$ is slightly perturbed.

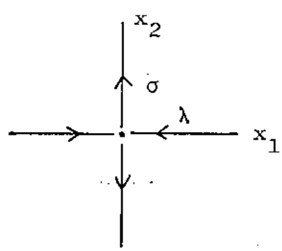
# CHAPTER II

## DYNAMIC CONSEQUENCES OF A TRANSVERSE HOMOCLINIC INTERSECTION

In this chapter we analyse the dynamical complexity due to one transverse homoclinic orbit. Although our discussion refers to the two-dimensional situation, the results and their proofs can be extended to arbitrary dimensions.

### 1. Description of the situation - linearizing coordinates.

Let $\varphi: M \to M$ be a $C^2$ diffeomorphism of a surface $M$ to itself and let $p \in M$ be a hyperbolic fixed point of saddle type, i.e. $\varphi(p) = p$ and $(d\varphi)(p)$ has two real eigenvalues $\lambda$ and $\sigma$ with $0 < |\lambda| < 1 < |\sigma|$. For simplicity we assume that

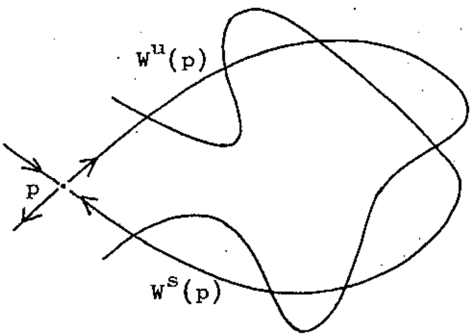



these eigenvalues are positive, so $0 < \lambda < 1 < \sigma$. From the theory of hyperbolicity (see Appendix) we know that:

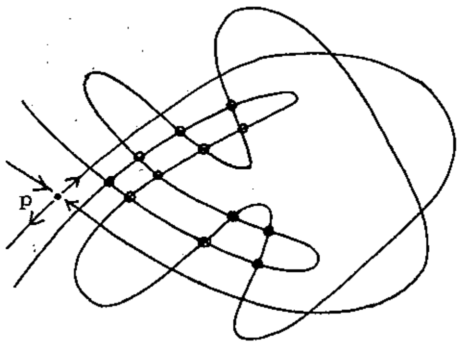
- the stable and unstable separatrices of $p$, $W^s(p)$ and $W^u(p)$, are $C^2$;

- there are $C^1$ linearizing coordinates in a neighbourhood of $p$, i.e. $C^1$ coordinates $x_1, x_2$ such that $p = (0,0)$ and such that $\varphi(x_1,x_2) = (\lambda.x_1, \sigma.x_2)$.

We assume that $W^s(p)$ and $W^u(p)$ have points of transverse intersection different from $p$ - such points, or their orbits, are called <u>homoclinic</u> or <u>biassymptotic</u> to $p$. Let $q$ denote one of these intersections. We assume that $q$ is <u>primary</u> in the sense that the arcs $\ell^u$, joining $p$ and $q$ in $W^u$, and $\ell^s$, joining $p$ and $q$ in $W^s$, form a double point free closed curve.




All indicated homoclinic points are primary.



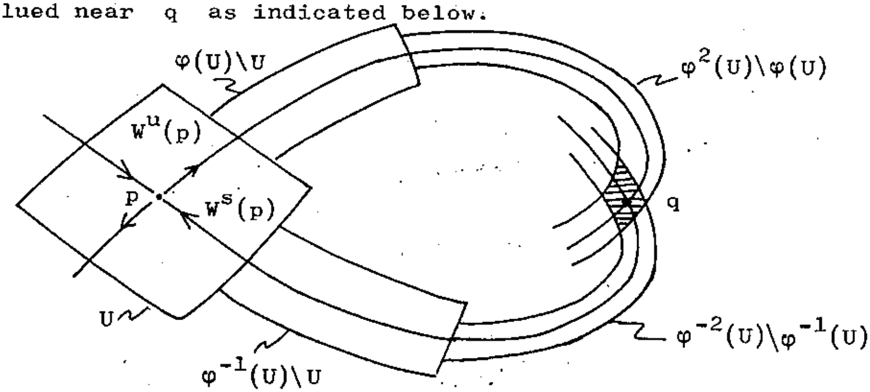

The encircled homoclinic points are not primary.

Note that whenever $p$ has homoclinic points, it has primary homoclinic points - if all intersections of $W^u(p)$ and $W^s(p)$ are transverse, then the number of primary homoclinic orbits is finite.

Let the linearizing coordinates be defined on $U$ and let their image be the square $(-1,+1) \times (-1,+1) \subset \mathbb{R}^2$. We consider extensions of the domain of definition of these linearizing coordinates. Identifying points in $U$ with the corresponding points in $\mathbb{R}^2$, we have: if $\varphi^{-1}([\lambda,1) \times (-1,+1)) \cap U = \emptyset$, we can extend the domain of the linearizing coordinates $x_1$, $x_2$ to $\varphi^{-1}([\lambda,1) \times (-1,+1))$ using the formulas

$$x_1 = \lambda^{-1}.(x_1 \circ \varphi), \qquad x_2 = \sigma^{-1}.(x_2 \circ \varphi).$$

Repeating this construction one can extend the linearizing coordinates along any segment in $W^s(p)$ starting in $p$ - one only has to take the original domain $U$ sufficiently small. This follows from the fact that $W^s(p)$ has no self intersections. In the same way one can extend the domain of these linearizing coordinates along the unstable separatrix $W^u(p)$. Homoclinic intersections however form an obstruction to a simultaneous extension along both the stable and the unstable separatrix. In our situation where $q$ is a primary homoclinic point, we extend the linearizing coordinates both along $\ell^u$ and $\ell^s$, the arcs in $W^u(p)$ and $W^s(p)$ joining $p$ and $q$ - however these coordinates will be bivalued near $q$ as indicated below.



Situation in M. '/////, denotes the neighbourhood of q on which the linearizing coordinates are bivalued.

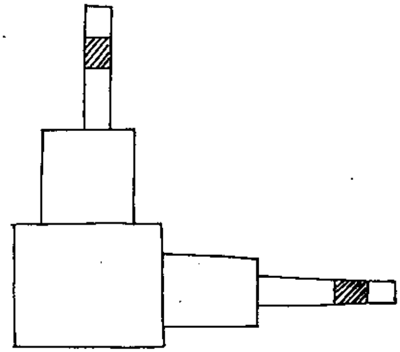


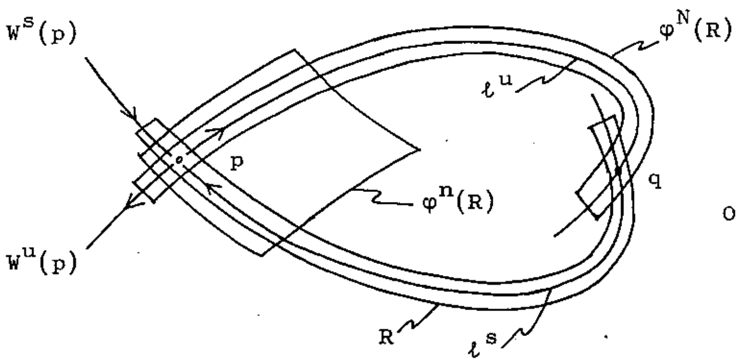
Image of the part of $\bigcup_{i=-2}^{2} \varphi^i(U)$ which the linearizing coordinates are defined. '////, denotes the two images of the above neighbourhood of q.

Now we consider in the domain of these extended coordinates a rectangle $R = \{-a \le x_1 \le b, \ -\alpha \le x_2 \le \beta\}$, $a,b,\alpha,\beta > 0$, containing $\ell^s$, the arc in $W^s(p)$ joining p and q, and such that for some N:

- $R \cap \varphi^n(R)$ consists of one rectangle containing p for $0 \le n < N$;
- $R \cap \varphi^N(R)$ consists of two connected components, one containing q, as indicated in the figure, i.e.

$$\{x_1 = b, \ -\alpha \le x_2 \le \beta\} \cap \varphi^N(R) = \emptyset$$

and $\varphi^N(\{-a \le x_1 \le b, \ x_2 = \beta\}) \cap R = \emptyset$.



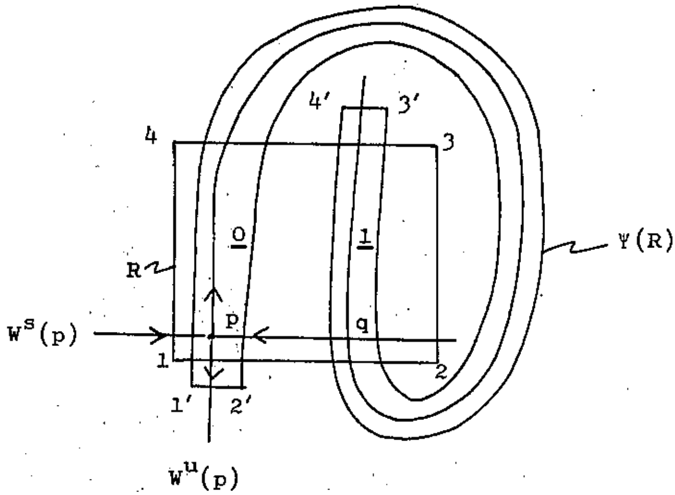

$0 < n < N$

For what follows it is important that one can choose R so that N is arbitrarily big - just take $\beta$ small. By taking $\beta$ small and hence N big, $\varphi^N(R)$ will become a very narrow strip around $\ell^u$. So transversality of $W^u(p)$ and $W^s(p)$ at q then implies transverse intersection of the sides of R and $\varphi^N(R)$.

The main object of interest in this chapter is the maximal invariant subset of R under $\varphi^N$, i.e. the set of those points $r \in R$ such that $\varphi^{k \cdot N}(r) \in R$ for all $k \in \mathbb{Z}$.

## 2. The maximal invariant subset of R - topological analysis.

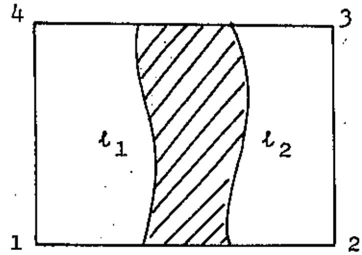
From now on we denote $\varphi^N$ by $\Psi$. We denote the maximal invariant subset of R under $\Psi$ by $\Lambda = \{r \in R \mid \Psi^k(r) \in R$ for all $k \in \mathbb{Z}\}$. Denoting the corners of R by 1, 2, 3, and 4 and their images in $\Psi(R)$ by 1', 2', 3', and 4', the relative positions of R and $\Psi(R)$ are as indicated in the figure, i.e. the




sides of R and Ψ(R) intersect transversally and the topology of the positions of R its sides and its corners relative to their images under Ψ are as in the figure. We denote the components of $R \cap \Psi(R)$ by $\underline{0}$ and $\underline{1}$, $\underline{0}$ containing p and $\underline{1}$ containing q.

__Theorem__. For any sequence $\{a_i\}_{i\in\mathbb{Z}}$, with $a_i = 0$ or 1, there is at least one point $r \in \Lambda$ such that $\Psi^i(r) \in \underline{a}_i$ for all $i \in \mathbb{Z}$.

__Proof__. We call a closed subset $S \subset R$ a vertical strip if it is bound (in R) by two disjoint continuous curves $\ell_1$ and $\ell_2$ connecting the side 1, 2 with the side 3, 4. If S is a vertical strip then $\Psi(S) \cap R$ contains two vertical strips, one in $\underline{0}$ and one in $\underline{1}$. Let now $\{a_i\}_{i\in\mathbb{Z}}$ be a sequence as in the




////// vertical strip.

theorem. We construct a nested sequence of vertical strips $S_o \supset S_1 \supset S_2 \supset \ldots$: $S_o = \underline{a}_o$; $S_1$ is a vertical strip in $\Psi(\underline{a}_{-1}) \cap S_o$; $S_2$ is a vertical strip in $\Psi^2(\underline{a}_{-2}) \cap S_1$ etc.; $S_\infty = \bigcap_{i \geq 0} S_i$. For each point $r \in S_\infty$, $\Psi^{-i}(r) \in \underline{a}_{-i}$, $i \geq 0$.

Horizontal strips are similarly defined and we have horizontal strips $T_1 \supset T_2 \supset T_3 \supset \ldots$ such that for $r \in T_\infty = \bigcap_{i \geq 1} T_i$, $\Psi^i(r) \in \underline{a}_i$ for all $i \geq 1$. Now $S_\infty \cap T_\infty \neq \phi$. Otherwise, for some $i_o$, $S_{i_o} \cap T_{i_o} = \phi$, but $S_{i_o}$ contains a line from side 1, 2 to the side 3, 4 and $T_{i_o}$ contains a line from 1, 4 to 2, 3. These lines have to intersect.

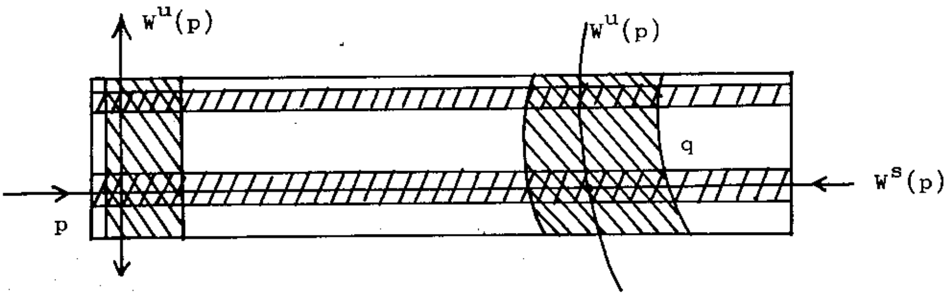
For any point $r \in S_\infty \cap T_\infty$, $\Psi^i(r) \in \underline{a}_i$ for all $i \in \mathbb{Z}$. From this it follows also that $r \in \Lambda$.

## 3. The maximal invariant subset of R - hyperbolicity and invariant foliations.

In this section we use more information about $\Psi = \varphi^N$ restricted to R. In the linearizing coordinates on a neighbourhood of $\ell^s$, the arc in $W^s(p)$ joining p and q, we have

$$R = \{-a \le x_1 \le b,\ -\alpha \le x_2 \le \beta\},$$

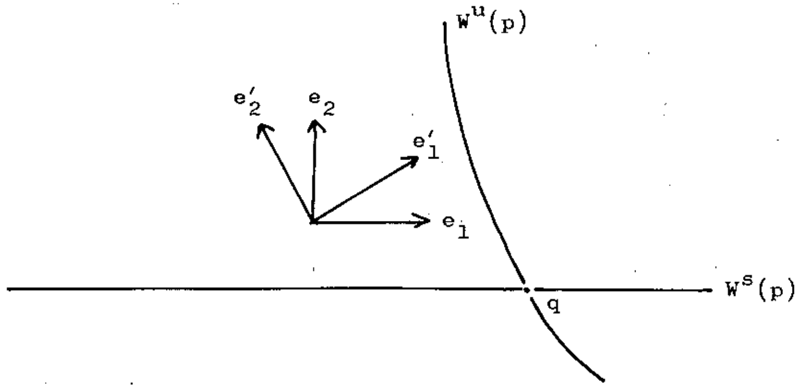
see Section 1. We only have to describe $\Psi$ in those points of R which are mapped back into R, i.e. in $\Psi^{-1}(R) \cap R$. In the




////// $\Psi^{-1}(R) \cap R$. \\\\\\\\ $\Psi(R) \cap R$.

component of $\Psi^{-1}(R) \cap R$ containing p and q, $\Psi$ is linear, in fact $\Psi(x_1,x_2) = (\lambda^N x_1, \sigma^N x_2)$ with $0 < \lambda < 1 < \sigma$.

The other component of $R \cap \Psi^{-1}(R)$ is mapped to the component of $R \cap \Psi(R)$ containing q. This component of $R \cap \Psi(R)$ is the region where the linearizing coordinates, constructed in Section 1, were bivalued, or rather where we have apart from the linearizing coordinates following $W^s(p)$ and which are in the above figure the Cartesian coordinates of the plane, also the linearizing coordinates following $W^u(p)$. We denote by $e_1$, $e_2$ the coordinate vector fields of the linearizing coordinates following $W^s(p)$ and by $e_1'$, $e_2'$ the coordinate vector fields of the linearizing coordinates following $W^u(p)$; see the figure below.




For $\Psi$, restricted to the component of $R \cap \Psi^{-1}(R)$ which is mapped on a neighbourhood of $q$, we have $(d\Psi)e_1(r) = \lambda^N . e_1'(\Psi(r))$ and $(d\Psi)e_2(r) = \sigma^N . e_2'(\Psi(r))$.

Due to the transversality of $W^u(p)$ and $W^s(p)$ and to the thinness of $\Psi(R)$, for $N$ big, $e_1$ and $e_2'$ are linearly independent. Also, by choosing $R$ and $\Psi(R)$ thin, we may assume that the matrix transforming $e_1$, $e_2$ into $e_1'$, $e_2'$ (or its inverse) is almost constant.

<u>Theorem</u>. For R sufficiently thin, and hence N big, the maximal invariant subset

$$\Lambda = \bigcap_{n \in \mathbb{Z}} \Psi^n(R) \text{ in } R \text{ is hyperbolic.}$$

<u>Proof</u>. A continuous cone field $C$ on $R \cap \Psi(R)$ is a map which assigns to each $r \in R \cap \Psi(R)$ a two-sided cone $C(r)$ in $T_r(M)$, given by two linearly independent vectors $w_1(r)$, $w_2(r)$:

$$C(r) = \{v \in T_r(M) \mid v = a_1.w_1(r) + a_2.w_2(r) \text{ with } a_1.a_2 \geq 0\}.$$

Continuity of $C$ means that $w_1$ and $w_2$ depend continuously on $r$. An unstable cone field is a continuous cone field on $R \cap \Psi(R)$ such that

- for each $r \in R \cap \Psi(R) \cap \Psi^{-1}(R)$, $\overline{(d\Psi)(C(r))} \subset \mathrm{Int}(C(\Psi(r))) \cup \{0_r\}$, (*)
- for each $r \in R \cap \Psi(R) \cap \Psi^{-1}(R)$ and $0 \neq v \in C(r)$, $\|d\Psi(v)\| > \|v\|$, where the norm is taken with respect to the basis $e_1$, $e_2$.

Below we construct such an unstable cone field. From the existence of such a cone field it follows that there is a continuous direction field $E^u(r)$, defined for $r \in \bigcap_{i \geq 0} \Psi^i(R)$, such that

- $E(r) \subset C(r)$;
- $d\Psi$ maps $E(r)$ to $E(\Psi(r))$ whenever $r \in \bigcap_{i \geq -1} \Psi^i(R)$, and for $0 \neq v \in E(r)$,

$$\|d\Psi(v)\| \geq \nu.\|v\| \text{ for some } \nu > 1 \quad (**)$$

---

(*) where $0_r$ denotes the zero vector in $T_r(M)$.

(**) $E^u$ is obtained by taking the intersections of the forward images of the cone field $C$ under $d\Psi$.

Replacing $\Psi$ by $\Psi^{-1}$, one constructs in the same way a stable cone field and the direction field $E^s$, which is invariant under and contracted by $d\Psi$. Then $T_\Lambda(M) = E^u_\Lambda \oplus E^s_\Lambda$ is the required splitting for hyperbolicity - see the Appendix.

Now we come to the construction of the unstable cone field on $R \cap \Psi(R)$. In the component of $R \cap \Psi(R)$ containing $p$ we simply take cones around $e_2$ extending $45^\circ$ to both sides. In the other component of $R \cap \Psi(R)$ there is (assuming $R$ and $\Psi(R)$ sufficiently thin) an angle $\alpha$, smaller than $90^\circ$, so that for each point $r$ in that component of $R \cap \Psi(R)$, the cone around $e_2(r)$, extending over an angle $\alpha$ to both sides, contains $e_2'(r)$ in its interior. This is due to the fact that $e_1(r)$ and $e_2'(r)$ are linearly independent. The unstable cone field $C$ is just defined as the field of cones, centred on $e_2$ and extending $45^\circ$, respectively $\alpha$, to both sides of $e_2$ depending on the component of $R \cap R(\Psi)$.

In order to show that this cone field has the required properties, we introduce constants $A$, $B$, and $B'$ so that: whenever $r \in R \cap \Psi(R)$ and $v = v_1.e_1(r) + v_2.e_2(r) \in T_r(M)$ then for $v \in C(r)$ we have $|v_1| \leq A.|v_2|$, on the other hand whenever $|v_1| \leq B.|v_2|$, then $v \in C(r)$; whenever $v = v_1'.e_1'(r) + v_2'.e_2'(r)$ and $|v_1'| \leq B_1'.|v_2'|$, then $v \in C(r)$. If $N$ is so big that $|\frac{\lambda}{\sigma}|^N.A < \min(B,B')$, then $d\Psi$ maps cones in the interior of cones. Also for $N$ sufficiently big, the lengths of vectors in our cones are strictly increased by $d\Psi$.

So for $N$ big enough our cone field has the required properties, <u>but</u> the cone - field was constructed <u>after</u> choosing $N$ - $\Psi$ is defined in terms of $N$: $\Psi = \varphi^N$ and the domain of the cone field is defined in terms of $\Psi$: domain = $R \cap \Psi(R)$. <u>However</u> the way to raize $N$, is to make $R$ thinner. This decreases the domain where the cone field has to be defined. The fact that $\Psi$ changes from $\varphi^N$ to $\varphi^{N'}$, $N' > N$, has no influence on the arguments: the vector fields $e_1$, $e_2$, $e'_1$ and $e'_2$ do not change. So $R$ and $N$ may be ajusted afterwards. This completes the proof of the hyperbolicity of $\Lambda$.

Observe that we proved slightly more: there are vector fields $e^u \in E^u_\Lambda$ and $e^s \in E^s_\Lambda$ and a constant $\nu > 1$, such that for all $r \in \Lambda$, $\|d\Psi(e^u(r))\| \geq \nu.\|e^u(r)\|$ and $\|d\Psi(e^s(r))\| \leq$ $\leq \nu^{-1}.\|e^s(r)\|$.

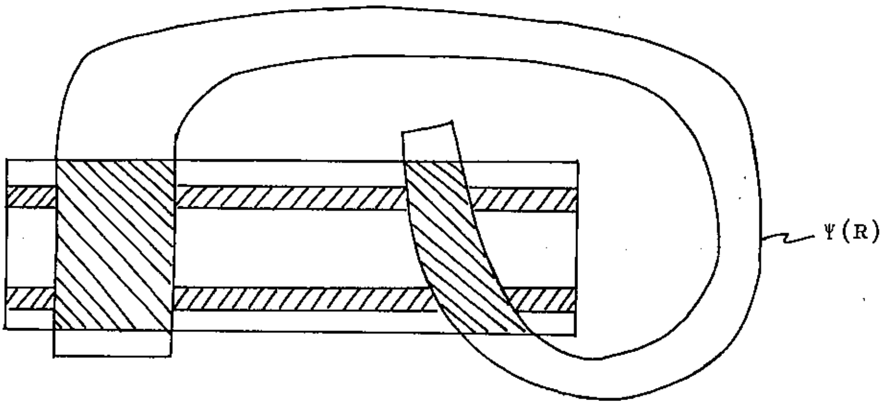
Now we come the second subject of this section. The cone fields, just constructed, will now be used to construct the stable and the unstable foliation. We only describe the construction of the unstable foliation. First the definition.

An <u>unstable foliation</u> for $\Lambda = \bigcap_{i\in\mathbb{Z}} \Psi^i(R)$ is a foliation $\mathcal{F}^u$ of a neighbourhood of $\Lambda$ (here we take $R \cap \Psi(R)$) such that

1. for each $r \in \Lambda$, $F^u(r)$, the leaf of $\mathcal{F}^u$ containing $r$, is tangent to $E^u(r)$;
2. for each $r$, sufficiently near $\Lambda$, $\Psi(F^u(r)) \supset F^u(\Psi(r))$.

We require the tangent directions of leaves of $\mathcal{F}^u$ to vary continuously. In fact, in the present case, where we deal with a $C^2$ diffeomorphism $\Psi$ and co-dimension one foliations, one can require them to be $C^{1+\epsilon}$, see the Appendix.

<u>Construction of the unstable foliation</u>. We recall the relative positions of $R$, $\Psi(R)$, and $\Psi^{-1}(R)$ in the figure below.




\\\\\\ $R \cap \Psi(R)$      ////// $(R \cap \Psi^{-1}(r))\setminus\Psi(R)$

We take a $C^2$ foliation $\tilde{\mathfrak{F}}^u$ (not yet the unstable foliation) on $(\Psi(R) \cup \Psi^{-1}(R)) \cap R$ so that:

1. in $R \cap \Psi(R)$ the tangent directions of leafs are contained in the unstable cones;
2. the image under $\Psi$ of leaves in $(R \cap \Psi^{-1}(R)) \cap \Psi(R)$ has tangent directions contained in the unstable cones;
3. the four arcs of $\partial R \cap \Psi^{-1}(R)$ are leafs of $\tilde{\mathfrak{F}}^u$, the union of these four arcs denoted by $E_o$;
4. the four arcs of $\partial(\Psi(R)) \cap R$ are leafs of $\tilde{\mathfrak{F}}^u$, the union of these four arcs is denoted by $E_1$;
5. $\Psi$ maps leaves of $\tilde{\mathfrak{F}}^u$ near $E_o$ to leaves of $\mathfrak{F}^u$ near $E_1$.

Since all the cones of the unstable cone field are centered around the vertical vector field $e_2$ and contain, where defined $e_2'$, it is clear that such a foliation $\tilde{\mathfrak{F}}^u$ exists.

For foliations as described above we define the operator $\Psi_*$. $\Psi_*(\tilde{\mathfrak{F}}^u)$ has the following leaves:

in $(R \cap \Psi^{-1}(R))\setminus\Psi(R)$, the leaves of $\tilde{\mathfrak{F}}^u$ and $\Psi_*(\tilde{\mathfrak{F}}^u)$ are the same;

in $(R \cap \Psi(R))$, the leaves of $\Psi_*(\tilde{\mathfrak{F}}^u)$ are connected components of $\Psi$ images of leaves of $\tilde{\mathfrak{F}}^u$ intersected with $(R \cap \Psi(R))$.

Due to the above conditions 3, 4, and 5, also $\Psi_*(\tilde{\mathfrak{F}}^u)$ is $C^2$. From invariant manifold theory it follows that the limit

$$\lim_{i\to\infty} \Psi_*^i(\tilde{\mathfrak{F}}^u) = \mathfrak{F}^u$$

exists. This limit depends on the choice of the "initial foliation" $\tilde{\mathfrak{F}}^u$. The limit is $C^1$; if $\varphi$ is $C^3$, this limit is $C^{1+\epsilon}$, see the Appendix.

Observe that we can extend our vector fields $e^u$ and $e^s$ in $E^u$, respectively $E^s$, to tangent vector fields of $\mathfrak{F}^u$ and $\mathfrak{F}^s$ ($\mathfrak{F}^s$ is just an unstable foliation for $\Psi^{-1}$) so that for some constant $\tilde{\nu} > 1$, and all $r \in R \cap \Psi(R) \cap \Psi^{-1}(R)$,

$$\|d\Psi(e^u(r))\| \geq \tilde{\nu}.\|e^u(r)\|$$

and

$$\|d\Psi(e^s(r))\| \leq \tilde{\nu}.\|e^s(r)\|.$$

Stable and unstable foliations can be constructed for any basic set of a $C^1$-diffeomorphism in dimension 2 [16]. In higher dimensions the existence of such foliations is not known. The fact that for $C^3$ diffeomorphisms in dimension 2 these foliations are $C^{1+\epsilon}$ will not be used in this chapter but will be quite essential in the following chapters.

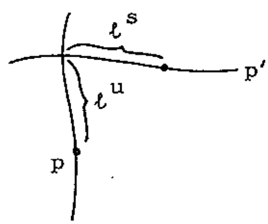
4. The maximal invariant subset of R - structure.

We divide $\Lambda$, the maximal invariant subset of R, in blocks. For each sequence $A = (a_{-k}, a_{-k+1}, \ldots, a_{k-1}, a_k)$ with $a_i = 0$ or $1$, we define the A-block as $\Lambda_A = \{r \in \Lambda \mid \Psi^i(r) \in \underline{a}_i$ for $i = -k, \ldots, k\}$; we call K the radius of A.

As we saw in the last section, expansions and contractions of vectors along unstable, respectively stable foliations are at least by a factor $\tilde{\nu} > 1$, respectively $\tilde{\nu}^{-1}$. Let c be the maximal length of a component of a stable or unstable leaf in $R \cap \Psi(R)$.

Proposition. Let $\Lambda_A$ be an A-block and let A have radius K. Then the diameter of $\Lambda_A$ is at most $2.c.\tilde{\nu}^{-K}$; for c, $\tilde{\nu}$ see above.

Proof. For any two points $p'$, $p''$ in the same component of $R \cap \Psi(R)$ there are unique arcs $\ell^u(p', p'')$ and $\ell^s(p', p'')$ in leaves of $\mathfrak{F}^u$, respectively $\mathfrak{F}^s$, jointing $p'$, respectively $p''$,




with the intersection of the unstable leaf through $p'$ and the stable leaf through $p''$. When $p'$, $p''$ are both in $\Lambda_A$, then this whole configuration will remain in the same component of $R \cap \Psi(R)$ when we apply $\Psi^i$, $i = -K, \ldots, +K$. This implies that the lengths of $\ell^u(p',p'')$ and $\ell^s(p',p'')$ and $\ell^s(p',p'')$ both at most $c.\tilde{\nu}^{-K}$. This implies the proposition.

From the above proposition and the result of section two we obtain:

<u>Theorem</u>. The size of an A-box $\Lambda_A$ goes to zero as the radius of A goes to infinity. For each infinite sequence $\ldots, a_{-2}, a_{-1}, a_o, a_1, a_2, \ldots,$ $a_i = 0$ or $1$, there is exactly one point $r \in \Lambda$ such that $\Psi^i(r) \in \underline{a}_i$ for all $i$. There is a homeomorphism $h: \Lambda \to (\mathbb{Z}_2)^{\mathbb{Z}}$ (product topology on $(\mathbb{Z}_2)^{\mathbb{Z}}$) such that for $r \in \Lambda$, $h(r) = \ldots, a_{-2}, a_{-1}, a_o, a_1, a_2, \ldots$ with $\Psi^i(r) \in \underline{a}_i$. If $\sigma: (\mathbb{Z}_2)^{\mathbb{Z}} \to (\mathbb{Z}_2)^{\mathbb{Z}}$ is the shift operator, i.e. $\sigma(\{a_i\}_{i\in\mathbb{Z}}) =$ $= \{a'_i\}_{i\in\mathbb{Z}}$ with $a'_i = a_{i+1}$, then

$$\begin{array}{ccc} \Lambda & \xrightarrow{\Psi|\Lambda} & \Lambda \\ \downarrow h & & \downarrow h \\ (\mathbb{Z}_2)^{\mathbb{Z}} & \xrightarrow{\sigma} & (\mathbb{Z}_2)^{\mathbb{Z}} \end{array}$$

commutes.

<u>Remark 1</u>. It follows from the above theorem and its proof that if $\tilde{\Psi}$ is $C^1$ close to $\Psi$, then the same conclusions hold for the maximal invariant set $\tilde{\Lambda}$ of $\tilde{\Psi}$ in R. Namely if C is an unstable cone field for $\Psi$ whose domain is slightly extended beyond $R \cap \Psi(R)$, then C is also an unstable cone field for $\tilde{\Psi}$ if $\tilde{\Psi}$ is sufficiently $C^1$ close to $\Psi$. Then all the above arguments apply, with the obvious modifications, to $\tilde{\Psi}$. This implies

that for such $\tilde{\Psi}$, $\tilde{\Lambda}$, there is a conjugacy $H: \Lambda \to \tilde{\Lambda}$, i.e. a homeomorphism such that the diagram below commutes.

$$\begin{array}{ccc} \Lambda & \xrightarrow{\Psi|\Lambda} & \Lambda \\ H \downarrow & & \downarrow H \\ \tilde{\Lambda} & \xrightarrow{\tilde{\Psi}|\tilde{\Lambda}} & \tilde{\Lambda} \end{array}$$

This is a special case of structural stability of basic sets.

<u>Remark 2</u>. The periodic points are dense in $\Lambda$. This follows from the corresponding statement for $(\mathbb{Z}_2)^{\mathbb{Z}}$ and $\sigma$: any finite sequence can be completed to an infinite periodic sequence. It is clear that all these periodic points are of saddle type (one expanding and one contracting direction). For the use in the next chapters, we add one more observation about these periodic orbits. In general we say that a fixed point $p$ of a diffeomorphism $\varphi$ is <u>dissipative</u> if $|\det(d\varphi)(p)| < 1$. The same applies to periodic points, say of period $k$: just replace $\varphi$ by $\varphi^k$. Now if the fixed point $p$ with which we started this Chapter (see Section 1) is dissipative, then, for $R$ sufficiently thin (or $N$ big), all the periodic points in $\Lambda$ will be dissipative. If $p' \in \Lambda$ is a periodic point of $\Psi$, and hence of $\varphi$, and if $R$ is thin the $p'$ will spend most of each orbit (under iteration of $\varphi$) in a small neighbourhood of $p$. The dissipativeness then follows from

$$\det(d\varphi^k)(p') = \prod_{i=0}^{k-1} \det(d\varphi)(\varphi^i(p')).$$

<u>Remark 3</u>. It follows also from the above constructions that $\Lambda$ has "local product structure" in the sense that if $r, r' \in \Lambda$ are

in the same component of $R \cap \Psi(R)$, then the intersection of the stable leaf $F^s(r)$ through $r$ and the unstable leaf $F^u(r')$ through $r'$ also belongs to $\Lambda$. In fact, if $h(r) = \{a_i\}_{i\in\mathbb{Z}}$ and $h(r') = \{a'_i\}_{i\in\mathbb{Z}}$, then $F^s(r)$ corresponds to the sequences

$$\{ \{b_i\}_{i\in\mathbb{Z}} \mid b_i = a_i \quad \text{for} \quad i \geq 0\}$$

and $F^u(r')$ corresponds to the sequences

$$\{ \{b'_i\}_{i\in\mathbb{Z}} \mid b'_i = a'_i \quad \text{for} \quad i \leq 0\}$$

Since $r, r'$ are in the same component of $R \cap \Psi(R)$, $a_o = a'_o$, so that the point in $F^s(r) \cap F^u(r)$ corresponds to the sequence

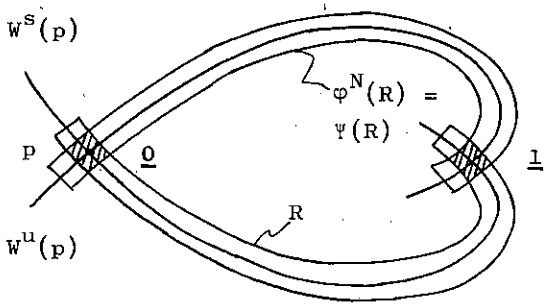
$$\ldots, a_{-2}, a_{-1}, a'_o = a_o, a_1, a_2, \ldots$$

## 5. <u>Conclusions for the dynamics near a transverse homoclinic orbit</u>.

We return to the diffeomorphism $\varphi$ (see Section 1) and discuss the consequences of the results in Sections 2 to 4 on $\Psi = \varphi^N$. We found and analysed the maximal closed invariant subset $\Lambda$ in $R$ under the map $\Psi$. This set $\Lambda$ is contained in $R \cap \Psi(R)$ whose components are denoted by $\underline{0}$ and $\underline{1}$. A corresponding invariant set for $\varphi$ is defined as $\hat{\Lambda} = \bigcup_{i=0}^{N-1} \varphi^i(\Lambda)$.

<u>Proposition</u>. The set $\hat{\Lambda}$, as defined above is the disjoint union of $\{p\}$, $\Lambda-\{p\}$, $\varphi(\Lambda-\{p\}),\ldots,\varphi^{N-1}(\Lambda-\{p\})$.

<u>Proof</u>. We only have to show that for $0 < i < N$, $\Lambda \cap \varphi^{-i}(\Lambda) = \{p\}$. In fact let $r \in \Lambda$, and $\varphi^i(r) \in \Lambda$ for some $0 < i < N$. Then




$r \notin \underline{1}$ and also $\varphi^{k.N}(r) = \Psi^k(r)$ has the same properties, i.e $\Psi^k(r) \in \Lambda$ and $\varphi^i(\Psi^k(r)) \in \Lambda$. This implies that $\Psi^k(r) \in \underline{0}$ for all $k$ and hence that $r = p$. This proves the proposition.

It is clear that $\hat{\Lambda}$ is a hyperbolic set for $\varphi$, and that the periodic orbits are dense in $\hat{\Lambda}$. As we have observed before a (transverse) homoclinic orbit implies great complexity of the patterns formed by the corresponding separatrices. In this direction we can prove.

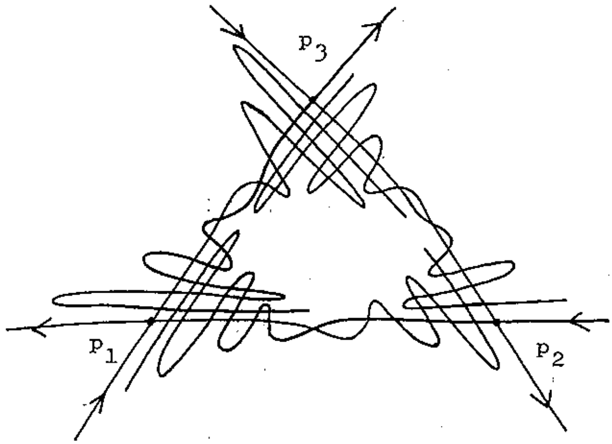
__Proposition__. In the above situation, $\hat{\Lambda}$ is contained in the closure of both the stable and the unstable manifold of $p$.

__Proof__. Since the periodic orbits are dense in $\hat{\Lambda}$ it is enough to prove that each periodic point of $\hat{\Lambda}$ is contained in $\overline{W^s(p)}$ (and in $\overline{W^u(p)}$). Since both $\overline{W^s(p)}$ (and $\overline{W^u(p)}$) are invariant under $\varphi$, it is enough to prove that the periodic points of $\Lambda$ are in $\overline{W^s}$. For any periodic point $r \in \Lambda$, the unstable separatrix $W^u(r)$ contains a leaf of the unstable foliation and hence intersects $W^s(p)$. Then it follows (iterate $\varphi^{-1}$) that this periodic point $r$ is contained in the closure of $W^s(p)$. In the same way one proves that it is contained in the closure of $W^u(p)$.

## 6. Homoclinic points of periodic orbits.

Let again $\varphi: M \to M$ be a diffeomorphism but now with a periodic orbit $\{p_o, p_1, \ldots, p_{k-1}\}$, $\varphi(p_i) = p_{i+1}$ mod $k$, with period $k$ which is of saddle type. Stable and unstable manifolds are denoted by $W^s(p_i)$ and $W^u(p_i)$. There are two types of homoclinic orbits, namely intersections of $W^s(p_i)$ and $W^u(p_i)$ - they are just homoclinic orbits of a hyperbolic saddle fixed point for $\varphi^k$ - and intersections of $W^s(p_i)$ and $W^u(p_j)$, $i \neq j$ mod $k$. For $t = j-i$, we then have also intersections of $W^s(p_j)$ and $W^u(p_{j+t})$ etc.. This means that we get something like a cycle whose "period" is the smallest number $\ell$ such that $\ell.t$ is a multiple of $k$. If the intersection of $W^s(p_i)$ and $W^u(p_j)$ is transverse, so are the intersection of $W^s(p_j)$ and $W^u(p_{j+t})$, of $W^s(p_{j+t})$ and $W^u(p_{j+2t})$, etc.. By the $\lambda$-lemma [22], see also the Appendix, this means that $W^s(p_i)$ is accumulating on $W^s(p_j)$ and hence intersecting $W^u(p_{j+t})$ transversally, hence accumulating on $W^s(p_{j+t})$ etc., so that we finally get a transverse intersection of $W^s(p_i)$ with $W^u(p_i)$ anyway.

An example of this last phenomenon occures in any generic 2-parameter family of diffeomorphisms $\varphi_\mu: \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \in \mathbb{R}^2$, such that $\varphi_o(0) = 0$ and such that $(d\varphi_o)(0)$ has eigenvalues $e^{\pm 2\pi i/3}$ - this is the subharmonic bifurcation with resonance 1:3 [3]. Stable and unstable separatrices are then as indicated below.




## 7. Transverse homoclinic intersections in arbitrary dimensions.

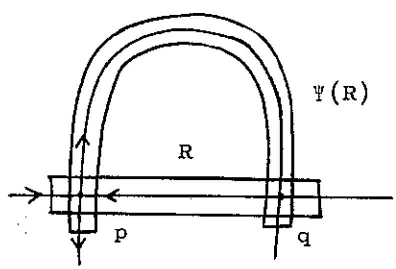
As we remarked in the beginning of this chapter, all results and proofs can be extended to diffeomorphisms $\varphi: M \to M$, where $M$ is an n-dimensional manifold. So one obtains:

Theorem. Let $\varphi: M \to M$ be a $C^1$ diffeomorphism with a hyperbolic fixed point $p$. Let $q$ be a point of transverse intersection of $W^u(p)$ and $W^s(p)$. Then there is a neighbourhood $U$ of the closure of the orbit $\overline{O(q)} = \overline{\bigcup_{i\in\mathbb{Z}} \varphi^i(q)}$ such that the maximal invariant set $\hat{\Lambda}$ under $\varphi$ in $U$ is a nontrivial basic set (see the Appendix). Also, there are neighbourhoods $V_p$ and $V_q$ of $p$ and $q$ and there is an integer $N$, such that the maximal invariant set $\Lambda$ under $\varphi^N$ in $V = V_p \cup V_q$ is also a nontrivial basic set and such that $\varphi^N|\Lambda$ is conjugated with the shift on $(\mathbb{Z}_2)^{\mathbb{Z}}$ as in Section 4.

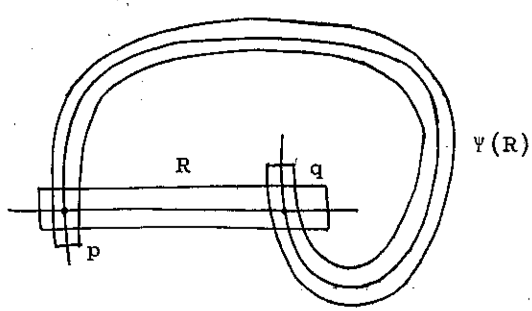
8. Historical note.

The main ideas in the chapter were developed by Poincaré [27], who realized that homoclinic points are accumulated by homoclinic points, by G.D. Birkhoff [4] who showed that homoclinic points are accumulated by periodic points, and by S. Smale [29] who essentially obtained the main theorem of Section 4. Transverse homoclinic orbits were also studied in [16-B].

The maximal invariant set in R is often called a horseshoe and the map $\Psi|R$ a horseshoe map. Due to the topology of $R^2$ there are two types of transverse homoclinic orbits but for both cases the analysis is the same; our figures refer to the less conventional case in which one does not "see" a horseshoe. In the conventional case one has:




instead of



# CHAPTER III

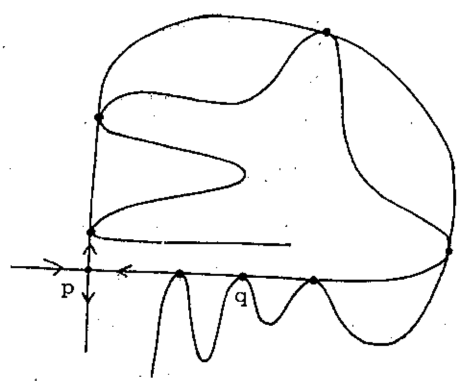
## HOMOCLINIC TANGENCIES: CASCADE OF BIFURCATIONS, SCALING AND QUADRATIC MAPS

### 1. General comments. Cascade of homoclinic tangencies.

In this chapter we discuss the unfolding of a homoclinic tangency for one-parameter families $\{\varphi_\mu\}$ of diffeomorphisms of a surface $M^2$. As a consequence of such unfolding the dynamics (orbit structure) of the diffeomorphisms undergoes a great number of changes (bifurcations) as the parameter evolves near the value say $\mu = 0$ corresponding to the homoclinic tangency. In particular the homoclinic tangency is accumulated by other homoclinic tangencies for values of $\mu$ approaching zero. Also many periodic points appear (or disappear) or lose hyperbolicity and change index (i.e., dimension of stable manifold). Although the expression "chaotic" is not in general well defined, sometimes authors apply it (among other situations) to a small neighbourhood of $\mu = 0$ in which these bifurcations occur calling it a chaotic region. Still, it may happen that in certain cases a large portion of this region is occupied by values of $\mu$ for which $\varphi_\mu$ has persistently a hyperbolic limit set (and thus it has a persistent or stable dynamics) - this is discussed in Chapter V. Here, in the first three sections, we describe the bifurcation phenomena mentioned above starting with a quadratic homoclinic tangency and unfolding it as to create homoclinic orbits. And in the last section we relate this unfolding with the well known family of quadratic maps of the interval of $f_\mu(y) = y^2+\mu$.

Let $\phi: M\times\mathbb{R} \to M$ be a $C^3$ map such that $\varphi_\mu = \phi: M\times\{\mu\} \to M$ is a diffeomorphism for each $\mu \in \mathbb{R}$. We shall denote such a family of diffeomorphisms simply by $\{\varphi_\mu\}$ or just $\varphi_\mu$. The reason we take the family to be $C^3$ (and not $C^1$ or $C^2$) come from the discussion of the period doubling bifurcation (or flip) to be presented in the next section of this chapter and is also due to the degree of differentiability of the leaves of the stable and unstable foliations that we need in our setting - see Chapter II. These foliations may themselves be differentiable: if they are codimension one then they are $C^1$ if the map is $C^2$ and $C^{1+\varepsilon}$ ($C^1$ plus Hölder) if the map is $C^3$; see Chapter II and IV.

Let us start studying homoclinic tangencies and their unfolding. Let $p = p_o$ be a fixed point for $\varphi_o$ and let $q$ be a homoclinic tangency related to $p$, that is $q$ is a point of tangency between $W^s(p)$ and $W^u(p)$. We assume that $W^s(p)$ and $W^u(p)$ have a quadratic (parabolic) contact at $q$, and just call $q$ or its orbit $O(q)$ a quadratic homoclinic tangency.



We choose local coordinates $(x_1,x_2)$ near $q$ so that we can express the local components of $W^s(p)$ and $W^u(p)$ containing $q$ by

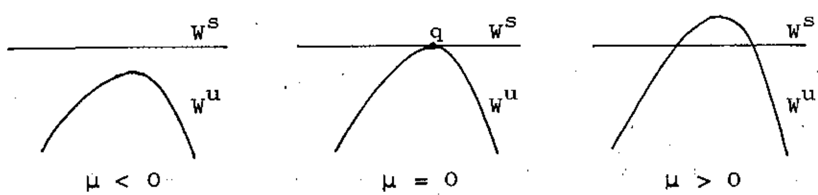
$$W^s(p) = \{(x_1,x_2);\ x_2 = 0\}$$
$$W^u(p) = \{(x_1,x_2);\ x_2 = ax_1^2\} \qquad (1)$$

where $a \neq 0$. Since $p$ is hyperbolic, we have for $\mu$ small a unique fixed point $p_\mu$ near $p$ and the mapping $\mu \to p_\mu$ is differentiable (Implicit Function Theorem). Also the local graphs of $W^s(p_\mu)$ and $W^u(p_\mu)$ near $q$ and $\mu$ near zero, depend differentiably on $\mu$.

Assuming $W^u(p_\mu)$, as function of $\mu$, to move transversaly with respect to $W^s(p_\mu)$ we can write (changing coordinates) that $W^s(p_\mu)$ is given by $x_2 = 0$ and $W^u(p_\mu)$ by

$$x_2 = ax_1^2 + b\mu, \quad a \neq 0 \text{ and } b \neq 0 \qquad (2)$$

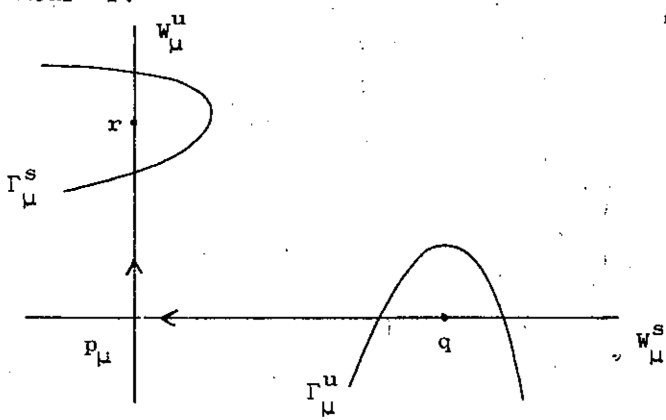
and say that the quadratic homoclinic tangency unfolds generically. These assumptions correspond to mild nondegeneracy conditions; they are satisfied by most (Baire second category) 1-parameter families in our setting. Taking $a < 0$ and $b > 0$ in (2) above, we get for the relative positions of the local components of $W^s(p_\mu)$ and $W^u(p_\mu)$:



We now prove one of the statements in the introduction of this chapter.

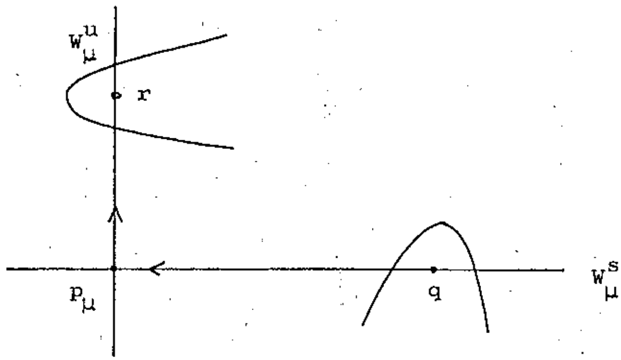
<u>Theorem</u>. Let $\{\varphi_\mu\}$ be a one-parameter family as above with a quadratic homoclinic tangency $q$ at $\mu = 0$ related to the fixed (periodic) saddle $p$ and suppose it unfolds generically. Then there is a sequence $\mu_n \to 0$ such that $\varphi_{\mu_n}$ has homoclinic tangencies $q_{\mu_n} \to q$ related to $p_{\mu_n} \to p$.

<u>Proof</u>. Let $r = \varphi^{-N}(q)$ for some large $N > 0$ and suppose the tangency unfolds into transversal homoclinic points for $\mu > 0$. Given $\mu > 0$ and small, there are small pieces of parabolas (see discussion on unfolding of homoclinic tangencies) $\Gamma_\mu^u \subset W_\mu^u$ near $q$ and $\Gamma_\mu^s \subset W_\mu^s$ near $r$.




Now take $\mu = \hat{\mu}$ arbitrarily small. Clearly, if $n > 0$ is large then $\varphi_{\hat{\mu}}^{-n}(\Gamma_{\hat{\mu}}^s)$ intersects $\Gamma_{\hat{\mu}}^u$. But if we take $\mu > 0$ much smaller than $\hat{\mu}$ we have, for the same integer $n$, that $\varphi_\mu^{-n}(\Gamma_\mu^s) \cap \Gamma_\mu^u = \emptyset$. Since $\varphi_\mu^{-n}(\Gamma_\mu^s)$ and $\Gamma_\mu^u$ depend $C^2$ on $\mu$, there is some $\mu_1 < \hat{\mu}$ for which $\varphi_{\mu_1}^{-n}(\Gamma_{\mu_1}^s)$ and $\Gamma_{\mu_1}^u$ are tangent say at $q_1$.

We can repeat the argument for smaller values of $\hat{\mu}$ and so we can construct the sequences $\mu_n$, $q_{\mu_n}$ as desired, proving the result in the case indicated in the figure above. The reader can easily adapt the arguments to other cases, like the one in the figure below.




<u>Remark</u>. In the proof of the theorem we can take the homoclinic tangencies $q_{\mu_n}$ to be of quadratic contact: due to different curvatures, $\varphi_\mu^{-n}(\Gamma^s_\mu)$ and $\Gamma^u_\mu$ have a quadratic contact at their last tangency for decreasing values of $\mu$. One can even show that these homoclinic tangencies can be chosen to unfold generically.

## 2. Saddle-node and period doubling bifurcations.

We will now show that while creating a horseshoe the family of diffeomorphisms goes through other bifurcations besides infinitely many homoclinic tangencies.

Let us briefly recall two of the three other bifurcations (besides homoclinic of heteroclinic tangencies) that can occur for generic 1-parameter families of diffeomorphisms. Let

$\{\varphi_\mu\}_{\mu\in R}$ be a family and $x_o$ a hyperbolic fixed point for $\varphi_o$. Then for $\mu$ small, $\varphi_\mu$ has a fixed point $x_\mu$, called the continuation of $x_o$, which is near $x_o$ for small $\mu$ and has the same index as $x_o$. Thus for $x_{\mu_o}$ to be a bifurcating orbit, we must have one or two eigenvalues say $\rho_1, \rho_2$ of $d\varphi_{\mu_o}$ at $x_{\mu_o}$ to have norm one. For generic families, we have three possible cases:

a) $\rho_1 = 1$ and $|\rho_2| < 1$ (or $|\rho_2| > 1$),

b) $\rho_1 = -1$ and $|\rho_2| < 1$ (or $|\rho_2| > 1$),

c) $\rho_1 = e^{i\theta}$, $\rho_2 = e^{-i\theta}$ for some real $\theta$.

Case (c), with further generic assumptions, corresponds to the so called Hopf bifurcation and it will not be considered here since we will impose that our mappings will be area contracting. In cases (a) and (b) there is a $C^3$ $\varphi_\mu$-invariant line $W^c_\mu$ or $W^c(p_\mu)$ which is tangent at $\mu = \mu_o$ to the eigenspace associated to $\rho_1 = 1$ or $\rho_1 = -1$. This is called the central manifold for $\varphi_\mu$ (see the Appendix). Thus if we let $f_\mu = \varphi_\mu/W^c_\mu$, we have the following expressions:

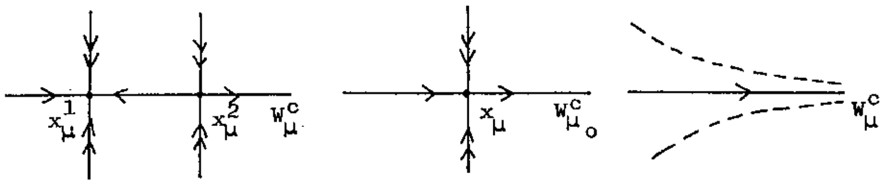
$$f_\mu(x) = x+ax^2+b(\mu-\mu_o)+\text{h.o.t.} \qquad (3)$$

$$f_\mu(x) = -x+ax^3+b(\mu)x+\text{h.o.t}, \; b(0) = 0 \qquad (4)$$

where (3) corresponds to the first case and (4) to the second; h.o.t. stands for higher order terms.

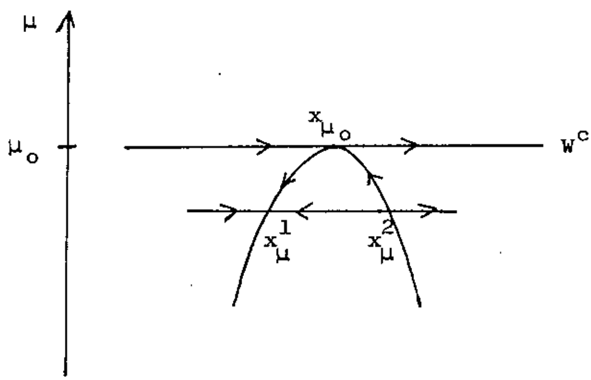
In (3) we take $a \neq 0$ call the orbit a <u>saddle-node</u>. We also take $b \neq 0$ and say that the saddle-node <u>unfolds generically</u>.

These are conditions that are clearly satisfied by generic families. It is easy to see from (3) that $f_\mu$, and thus $\varphi_\mu$, has two hyperbolic fixed points for $\mu < \mu_o$ and none for $\mu > \mu_o$ or vice-versa. If we consider $a > 0$, $b > 0$ and $|\rho_2| < 1$ we have the following unfolding of the saddle-node: a sink and a saddle collapse and then disappear, as shown in the figures:



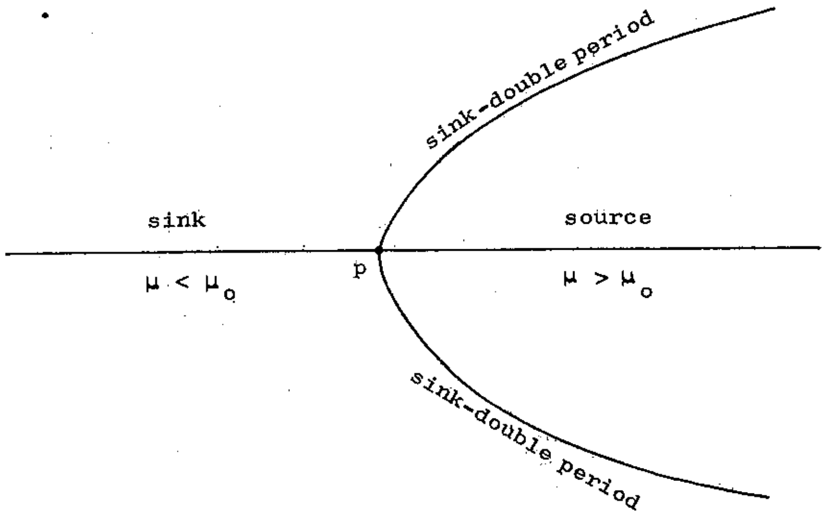

$\mu < \mu_o$ $\mu = \mu_o$ $\mu > \mu_o$

The double arrows in the figures mean that the normal contraction is stronger than along $W_\mu^c$. If we consider the curves $\mu \to x_\mu^1$, $\mu \to x_\mu^2$ of fixed points, we get:



Notice that the two curves are differentiable for $\mu$ small, $\mu < \mu_o$. If we follow the curve $\mu \to x_\mu^2$ for $\mu \uparrow \mu_o$, we can then return along $\mu \to x_\mu^1$ with decreasing values of $\mu$. So the two branches can naturally be oriented as above (or vice-versa). In words: if we follow the curve of saddles for increasing values of $\mu$, up to $\mu = \mu_o$, we then return along the curve of sinks for decreasing values of $\mu$. This fact will play a role in the next theorem.

Now we consider the expression (4) above corresponding to the eigenvalue $\rho_1 = -1$. Similarly to what we have done before in (2) and (3), we take $a \neq 0$ (which is a generic condition) and call the orbit a <u>period doubling bifurcation</u> (or flip) and we say that it <u>unfolds generically</u> if $\left.\frac{db}{d\mu}\right|_{\mu=\mu_o} \neq 0$ (another generic condition!). when $a > 0$ and $\left.\frac{db}{d\mu}\right|_{\mu=\mu_o} < 0$, we can easily show that for $\mu < \mu_o$ there exists a unique fixed point which is a sink and for $\mu > \mu_o$ a source (both with <u>negative eigenvalues</u>) and a <u>period two</u> sink (with <u>positive eigenvalue</u>). Thus the name period doubling bifurcation. The results are of course similar in the other cases, where $a$ and $\left.\frac{db}{d\mu}\right|_{\mu_o}$ may have signs different from the ones above. The assumptions and results are also similar for period doubling bifurcations of periodic orbits by just considering the power of the map equal to the period. For instance, a sink of period $k$ may bifurcate into a source of period $k$ (both with corresponding negative eigenvalues) and a sink with twice the period (and positive eigenvalues). All these considerations are of course along the center manifold; normally to it we may have a contraction or expansion. The unfolding and the curves of fixed and period two periodic orbits are as follows:




In the second figure, if we identify points on the same orbit of the double period sinks we obtain a topological 1-manifold - the curve of sinks for $\mu < \mu_o$ - branching off into two topological 1-manifolds one formed by the curve of sources and the other by the curve of double period sinks. Notice that the sink to the left and the source to the right both have the same period (and a corresponding negative eigenvalue for $df_{\mu}^{k}$, $k$ being the period).

## 3. Cascades of period doubling bifurcations and sinks.

We now discuss the definitions and assumptions of the next theorem showing the existence of many sinks (or sources) and period doubling bifurcations while creating a horseshoe.

Let $R$ be a rectangle in $\mathbb{R}^2$ and $\{\varphi_\mu\}$ a family of diffeomorphisms of $R$ into $\mathbb{R}^2$ such that

1) $\varphi_{-1}(R) \cap R = \emptyset$,

2) $\varphi_\mu / R$ is dissipative (area contracting) for $-1 \lneq \mu \leq 1$, that is $|\det(d\varphi_\mu)| < 1$ on $R$,

3) $\varphi_1$ has periodic points and they are all saddles,

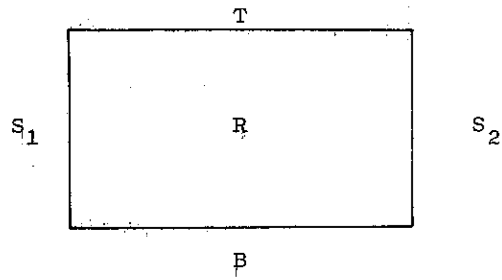
4) $\varphi_\mu(R) \cap S_1 = \emptyset$, $\varphi_\mu(R) \cap S_2 = \emptyset$, $-1 \leq \mu \leq 1$, where $S_1$, $S_2$ are two opposite sides in the boundary of $R$, say the vertical sides,

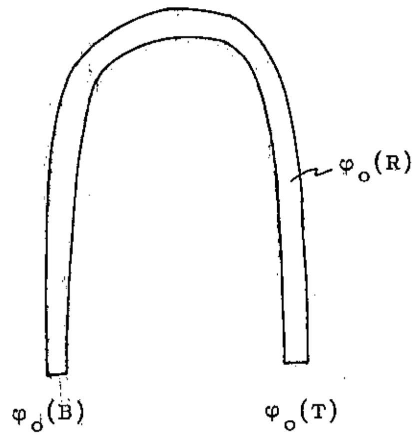
5) $\varphi_\mu(T) \cap R = \emptyset$, $\varphi_\mu(B) \cap R = \emptyset$, $-1 \leq \mu \leq 1$, where $T$ is the top side of $R$ and $B$ is the bottom side.

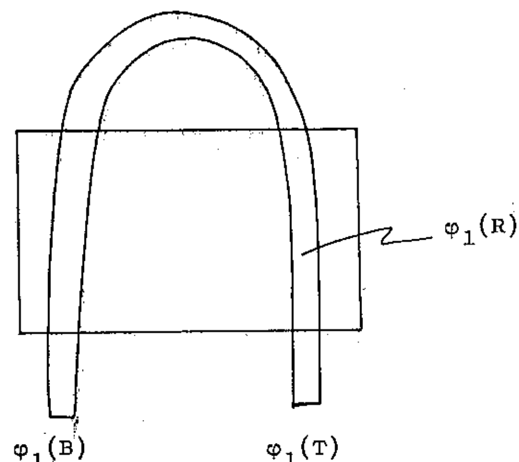
We also consider the following generic (Baire second category) condition on the family $\{\varphi_\mu\}$ which we assumed to be dissipative on $R$

6) $\varphi_\mu$ has at most one nonhyperbolic periodic orbit for each $-1 \leq \mu \leq 1$ and this orbit must be either a saddle-node or a period doubling bifurcation which must unfold generically.

(Because $\varphi_\mu$ is area contracting there is no Hopf bifurcation.) Although we did not formally require $\varphi_1$ to be a horseshoe mapping like in Chapter II, that is precisely the situation we have in mind. In this case, we say we have a family creating a horseshoe, like in the pictures,



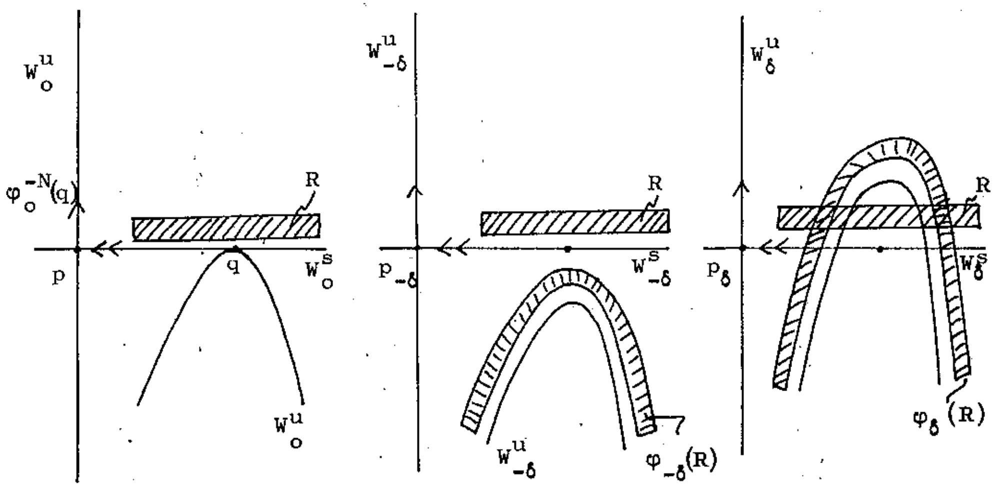



It is also important to notice that when we unfold a generic homoclinic tangency we do obtain a family of diffeomorphisms that creates a horseshoe. Of course, to get the area decreasing property, we assume the Jacobian of the map at the fixed (or periodic) saddle with a homoclinic tangency to have norm less than one. To see this let $\varphi_\mu$ be such that

(i) $\varphi_o$ has a fixed saddle $p$ and $|\det(d\varphi_o)_p| < 1$

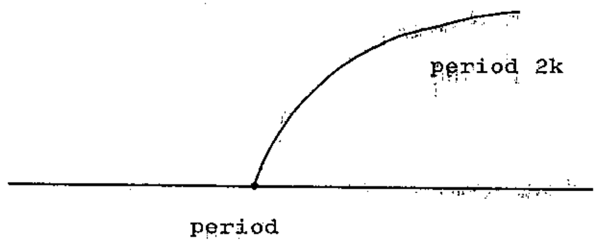
(ii) there is generic homoclinic tangency $q$ associated to $p$.

We then clame that there exists a rectangle $R$, a number $\delta > 0$ and an integer $N > 0$ such that $\varphi_\mu^N/R$ creates a horseshoe for $-\delta < \mu < \delta$: take $R$ to be a very thin rectangle very near $q$ and parallel to the local component of $W^s(p)$, as in the figure



The fact that we can choose $R$, $\delta$ and $N$ as wished comes from the following considerations. In $C^1$ coordinates linearizing $\varphi_o$ and defined in a neighbourhood of $\widehat{pq}$ in $W^s_o$, we choose $R$ to be thin and sufficiently close to $W^s_o$ so that its projection on $W^u_o$ parallel to $W^s_o$ contains $\varphi_o^{-N}(q)$ for some large $N$ as well as a fundamental domain $\widehat{z,\varphi_o(z)}$ in $W^u_o$. One can then apply arguments similar to the ones in Chapter II to show that $\varphi_\mu^N/R$ is area decreasing for $-\delta \le \mu \le \delta$ and that $\varphi_\delta^N/R$ has its maximal invariant set hyperbolic with dense subset of periodic orbits. We point out that the construction above also follows directly from the last section of the present chapter. In fact we observe that although the configuration $R$, $\varphi_\delta(R)$ resembles the situation in Chapter II, the rectangles considered are quite different: there we had a loog rectangle containing $p$ and $q$; here the rectangle is contained in a small neighbourhood of $q$.

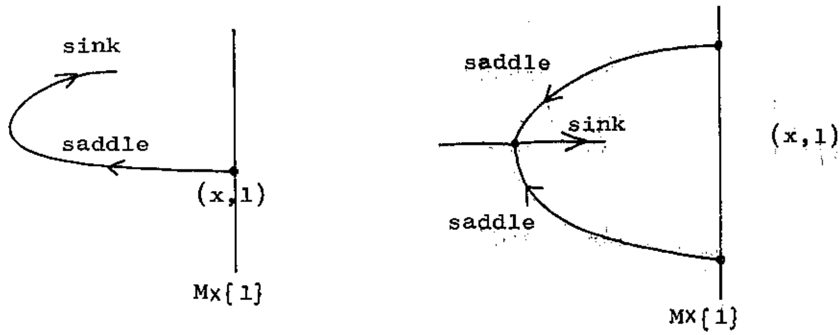
For $\varphi_\mu: R \to \mathbb{R}^2$ as above, let $Per(\varphi_\mu)$ be the set of periodic orbits of $\varphi_\mu$ and $P = \{(x,\mu);\ x \in Per(\varphi_\mu)\}$. We now define the topological space $\tilde{P} = P/\sim$, where the equivalence relation $\sim$ is the identification of points in the same orbit. A component of $\tilde{P}$ through $(O(x),\mu)$ is a continuous curve in $\tilde{P}$ passing through $(O(x),\mu)$, where $O(x)$ denotes the orbit of $x$. Notice that $\tilde{P}$ looks locally like a curve except at period doubling (or undoubling) bifurcation where it branches and looks like



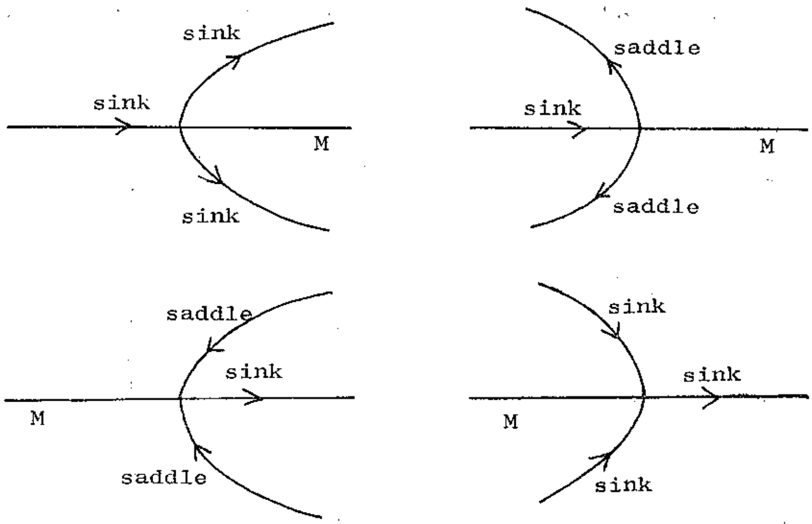

<u>Theorem</u> [35]. Let $\varphi_\mu : R \to \mathbb{R}^2$ be a family of diffeomorphisms satisfying conditions (1) to (6). Then each $(O(x),1) \in \tilde{P}$ has a component containing attracting periodic orbits (sinks) of period $2^n k$ for each $n \geq 0$, where $k$ is the period of $x$ for $\varphi_1$.

<u>Proof</u>. Let $(O(x),1) \in \tilde{P}$ and assume first that $d\varphi_1^k(x)$ has positive eigenvalues, $k$ being the period of $x$ (for simplicity we are assuming that $\varphi_\mu$ preserves orientation). By the Implicit Function Theorem, there is a (unique) continuous path $\Gamma$ on $\tilde{P}$ through $(O(x),1)$ which we follow for decreasing values of $\mu$. We then must reach either a saddle-node or a period undoubling bifurcation for otherwise we could follows $\Gamma$ up to $M \times \{-1\}$; by conditions (4) and (5) the maximal invariant set of $\varphi_\mu$ in $R$ is bounded away from $\partial R$ (periodic points can not escape through $\partial R$) and we can not terminate $\Gamma$ in $M \times (-1,1)$ because we can always prolong a path of saddle points. But $\varphi_{-1}$ has no periodic points in $R$ and so we must reach a saddle-node or a period undoubling bifurcation. In both cases we then follow the path of sinks that emanates from the bifurcating orbit changing the direction relative to $\mu$ - i.e., increasing values of $\mu$ (see dis-

cussions before on saddle-nodes and period doubling bifurcations):

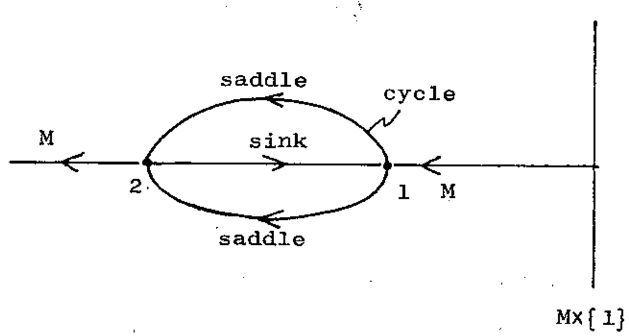



In what follows we alsays prolong $\Gamma$ at a period doubling or undoubling bifurcation avoiding saddles with corresponding negative eigenvalues (Moebius paths) and orient the path + (or to the right or to increasing values of $\mu$) if it is a path of sinks and - if it is a path of saddles. Now we can not reach back $M \times \{1\}$ since $\varphi_1$ has only periodic saddles and also we can not have a cycle, i.e. a closed oriented path of periodic orbits, since the initial path starting at $M \times \{1\}$ could not belong to it (see figures above). So prolonging $\Gamma$ we must reach either a saddle-node and we can again prolong $\Gamma$ following the orientation above or we reach a period doubling or undoubling bifurcation, in which case we can also prolong $\Gamma$ in a unique way avoiding Moebius paths, as in the figures




It is then clear, following the above procedure, that we do not return to $M \times \{1\}$ for the reasons we explained before. We also claim we can not terminate $\Gamma$ in $M \times (-1,1)$ if we go through only finitely many bifurcating orbits or even infinitely many ones say $(x_i,\mu_i)$ with bounded periods. In fact, in the first case, from our discussions on hyperbolic and generic bifurcating periodic orbits, we could clearly prolong $\Gamma$. In the second case we can consider a limiting point $(\tilde{x},\mu)$ of $(x_i,\mu_i)$ and argue that $(\tilde{x},\mu) \in P$ and then by the genericity assumption (6) on $\varphi_\mu$, $(\tilde{x},\mu)$ had to be locally isolated as a bifurcating orbit of bounded period which is not the case. Thus we must go through infinitely many bifurcating periodic orbits with unbounded periods. This can be achieved only if we go through infinitely many period doubling bifurcations with unbounded periods, which then clearly implies the result in this case where we started the path at a saddle $(O(x),1) \in \tilde{P}$ with positive eigenvalues.

Let us now begin with $(O(x),1) \in \tilde{P}$ such that the eigenvalues of $d\varphi_1^k(x)$ are negative, $k$ the period of $x$, and a path through it in $\tilde{P}$ - a Moebius path. We will show that the result is also true in this case. Before we do that, let us again orient our paths in the positive $\mu$-direction along a path of sinks and in the negative $\mu$-direction both along a path of saddles with positive eigenvalues (which we just call path of saddles) and along a Moebius path. So let $\Gamma$ be a Moebius path starting at $M \times \{1\}$. As argued before, $\Gamma$ must go through bifurcating orbits and the first one must be a period doubling bifurcation. We then follow the path of saddles of twice the period that emanates from it. At the next bifurcating orbit we repeat the procedure of prolonging $\Gamma$ along the unique non Moebius path emanating from it. But already at this point we may get a cycle! That is, a closed oriented path of periodic orbits not containing any Moebius curves. The figure below illustrates this possibility



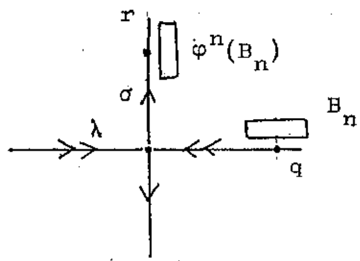
In this case, we proceed with the Moebius path in the left. This path can not return to the cycle or to $M \times \{1\}$ and so it must go through a new bifurcating orbit. So we can repeat the same procedure of prolonging $\Gamma$ with non Moebius paths. We then may or may not create a new cycle. If not, we argue as in the first case and obtain the result. If we create a new cycle then necessarily we must have the same number of period doubling and period undoubling bifurcations - see previous figures. Each one has a Moebius path emanating from it. We claim that one of these Moebius path must go through a new - that is, not in the cycles - period doubling bifurcating orbit: if not, then each Moebius path in the cycles must have its end points in a well defined pair period undoubling - period doubling bifurcations, but this contradicts the fact that the first one is connected to $M \times \{1\}$ (and so it is not attached to any period undoubling bifurcation). This shows that, with the procedure we have stablished, we can always prolong $\Gamma$ in a way that new bifurcating orbits are introduced. As before their periods must be unbounded, which implies the result also in this case. The proof of the theorem is complete.

## 4. Homoclinic tangencies, scaling and quadratic maps.

We consider a one-parameter family of diffeomorphisms $\varphi_\mu : M \to M$, $M$ a 2-manifold, which has for $\mu = 0$ a homoclinic tangency. Let $p_\mu$ denote the saddle point of $\varphi_\mu$ which is related, for $\mu = 0$, to this tangency. We assume the tangency of

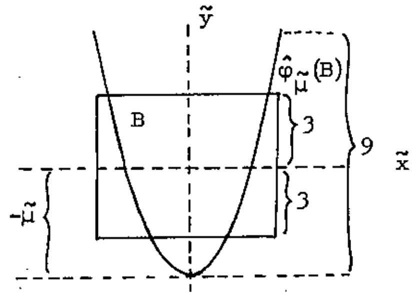
$W^u(p_o)$ and $W^s(p_o)$ to be generic (parabolic contact) and also to unfold generically.

Before we go into technicalities, we want to give a heuristic idea of the construction to be described in this section and its consequences. Near $p_o$ we take linearizing coordinates x, y so that $\varphi(x,y) = (\lambda.x, \sigma.y)$ with $0 < |\lambda| < 1 < |\sigma|$. We assume $\lambda$ and $\sigma$ to be positive (otherwise we replace $\varphi$ by $\varphi^2$), and that $\lambda.\sigma < 1$ (if $\lambda.\sigma > 1$ we replace $\varphi$ by. $\varphi^{-1}$ and if $\lambda.\sigma = 1$ our construction does not work). Let q and r be




points on the orbit of tangency in the domain of the linearizing coordinates as indicated. So, for some N, $\varphi^N(r) = q$. For each sufficiently big n, we take a box $B_n$ near q such that $\varphi^n(B_n)$ is a box near r as indicated. We consider $\varphi_\mu^{n+N}(B_n)$, and especially its position relative to $B_n$. As was already announced earlier, if one chooses $B_n$ carefully then, for n sufficiently big, $\varphi_\mu^{n+N}(B_n)$ will cross over $B_n$ so as to create a horseshoe. We shall not only prove this but even show that, after applying n-dependent coordinate transformations to both the x, y variables and the $\mu$ variable (denoting the new variables by $\tilde{x}$, $\tilde{y}$, and $\tilde{\mu}$), $\varphi_{\tilde{\mu}}^{n+N}$ converges for $n \to \infty$ to the map $\hat{\varphi}_{\tilde{\mu}}$, given by $\hat{\varphi}_{\tilde{\mu}}(\tilde{x},\tilde{y}) = (\tilde{y}, \tilde{y}^2+\tilde{\mu})$. (*)

Taking the box $B_n$ in these (n-dependent) coordinates equal to $B = \{(\tilde{x},\tilde{y}) \mid |\tilde{x}| \le 3, |\tilde{y}| \le 3\}$ we get the horseshoe




formation when $\tilde{\mu}$ decreases from say 4 to -4, at least for n sufficiently big.

Note that this limiting map is not a diffeomorphism any more. This is related to the fact that $\varphi_\mu$ is area contracting at $p_\mu$ and hence that $\varphi^{n+N}$, for $n \to \infty$, becomes more and more area contracting.

For the limiting map, the value $\tilde{x}$ is unimportant. Restricting to the $\tilde{y}$ variable we have

$$\tilde{y} \longmapsto \tilde{y}^2 + \tilde{\mu},$$

which is the well known one-parameter family of quadratic one-dimensional maps which was studied e.g. in [6].

This being the limiting map, $\varphi_\mu^{n+N}$ "contains" approxima-

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(*) In the final statement of the result, we replace n+N by n.

tions of this one-parameter family and hence contains much of its complexity, see [34]: hyperbolic sets, period doubling etc..

Because it will be used later, we give here one example of extending a fact about quadratic maps to the one-parameter families like $\varphi_\mu$. For $\tilde{\mu}$ near zero, the map $\tilde{y} \mapsto \tilde{y}^2 + \tilde{\mu}$ has an attracting fixed point near zero. Let $\mu_n \to 0$ be the sequence of $\mu$-values corresponding to $\tilde{\mu} = 0$ in the different reparametrizations of the $\mu$-variable. Then for $n$ sufficiently big and $\mu$ near $\mu_n$, $\varphi_\mu^{n+N}$ has an attracting fixed point.

Now we give a more formal and complete description of the result. First we have to state some extra assumptions on the 1-parameter family $\varphi_\mu$. As we mentioned already we assume that the eigenvalues $\lambda$, $\sigma$ of $(d\varphi_o)p_o$ are positive and satisfy $\lambda.\sigma < 1$. Also we need $C^2$ linearizing coordinates of $p_\mu$. For this reason we require $\varphi_\mu(x)$ to be $C^\infty$ in $(\mu,x)$. The $C^2$ linearizing coordinates ($\mu$-dependent) then exist, provided some generic (even open and dense) conditions are satisfied by the eigenvalues $\lambda$ and $\sigma$, see [32].

<u>Theorem</u>. For a one-parameter family $\varphi_\mu$ as above, with $q$ a point on the orbit of tangency for $\mu = 0$, there are for each positive integer $n$ reparametrizations $\mu = M_n(\tilde{\mu})$ of the $\mu$ variable and $\tilde{\mu}$-dependent coordinate transformations

$$(\tilde{x},\tilde{y}) \mapsto \Psi_{n,\tilde{\mu}}(\tilde{x},\tilde{y})$$

such that:

- for each compact set $K$ in the $\tilde{\mu}$, $\tilde{x}$, $\tilde{y}$ space the images of $K$ under the maps

$$(\tilde{\mu},\tilde{x},\tilde{y}) \longmapsto (M_n(\tilde{\mu}),\Psi_{n,\tilde{\mu}}(\tilde{x},\tilde{y}))$$

converge, for $n \to \infty$, to $(0,q)$;

- the domains of the maps

$$(\tilde{\mu},\tilde{x},\tilde{y}) \longmapsto (\tilde{\mu},(\Psi^{-1}_{n,\tilde{\mu}}\circ\varphi^{n}_{M_n(\tilde{\mu})}\circ\Psi_{n,\tilde{\mu}}))$$

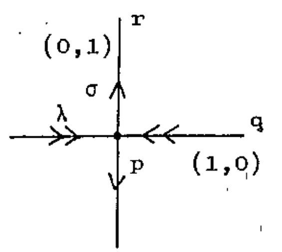
converge, for $n \to \infty$ to all of $\mathbb{R}^3$, and the maps converge, for $n \to \infty$, to the map

$$(\tilde{\mu},\tilde{x},\tilde{y}) \longmapsto (\tilde{\mu},\hat{\varphi}_{\tilde{\mu}}(\tilde{x},\tilde{y}))$$

with $\hat{\varphi}_{\tilde{\mu}}(\tilde{x},\tilde{y}) = (\tilde{y},\tilde{y}^2+\tilde{\mu})$.

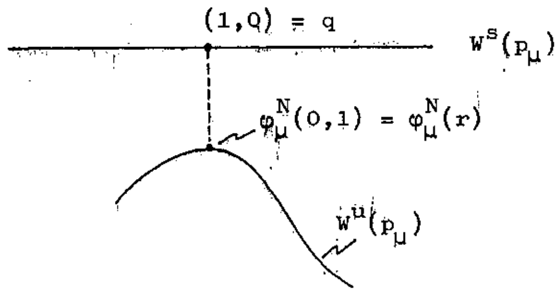
This theorem is an expanded version of a remark in §6.7 (p.336) of [7].

Proof. We start with carefully choosing $\mu$-dependent $C^2$ linearizing coordinates near $p_\mu$. We denote them by $(x,y)$. For $\mu=0$ we have $\varphi_0(x,y) = (\lambda x,\sigma y)$ with $0 < \lambda < 1 < \sigma$ and $\lambda.\sigma < 1$. Let $q$ be a point on the orbit of tangency in the "local" stable




manifold of $p$ and $r$ such a point on the "local" unstable manifold of $p$. By multiplying $x$, $y$ with constants, we arrange that $q = (1,0)$ and $r = (0,1)$. Since both $r$ and $q$ are on the orbit of tangency, there is an $N$ such that $\varphi_o^N(r) = q$. For $\mu$ near zero we adapt our linearizing coordinates so that:

- $\varphi_\mu^N(0,1)$ is a local maximum of the y-coördinate restricted to $W^u(p_\mu)$;
- the x-coordinate of $\varphi_\mu^N(0,1)$ is 1.




We "reparametrize" our bifurcation parameter $\mu$ in such a way that the y-coordinate of $\varphi_\mu^N(0,1)$ is $\mu$.

With all these preparations we can write $\varphi_\mu^N$, near $(0,1)$, as

$$(x,1+y) \mapsto (1,0) + (H_1(\mu,x,y), H_2(\mu,x,y))$$

with

$$H_1(\mu,x,y) = \alpha . y + \tilde{H}_1(\mu,x,y),$$

$$H_2(\mu,x,y) = \beta . y^2 + \mu + \gamma . x + \tilde{H}_2(\mu,x,y)),$$

where $\alpha$, $\beta$, $\gamma$ are non-zero constants, and where, for $\mu = x = y = 0$:

$$* \quad \begin{cases} \tilde{H}_1 = \partial_y \tilde{H}_1 = \partial_\mu \tilde{H}_1 = 0, \\ \tilde{H}_2 = \partial_x \tilde{H}_2 = \partial_y \tilde{H}_2 = \partial_\mu \tilde{H}_2 = \partial_{yy} \tilde{H}_2 = \partial_{y\mu} \tilde{H}_2 = \partial_{\mu\mu} \tilde{H}_2 = 0. \end{cases}$$

The functions $H_i$ and $\tilde{H}_i$ are clearly $C^2$ since $\varphi_\mu$ is $C^\infty$ and the $x,y$-coordinates are $C^2$.

Next we define an n-<u>dependent</u> reparametrization of $\mu$ and $\mu$-dependent coordinate transformation by the following formulas:

$$\mu = \sigma^{-2n}.\bar{\mu} - \gamma.\lambda^n + \sigma^{-n} \qquad \bar{\mu} = \sigma^{2n}.\mu + \gamma.\lambda^n.\sigma^{2n} - \sigma^n$$

$$x = 1 + \sigma^{-n}.\bar{x} \qquad \bar{x} = \sigma^n.(x-1)$$

$$y = \sigma^{-n} + \sigma^{-2n}.\bar{y} \qquad \bar{y} = \sigma^{2n}.y - \sigma^n.$$

Note that these are <u>not</u> yet the final reparametrizations and coordinate transformations but the final ones will be easily deducible from these.
Note also that $\sigma$ and $\lambda$ depend on $\mu$ and hence on $\bar{\mu}$ although this is not expressed in the above formulas.
Finally note that a fixed box in the $\bar{x}$, $\bar{y}$ coordinates, gives for $n \to \infty$ boxes converging to $q$ in the $x$, $y$ coordinates.

We now start with our main calculation: expressing $\varphi_\mu^{n+N}$ in terms of $\bar{\mu}$, $\bar{x}$ and $\bar{y}$.

Let $(\bar{\mu},\bar{x},\bar{y})$ denote a point. The $(\mu,x,y)$ variables of this point are (see above):

$$\mu = \sigma^{-2n}.\bar{\mu} - \gamma.\lambda^n + \sigma^{-n} \qquad x = 1 + \sigma^{-n}.\bar{x}$$

$$y = \sigma^{-n} + \sigma^{-2n}.\bar{y}.$$

After applying $\varphi_\mu^n$ to this point we get as $x$, $y$ coordinates ($\mu$ does not change):

$$x = \lambda^n.(1+\sigma^{-n}\bar{x}) \qquad y = 1 + \sigma^{-n}.\bar{y}$$

Next we apply $\varphi_\mu^N$ and find:

$$x = 1 + \alpha.\sigma^{-n}.\bar{y} + \tilde{H}_1(\mu,\lambda^n.(1+\sigma^{-n}\bar{x}),\sigma^{-n}.\bar{y})$$

$$y = \beta.\sigma^{-2n}.\bar{y}^2 + (\sigma^{-2n}\bar{\mu} - \gamma.\lambda^n + \sigma^{-n})$$

$$+ \gamma.\lambda^n.(1+\sigma^{-n}\bar{x}) + \tilde{H}_2(\mu,\lambda^n.(1+\sigma^{-n}),\sigma^{-n}.\bar{y}).$$

Transforming this back to the $\bar{x}$, $\bar{y}$ coordinates, and denoting the values of these coordinates of the new point by $\bar{\bar{x}}$, $\bar{\bar{y}}$ we have:

$$** \begin{cases} \bar{\bar{x}} = \alpha\bar{y}+\sigma^n.\tilde{H}_1(\sigma^{-2n}.\bar{\mu}-\gamma.\lambda^n+\sigma^{-n},\ \lambda^n.(1+\sigma^{-n}\bar{x}),\ \sigma^{-n}.\bar{y}) \\ \bar{\bar{y}} = \beta\bar{y}^2+\bar{\mu}+\gamma.\lambda^n.\sigma^n.\bar{x}+\sigma^{2n}.\tilde{H}_2(\sigma^{-2n}.\bar{\mu}-\gamma.\lambda^n+\sigma^{-n}, \\ \qquad \lambda.(1+\sigma^{-n}\bar{x}),\sigma^{-n}.\bar{y}). \end{cases}$$

Next we need to show that in the above expression certain parts converge to zero for $n \to \infty$ in the $C^2$ topology (uniformly on compacta in the $\bar{\mu}$, $\bar{x}$, $\bar{y}$ coordinates).

In the expression for $\bar{\bar{y}}$, the term $\gamma.\lambda^n.\sigma^n.\bar{x}$ goes clearly to zero because $\lambda\sigma < 1$. The terms involving $\tilde{H}_i$ are more complicated. We first observe that when

$$(\bar{\mu},\bar{x},\bar{y})$$

remains bounded, the corresponding values of

$$(\mu,x,(y-1))$$

which are substituted in $\tilde{H}_i$ satisfy:

$$\left.\begin{aligned} \mu &= O(\sigma^{-n}) \\ x &= O(\lambda^n) \\ y-1 &= O(\sigma^{-n}) \end{aligned}\right\} \quad ***$$

as $n$ goes to infinite.

Next we define

$$\bar{H}_1(\bar{\mu},\bar{x},\bar{y}) = \sigma^n.\tilde{H}_1(\sigma^{-2n}.\bar{\mu}-\gamma.\lambda^n+\sigma^{-n},\ \lambda^n.(1+\sigma^{-n}.\bar{x}),\sigma^{-n}.\bar{y}).$$

Then

$$\bar{H}_1(0,0,0) = \sigma^n.\tilde{H}_1(-\gamma.\lambda^n+\sigma^{-n},\lambda^n,0) = \sigma^n.(O(\lambda^n)+O(\lambda^{2n})),$$

which converges to zero for $n \to \infty$. Next, the first and second order derivatives of $\bar{H}_1(\bar{\mu},\bar{x},\bar{y})$ converge to zero (uniformly on compacta); this follows from $*$ , $*^*{*}$ , and $0 < \lambda\sigma < 1$ - in fact the derivatives of $\bar{H}_1$ are easier to estimate than $\bar{H}_1$ itself. This is the way in which one proves that $\bar{H}_1$ goes to zero as announced. The same procedure works for the corresponding expression in the formula for $\bar{\bar{y}}$.

So for $n \to \infty$ the transformation formulas $**$ converge to

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} \mapsto \begin{pmatrix}\alpha\bar{y}\\ \beta\bar{y}^2 + \bar{\mu}\end{pmatrix}.$$

By the substitution

$$\bar{\mu} = \beta^{-1}\tilde{\mu}$$

$$\bar{x} = \alpha\beta^{-1}\tilde{x}$$

$$\bar{y} = \beta^{-1}\tilde{y},$$

this limiting transformation becomes

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} \longmapsto \begin{pmatrix}\tilde{y}\\ \tilde{y}^2+\tilde{\mu}\end{pmatrix}.$$

Now the theorem is essentially proved: we have the announced transformation as limit of $\varphi_\mu^{n+N}$, composed with suitable coordinate transformations and reparametrizations of $\mu$; the difference between $\varphi^n$ in the statement of the theorem and $\varphi^{n+N}$ in the proof is immaterial - N is fixed and the conclusion in only for n, or N+n, big.

# CHAPTER IV

# CANTOR SETS

As already indicated in earlier chapters, the closure of a set of homoclinic intersections is often a Cantor set. In the following chapters we shall have to impose, in the formulations of several theorems, conditions on such Cantor sets. These conditions will involve numerical invariants which we discuss in this chapter. Since the Cantor sets occuring are not of the most general type we begin our discussion with the description of "dynamically defined Cantor sets": they form the class of Cantor sets with which we mainly deal.

## 1. Dynamically defined Cantor sets.

We consider the following situation: $\varphi: M \to M$ is a $C^3$-diffeomorphism of a 2-manifold $M$ which has a hyperbolic fixed point $p$ of saddle type which is part of a non-trivial basic set $\Lambda$, i.e. of a hyperbolic invariant set, whose periodic points are dense, and which has a dense orbit - as an example one may think of the "maximal invariant subset of $R$" as analysed in Chapter II. By definition, $\Lambda \cap W^s(p)$, as a subset of $W^s(p)$ is a dynamically defined Cantor set. With this we mean the following. Let $\alpha: \mathbb{R} \to W^s(p)$ be a smooth identification, say such that $\alpha^{-1}\circ(\varphi|W^s(p))\circ\alpha$ is a linear contraction (see [31]) and let $K$ be an open and compact neighbourhood of $0$ in $\alpha^{-1}(W^s(p) \cap \Lambda)$, then $K$ is called a <u>dynamically defined Cantor set</u>. Usually we even assume that $K$ is obtained by intersecting $\alpha^{-1}(W^s(p) \cap \Lambda)$ with an interval $K_o \subset \mathbb{R}$, containing $0$, and whose boundary points are not contained in $\alpha^{-1}(W^s(p) \cap \Lambda)$. We discuss some of the main properties of these Cantor sets; we continue to refer to the above situation.

<u>Scaling</u>. If $\alpha^{-1}\circ(\varphi|W^s(p))\circ\alpha$ is a linear contraction by $\lambda$, which we assume positive, then, since $\Lambda$ is invariant under $\varphi$,

$$\lambda.K = K \cap (\lambda.K_o)$$

where, for $A \subset \mathbb{R}$ and $\lambda \in \mathbb{R}$, $\lambda.A = \{\lambda.a \mid a \in A\}$. This means that the choice of the interval $K_o$ is not very essential: in each interval $[\lambda.a,a]$ one has the same geometry.

<u>Expanding structure</u>. There is a smooth expanding map $\Psi: K \to K$ with some remarkable properties. We first construct this map. As in Chapter II we choose an unstable foliation $\mathfrak{F}^u$, defined on a neighbourhood $U$ of $\Lambda$. Since the diffeomorphism $\varphi$ is $C^3$, this foliation is $C^{1+\varepsilon}$. If the interval $K_o$ is sufficiently big, then we have a projection $\pi$, along leaves of $\mathfrak{F}^u$, of a neighbourhood $U'$ of $\Lambda$ to $\alpha(K_o)$; clearly $\pi(\Lambda) = \alpha(K)$. This projection is in general not unique: one leaf of $\mathfrak{F}^u$ may have more than one intersection with $\alpha(K_o)$. Since $\Lambda$ is totally disconnected, one can still make $\pi$, on a small neighbourhood of $\Lambda$, continuous and hence differentiable (in fact $C^{1+\varepsilon}$). The derivative of $\pi|(W^s(p) \cap U')$ is bounded and bounded away from zero since the components of $W^s(p) \cap U'$ are leafs of the stable

foliation, which is transverse to $\mathcal{F}^u$. For $N$ sufficiently big,

$$\Psi = \alpha^{-1} \circ \pi \circ \varphi^{-N} \circ \alpha : K_o \to K_o$$

is, where defined, expanding in the sense that the derivative has norm bigger than one. Indeed, the possible contractions in $\pi | (W^s(p) \cap U')$ are compensated by $\varphi^{-N}$. From the above construction it follows that $\Psi(K) = K$ (we also denote $\Psi|K$ by $\Psi$) and that $\Psi$ is $C^{1+\epsilon}$ on a neighbourhood of $K$.

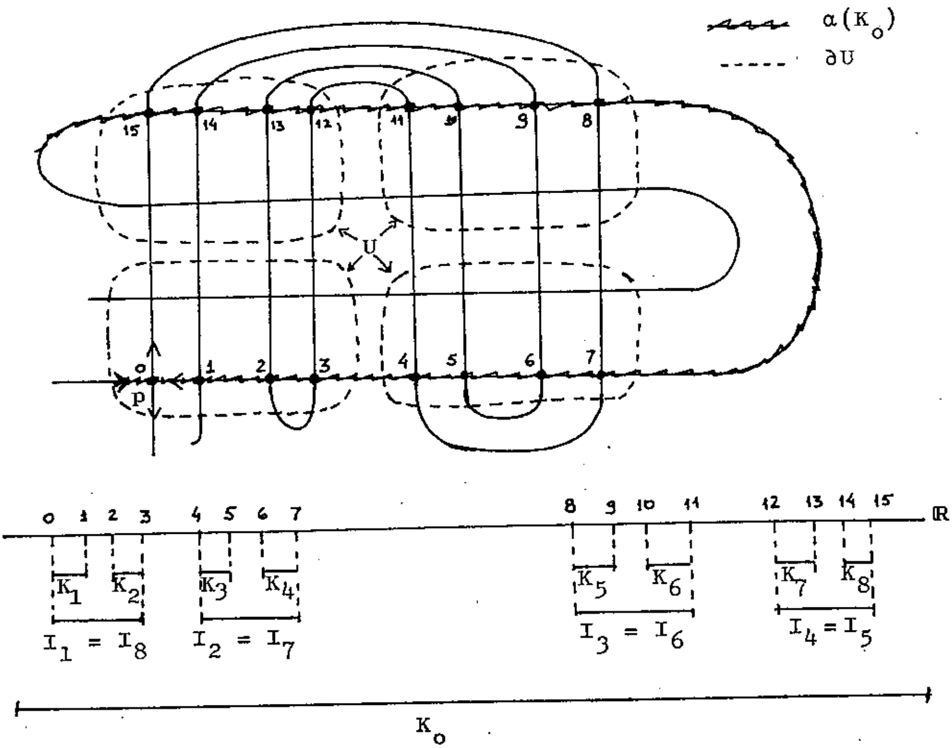
Our assumption that $K_o$ has to be sufficiently gib is no real restriction due to the scaling property. The non-uniqueness of $\Psi$, due to the non-uniqueness of $\pi$ is still a problem to which we shall return in the discussion of Markov partitions.

Markov partitions. For a Cantor set $K$ and an expanding map $\Psi$ as above, we define a Markov partition as a finite set of disjoint intervals $K_1, \ldots, K_k \subset K_o$ such that

- $\Psi$ is defined on a neighbourhood of each $K_i$, $i \geq 1$;
- $K$ is contained in $\bigcup_{i=1}^{k} K_i$, and the boundary of each $K_i$ is contained in $K$;
- for each $i \geq 1$, $\Psi(K_i)$ is an interval, which is the convex hull (in $\mathbb{R}$) of a finite collection of the intervals of the Markov partition;
- for each $i \geq 1$ and $n$ sufficiently big, $\Psi^n(K \cap K_i) = K$.

For a given Cantor set $K$ as above there are Markov partitions; one can even make the intervals $K_i$ as small as one wishes. For the construction of such Markov partitions one needs however to make special choices when defining $\Psi$. The existence of these Markov partitions follows from the construction of Markov partitions for basic sets of diffeomorphisms, applied to the basic set $\Lambda$ of $\varphi$, see [5].

Here we only indicate how to make such a Markov partition when our basic set $\Lambda$ is the horseshoe (see Chapter I). In this case $W^u(p)$ and $W^s(p)$ are as indicated below; $\Lambda = \overline{W^u(p) \cap W^s(p)}$.



In $W^s(p)$ we indicated 16 intersections with $W^u(p)$ (numbered from 0 to 15), they are all in $\alpha(K_o)$. In a "separate copy of $\mathbb{R}$" we indicate the inverse images of these points by $\alpha$ and indicate the intervals $K_1,\dots,K_8$ of the Markov partition.

In the figure with $W^u(p)$ and $W^s(p)$ we indicated $\alpha(K_o)$ and U, the neighbourhood of $\Lambda$ on which we assume $\mathfrak{F}^u$ to be defined; note that with this choice of U and $\alpha(K_o)$, the projection $\pi$ (projecting U along fibres of $\mathfrak{F}^u$ to $\alpha(K_o)$) is uniquely defined. As expanding map we take

$$\Psi = \alpha^{-1}\circ\pi\circ\varphi^{-1}\circ\alpha .$$

The action of $\Psi$ on the points $0,\dots,16$ is then given by:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| 0 | 3 | 4 | 7 | 8 | 11 | 12 | 15 |
| ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 |

Taking as intervals of the Markov partition $K_1,\dots,K_8$ as indicated in the above figure, their images are $\Psi(K_i) = I_i$. From this it is simple to verify that $K_1,\dots,K_8$ forms indeed a Markov partition.

Observe that if $K_1,\dots,K_k$ is a Markov partition of a dynamically defined Cantor set K with expanding map $\Psi$, one gets a Markov partition with more and shorter intervals by just taking as new intervals connected components of $\Psi^{-1}(K_i)$. So one can refine a Markov partition without redefining $\Psi$.

So far we have seen the basic properties of dynamically defined Cantor sets. For the purpose of what follows it is convenient to use these properties as definition.

<u>Definition</u>. A <u>dynamically defined Cantor set</u> is a Cantor set $K \subset R$, together with an interval $K_0$, $K \subset K_0 \subset \mathbb{R}$, a real number $\lambda$, $0 < |\lambda| < 1$, and a $C^{1+\varepsilon}$ map $\Psi: K \to K$, which can be extended to a neighbourhood of $K$, so that all the above assertions about scaling, expanding structure and Markov partitions hold.

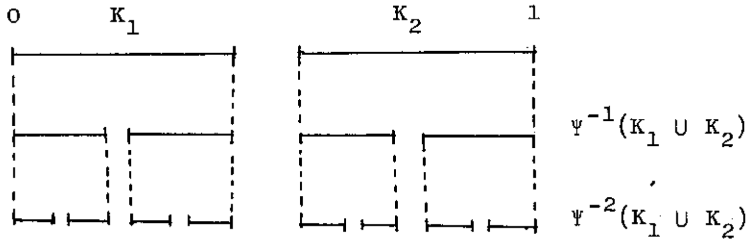
<u>Examples</u>. In each of the examples below we <u>define</u> the Cantor set by a Markov partition and expanding map. Observe that with a Markov partiton $\{K_1,\ldots,K_k\}$ and expanding map $\Psi$, we have as Cantor set $K = \bigcap_{i=0}^{\infty} \Psi^{-i}(K_1 \cup \ldots \cup K_k)$. Further, we consider only examples where $\Psi|K_i$ is affine, i.e. has constant derivative.

Our first example is the <u>mid-$\alpha$-Cantor set</u>. In this case

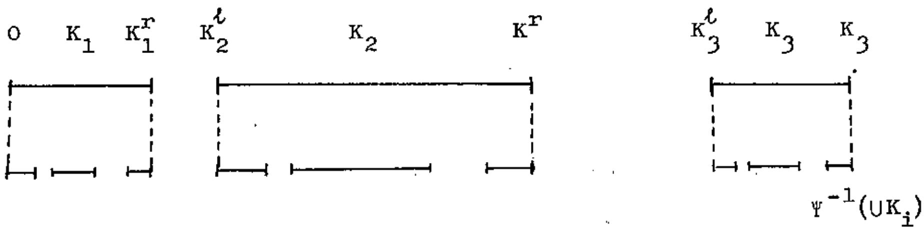
$K_1 = [0, \frac{1}{2}(1-\alpha)]$

$K_2 = [\frac{1}{2}(1+\alpha), 1]$

and $\Psi|K_i$ maps $K_i$ affinely to $[0,1]$; the scaling constant can be taken as $\lambda = \frac{1}{2}(1-\alpha)$. For $\alpha = 1/3$ this is the most well known Cantor set; in any case, for this construction one needs $0 < \alpha < 1$.




The second example, or rather a class of examples, covers the affine Cantor sets. They are defined by a sequence of intervals $K_1,\ldots,K_k$ with endpoints $K_i^\ell$, $K_i^r$ so that $0 = K_1^\ell < K_1^r < K_2^\ell < K_2^r < K_3^\ell < \ldots < K_k^r$; $\Psi|K_i$ maps $K_i$ affinely onto $[0,K_k^r]$; as scaling constant one can take $\lambda = K_k^r/K_1^r$.

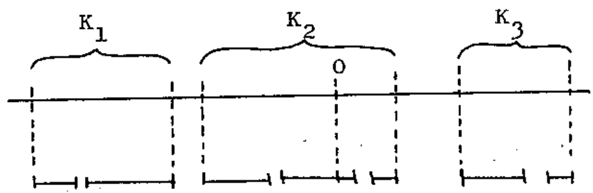



Finally we define generalized affine Cantor sets. They are obtained as the affine Cantor sets, only now the image $\Psi(K_i)$ may be smaller. If we denote the endpoints of $K_i$ as above by $K_i^\ell$ and $K_i^r$ with $.. < K_i^\ell < K_i^r < K_{i+1}^\ell < ..$ then $\Psi(K_i)$ should just be an interval of the form $[K_{j_i}^\ell, K_{j'_i}^r]$ with $j_i \le j'_i$. In this case one still has to verify whether $\Psi$ is expanding, whether there is scaling and whether $\Psi^n(K_i \cap K) = K$ for big $n$. In the special example below we have

$\Psi(K_1 \cap K) = (K_1 \cup K_2) \cap K$

$\Psi(K_2 \cap K) = K$

$\Psi(K_3 \cap K) = (K_2 \cup K_3) \cap K$




One sees that $\Psi$ is expanding, $\Psi^n(K_i \cap K) = K$ for $n \geq 2$. In order to have scaling one needs 0 to be fixed point of the affine map $\Psi | K_2$.

Bounded distorsion property. The above examples are special in the sense that $\Psi$ was affine on each $K_i$. This is of course in general not the case. However, as we shall see, the "distorsions" due to the fact that the derivative of $\Psi$, and its iterates, is not locally constant, are bounded in a very strong sense. It is for these estimates that we required $\Psi$ to be $C^{1+\varepsilon}$.

Theorem. Let $K \subset \mathbb{R}$ be a dynamically defined Cantor set with expanding map $\Psi$. Then, for every $c > 0$ there is a $\delta > 0$ such that, whenever $q_0, q_{-1}, q_{-2}, \ldots$ and $\tilde{q}_0, \tilde{q}_{-1}, \tilde{q}_{-2}, \ldots$ are two sequences in $K$ such that

$$|q_o - \tilde{q}_o| < \delta;$$

$$\Psi(q_{-i}) = q_{-i+1}, \qquad \Psi(\tilde{q}_{-i}) = \tilde{q}_{-i+1};$$

the interval $(q_{-i}, \tilde{q}_{-i})$ is contained in the domain of $\Psi$; it follows that

$$\left| \ln |(\Psi^i)'(q_{-i})| - \ln|(\Psi^i)')(\tilde{q}_{-i})| \right| < c$$

for all $i > 0$.

Proof. From the fact that $\Psi$ is expanding it follows that for some $\sigma > 1$, $|q_{-i}-\tilde{q}_{-i}| < \delta.\sigma^{-i}$. Since $\Psi$ is $C^{1+\epsilon}$, and $\Psi'$ is bounded away from zero, there is a constant $C$ such that $\left|\ln |\Psi'(q_{-i})| - \ln|\Psi'(\tilde{q}_{-i})| \right| \le C.|a_{-i}-\tilde{q}_{-i}|^{\epsilon} < C.\delta^{\epsilon}.(\sigma^{-\epsilon})^{i}$. From this, and the fact that $(\Psi^i)'(q_{-i}) = \prod_{j=i}^{1} \Psi'(q_{-j})$, it follows that $\left|\ln |(\Psi^i)'(q_{-i})| - \ln|(\Psi^i)'(\tilde{q}_{-i})| \right| \le C.\delta^{\epsilon} . \frac{\sigma^{-\epsilon}}{1-\sigma^{-\epsilon}}$. By choosing $\delta$ small we can make this last expression smaller than $c$.

We want to make two remarks on this last theorem. First, it is clear that we used $\Psi$ to be $C^{1+\epsilon}$; in order to have this, we required our 2-dimensional diffeomorphism $\varphi$ to be $C^3$. If $\Psi$ is only $C^1$, the above theorem is not valid. However, if we would have started our constructions with a $C^2$ diffeomorphism $\varphi$, and hence obtained a $C^1$ expanding map $\Psi$, the above theorem would still be true for <u>this</u> map $\Psi$. The proof of this, which is of course based on the fact that $\Psi$ is obtained from a $C^2$ basic set, is contained in the proof of "continuity of thickness" in [17] and [20].

Second, for $K$ and $\Psi$ as in the above theorem we can construct very fine Markov partitions $K_1,\dots,K_k$ of $K$. Then the distorsion of $\Psi^i$, on an interval of its domain of definition which is mapped by $\Psi^i$ into one of the $K_j$'s, due to the variation of the derivative, is bounded, independent of $i$, and can be made smaller by refining the partition. This means that by taking a sufficiently fine Markov partition one gets a situation which is well approximated by a generalized affine Cantor set (see the examples). This means that results, true for generalized affine Cantor sets, are often true for all dynamically defined Cantor sets. This idea is formalized in the thermodynamic formalism [5].

## 2. Numerical invariants of Cantor sets.

In this section we define three numerical invariants for Cantor sets, namely the Hausdorff dimension, the limit capacity and the thickness. Then we discuss the Lebesgue measure of the difference of two Cantor sets in the real line, in terms of these invariants and finally we give some relations between these invariants when applied to the same Cantor set.

Before we can define the Hausdorff dimension, we need to introduce some preliminary notions. Let $K \subset \mathbb{R}$ be a Cantor set and $\mathfrak{U} = \{U_i\}_{i\in I}$ a finite covering of $K$ with open intervals in $\mathbb{R}$. We define the diameter $d(\mathfrak{U})$ of $\mathfrak{U}$ as the maximum of $\ell(U_i)$, $i \in I$, where $\ell(U_i)$ denotes the length of $U_i$. Define $H_\alpha(\mathfrak{U}) = \sum_{i\in I} \ell_i^\alpha$. Then the Hausdorff $\alpha$ measure of $K$ is

$$m_\alpha(K) = \lim_{\epsilon\to 0} \Big( \inf_{\substack{\mathfrak{U} \text{ covers } K \\ d(\mathfrak{U}) < \epsilon}} H_\alpha(\mathfrak{U}) \Big).$$

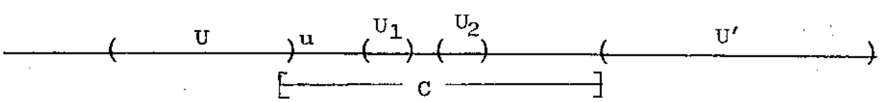
It is not hard to see that there is a unique number, the <u>Hausdorff dimension</u> of $K$, denoted by $HD(K)$, such that for $\alpha < HD(K)$, $m_\alpha(K) = \infty$ and for $\alpha > HD(K)$, $m_\alpha(K) = 0$.

In order to define the limit capacity, let $N_\epsilon(K)$, $K$ again a Cantor set in $\mathbb{R}$, be the minimal number of intervals of length $\epsilon$ needed to cover $K$. Then the <u>limit capacity</u> of $K$, denoted by $d(K)$, is defined as

$$d(K) = \limsup_{\epsilon\to 0} \frac{\ln N_\epsilon(K)}{-\ln \epsilon} .$$

To define the thickness, we consider the <u>gaps</u> of $K$: a gap of $K$ is a connected component of $\mathbb{R}\setminus K$; a bounded gap is a bounded connected component of $\mathbb{R}\setminus K$. Let $U$ be any bounded gap and $u$ be a boundary point of $U$, so $u \in K$. Let $C$ be the maximal interval in $\mathbb{R}$ such that:

- $u$ is a boundary point of $C$;
- $C$ contains no point of a gap $U'$ whose length $\ell(U')$ is at least the length of $U$.




$U$, $U'$, $U_1$, $U_2$ are gaps of $K$; $\ell(U') > \ell(U)$, $\ell(U_i) < \ell(U)$.

The thickness of $K$ at $u$ is defined as $\tau(K,u) = \ell(C)/\ell(U)$. The <u>thickness</u> of $K$, denoted by $\tau(K)$, is the infimum over these $\tau(K,u)$ for all boundary points $u$ of bounded gaps.

Now we come to the discussion of the Lebesgue measure of the difference of two Cantor sets. Let $K_1$, $K_2$ be subsets of R. We define their <u>difference</u> as

$$K_1 - K_2 = \{t \in R \mid \exists\, k_1 \in K_1,\ k_2 \in K_2, \text{ such that } k_1 - k_2 = t\}$$

<u>Proposition</u> [ ]. Let $K_1, K_2 \subset \mathbb{R}$ be Cantor sets with limit capacity $d_1$ and $d_2$. If $d_1 + d_2 < 1$ then the Lebesgue measure of $K_1 - K_2$ is zero.

<u>Proof</u>. Let $d_1'$, $d_2'$ be numbers such that $d_1 < d_1'$, $d_2 < d_2'$ and $d_1' + d_2' < 1$. Then there is an $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$, $K_i$ can be covered with $\varepsilon^{-d_i'}$ intervals of length $\varepsilon$; this follows directly from the definition of limit capacity. The difference of two intervals of length $\varepsilon$ is an interval of length $2\varepsilon$. So $K_1 - K_2$ is contained in $\varepsilon^{-d_1'} \cdot \varepsilon^{-d_2'}$ intervals of length $2\varepsilon$. The total length of these intervals, disregarding overlap, is $2 \cdot \varepsilon^{1 - d_1' - d_2'}$. Since $d_1' + d_2' < 1$, this can be made arbitrarily small by choosing $\varepsilon$ small. Hence the Lebesgue measure of $K_1 - K_2$ is zero.

<u>Proposition</u> [17]. Let $K_1, K_2 \subset \mathbb{R}$ be Cantor sets with thickness $\tau_1$ and $\tau_2$. If $\tau_1 \cdot \tau_2 > 1$ then one of the following three alternatives occurs: $K_1$ is contained in a gap of $K_2$; $K_2$ is contained in a gap of $K_1$; $K_1 \cap K_2 \neq \emptyset$. As a consequence, $K_1 - K_2$ has positive Lebesgue measure and it even contains nontrivial intervals.

<u>Proof</u>. We assume that neither of the two Cantor sets is contained in a gap of the other and we assume that $K_1 \cap K_2 = \emptyset$, and derive

a contradiction from this. If $U_1$, $U_2$ are bounded gaps of $K_1$, $K_2$, we call $(U_1,U_2)$ a gappair if $U_2$ contains exactly one boundary point of $U_1$ (and vice-versa). Since neither of the Cantor sets is contained in a gap of the other and since they are disjoint, there is a gappair. Given such a gappair $(U_1,U_2)$ we construct:

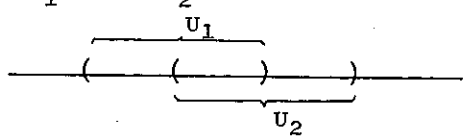
either a point in $K_1 \cap K_2$,

or a different gappair $(U_1',U_2)$ with $\ell(U_1') < \ell(U_1)$,

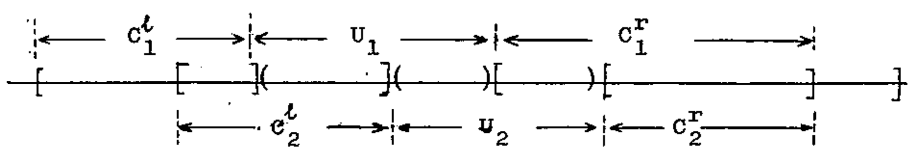
or a different gappair $(U_1,U_2')$ with $\ell(U_2') < \ell(U_2)$.

This leads to a contradiction: even if we don't find a point in $K_1 \cap K_2$ after applying this construction a finite number of times, we get a sequence of gappairs $(U_1^{(i)},U_2^{(i)})$ such that $\ell(U_1^{(i)})$ or $\ell(U_2^{(i)})$ goes to zero; assuming $\ell(U_1^{(i)})$ goes to zero, take $q_i \in U_1^{(i)}$ and any accumulation point of $\{q_i\}$ belongs to $K_1 \cap K_2$.

Now we come to the announced construction. Let the relative position of $U_1$ and $U_2$ be as indicated.




Let $C_i^\ell$ and $C_i^r$ be maximal intervals "around" $U_i$ (to the left and to the right) containing no points of gaps of $K_i$ which are at least as long as $U_i$ (see the definition of thickness and the figure below).



Since $\tau_1 . \tau_2 > 1$, $\frac{\ell(C_1^r)}{\ell(U_1)}$, $\frac{\ell(C_2)}{\ell(U_2)} > 1$. So $\ell(C_1^r) > \ell(U_2)$ or $\ell(C_2^\ell) > \ell(U_1)$, or both. So the right endpoint of $U_2$ is in $C_1^r$ or the left endpoint of $U_1$ is in $C_2^\ell$, or both. Suppose the first. Let u be the right endpoint of $U_2$. If $u \in K_1$ then we are done, since $u \in K_2$ anyway. If $u \notin K_1$, then u is contained in a gap $U_1'$ of $K_1$ with $\ell(U_1') < \ell(U_1)$ and $(U_1', U_2)$ is the required gappair. This completes the proof.

Proposition. Let $K_1$ and $K_2$ be Cantor sets in $\mathbb{R}$ with Hausdorff dimension $h_1$ and $h_2$. If $h_1 + h_2 > 0$, then, for almost any scalar $\lambda \in \mathbb{R}$, the Lebesgue measure of $K_1 - \lambda . K_2$ is positive. Almost any scalar means here that we only have to exclude from $\mathbb{R}$ a set of Lebesgue measure zero.

For the proof of this proposition, which is much more complicated than the proofs of the first two propositions, we refer to [14].

Next we come to the relations between the different invariants when applied to the same Cantor set.

Proposition. Let $K \subset \mathbb{R}$ be a Cantor set. Then $d(K) \geq HD(K)$.

Proof. For any $d' > d(K)$ and $\varepsilon$ sufficiently small, there is a covering of K with $\varepsilon^{-d'}$ intervals of length $\varepsilon$. For such a covering $\mathcal{U}$, and $d'' > d'$, we have $H_{d''}(\mathcal{U}) = \varepsilon^{-d'} . \varepsilon^{d''}$. For $\varepsilon$ going to zero, this last expression goes to zero. This means that for any $d'' > d(K)$, the Hausdorff $d''$ measure of K is zero, and the proposition follows.

<u>Proposition</u>. Let $K \subset \mathbb{R}$ be a dynamically defined Cantor set. Then $d(K) = HD(K)$.

<u>Outline of the proof</u>. We prove the proposition only for affine Cantor sets (see the examples in the previous section). For the extension to the general case we give some indications and references to the literature. For the affine case we can even prove more.

<u>Lemma</u>. Let $K$ be an affine Cantor set defined by a Markov partition $K_1,\dots,K_k$ (see the examples in the previous section) with $\ell(K_i) = \lambda_i$ and such that the convex hull of $K$ (or of $K_1 \cup\dots\cup K_k$) is a unit interval. Let $0 < d < 1$ be the unique number such that $\Sigma\, \lambda_i^d = 1$. Then $d(K) = HD(K) = d$.

<u>Proof</u>. We first derive a contradiction from the assumption that $d(K) > d$. Take $d' > d(K) > d$. Let $\varepsilon_o$ be so that for $0 < \varepsilon < \varepsilon_o$, $K$ can be covered with $\varepsilon^{-d'}$ intervals of length $\varepsilon$. Hence $K_i$ can be covered with $\varepsilon^{-d'}$ intervals of length $\lambda_i \cdot \varepsilon$ (for $\varepsilon \le \varepsilon_o$); in other words, for $\varepsilon < \lambda_i \varepsilon_o$, $K_i$ can be covered with $(\varepsilon \cdot \lambda_i^{-1})^{-d'}$ intervals of length $\varepsilon$. Applying this to all the intervals $K_i$, we obtain: For $\varepsilon < \min_i (\lambda_i \cdot \varepsilon_o)$, $K$ can be covered by $\varepsilon^{-d'} \cdot (\Sigma\, \lambda_i^{d'})$ intervals of length $\varepsilon$. Now, since $d' > d$, $\Sigma\, \lambda_i^{d'} < 1$, say $\Sigma\, \lambda_i^{d'} = \alpha(d')$. Note that, since we assume $d(K) > d$, even $\alpha(d(K)) < 1$. Letting $\tilde{\lambda} = \min_i \lambda_i$, we derived:

For $\varepsilon < \tilde{\lambda} \cdot \varepsilon_o$, $K$ can be covered by $\alpha(d') \cdot \varepsilon^{-d'}$ intervals of length $\varepsilon$. By repeated application of this argument we get, for $N > 0$:

For $\varepsilon < \tilde{\lambda}^N \cdot \varepsilon_o$, $K$ can be covered by $(\alpha(d'))^N \cdot \varepsilon^{-d'}$ intervals of length $\epsilon$. So the limit capacity is at most

$$\lim_{N\to\infty} \frac{\ln((\alpha(d'))^N\cdot(\tilde{\lambda}^N\cdot\epsilon_o)^{-d'})}{-\ln(\tilde{\lambda}^N\cdot\epsilon_o)} = \frac{d'\cdot\ln\tilde{\lambda} - \ln\alpha(d')}{\ln\tilde{\lambda}}.$$

For $d'$ sufficiently near $d(K)$, this last expression is smaller than $d(K)$; this is the required contradiction.

Next we derive a contradiction from the assumption $HD(K) < d$. Take $HD(K) < d' < d$. There are coverings $\mathfrak{U} = \{U_1,\ldots,U_N\}$ of $K$ with arbitrarily small diameter for which $H_{d'}(\mathfrak{U}) = \Sigma\, \ell(U_i)^{d'}$ is also arbitrarily small. We assume the covering $\mathfrak{U}$ to be such that no gap between differenti $K_i$'s is contained in one element of the covering. This will be the case if we require $H_{d'}(\mathfrak{U})$ to be smaller than some $\epsilon(d')$. From these assumptions it follows that there are $k$ disjoint subsets $I(1),\ldots,I(k)$ of $\{1,\ldots,N\}$ such that for each $j = 1,\ldots,k$ $\mathfrak{U}^{(j)} = \{U_i\}_{i\in I(j)}$, is a covering of $K_j$. Using the affine transformation $\Psi_j = \Psi|K_j$, we get for each $1 \le j \le k$ a new covering of $K$ whose elements are $\{\Psi_j(U_i)\}_{i\in I(j)}$.
Let $m_j = \sum_{i\in I(j)} \ell(\Psi_i(U_j))^{d'} = \lambda_j^{-d'}\cdot \sum_{i\in I(j)} \ell(U_i)^{d'}$. Let $m_{j_o} = \min(m_1,\ldots,m_k)$. We claim that $m_{j_o} \le \sum_{i=1}^{N} \ell(U_i)^{d'}$. Otherwise we would have for each $j=1,\ldots,k$, $\sum_{i\in I(j)} \lambda_j^{-d'}\cdot\ell(U_i)^{d'} > \sum_{i=1}^{N} \ell(U_i)^{d'}$, hence $\sum_{i\in I(j)} \ell(U_i)^{d'} > \lambda_j^{d'}\cdot\sum_{i=1}^{N}\ell(U_i)^{d'}$, so $\sum_{i=1}^{N}\ell(U_i)^{d'} > (\Sigma\,\lambda_j^{d'})\cdot\sum_{i=1}^{N}\ell(U_i)^{d'}$; but this contradicts the fact that $\Sigma\,\lambda_j^{d'} > 1$.

So, starting with a (finite) covering $\mathfrak{U}$ with $H_{d'}(\mathfrak{U}) <$

$< \varepsilon(d')$ we obtain, by restricting to a well chosen $K_i$ and then enlarging it by $\Psi_i = \Psi|K_i$, a new covering $\mathfrak{U}'$ with:

- $H_{d'}(\mathfrak{U}') \leq H_{d'}(\mathfrak{U}) < \varepsilon(d')$;
- $\mathfrak{U}'$ has less elements than $\mathfrak{U}$.

Repeating this construction we finally get a covering with no elements; this is the required contradiction and concludes the proof of the lemma.

As to the proof of the general case, we observe that the first estimate, relating the limit capacity $d(K)$ with the contraction rates $\lambda_1,\ldots,\lambda_k$ can easily be extended to generalized affine Cantor sets and then can be treated by the termodynamic formalism; see the remark at the end of Section 1. On the other hand, a calculation of the Hausdorff dimension of dynamically defined Cantor sets by the termodynamical formalism was undertaken in [15]. In both cases the result is the same:

The Hausdorff dimension and the limit capacity of $K$ are both equal to the constant $d$, for which the pressure of $-d.\ell n(|\Psi'|)$ is zero, where $\Psi'$ is the derivative of the expanding map defining $K$, se [15, 5].

<u>Remark</u>. The above proposition is a consequence of the regularity of dynamically defined Cantor sets. It makes that the propositions on the measure of the difference of two Cantor sets, in terms of limit capacity and Hausdorff dimension, cover, for dynamically defined Cantor sets, almost all cases - the exceptions being $d(K_1) + d(K_2) = 1$ and $K_1 - \lambda K_2$ for exceptional values of $\lambda$.

<u>Proposition</u>. If $K$ is a Cantor set in $\mathbb{R}$ with limit capacity $\alpha$, then the thickness of $K$ is at most $2^{1/(1-\alpha)} - 2$.

<u>Proof</u>. It follows easily from the definitions and the previous discussions, that for the middle $\beta$ Cantor set we have

$$\tau(K_\beta) = (1-\beta)/2\beta$$

and

$$d(K_\beta) = (\ln 2)/(\ln 2 - \ln(1-\beta)).$$

Let now $K \subset \mathbb{R}$ be a Cantor set with limit capacity $d(K) = \alpha$. Then for

$$\alpha + \ln 2/(\ln 2 - \ln(1-\beta)) < 1 \tag{1}$$

we have that the Lebesgue measure of $K - K_\beta$ is zero. The inequality (1) is equivalent with $\beta > 1 - 2^{\alpha/(\alpha-1)}$. For these values of $\beta$, the thickness of $K_\beta$ satisfies

$$\tau(K_\beta) < \frac{2^{\alpha/(\alpha-1)}}{2.(1-2^{\alpha/(\alpha-1)})}.$$

For these values of $\beta$ the Lebesgue measure of $K - K_\beta$ is zero, so

$$\tau(K) < \frac{2.(1-2^{\alpha/(\alpha-1)})}{2^{\alpha/(\alpha-1)}} = 2^{1/(1-\alpha)} - 2.$$

This proves the proposition.

## 3. <u>Local invariants and continuity</u>.

We conclude this chapter with some remarks on localized versions of the numerical invariants for Cantor sets introduced so far, and on the (continuous) dependence of these invariants on the Cantor set, at least for dynamically defined Cantor sets.

We give the definition of local thickness; local Hausdorff dimension and local limit capacity are similarly defined. Let $K \subset \mathbb{R}$ be a Cantor set and $k \in K$. The local thickness $\tau(K,k)$ of $K$ at $k$ is defined as

$$\tau(K,k) = \lim_{\varepsilon \to 0} \Big( \sup_{\substack{\tilde{K} \subset K \\ \tilde{K} \text{ contained in an} \\ \varepsilon\text{-neighbourhood of } k}} \tau(\tilde{K}) \Big).$$

For dinamically defined Cantor sets the notions have some additional properties. Let $K$ be a dynamically defined Cantor set with expanding map $\Psi$. Then for every $U \subset K$, $U$ open, there is some $n$ so that $\Psi^n(U) = K$. From this and the bounded distorsion property it follows that the local invariants $\tau(K,k)$, $HD(K,k)$, and $d(K,k)$ are, in the dynamically defined case, all independent of $k$. Also, since the limit capacity and the Hausdorff dimension are invariant under diffeomorphisms, one has in this case $HD(K,k) = HD(K) = d(K) = d(K,k)$. The thickness is not invariant under diffeomorphisms, and we may have $\tau(K) < \tau(K,k)$.

This localization was introduced in [20], at least for the thickness.

For a discussion of the continuous dependence of the invariants on the Cantor set, we restrict completely to the dynamically defined case. Let $K$ be a Cantor set with Markov partition $K_1,\dots,K_\ell$ and expanding map $\Psi$ which is $C^{1+\varepsilon}$ with Hölder constant $C$, i.e. with $\frac{|\Psi'(p)-\Psi'(q)|}{|p-q|^\varepsilon} < C$ whenever $p \neq q$. We say that the Cantor set $\tilde{K}$ is near $K$, if $\tilde{K}$ admits a Markov partition $\tilde{K}_1,\dots,\tilde{K}_\ell$ and expanding map $\tilde{\Psi}$ such that:

- $\{\tilde{K}_i\}$ is near $\{K_i\}$ in the sense that corresponding endpoints are near;
- $\tilde{\Psi}$ is $C^1$ near $\Psi$;
- $\tilde{\Psi}$ is $C^{1+\epsilon}$ with the same Hölder constant $C$.

Using again bounded distorsion it follows that the (local) thickness depends continuously on $K$, see [20] for details. The corresponding continuity of the Hausdorff dimension, and hence of the limit capacity was obtained in [15] as a consequence of the variational principle.

Observe that if $\varphi$ is a $C^3$ diffeomorphism in dimension two with basic set $\Lambda$ and saddle point $p$, then for a $C^3$ nearby diffeomorphism $\tilde{\varphi}$ with basic set $\tilde{\Lambda}$ and saddle point $\tilde{p}$, the dynamically defined Cantor sets $W^u(p) \cap \Lambda$ and $W^u(\tilde{p}) \cap \tilde{\Lambda}$ will be near in the above sense (if $W^u(p)$ and $W^u(\tilde{p})$ are suitably parametrized). This follows from the continuous dependence of basic sets and their $C^{1+\epsilon}$ stable and unstable foliations on diffeomorphisms.

# CHAPTER V

# HOMOCLINIC TANGENCIES, CANTOR SETS AND MEASURE OF BIFURCATION SETS

In this and the following chapters we discuss results relating the invariants of Cantor sets from the previous chapter with homoclinic bifurcations. Here we deal especially with questions about the (relative) measure of the bifurcation set. We begin with the definition of hyperbolic or non-bifurcating diffeomorphisms. Then we construct a special family of diffeomorphisms $\varphi_\mu$ (in fact a class of such families) which has a homoclinic bifurcation for $\mu = 0$. This 1-parameter family will be used to illustrate the results, which will be stated in general, and to indicate how to prove these results.

## 1. Hyperbolic diffeomorphisms.

In the present section we deal with diffeomorphisms which are not bifurcating. First we discuss the definition of these so called hyperbolic diffeomorphisms, then we say why these are just the non-bifurcating diffeomorphisms, and finally give an example.

Let $\varphi: M \to M$ be a diffeomorphism on a compact manifold. In the present section we take $\varphi$ to be $C^1$ and $M$ of arbitrary dimension. For $x \in M$, we define the $\underline{\alpha}$ and $\underline{\omega}$ limit as

$$\alpha(x) = \{y \in M \mid \exists\, n_i \to -\infty \text{ such that } \varphi^{n_i}(x) \to y\}$$

$$\omega(x) = \{y \in M \mid \exists\, n_i \to +\infty \text{ such that } \varphi^{n_i}(x) \to y\}.$$

The <u>positive limit set</u> of $\varphi$ is then defined as

$$L^+(\varphi) = \overline{\bigcup_{x\in M} \omega(x)}.$$

From the definition it is clear that $L^+(\varphi)$ is a compact <u>invariant</u> set, invariant in the sense that $\varphi(L^+(\varphi)) = L^+(\varphi)$, and that for each $x \in M$, $\varphi^n(x)$ approaches $L^+(\varphi)$ as $n$ goes to infinite. So $L^+(\varphi)$ and $\varphi|L^+(\varphi)$ describe the assymptotic behaviour of orbits (i.e. sequences $\{\varphi^n(x)\}_{n\in\mathbb{Z}}$) in $M$ for $n \to \infty$. Similarly, one can define $L^-(\varphi)$. Although not needed here, we also recall that a point $x$ is called nonwandering if for any neighbourhood of it there is an integer $n$ such that $\varphi^n(U) \cap U \neq \neq \phi$. The union of such points is a closed set denoted by $\Omega(\varphi)$ and we have $L^+(\varphi) \cup L^-(\varphi) \subset \Omega(\varphi)$; also any homoclinic point is nonwandering.

We say that a compact set $K \subset M$, invariant under the action of $\varphi$, is <u>hyperbolic</u> if there is a continuous splitting of $T_K(M)$ as $T_K(M) = E^u_K \oplus E^s_K$, i.e. for each $x \in K$ a splitting $T_x(M) = E^u_x \oplus E^s_x$ depending continuously on $x$, such that

- the splitting is invariant under the action of $d\varphi$;
- there are constants $\sigma > 1$ and $C > 0$ such that for $v \in E^u_K$ and $n > 0$,

  $$\|d\varphi^n(v)\| \geq C.\sigma^n.\|v\|$$

  and such that for $v \in E^s_K$ and $n > 0$,

  $$\|d\varphi^n(v)\| \leq C^{-1}.\sigma^{-n}.\|v\|.$$

Here $\|\ \|$ denotes the length of a vector with respect to some Riemannian metric.

We say that a compact invariant set $K$ as above has a *cycle* if there are points $x_1,\ldots,x_k = x_o$, not all in $K$, such that

- $\alpha(x_i)$ and $\omega(x_i)$ are contained in $K$ for all $i=1,\ldots,k$;
- $\omega(x_i) \cap \alpha(x_{i+1}) \neq \emptyset$ for $i=0,\ldots,k-1$.

If $K$ has no cycle we say that $K$ has the *no-cycle property*.

Finally we say that the diffeomorphism $\varphi$ is *hyperbolic* if its positive limit set is hyperbolic and has the no-cycle property.

It follows from [18] that this notion is equivalent with: the $\Omega$-limit set of $\varphi$ is hyperbolic, is the closure of its periodic orbits and has no cycles. In this situation we have the $\Omega$-stability theorem [30] which in our terminology can be formulated as:

*Theorem*. Let $\varphi\colon M \to M$ be hyperbolic. Then there is a neighbourhood $U$ of $\varphi$ in $\mathrm{Diff}^1(M)$, the space of $C^1$ diffeomorphisms, such that any $\tilde{\varphi} \in U$ is hyperbolic and there is a homeomorphism $h\colon L^+(\varphi) \to L^+(\tilde{\varphi})$ which makes the following diagram commutative:

$$\begin{array}{ccc} L^+(\varphi) & \xrightarrow{\ \varphi\ } & L^+(\varphi) \\ \downarrow h & & \downarrow h \\ L^+(\tilde{\varphi}) & \xrightarrow{\ \tilde{\varphi}\ } & L^+(\tilde{\varphi}). \end{array}$$

We then say that the hyperbolicity of $\varphi$ implies that $\varphi$ is $L^+$-stable. In other words, the *topology* of the *dynamics*, at least when restricted to the psotive limit set, does not change

if we perturb $\varphi$ inside U. This is the opposite of what happens at a bifurcating diffeomorphism, where the topology of the dynamics may change as a consequence of an arbitrarily small perturbation. Moreover, by results of Liao [11] and Mañé [12] in dimension two the assumption of hyperbolicity in the above theorem turns out to be not only sufficient, but also necessary for the conclusion. That the no-cycle property is necessary in any dimension had been proved in [23]. The $\Omega$-stability Conjecture states precisely that f is $\Omega$-stable iff $\Omega(f)$ is hyperbolic and has no cycles*. So for the purpose of this chapter, we define a bifurcating diffeomorphism simply as a diffeomorphism which is not hyperbolic. Also if $\varphi_\mu$ is a 1-parameter set of diffeomorphisms, we define its bifurcation set B as
$B = \{\mu \mid \varphi_\mu$ is not hyperbolic$\}$.

Note that bifurcating diffeomorphisms include all diffeomorphisms which have at least one periodic orbit which is nonhyperbolic (see Chapter III) and also all diffeomorphisms which have a homoclinic tangency.

Finally we indicate one non-trivial example of a hyperbolic diffeomorphism on the 2-sphere $S^2$. It is in fact the horseshoe example from the introduction but now in $S^2 \simeq \mathbb{R}^2 \cup \infty$ instead of in $\mathbb{R}^2$. We take in $S^2$ the diffeomorphic image of a square Q with two semi-circular discs $D_1$ and $D_2$ attached as indicated.

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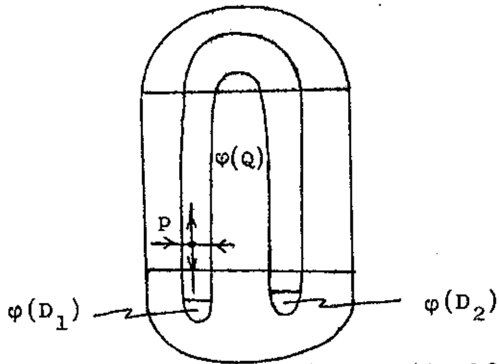
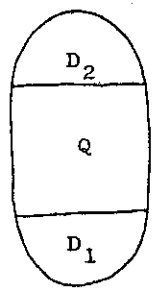
* After these notes were written, a proof of the $C^1$ $\Omega$-stability Conjecture in any dimension was presented in [26], based in Mañé's solution of the $C^1$ Stability Conjecture.




We let $\varphi$ map $Q \cup D_1 \cup D_2$ inside itself as indicated, i.e. so that in $Q$ we have the horseshoe example and in $D_1$



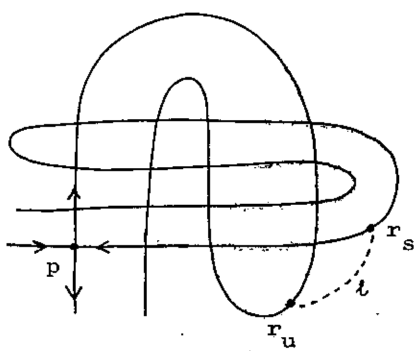

we have one hyperbolic sink $S_i$, attracting all points in $D_1$. We extend $\varphi$ to the complement of $Q \cup D_1 \cup D_2$ in $S^2$ in such a way that there is only one hyperbolic source $S_o$ and such that for each $x \in S^2 - (Q \cup D_1 \cup D_2)$, $\lim_{n\to\infty} \varphi^{-n}(x) = S_o$. It is easy to verify that in this case the positive limit set consists of $S_i$, $S_o$, and the maximal invariant subset in $Q$. This last set can be analysed as in Chapter II and is hyperbolic.

For later reference we let $p$ denote the indicated saddle point in $Q$, see also the introduction; $\Lambda$ denotes the maximal invariant subset of $Q$. The Cantor sets $W^s \cap \Lambda$ and $W^u \cap \Lambda$ are clearly dynamically defined (if $\varphi$ is $C^3$ and if we take a correct identification of $W^s(p)$ and $W^u(p)$ with $\mathbb{R}$); these are the Cantor sets to which the results of Chapter IV shall be applied.

## 2. Construction of a bifurcating family of diffeomorphisms.

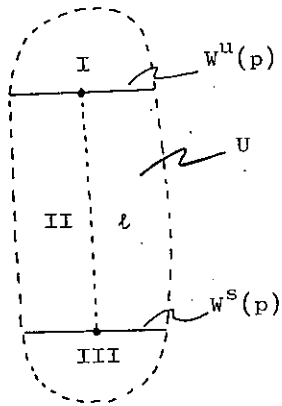
The one-parameter family of diffeomorphisms which we construct in this section, is based on the example in the last section of a diffeomorphism in $S^2$ with a horseshoe, a source, and a sink. As before, we denote such $S^2$ diffeomorphism by $\varphi$; here we assume this diffeomorphism $\varphi$ to be $C^3$.

Let $\ell$ be a curve from $r_s \in W^s(p)$ to $r_u \in W^u(p)$ as




indicated in the figure. $U$ denotes a small neighbourhood of $\ell$ which is devided by the local components of $W^s(p) \cap U$ and $W^u(p) \cap U$ containing $r_s$ and $r_u$ in the regions $U_I$, $U_{II}$, and

$U_{III}$. We shall obtain our one-parameter family by modifying the



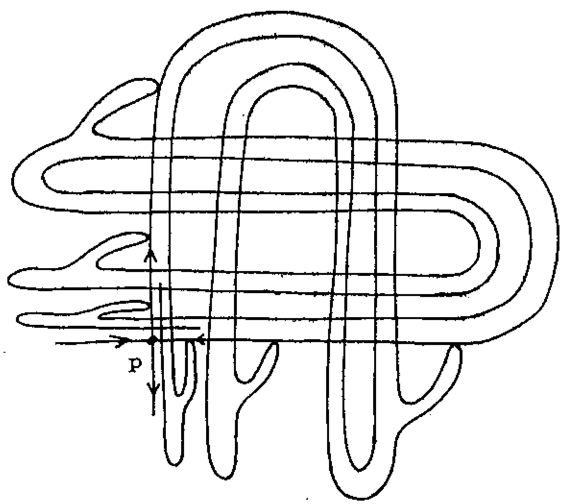

map $\varphi$ in U, i.e. by composing $\varphi$ with $\Psi_\mu$, $\Psi_\mu$ a one-parameter family of diffeomorphisms which are, outside U, equal to the identity.

Before we describe $\Psi_\mu$, we analyse the dynamic properties of orbits passing through U; we assume that this neighbourhood U of $\ell$ is sufficiently small so that the following considerations are valid. If $x \in U_I \cup U_{II}$ then $\varphi^n(x)$ tends, for $n \to +\infty$, to the sink $S_i$ and if $x \in U_{II} \cup U_{III}$ then $\varphi^n(x)$ tends, for $n \to -\infty$, to the source $S_o$. For $x \in U_{III}$, the positive iterates $\varphi^n(x)$, $n \to +\infty$, will stay near $W^u(p)$, but apart from that they may go to the sink or may stay near $\Lambda$ ($\Lambda$, see the <u>previous section</u>, can be defined as $\Lambda = \overline{W^u(p) \cap W^s(p)}$); in any case there are points $x \in U_{III}$ such that $\varphi^n(x) \in U_I$ for some positive n. Similarly for $x \in U_I$, the negative iterates $\varphi^{-n}(x)$, $n \to +\infty$, will stay near $W^s(p)$, but apart from that they may go to the soure or may stay near $\Lambda$; in any case there are points $x \in U_I$ such that $\varphi^{-n}(x) \in U_{III}$ for some $n > 0$.

Now we come to the description of the one-parameter family $\Psi_\mu$ moving the points in $U$. We take $\Psi_\mu$ so that:

- for $\mu \le -1$, $\Psi_\mu$ is the identity;
- for $\mu > -1$, $\Psi_\mu$ pushes points in $U$ down (in the direction of $U_{III}$) so that for $\mu < 0$ $U_I$ is still mapped inside $U_I \cup U_{II}$;
- for $\mu = 0$ there is a tangency of $\Psi_o(W^u(p))$ and $W^s(p)$, or more precisely of $\Psi_o(U_I \cap U_{II})$ and $U_{II} \cap U_{III}$; this tangency has parabolic order of contact and unfolds generically for $\mu > 0$ in two intersections.

Below we indicate the stable and unstable manifold of $p$ for the diffeomorphism $\Psi_\mu \circ \varphi$, $\mu = 0$.



From the discussion of the dynamics of the points in $U$ under iterations of $\varphi$, it follows that for $\mu < 0$, the positive limit set of $\varphi_\mu = \Psi_\mu \circ \varphi$ is the same as the positive limit set of $\varphi$. This implies that the bifurcation set of the one-parameter family $\varphi_\mu = \Psi_\mu \circ \varphi$ is contained in $R_+ = \{\mu \mid \mu \geq 0\}$.

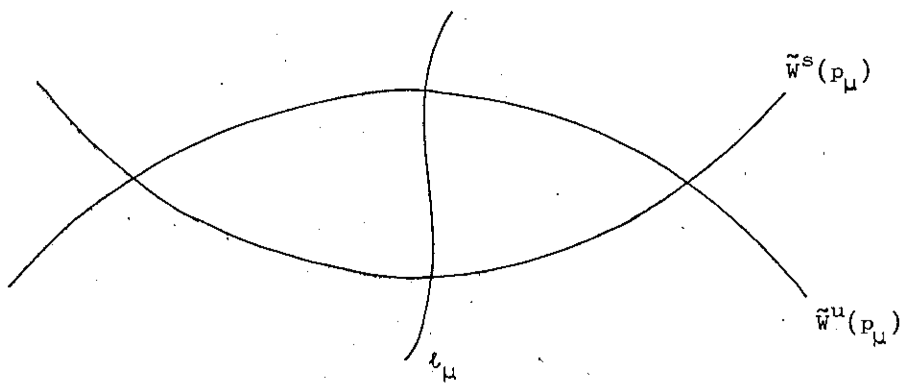
In this case we say that the homoclinic tangency for $\mu = 0$ is a <u>first bifurcation</u>.

For $\mu = 0$, $L^+(\varphi_0) = L^+(\varphi)$, but now $L^+(\varphi_0)$ has a cycle. For $\mu > 0$, the positive limit set of $\varphi_\mu$ is strictly bigger than the positive limit set of $\varphi$. The orbits in $L^+(\varphi_\mu) - L^+(\varphi)$ are limits of orbits which go through $U$. From the above discussion of the dynamics of points in $U$ under iteration of $\varphi$, it follows even that these $\varphi_\mu$-orbits have to pass infinitely often through $\Psi_\mu(U_I) \cap U_{III}$.

We observe that the main properties of the one-parameter family $\varphi_\mu$ remain after a small perturbation (small in the $C^3$ topology) as long as $\mu = 0$ remains the first bifurcation. So in fact we constructed a whole open class of bifurcating arcs of diffeomorphisms. What follows is applicable to any element of this class.

Let $\Lambda_\mu$ denote the horseshoe of $\varphi_\mu$, i.e. for $\mu \leq 0$ $\Lambda_\mu$ is the positive limit set of $\varphi_\mu$ without the source and the sink; for $\mu > 0$, $\Lambda_\mu$ is the continuation of $\Lambda_0$ which can be obtained, for small positive $\mu$, by taking a sufficiently small neighbourhood $V$ of $\Lambda_0$ and then taking $\Lambda_\mu$ the maximal invariant subset of $V$ for $\varphi_\mu$. For $\varphi_\mu$ as constructed, $\Lambda_\mu$ is independent of $\mu$. But this does not persist under small perturbations of the one-parameter family. So in general we must allow for a $\mu$-dependence of $\Lambda$ - the same can be said concerning the $\mu$-dependence of the saddle point $p$.

As in Chapter II, we construct the stable and the unstable foliations $\mathcal{F}^s_\mu$ and $\mathcal{F}^u_\mu$ on a neighbourhood of $\Lambda_\mu$. These foliations are $C^{1+\varepsilon}$ (in this chapter we use only $C^1$) and depend differentiably on $\mu$. We can if necessary extend the domains of definition of $\mathcal{F}^s_\mu$ and $\mathcal{F}^u_\mu$ by applying $\varphi_\mu^{-1}$ respectively $\varphi_\mu$. In this way we can obtain that both $\mathcal{F}^s_\mu$ and $\mathcal{F}^u_\mu$ are defined in a neighbourhood of the point of first tangency in $U$. In particular we want both $\mathcal{F}^s_\mu$ and $\mathcal{F}^u_\mu$ for $\mu$ positive and small, be defined on the disc bounded by the two components of $W^s(p_\mu) \cap U$ and $W^u(p_\mu) \cap U$ which made the first tangency. From now on we denote these components, which are also leaves of $\mathcal{F}^s_\mu$ and $\mathcal{F}^u_\mu$ by $\tilde{W}^s(p_\mu)$ and $\tilde{W}^u(p_\mu)$.




The line $\ell_\mu$ is defined as the set of points where the leaves of

$\mathcal{F}_\mu^s$ and $\mathcal{F}_\mu^u$ are tangent. From the fact that the contact of $\tilde{W}^u(p_o)$ and $\tilde{W}^s(p_o)$ is parabolic and the fact that the foliations $\mathcal{F}_\mu^u$ and $\mathcal{F}_\mu^s$ are $C^1$, it follows that $\ell_\mu$ is a smooth curve, depending differentiably on $\mu$.

Both in $\mathcal{F}_\mu^s$ and $\mathcal{F}_\mu^u$ there is a collection of exceptional leaves - the stable and unstable manifolds of points of $\Lambda_\mu$. The collection of these leaves is denoted by $\mathcal{F}^s(\Lambda_\mu)$, $\mathcal{F}^u(\Lambda_\mu)$: they are just the leaves of $\mathcal{F}_\mu^s$, $\mathcal{F}_\mu^u$ passing through points of $\Lambda_\mu$.

In the following sections we mainly deal with the geometry of the configuration formed by $\mathcal{F}_\mu^s$, $\mathcal{F}_\mu^u$, $\mathcal{F}^s(\Lambda_\mu)$, $\mathcal{F}^u(\Lambda_\mu)$, $\ell_\mu$, when restricted to the disc bounded by $\tilde{W}^s(p_\mu)$ and $\tilde{W}^u(p_\mu)$.

## 3. Homoclinic tangencies with bifurcation set of small relative measure - statement of the result.

The result which we discuss in this section has for the one-parameter family (or class of one-parameter families) constructed in the last section the following consequence.

Theorem. Let $\varphi_\mu$ be a one-parameter family as constructed in the last section. Let $d^u$ and $d^s$ be the (local) limit capacity of $W^u(p_o) \cap \Lambda_o$ and $W^s(p_o) \cap \Lambda_o$. If $d^u+d^s < 1$, then

$$\lim_{\mu_o \to 0} \frac{m(B(\varphi_\mu) \cap [0,\mu_o])}{\mu_o} = 0,$$

where $B(\varphi_\mu)$ is the bifurcation set of the one-parameter family $\varphi_\mu$: $B(\varphi_\mu) = \{\mu \in \mathbb{R} \mid \varphi_\mu$ is not hyperbolic$\}$, and where $m(\ )$ denotes the Lebesgue measure.

In the next section we shall describe the structure of the proof of this theorem which is taken from [25]. In this section we discuss more general assumptions under which the conclusion of the theorem still holds.

First, we restrict our one-parameter families to $C^2$-diffeomorphisms $\varphi_\mu : M \to M$ of a compact 2-manifold $M$. It is clear that we have to assume that our diffeomorphisms are $C^2$: otherwise we even cannot say what a generic tangency is. We actually assumed $\varphi$ to be $C^3$ but this is not really necessary for the present theorem. If $\varphi$ is only $C^2$ the stable and unstable foliations are still $C^1$. The restriction to 2-manifolds is somewhat more arbitrary and in fact the results can be extended to higher dimensions, but for this we would have to introduce several new concepts, see also [33].

Second, we restrict to those one-parameter families $\varphi_\mu$ which have a homoclinic tangency for $\mu = 0$ which is generic, which unfolds generically, and which is a first bifurcation, i.e. for $\mu < 0$, $\varphi_\mu$ is hyperbolic. Let $L^+(\varphi_\mu)$ be the positive limit set of $\varphi_\mu$. We also assume that $\lim_{\mu \nearrow 0} L^+(\varphi_\mu) = L_o^+$ exists and that it is a compact invariant hyperbolic subset for $\varphi_o$. This means that the homoclinic tangency should <u>not</u> take place in the limit $L_o^+$ of positive limit sets.

These restrictions are imposed by the methods of proof which will be discussed in the next section. It is not clear whether these conditions are really necessary for the conclusion we are interested in; small relative measure of the bifurcation set near zero. In this context it is worth remembring the example of a homoclinic bifurcation which we gave in the introduction: a horseshoe map composed with a "translation downwards". It is not known whether the first tangency of stable and unstable manifold of $p_\mu$, as indicated in figure 2, is a first bifurcation. If it is, it is a tangency inside the positive limit set. We have no information on the relative measure of the bifurcation set in this case. For a discussion of the last assumptions and the restrictions which they impose on the global topology of the dynamics of $\varphi_o$, see [24].

Thirdly, and finally, we impose some generic conditions: for $\mu = 0$, $\varphi_o$ should only have one orbit of tangency of a stable and an unstable manifold. Also, in some cases depending on the global dynamics, we have to require that $|\det(d\varphi_o(p_o))| \neq 1$, where $p_o$ denotes the saddle point related to the homoclinic tangency. (For the one-parameter family constructed in the previous section one did not have to impose this last condition on the determinant of $d\varphi_o$.)

For a one-parameter family $\varphi_\mu$ as above and with $p_o$ the saddle point of $\varphi_o$ related with the homoclinic tangency, we define the stable and the unstable limit capacity as the (local) limit capacity of $W^s(p_o) \cap L_o^+$ and $W^u(p_o) \cap L_o^+$, where as before $L_o^+$ is the limit of $L^+(\varphi_\mu)$ for $\mu$ approaching 0 from below.

Then under these more general assumptions we have the same conclusion:

<u>Theorem</u>. If, in the above situation, the sum of stable and unstable limit capacities is smaller than one, then

$$\lim_{\mu_o \to 0} \frac{m(B(\varphi_\mu) \cap [0,\mu_o])}{\mu_o} = 0.$$

## 4. Homoclinic tangencies with bifurcation set of small relative measure - idea of the proof.

In this section we want to outline the proof of the first theorem in the previous section (for the one-parameter families $\varphi_\mu$ as constructed in Section 2); a detailed exposition of this proof is in [25] but we hope that the present outline is even usefull as a preparation for reading the complete proof.

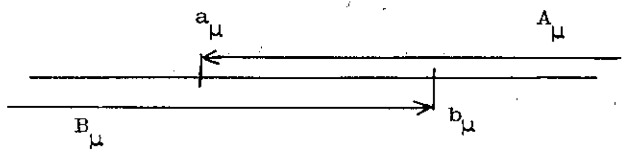
As a first step we have

Proposition 1. For each $c > 0$, there is a $\mu_1(c) > 0$ so that for every $0 < \mu \le \mu_1(c)$ such that the distance in $\ell_\mu$ (see Section 2) between points of $\mathfrak{F}^s(\Lambda_\mu)$ and $\mathfrak{F}^u(\Lambda_\mu)$ is at least $c.\mu_1(c)$, we have that $\varphi_\mu$ is hyperbolic.

The proof of this first proposition is based on the fact that this minimal distance $c.\mu_1(c)$ implies a minimal angle of intersection between leaves of $\mathfrak{F}^s(\Lambda_\mu)$ and $\mathfrak{F}^u(\Lambda_\mu)$ which enables us to prove hyperbolicity by the cone construction described in Chapter II.

For the next considerations we use an identification of $\ell_\mu$ with $\mathbb{R}$. We denote the points in $\mathbb{R}$ corresponding to $\ell_\mu \cap \tilde{W}^s(p_\mu)$ and $\ell_\mu \cap \tilde{W}^u(p_\mu)$ by $a_\mu$ and $b_\mu$, and denote the subsets in $\mathbb{R}$ corresponding to $\ell_\mu \cap \mathfrak{F}^s(\Lambda_\mu)$ and $\ell_\mu \cap \mathfrak{F}^u(\Lambda_\mu)$ by $A_\mu$ and $B_\mu$.

The relative positions are as indicated below.




The set $A_\mu$ is the diffeomorphic image of a scaled set in $W^u(p_\mu)$, the diffeomorphism being the $C^1$ projection along leaves of $\mathcal{F}^s_\mu$ to $\ell_\mu$. Let $\sigma(\mu)$ be the expanding eigenvalue of $d\varphi_\mu$ at $p_\mu$. Then we define the corresponding "linearized set" $L(A_\mu)$ as

$$L(A_\mu) = \lim_{n\to\infty} (H_{a_\mu,\sigma(\mu)})^n(A_\mu),$$

where $H_{a_\mu,\sigma(\mu)}$ is the affine transformation fixing $a_\mu$ and multiplying distances with $\sigma(\mu)$; the convergence is with respect to the Hausdorff metric on compact parts. In the same way one defines $L(B_\mu)$.

<u>Proposition 2</u>. For each $c > 0$, there is a $\mu_2(c) > 0$ so that for every $0 < \mu \le \mu_2(c)$, $A_\mu \cap (a_\mu, b_\mu)$ is contained in a $(c.\mu_2(c))$ - neighbourhood of $L(A_\mu)$ and $B_\mu \cap (a_\mu, b_\mu)$ is also contained in a $(c.\mu_2(c))$ - neighbourhood of $L(B_\mu)$.

The proof of this proposition is quite elementary and uses only the definition of derivative.

For the next proposition we use the notation $A+t$, where $A \subset \mathbb{R}$ and $t \in \mathbb{R}$, to denote the set $A$ shifted over a distance $t$ to the right.

<u>Proposition 3</u>. For each $c > 0$ there is a $\mu_3(c) > 0$ so that for any $0 < \mu \le \mu_3(c)$, $L(A_\mu) \cap (a_\mu, b_\mu)$ is contained in a $(c.\mu_3(c))$ neighbourhood of $L(A_o) + (a_\mu - a_o)$, and $L(B_\mu) \cap (a_\mu, b_\mu)$ is contained in a $(c.\mu_3(c))$ neighbourhood of $L(B_o) + (b_\mu - b_o)$.

This proposition is somewhat complicated to prove especially since the scaling factors of $L(A_\mu)$ and $L(B_\mu)$ depend on $\mu$. The proof is mainly analytical.

In the last proposition we assume that the sum of the limit capacities of $L(A_o)$ and $L(B_o)$ is smaller than one.

<u>Proposition 4</u>. For each $\delta \in (0,1)$ there is a $c(\delta)$ such that for every $0 < c < c(\delta)$, there is a $\mu_4(c,\delta) > 0$, such that for each $0 < \bar{\mu} \le \mu_4(c,\delta)$,

$$\frac{m(\{\mu \in [0,\bar{\mu}] \mid \mathrm{dist}((L(A_o)+(a_\mu - a_o)),(L(B_o)+(b_\mu - b_o))) < c.\bar{\mu}\})}{\bar{\mu}} < \delta .$$

The proof of this proposition is based on the fact that if $A \subset \mathbb{R}$ is a scaled set with (local) limit capacity $d$ and if $d' > d$ then there is an $\epsilon_o > 0$ such that for any $\alpha > 0$ and $0 < \epsilon < \alpha\epsilon_o$, $A \cap [-\alpha, +\alpha]$ can be covered by $\alpha^{d'}.\epsilon^{-d'}$ intervals of length $\epsilon$.

The final result, the first theorem of Section 3, follows from the above propositions and the observation that the limit capacities of $L(A_o)$ and $L(B_o)$ are equal to the limit capacities of $W^u(p_o) \cap \Lambda_o$ and $W^s(p_o) \cap \Lambda_o$. In fact $L(A_o)$ and $L(B_o)$ are equal to these sets, up to an affine transformation.

# CHAPTER VI

## INFINITELY MANY SINKS

In this chapter we present a proof of the following remarkable fact.

Theorem [19]. There are subsets $\mathcal{U} \subset \mathcal{V}$ of $\mathrm{Diff}^2(S^2)$, $\mathcal{V}$ open and $\mathcal{U}$ of second category in $\mathcal{V}$, such that each $\varphi \in \mathcal{U}$ has infinitely many sinks.

By 1970, Newhouse [17] had exhibited an open set in $\mathrm{Diff}^2(S^2)$ of nonhyperbolic diffeomorphisms. Similar results were already known in higher dimensions [1] and even in the $C^1$ setting (the $C^1$ question on $S^2$ is still open). The novelty to obtain the $(C^2)$ result on $S^2$ was the use of thickness of Cantor sets - see Chapter IV - to get full intervals in their arithmetic difference. (Curiously, M. Hall had used the same concept and obtained a similar result but in the number theoretic contex [8]). Here the Cantor sets are intersection points of leaves of foliations $\mathcal{F}^s(\Lambda)$ and $\mathcal{F}^u(\Lambda)$, $\Lambda$ a basic set of a diffeomorphism of $S^2$, with a line of points of tangency between the leaves of extended stable and unstable foliations defined in a neighbourhood of $\Lambda$ - see Chapters II, V. Persistence of intersections of Cantor sets leads then to persistence of orbits of tangency between stable and unstable leaves of a basic set, which in turn yields an open set of nonhyperbolic diffeomorphisms. A few years later this construction was blended together with the appearence of a sink when unfolding a homoclinic tangency and the theorem stated above was obtained.

1. Proof of the theorem.

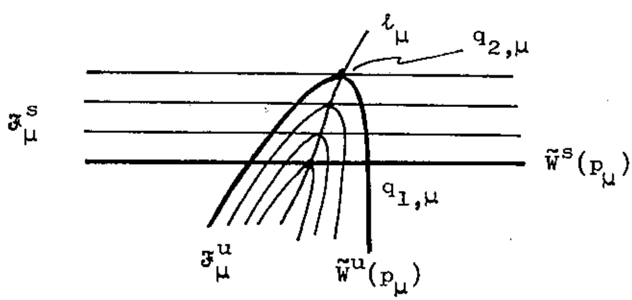
To shorten somewhat the exposition, we will start the construction of the sets $\mathcal{U}$, $\mathcal{V}$ in $\mathrm{Diff}^2(S^2)$ considering a $C^\infty$ family of diffeomorphisms: the family $\varphi_\mu$ exhibited in §3 of the previous chapter. Recall that $\varphi_\mu$ has a basic set $\Lambda_\mu$ (a horseshoe) with a fixed point $p_\mu \in \Lambda_\mu$ and, as in §3 - Chapter III, we assume that $\varphi_\mu$ is $C^2$ linearizable in a neighbourhood of $p_\mu$, $\mu$ small, which is a generic assumption. Also, $\varphi_o$ has a parabolic orbit of tangency $\Theta(q)$ related to $p = p_o$ which unfolds for $\mu > 0$ into two transverse homoclinic orbits. We also add the simple requirement that $|\det d\varphi_o(p)| < 1$. As before, we denote by $\tilde{W}^s(p)$ and $\tilde{W}^u(p)$ the local components of $W^s(p)$ and $W^u(p)$ to which $\Theta(q)$ belongs and by $\tilde{W}^s(p_\mu)$ and $\tilde{W}^u(p_\mu)$ their continuation for $\mu$ near zero. The stable and unstable foliations of $\Lambda_\mu$ are denoted by $\mathcal{F}^s(\Lambda_\mu)$ and $\mathcal{F}^u(\Lambda_\mu)$ and simply by $\mathcal{F}^s_\mu$ and $\mathcal{F}^u_\mu$ their extended ($C^1$ and $\varphi_\mu$-invariant) foliations defined in a full neighbourhood of $\Lambda_\mu$. These foliations depend continuously on $\mu$ (on compact parts) and their leaves, which are $C^2$ (actually $C^\infty$) curves, depend differentiably on $\mu$.

We shall add to $\varphi_\mu$ the following key condition on thickness

$$\tau(W^u(p) \cap \Lambda_o) \cdot \tau(W^s(p) \cap \Lambda_o) > 1.$$

It is clear that we can construct the family $\varphi_\mu$ as before with this further condition being imposed.

Consider now the line of tangency $\ell_\mu$, near q and for $\mu$ small, between the leaves of $\mathfrak{F}^s_\mu$ and $\mathfrak{F}^u_\mu$. As observed in §2 of the previous chapter, and as a consequence of the differentiability of $\mathfrak{F}^s_\mu$ and $\mathfrak{F}^u_\mu$ and continuous dependence on $\varphi_\mu$ (and in particular on $\mu$), we have that $\ell_\mu$ is a $C^1$ curve and depends continuously on $\mu$ in the $C^1$ topology. Let $q_{1,\mu} = \tilde{W}^s(p_\mu) \cap \ell_\mu$, $\mu$ small and $\widehat{p_\mu q_{1,\mu}}$ close to $\widehat{pq} \subset \tilde{W}^s(p)$, and $q_{2,\mu} = \tilde{W}^u(p_\mu) \cap \ell_\mu$, $\widehat{p_\mu q_{2,\mu}} \subset \tilde{W}^u(p)$ as in the figure.




Let $N^s_\mu$, $N^u_\mu$ be small neighbourhoods of $p_\mu$ in $W^s(p_\mu)$ and $W^u(p_\mu)$, respectively. Define the projections $\pi^s_\mu: N^u_\mu \to \ell_\mu$, $\pi^u_\mu: N^s_\mu \to \ell_\mu$ along the leaves of $\mathfrak{F}^s_\mu$ and $\mathfrak{F}^u_\mu$; they are $C^1$ and they take the Cantor sets $K^u_\mu = N^u_\mu \cap \Lambda_\mu$ and $K^s_\mu = N^s_\mu \cap \Lambda_\mu$ onto neighbourhoods of $q_{1,\mu}$ in $\mathfrak{F}^u(\Lambda_\mu) \cap \ell_\mu$ and of $q_{2,\mu}$ in $\mathfrak{F}^s(\Lambda_\mu) \cap \ell_\mu$. Notice that $\tau(K^u_o) = \tau(W^u(p) \cap \Lambda_o)$ and $\tau(K^s_o) = \tau(W^s(p) \cap \Lambda_o)$.

Now, if $N_\mu^s$, $N_\mu^u$ are small neighbourhoods, and $\mu$ is positive and small, then

(1) the Cantor sets $\pi_\mu^s(K_\mu^u)$ and $\pi_\mu^u(K_\mu^s)$ are not in the gaps of one another nor in $\ell_\mu/\pi_\mu^u(K_\mu^s)$ and $\ell_\mu/\pi_\mu^s(K_\mu^u)$, respectively,

(2) $\tau(K_\mu^u)$ is close to $\tau(K_o^u)$ and $\tau(K_\mu^s)$ is close to $\tau(K_o^s)$ by continuity of thickness,

(3) $\tau(\pi_\mu^s(K_\mu^u))$ is close to $\tau(K_\mu^u)$ because the derivative of $\pi_\mu^s$ is almost constant since the neighbourhood $N_\mu^u$ is small; similarly, $\tau(\pi_\mu^u(K_\mu^s))$ is close to $\tau(K_\mu^s)$.

From (2) and (3) we get $\tau(\pi_\mu^s(K_\mu^u)).\tau(\pi_\mu^u(K_\mu^s)) > 1$, and from this and (1) we conclude that $\pi_\mu^s(K_\mu^u) \cap \pi_\mu^u(K_\mu^s) \neq \phi$. This means that, for all $\mu > 0$ and small say $0 < \mu < \delta$ for some $\delta > 0$, there are in $\ell_\mu$ points of tangency between the leaves of $\mathfrak{F}^s(\Lambda_\mu)$ and $\mathfrak{F}^u(\Lambda_\mu)$. Since $W^s(p_\mu)$ is dense in $\mathfrak{F}^s(\Lambda_\mu)$ and $W^u(p_\mu)$ is dense in $\mathfrak{F}^u(\Lambda_\mu)$, densily in $(0,\delta)$ we have values of $\mu$ so that $\varphi_\mu$ has homoclinic tangency near $q$. These homoclinic tangencies are parabolic since near $q$ each leaf of $\mathfrak{F}^u(\Lambda_o)$ and each leaf of $\mathfrak{F}^s(\Lambda_o)$ have different curvatures and so the same is true for the leaves of $\mathfrak{F}^u(\Lambda_\mu)$ and $\mathfrak{F}^s(\Lambda_\mu)$ for $\mu$ small. Even more so they unfold generically: near $q$ and for small values of $\mu$, the leaves of $\mathfrak{F}_\mu^u$, as differentiable functions of $\mu$, more transversely across the leaves of $\mathfrak{F}_\mu^s$ since this is the case for the leaves $\tilde{W}^u(p)$ and $\tilde{W}^s(p)$ as we imposed when constructing the family $\varphi_\mu$.

Let us show that there is a generic (second category) set

$G \subset (0,\delta)$ so that if $\mu \in G$ then $\varphi_\mu$ has infinitely many sinks. For that it is enough to show that for each integer $N > 0$ the set $\beta_N$ of values of $\mu$ for which $\varphi_\mu$ has at least $N$ sinks is open and dense in $(0,\delta)$. Since sinks are persistent under small $C^r$ $(r \geq 1)$ perturbations of the map, openess of $\beta_N$ is clear. To show density, consider any $\mu_o \in (0,\delta)$ and any neighbourhood $N_o$ of $\mu_o$. We have shown above that there exists $\tilde{\mu}_1 \in N_o$ such that $\varphi_{\tilde{\mu}_1}$ has a parabolic homoclinic tangency which unfolds generically. Then, as proved in §3 - Chapter III, there exists $\mu_1$ arbitrarily close to $\tilde{\mu}_1$ such that $\varphi_{\mu_1}$ has a sink $S_1$. Now we take a small neighbourhood $N_1$ of $\mu_1$ in $N_o$ so that, for every $\mu \in N_1$, $\varphi_\mu$ has a sink $S_{1,\mu}$ near $S_1$. Again in $N_1$ we find $\tilde{\mu}_2$ such that $\varphi_{\tilde{\mu}_2}$ has a parabolic homoclinic tangency which unfolds generically. That yields $\mu_2 \in N_1$ such that $\varphi_\mu$ has a new sink $S_2$ besides $S_{1,\mu_2}$. We can proceed by induction in exactly the same manner, proving our statement. Thus, we point out, we have obtained a parametrized version of the theorem (although this was not demanded).

Finally, we can take $\mathcal{U}$ - an open set as required - as a sufficiently small neighbourhood in $\mathrm{Diff}^2(S^2)$ of $\varphi_{\mu^*}$ for some $\mu^* \in (0,\delta)$. If $\mathcal{U}$ is small then any $g \in \mathcal{U}$ has a basic set $\Lambda_g$ close to $\Lambda_{\mu^*}$ and having a fixed point $p_g$ close to $p_{\mu^*}$ and, by continuity of thickness in the $C^2$ topology, $\tau(W^s(p_g) \cap \Lambda_g)$ is close to $\tau(W^s(p_{\mu^*}) \cap \Lambda_{\mu^*})$ and $\tau(W^u(p_g) \cap \Lambda_g)$ is close to $\tau(W^u(p_{\mu^*}) \cap \Lambda_{\mu^*})$, implying that their product is bigger than one. Again the extended foliations $\mathcal{F}_g^u$ and $\mathcal{F}_g^s$, defined in a neighbourhood of $\Lambda_g$, are $C^1$ close to $\mathcal{F}^u_{\mu*}$ and $\mathcal{F}^s_{\mu*}$ and define near $q$ a line of tangencies of their leaves say $\ell_g$ which is $C^1$ close to $\ell_{\mu*}$. Also, we can define $C^1$ projections $\pi^s_g: N^u_g \cap \Lambda \to \ell_g$ and $\pi^u_g: N^s_g \cap \Lambda_g \to \ell_g$ $C^1$ close to $\pi^s_{\mu*}$ and $\pi^u_{\mu*}$, respectively, and thus both of them having almost constant derivatives. So the Cantor sets obtained as intersections of the leaves of $\mathcal{F}^u(\Lambda_g)$ and $\mathcal{F}^s(\Lambda_g)$ with $\ell_g$ have a point in common and consequently there is near $q$ a point of tangency between leaves of these foliations. We can now proceed as above to show that for each $N > 0$ the subset of $\mathcal{V}$ with at least $N$ sinks is open and dense. The only point to be careful about is when arguing on the density: one approximates a given diffeomorphism in $\mathcal{V}$ by a $C^\infty$ one which is $C^2$ linearizable in a neighbourhood of its fixed saddle, the continuation of $p$, so that we can apply §3 of Chapter III as to obtain sinks. We conclude that there exists a second category subset $\mathcal{U}$ of $\mathcal{V}$ such that if $\varphi \in \mathcal{U}$ then $\varphi$ has infinitely many sinks. The theorem is proved.

<u>Comments and questions</u>.

The first comment is that in the open set $\mathcal{V} \subset \mathrm{Diff}^2(S^2)$, constructed in the proof of the theorem, all diffeomorphisms are nonhyperbolic as we defined in §1 - Chapter V. This follows from the fact that densily in $\mathcal{V}$ the diffeomorphisms have a homoclinic tangency and such an orbit is either in the limit set, and then the limit set is not hyperbolic, or else it corresponds to a cycle in the limit set, and this again implies nonhyperbolicity. Notice that our definition of hyperbolicity for $\varphi \in \mathrm{Diff}^1(M)$ concerns

$L^+_{(\varphi)}$ - the positive limit set - and includes the no-cycle condition besides the usual notion of hyperbolicity and therefore in our sense hyperbolicity becomes an "open property" in $\text{Diff}^1(M)$ (and hence in $\text{Diff}^r(M)$, $r \geq 1$). We can also consider hyperbolicity for the nonwandering set $\Omega(\varphi)$, $\varphi \in \mathcal{V}$. Again, all $\varphi$'s in V are not $\Omega$-hyperbolic: they all have an orbit of tangency between stable and unstable leaves of a basic set (the horseshoe which is the continuation of $\Lambda_o$), and since such orbit is non-

Our second comment concerns stability of $\varphi \in \mathcal{V}$, more specifically $L^+$-stability and the similar notion of $\Omega$-stability: $\varphi$ is $C^r$ $\Omega$-stable if for any $\tilde{\varphi}$ $C^r$ near $\varphi$ there is a homeomorphism $h:\Omega(\varphi)\to\Omega(\tilde{\varphi})$ such that $C^r$ near $\varphi$ there is a homeomorphism $h: \Omega(\varphi) \to \Omega(\tilde{\varphi})$ such that $h\varphi(x) = \tilde{\varphi}h(x)$ for all $x \in \Omega(\varphi)$. We make a little digression to point out that the two notions are the same in the $C^1$ topology. This follows from the fact that if $\varphi$ is either $C^1$ $L^+$-stable or $\Omega$-stable, then $L^+(\varphi) = \Omega(\varphi) = \overline{P(\varphi)}$, where $P(\varphi)$ indicates the set of periodic points. This in turn follows from Pugh's closing lemma: if $x \in \Omega(\varphi)$ or $x \in L^+(\varphi)$ and is isolated from $P(\varphi)$ then we can $C^1$ perturb $\varphi$ away from $\overline{P(\varphi)}$ so to obtain a new periodic orbit through x. Of course $\varphi$ and its perturbation cannot be $L^+$-conjugate nor $\Omega$-conjugate. The corresponding question in the $C^r$ topology, $r \geq 2$, is open (and very difficult). Returning to the open set $\mathcal{V} \subset \text{Diff}^2(S^2)$, it follows the result of Liao and Mané (§1 - Chapter V) that since no $\varphi \in \mathcal{V}$ is hyperbolic then it cannot be $C^1$ $L^+$-stable or $\Omega$-stable. In the present case this can be proved directly using that the diffeomorphisms with homoclinic tangencies are dense in $\mathcal{U}$. It also follows from the more interesting fact that no $\varphi \in \mathcal{U}$ is $L^+$-stable (or $\Omega$-stable). To show that $\varphi \in \mathcal{U}$ is not $C^2$ $L^+$-stable it is enough to show that densily in $\mathcal{U}$ the elements are not $C^2$ $L^+$-stable. By playing with gaps of Cantor sets in our construction of $\mathcal{U}$, one can show that densily in $\mathcal{U}$ the diffeomorphisms have a homoclinic tangency which is isolated in the nonwandering set and so it is either isolated in the positive limit set $L^+$ (if it belongs to it) or it is far from it: in either case with a $C^2$ perturbation away from the rest of $L^+$ we introduce a new horseshoe and so many new periodic orbits. Thus the initial diffeomorphism with the homoclinic tangency cannot be $L^+$-conjugate to the perturbed one. Another argument: when unfolding a homoclinic tangency to create a sink as in the proof of the theorem, we must create in this process a nonhyperbolic periodic orbit generically a saddle-node. Thus, densily in the $C^2$ topology in $\mathcal{U}$ the diffeomorphisms have periodic saddle-nodes and, again, they cannot be $L^+$-stable because with small $C^2$ perturbation we can change the number of periodic orbits of a given period (see §2 - Chapter III). Similarly, no $\varphi \in \mathcal{U}$ is $C^2$ $\Omega$-stable.

Related to the comments above, we want to pose the following very interesting questions:

(1) Are the hyperbolic diffeomorphisms dense in $\mathrm{Diff}^1(S^2)$ ?

(2) Are the hyperbolic diffeomorphisms dense in $\mathrm{Diff}^1(T^2)$, $T^2$ the 2-torus?

These questions (dating back to more than 25 years ago, and due to Smale) were sharply revived due to possible applications of techniques developed recently by Mañé in his recent and remarkable solution of the $C^1$ stability conjecture (an n-dimensional diffeomorphism is globally stable if and only if it is hyperbolic and all stable and unstable manifolds are in general position) [13]. For instance, using these techniques A.L. Araujo [2] showed that densily in $\mathrm{Diff}^1(M^2)$ there are hyperbolic attractors and repellors.

We now want to relate the main results in Chapters V and VI, discuss some extensions of them as well as some open problems.

First of all, it might be asked why in unfolding a homoclinic tangency for a family $\varphi_\mu$, say at $\mu = 0$, one uses "small" limit capacities (or Hausdorff dimensions) to get mostly hyperbolic diffeomorphisms for values of $\mu$ near zero, and, on the other hand, one uses "big" thickness to get an interval say $[0,\delta]$ and a dense subset $\mathcal{U} \subset [0,\delta]$ so that $\varphi_\mu$ has infinitely many sinks for all $\mu \in \mathcal{U}$? Is there any way of relating to some extend these results? In this direction we suggest the following conjectures:

<u>Conjecture I</u>. For generic $\varphi_\mu$, if $\varphi_0$ has a homoclinic tangency related to a basic set $\Lambda_0$ whose stable and unstable limit capacities satisfy $d^s+d^u > 1$, then

$$\liminf_{\mu_0 \to 0} \frac{m(B(\varphi_\mu) \cap [0,\mu_0])}{\mu_0} > 0$$

where, as before, $B(\varphi_\mu) = \{\mu \in \mathbb{R} \mid \varphi_\mu \text{ is not hyperbolic}\}$.

A weaker, still important, version of this conjecture is to consider lim sup in the above expression.

Conjecture II. With the same setting as in the first conjecture, let $\mathcal{U} = \{\mu \in \mathbb{R} \mid \varphi_\mu$ has infinitely many sinks$\}$. Then,

$$\limsup_{\mu_o \to 0} \frac{m(\bar{\mathcal{U}} \cap [0,\mu_o])}{\mu_o} > 0.$$

A result concerning measure of bifurcation set, similar to that in Chapter V, is also true for cycles instead of just homoclinic orbits (i.e. 1-cycles). This is discussed in the Appendix, where we also pose conjectures similar to the above. More recently, the result has been extended to homoclinic orbits in higher dimensions. We notice, however, that it is an interesting open question whether in higher dimensions the local stable (or unstable) limit capacity is the same at all points of a basic set.

The theorem in the present chapter was extended considerably by Newhouse [19] (see also [27-A]), since in the new version there is no a priori assumption on thickness: a family $\varphi_\mu$ having a fixed (periodic) saddle whose Jacobian has norm less than one and a related quadratic homoclinic tangency unfolding generically gives rise to homoclinic tangencies related to horseshoes of arbitrarily big thickness. As a consequence, if $\varphi_\mu$ has a homoclinica tangency as above at $\mu = 0$, then there are intervals $I_i$ arbitrarily close to $\mu = 0$ in R and dense subsets $\mathcal{U}_i \subset I_i$ such that $\varphi_\mu$ has infinitely many sinks for each $\mu \in \mathcal{U}_i$. (The last argument of a key lemma in [19] seems incomplete; however, C. Robinson very recently asserted its completion).

Notice that under the assumptions of Chapter V for a family $\varphi_\mu$ (actually an open class of families), specially the hypothesis $d^s+d^u < 1$ on capacities or dimensions of the basic set related to the homoclinic tangency, the Lebesgue measure of the intervals $I_i$ above would be assymptotically very small relative to the interval $[0,\mu_o]$ we consider in the $\mu$-space. On the other hand, they would be quite sizable when $d^s+d^u > 1$ if Conjecture II is true.

# APPENDIX I

## 1. Hyperbolicity. Stable manifolds and foliations.

The purpose of this appendix is to collect a number of results from the literatura, mainly from [9], which were used in the previous chapters. These results are all related to the notion of hyperbolicity (norms of eigenvalues being different from one) and are based on the construction of objects which are invariant under a diffeomorphism $\varphi$ by applying $\varphi^n$ to a non-invariant object and then taking a limit.

Stable manifold theorem. Let $\varphi: M \to M$ be a $C^k$ diffeomorphism, $k \geq 1$, and let $p \in M$ be a fixed point, i.e. $\varphi(p) = p$, such that $(d\varphi)_p$ has no eigenvalue of norm one (in this case one calls $p$ a hyperbolic fixed point). Then the stable manifold

$$W^s(p) = \{x \in M \mid \lim_{n\to\infty} \varphi^n(x) = p\}$$

is an injectively immersed $C^k$ submanifold of $M$. If $\tilde{\varphi}$ is $C^k$ near $\varphi$, $\tilde{\varphi}$ has a hyperbolic fixed point $\tilde{p}$ near $p$ and the stable manifold $W^s(\tilde{p})$ is near $W^s(p)$ at least if we restrict ourselves to compact neighbourhoods of $\tilde{p}$ and $p$ in $W^s(\tilde{p})$ and $W^s(p)$.

Applying the same theorem to $\varphi^{-1}$, we obtain the unstable manifold $W^u(p)$. One way to prove this theorem is to take a "generic" submanifold $W$ of $M$ of the right dimension and containing $p$, and proving that $\varphi^{-n}(W)$ has a limit for $n \to +\infty$

and that this limit is the stable manifold. If $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is linear and $p = 0$, then one can easily see how this works. This idea to construct the stable manifold is also the basic idea of the so-called $\lambda$-lemma.

<u>Theorem</u>. Let $\varphi: M \to M$ be a $C^k$ diffeomorphism, $k \geq 1$, with a hyperbolic fixed point $p$. Let $V \subset M$ be a $C^k$ submanifold such that:

- $\dim(V) = \dim(W^s(p))$;
- $V$ has a point $q$ of transverse intersection with $W^u(p)$.

Then $\varphi^{-n}(V)$ converges to $W^s(p)$ in the following sense. For each $n$ one can choose a disc $D_n \subset \varphi^{-n}(V)$, which is a neighbourhood of $\varphi^{-n}(q)$ in $\varphi^{-n}(V)$ such that

$$\lim_{n\to\infty} D_n = D,$$

where $D$ is a disc-neighbourhood of $p$ in $W^s(p)$ and where the convergence is in the sense of $C^k$ submanifolds.

There is a first generalization of the stable manifold theorem where the fixed point is replaced by a so-called <u>hyperbolic set</u> of which we shall now give a definition. Let $\varphi: M \to M$ be a $C^k$ diffeomorphism, $k \geq 1$. A compact set $\Lambda \subset M$ is a hyperbolic set for $\varphi$ if $\varphi(\Lambda) = \Lambda$ and if there is a splitting $T_x(M) = E^u_x \oplus E^s_x$ for each $x \in \Lambda$ such that:

- the splitting depends continuously on $x$;
- the splitting is invariant, i.e. $\varphi(E^u_x) = E^u_{\varphi(x)}$ and $\varphi(E^s_x) = E^s_{\varphi(x)}$;

- there are constants $\sigma > 1$ and $C > 0$ such that for any $v \in E_x^u$ and $n > 0$,

$$\|d\varphi^n(v)\| \geq C.\sigma^n.\|v\|,$$

and such that for any $v \in E_x^s$ and $n > 0$,

$$\|d\varphi^n(v)\| \leq C^{-1}.\sigma^{-n}.\|v\|.$$

Here $\| \ \|$ denotes the norm of tangent vectors of $M$ with respect to some fixed Riemannian metric on $M$.

Note that a hyperbolic fixed point is a special case of a hyperbolic set.

<u>Generalized Stable Manifold Theorem</u>. Let $\Lambda \subset M$ be a hyperbolic set for a $C^k$ diffeomorphism $\varphi$, $k \geq 1$. Then for each $x \in \Lambda$ the stable manifold

$$W^s(x) = \{y \in M \mid \lim_{n\to\infty} \rho(\varphi^n(x),\varphi^n(y)) = 0\}$$

(where $\rho$ denotes the distance with respect to some Riemannian metric on $M$) is an injectively immersed $C^k$ submanifold. $W^s(x)$ depends continuously on $x$ in the following sense.

Let $\mathrm{Emb}^k(D,M)$ be the space of $C^k$ embeddings of a disc $D$ (whose dimension equals the dimension of $E_x^s$ which we assume to be independent of $x$) in $M$. Then there is a <u>continuous</u> map $\Phi: \Lambda \to \mathrm{Emb}^k(D,M)$ such that for each $x \in \Lambda$, the image of $\Phi(x)$ is a neighbourhood of $x$ in $W^s(x)$.

In the above situation (hyperbolic sets and their collection of stable manifolds) it is often usefull to use <u>local stable manifolds</u> which are defined as follows

$$W^s_\epsilon(x) = \{y \in M \mid \lim_{n\to\infty} \rho(\varphi^n(x),\varphi^n(y)) = 0 \quad \text{and}$$
$$\rho(\varphi^n(x),\varphi^n(y)) \le \epsilon \quad \text{for all} \quad n \ge 0\}.$$

These local stable manifolds are, for $\epsilon$ sufficiently small, embedded disks which depend continuously on $x$.

We come now to a different generalization of the stable manifold theorem. It provides many more invariant manifolds for fixed points, not necessarily hyperbolic, of diffeomorphisms. In describing these results we prefer to use diffeomorphisms defined on $R^n$ instead of on a general manifold $M$.

<u>Theorem</u>. Let $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ be a $C^k$ diffeomorphism, $k \ge 1$, with $\varphi(0) = 0$. Let $\mathbb{R}^n = V_1 \oplus V_2$ be a splitting such that for some $a \in \mathbb{R}$,

- $d\varphi(V_i) = V_i$, $i = 1,2$;
- the norms of the eigenvalues of $d\varphi|V_1$ are smaller than $a$;
- the norms of the eigenvalues of $d\varphi|V_2$ are greater than $a$.

Then there is a locally invariant $C^1$-manifold $V$ such that $0 \in V$ and $T_0(V) = V_1$. $V$ is locally invariant in the sense that $V \cap \varphi(V)$ contains a neighbourhood of $0$ in $V$. The manifold $V$ is in general not unique and not $C^k$.

When we come to normally hyperbolic invariant manifolds, we shall say more about the exact amount of differentiability of these (locally) invariant manifolds like $V$ in the above theorem. But we first want to indicate how also the manifolds in the above theorem can be obtained by a limiting process.

Sketch of the construction of V. We can write the map $\varphi$ as $\varphi(x) = L(x) + \tilde{\varphi}(x)$, where $L$ is linear, in fact $L = (d\varphi)_o$, and where $\tilde{\varphi}(0) = 0$ and $(d\tilde{\varphi})_o = 0$. Next we choose a function $\Psi$ on $\mathbb{R}^n$ which is identically equal to one on a neighbourhood of the origin and which is equal to zero outside the unit disk. We define

$$\bar{\varphi}(x) = L(x) + \Psi(\tfrac{1}{\varepsilon}\cdot x)\cdot\tilde{\varphi}(x),$$

where $\varepsilon$ is a small positive number. For $\varepsilon$ small, $\bar{\varphi}$ is a diffeomorphism which, in a small neighbourhood of the origin is equal to $\varphi$. It can be shown that for $\varepsilon$ sufficiently small, the manifolds $\bar{\varphi}^{-n}(V_1)$ converge for $n \to \infty$ (remember that $V_1$ was invariant under $(d\varphi)_o = L$). This limit $V$ is an invariant manifold for $\bar{\varphi}$ and hence a locally invariant manifold for $\varphi$ as required.

As applications of the above theorem, we construct now the centre manifold, the centre-stable manifold and the centre-unstable manifold. First the centre-stable manifold. Let $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ be as in the above theorem. There is some $a > 1$ such that all eigenvalues of $(d\varphi)_o$ which are in norm smaller than $a$ are exactly all eigenvalues of $(d\varphi)_o$ which are in norm smaller than or equal to one. For this value $a > 1$ we construct the corresponding splitting $\mathbb{R}^n = V_1 \oplus V_2$ and apply the theorem. The resulting locally invariant manifold is called the centre-stable manifold and is denoted by $W^{cs}(0)$. If $\varphi$ is $C^k$, $k \geq 1$, $k$ finite, then one can choose $W^{cs}(0)$ to be $C^k$. The centre-unstable manifold for $\varphi$ is just the centre-stable manifold for $\varphi^{-1}$; it is denoted by $W^{cu}(0)$. The centre manifold is just the intersection of a centre-stable and a centre-unstable manifold: $W^c(0) = W^{cs}(0) \cap W^{cu}(0)$.

The (locally) invariant manifolds which we have seen so far are all examples of <u>normally hyperbolic invariant manifolds</u>. This notion is defined as follows. Let $\varphi: M \to M$ be a $C^k$ diffeomorphism, $k \geq 1$. A submanifold $V \subset M$ is said to be a normally hyperbolic invariant submanifold if $\varphi(V) = V$ and if there is a splitting $T_x(M) = T_x(V) \oplus N_x^s \oplus N_x^u$ for each $x \in V$ such that

- the splitting depends continuously on $x$;
- the splitting is invariant under $d\varphi$, i.e. $d\varphi(N_x^s) = N^s_{\varphi(x)}$ and $d\varphi(N_x^u) = N^u_{\varphi(x)}$;
- for some Riemannian metric, and constants $C > 0$, $\sigma > 1$, $r \geq 1$, one has for every triple of unit vectors $v \in T_x(V)$, $n^s \in N_x^s$, and $n^u \in N_x^u$ and any $n > 0$:

$$\frac{\|(d\varphi^n)n^u\|}{\|(d\varphi^n)v\|^r} \geq C.\sigma^n \qquad \text{and} \qquad \frac{\|(d\varphi^n)n^s\|}{\|(d\varphi^n)v\|} \leq C^{-1}.\sigma^{-n}.$$

In this case one call $V$ even r-normally hyperbolic.

This notion of normal hyperbolicity is of importance since it implies a certain minimal amount of differentiability of the invariant manifold. This is the content of the following

<u>Theorem</u>. Let $V \subset M$ be an r-normally hyperbolic invariant manifold for a $C^k$ diffeomorphism $\varphi$. If $k \geq [r] + 1$, $[r]$ is the integer part of $r$, and if $V$ is compact and $C^1$, then $V$ is

$C^r$ in the following sense. $V$ is locally the graph of a $C^{[r]}$ function whose $[r]^{th}$ derivative is $(r-[r])$-Hölder.

Although in [9] this theorem is only stated for $r$ an integer (theorem 4.1 on page 39) the general case follows from remark 2 (on page 38) dealing with Hölder sections.

In the above theorem we required $V$ to be compact. This means that this theorem cannot be immediately be applied to the locally invariant submanifolds with which the previous theorem was dealing. Still, one can prove in the same way that if in the previous theorem $V_1$ is an r-normally hyperbolic invariant submanifold for $(d\varphi)_o$ then, for any $r'$ such that $r' < r$ and such that $k \geq [r'] + 1$, one can choose $V$ to be $C^{r'}$.

Finally we have to say something about the stable and unstable foliations on a neighbourhood of a basic set of a diffeomorphism $\varphi: M \to M$ on a 2-manifold $M$. The construction of such a foliation was given in Chapter II (in the case of a horseshoe). Here we are concerned with its differentiability.

For diffeomorphisms $\varphi: M \to M$ as above, we construct a lift

$$\begin{array}{ccc} P(M) & \xrightarrow{P(\varphi)} & P(M) \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\varphi} & M \end{array}$$

where $P(M) = \{(x,L) \mid x \in M,\ L$ is a one dimensional linear subspace of $T_x(M)\}$ is the projectivised tangent bundel of $M$, and $P(\varphi)$ is the diffeomorphism induced by $\varphi$ on $P(\varphi)$. Clearly, if $\varphi$ is $C^k$ then $P(\varphi)$ is $C^{k-1}$.

Consider an unstable foliation $\mathfrak{F}^u$ as constructed in Chapter II. If its domain of definition is $U \subset M$, then we get a (locally) invariant manifold $\tilde{U} \subset P(M)$ by defining $\tilde{U} = \{(x,L) \mid x \in U,$ $L$ is the tangent space of the $\mathfrak{F}^u$ leaf through $x\}$. We call the foliation $\mathfrak{F}^u$ $C^\alpha$ whenever the corresponding manifold $\tilde{U}$ is $C^\alpha$. This is somewhat unconventional: if $\tilde{U}$ is $C^\alpha$ then the leaves of $\mathfrak{F}^u$ are $C^{\alpha+1}$. But the reason for this definition is that if $\tilde{U}$ is $C^\alpha$ then also the "projections along leaves", as used in the Chapters V and VI, are $C^\alpha$.

One can verify that $\tilde{U}$ is $(1+\epsilon)$-normally hyperbolic for $P\varphi$ for some $\epsilon > 0$. Although $\tilde{U}$ is not compact and only locally invariant, the conclusion of the last theorem is still valid (this is due to the careful definition of $\mathfrak{F}^u$ on $(R \cap \Psi^{-1}(R)) \setminus \Psi(R)$. Hence, if $\varphi$ is $C^3$, $P(\varphi)$ is $C^2$ and $\tilde{U}$ is $C^{1+\epsilon}$; if $\varphi$ is $C^2$, $\tilde{U}$ is only $C^1$.

# APPENDIX II

## 2. Markov partitions.

In Chapter IV we mentioned the existence of Markov partitions for dynamically defined Cantor sets and their relation with Markov partitions for hyperbolic diffeomorphisms in dimension two. Here we shall explain these points further.

Let $\varphi: M \to M$ be a hyperbolic diffeomorphism on a compact 2-manifold $M$. Let $\Lambda$ be a basic set for $\varphi$ of saddle type, i.e. $\Lambda$ is a compact invariant set, contained in the positive limit set $L^+(\varphi)$, which contains a dense orbit, and such that in the hyperbolic splitting, restricted to $\Lambda$, $E^u$ and $E^s$ are both one-dimensional. We assume $\Lambda$ to be non-trivial in the sense that it contains more than one periodic orbit. Also we assume $\Lambda$ not to be "periodic" in the following sense. We say that $\Lambda$ is periodic if for certain integers $n \neq 0$, $\Lambda$ has no dense orbit of $\varphi^n$. In this case there is an $n \neq 0$ such that $\Lambda$ can be decomposed in $n$ non-periodic basic sets of $\varphi^n$.

For $x \in \Lambda$ we define local stable and unstable manifolds:

$$W^u_\varepsilon(x) = \{y \in M \mid \lim_{n\to-\infty} \rho(\varphi^n(x),\varphi^n(y)) = 0 \quad \text{and}$$
$$\text{for all } n \le 0, \quad \rho(\varphi^n(x),\varphi^n(y)) \le \varepsilon\}$$

$$W^s_\varepsilon(x) = \{y \in M \mid \lim_{n\to+\infty} \rho(\varphi^n(x),\varphi^n(y)) = 0 \quad \text{and}$$
$$\text{for all } n \ge 0, \quad \rho(\varphi^n(x),\varphi^n(y)) \le \varepsilon\},$$

where $\rho$ denotes the distance with respect to some fixed Riemannian metric. From the local product theorem [5] we know that for $x, x' \in \Lambda$ sufficiently near, $W_\epsilon^u(x)$ and $W_\epsilon^s(x')$ have a unique point of intersection and that this point of intersection also belongs to $\Lambda$.

Since $\Lambda$ is non-trivial, and due to the local product theorem, $W_\epsilon^s(x) \cap \Lambda$ is a Cantor set for each $x \in \Lambda$. We say that $x$ is a <u>boundary point</u> of $\Lambda$ in the <u>unstable direction</u> if $x$ is a boundary point of $W_\epsilon^u(x) \cap \Lambda$, i.e. if $x$ is only accumulated from one side by points in $W_\epsilon^u(x) \cap \Lambda$. If $x$ is a boundary point of $\Lambda$ in the unstable direction, then, due to the local product theorem, the same holds for all points in $W_\epsilon^s(x) \cap \Lambda$. So the boundary points in the <u>unstable</u> direction are locally intersections of local <u>stable</u> manifolds with $\Lambda$. For this reason we denote the set of boundary points in the unstable direction by $\partial_s \Lambda$. The boundary points in the stable direction are similarly defined; the set of these boundary points is denoted by $\partial_u \Lambda$.

Our construction of Markov partitions for $\Lambda$ is based on the following theorem which we quote from [21].

<u>Theorem</u>. For a basic set $\Lambda$ as above there is a finite number of (periodic) saddle points $p_1^s, \ldots, p_{n_s}^s$ such that

$$\Lambda \cap (\bigcup_i W^s(p_i^s)) = \partial_s \Lambda;$$

Similarly, there is a finite number of (periodic) saddle points $p_1^u, \ldots, p_{n_u}^u$ such that

$$\Lambda \cap (\bigcup_i W^u(p_i^u)) = \partial_u \Lambda.$$

We note that this theorem, which is tipically two-dimensional, makes the construction of Markov partitions in dimension two much simpler than in higher dimensions, compare [5].

Now we come to the <u>definition</u> of a <u>Markov partition</u> for a basic set $\Lambda$ as introduced above. Such a Markov partition consists of a finite set of <u>boxes</u>, i.e. diffeomorphic images of the square $Q = [-1,+1]^2$, say $B_1 = \Psi_1(Q),\dots,B_\ell = \Psi_\ell(Q)$ such that

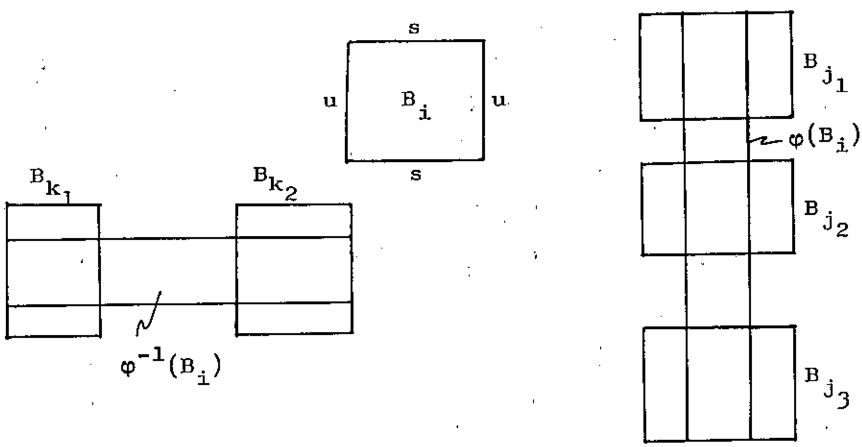
- $\Lambda \subset \bigcup_i B_i$ ;
- $\mathring{B}_i \cap \mathring{B}_j = \emptyset$ if $i \neq j$;
- $\varphi(\partial_s B_i) \subset \bigcup_j \partial_s B_j$ and

  $\varphi^{-1}(\partial_u B_i) \subset \bigcup_j \partial_u B_j$ , where

  $\partial_s B_i = \Psi_i(\{(x,y) \mid -1 \le x \le 1, \ |y| = 1\})$ and

  $\partial_u B_i = \Psi_i(\{(x,y) \mid |x| = 1, \ -1 \le y \le 1\})$.

The geometric consequences of this last condition are indicated below




all horizontal boundaries belong to $\partial_s$
and all vertical ones to $\partial_u$ .

Usually one also requires that $\varphi(B_i) \cap B_j$ is either empty or connected. For our present considerations this is not important, but one can always satisfy this last condition by taking the boxes of the Markov partition sufficiently small.

For the <u>construction</u> of a Markov partition we take arcs $I_1^s, \ldots, I_{n_s}^s$ in $W^s(p_1^s), \ldots, W^s(p_{n_s}^s)$ and arcs $I_1^u, \ldots, I_{n_u}^u$ in $W^u(p_1^u), \ldots, W^u(p_{n_u}^u)$, where $p_1^s, \ldots, p_{n_s}^s$ and $p_1^u, \ldots, p_{n_u}^u$ are (periodic) saddle points as in the above theorem, such that for each $i$, $\partial I_i^s \subset \bigcup_j I_j^u$ and $\partial I_i^u \subset \bigcup_j I_j^s$. It is possible to satisfy these last conditions since both $(\bigcup_j W^u(p_j^u)) \cap \Lambda$ and $(\bigcup_j W^s(p_j^s)) \cap \Lambda$ are dense in $\Lambda$.

We shall prove that if the arcs $I_1^s, \ldots, I_{n_s}^s$ and $I_1^u, \ldots, I_{n_u}^u$ are sufficiently long, they "divide" $\Lambda$ according to a Markov partition. To be more precise, we fix $\varepsilon > 0$ sufficiently small and say that $x \in \Lambda$ is enclosed by the above arcs if

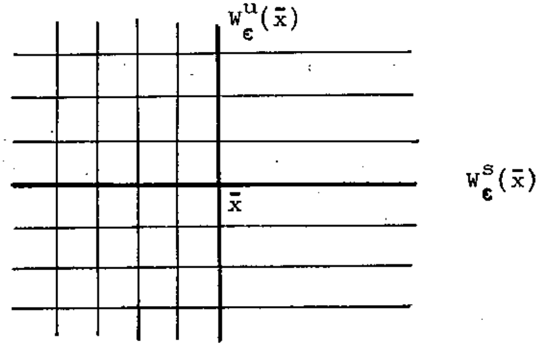
$W_\varepsilon^s(x) \cap (\bigcup_j I_j^u)$ contains $x$ or contains points on both sides of $x$,

and if

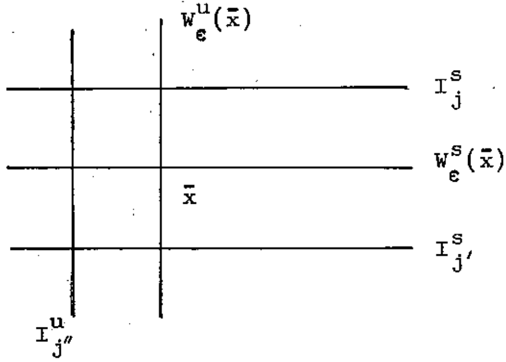
$W_\varepsilon^u(x) \cap (\bigcup_j I_j^s)$ contains $x$ or contains points on both sides of $x$.

We claim that if all $I_j^u$ and $I_j^s$ extend at least to length $\ell$ from the saddle point $p_j^u$, $p_j^s$ along $W^u(p_j^u)$, $W^s(p_j^s)$ in both directions, then for $\ell$ sufficiently big, they enclose all points of $\Lambda$.

We prove this claim by contradiction. Let $x_i$ be a point of $\Lambda$ which is not yet enclosed when $\ell = i$ for $i \in N$. By compactness $\{x_i\}$ has an accumulation point, say $\bar{x}$. We show that for some finite $\ell_o$ there is a neighbourhood of $\bar{x}$ such that all its points are enclosed whenever $\ell \geq \ell_o$. We have to distinguish between $\bar{x} \in \partial_u\Lambda$ and $\bar{x} \notin \partial_u\Lambda$ and also between $\bar{x} \in \partial_s\Lambda$ and $\bar{x} \notin \partial_s\Lambda$. We consider the case $\bar{x} \in \partial_u\Lambda$, $\bar{x} \notin \partial_s\Lambda$; the other cases can be treated similarly.




In the above figure we indicated local stable and unstable manifolds of points $x \in \Lambda$ near $\bar{x}$. Since $\bar{x} \in \partial_u\Lambda$, $\bar{x} \in \bigcup_j W^u(p^u_j)$ so for some $\ell_1$ and $\ell \geq \ell_1$, $\bar{x}$ belongs to $\bigcup_j I^u_j$. Since $(\bigcup_j W^u(p^u_j)) \cap \Lambda$ and $(\bigcup_j W^s(p^s_j)) \cap \Lambda$ are both dense in $\Lambda$, for some $\ell_2$ and $\ell \geq \ell_2$, $\bigcup_j I^s_j$ and $\bigcup_j I^u_j$ contain segments as indicated.



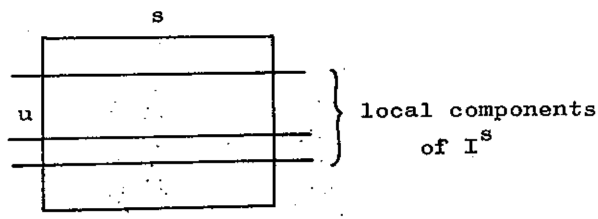

For all $\ell \geq \ell_o = \max(\ell_1, \ell_2)$ there is clearly a full neighbourhood of $\bar{x}$ in which all the points are surrounded. This gives the required contradiction and completes the proof of the claim.

From now on we assume that the arcs $I^s_j$ and $I^u_j$ are so long that all points of $\Lambda$ are enclosed. For $\Lambda - (\bigcup_j I^s_j \cup \bigcup_j I^u_j)$ we define an equivalence relation: $x \sim x'$ if we can join $x$ by $x'$ without crossing any of the arcs $I^s_j$ or $I^u_j$. Now it is easy to see that one can construct for each equivalence class a box, containing that equivalence class in its interior and whose boundary consists of segments of $\bigcup_j I^s_j$ and of $\bigcup_j I^u_j$ and such that the interior of the box contains no points of $\bigcup_j I^s_j$ or of $\bigcup_j I^u_j$. These boxes, one for each equivalence class form a Markov partition. Notice that $\varphi(\partial_s B_i) \subset \bigcup_j \partial_s B_j$ because $\varphi(\bigcup_j I^s_j) \subset \subset \bigcup_j I^s_j$, and $\varphi^{-1}(\partial_u B_i) \subset \bigcup_j \partial_u B_j$ because $\varphi^{-1}(\bigcup_j I^u_j) \subset \bigcup_j I^u_j$.

Observe that although points of $\Lambda$ may lie in the common boundary of two boxes, there is for each $x \in \Lambda$ a unique box $B$ such that $x \in \overline{(B^o \cap \Lambda)}$. In fact we can obtain a Markov partition with all the boxes disjoint by replacing each box $B$ by the smallest box $\tilde{B}$ such that:

- $\tilde{B} \cap \Lambda = \overline{(\mathring{\tilde{B}} \cap \Lambda)}$;
- $\partial\tilde{B}$ consists of parts of local stable and local unstable manifolds.

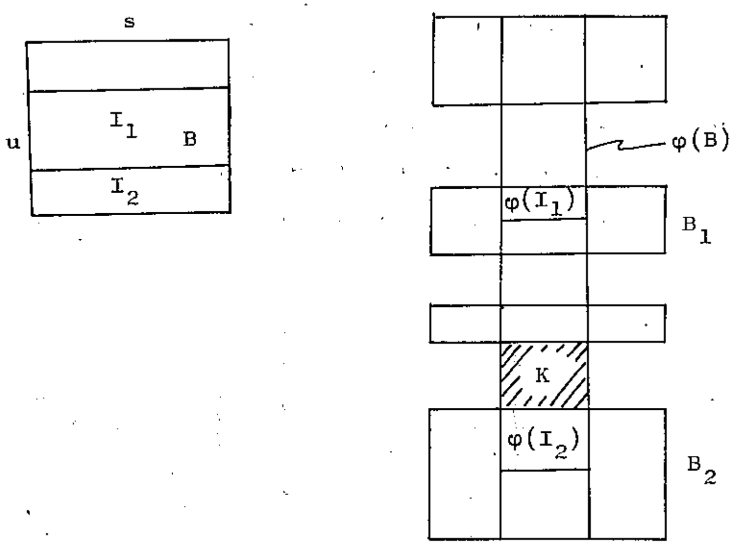
Now we choose a saddle point $p \in \Lambda$ and consider $W^s(p) \cap \Lambda$, or, more precisely for an arc $I^s$ in $W^s(p)$ we consider $I^s \cap \Lambda$. We take $I^s$ so long that it passes through all the boxes of the Markov partition. So for each box $B$, $B \cap I^s$ consists of a number, at least one, of arcs in the s-direction passing from one component of $\partial_u B$ to the other component of $\partial_u B$.




This fact, that one box $B$ can be crossed several times by $I^s$ is inconvenient because it makes the definition of a projection of $\Lambda$ on $\Lambda \cap I^s$ ambiguous. For this reason we shall describe how to refine the Markov partition for a fixed $I^s$, so as to obtain a new Markov partition in which each box is passed exactly once by $I^s$.

Let $I_1, I_2 \subset I^s$ be components of $B \cap I^s$. Let $\varphi(I_1)$ and $\varphi(I_2)$ be contained in the boxes $B_1$ and $B_2$; the component of $I^s \cap B_i$, containing $\varphi(I_i)$ is denoted by $\tilde{I}_i$. If $B_1 = B_2$ we start again with $B_1 = B_2$ instead of $B$ and $\tilde{I}_i$ instead of $I_i$. We repeat this until we get $B_1 \neq B_2$; this must finally happen since under positive iterations of $\varphi$, $I_1$ and $I_2$ get more and more separated in the u-direction.

We have now the following situation.




The region $K$ of $\varphi(B)$, indicated in the above figure is just a connected component of $\varphi(B) - (\bigcup_j B_j)$, where $\bigcup_j B_j$ is the union of all the boxes of the Markov partition. Now we refine our Markov partion by removing from $B$ the stip $\varphi^{-1}(K)$, thus splitting B in two smaller boxes. It is not hard to see that we obtain in this way a new Markov partition in which $I_1$ and $I_2$ are not in the same box any more. So we deminished the set of pairs of components of $I^s \cap (\bigcup_j B_j)$ by one. Repeating this construction sufficiently often we obtain a Markov partition in which each box is crossed exactly once by $I^s$.

Next we define a projection $\pi: \Lambda \to \Lambda \cap I^s$ by taking in each box of the Markov partition the projection along local unstable manifolds into the intersection of $I^s$ with that box. Then we define the expanding map $\Psi: I^s \cap \Lambda \to I^s \cap \Lambda$ as $\Psi = \pi \circ \varphi^{-N}$ (this is the same as in Chapter IV except for the parametrization $\alpha: \mathbb{R} \to W^s(p)$). Extending the projection $\pi$ from $\Lambda$ to the union of the boxes of the Markov partition of $\Lambda$, we get the map $\Psi$ defined on a set of intervals $K_1, \ldots, K_k$ in $I^s$; each interval is mapped by $\Psi$ diffeomorphically onto one of the intersections of $I^s$ with a box of the Markov partition of $\Lambda$. The intervals $K_1, \ldots, K_k$ form the Markov partition of $I^s \cap \Lambda$ with expanding map $\Psi$.

With these constructions it is clear that the ideas of the example in Chapter IV, namely the construction of Markov partitions when the basic set is the horseshoe, carry over to the general situation.

# APPENDIX III

## 3. Heteroclinic cycles.

We discuss here an extension of the theorem on unfolding of homoclinic tangencies and measure of bifurcation sets (Chapter V).

Again we consider a family of diffeomorphisms $\varphi_\mu$ on $M^2$ and assume that $\varphi_\mu$ is hyperbolic for $\mu < 0$; i.e. $L^+(\varphi_\mu)$ is a hyperbolic set with the no-cycle property for all $\mu \leq 0$. We can write [30], [18]

$$L^+(\varphi_o) = L_1 \cup \ldots \cup L_k$$

where $L_i$ is closed, $\varphi_o$-invariant and transitive (it has a dense orbit); also, the periodic orbits are dense in $L_i$. The sets $L_i$ are called basic sets. We again assume that we have a <u>cycle</u> $\Gamma$ in $L^+(\varphi_o)$ but we now assume the cycle to involve more than one basic set, say $j$ of them, $L_{i_1}, \ldots, L_{i_j}$. Recall that the notion of cycle was introduced in Chapter V - Section 1. We now define $d^s(\Gamma) = \max_{1 \leq k \leq j} d^s(L_{i_k})$ and $d^u(\Gamma) = \max_{1 \leq k \leq j} d^u(L_{i_k})$. Similar to the families of diffeomorphisms in Chapter V, we consider $\varphi_\mu$ such that:

- $\varphi_\mu$ is hyperbolic for $\mu \leq 0$
- $\varphi_o$ has a cycle $\Gamma$ with a unique orbit of tangency of stable and unstable manifolds, which is quadratic and unfolds generically.

As before, let $B(\varphi_\mu) = \{\mu \in \mathbb{R} \mid \varphi_\mu$ is not hyperbolic$\}$ and let m( ) denote the Lebesgue measure.

<u>Theorem</u> [24]. For a family $\{\varphi_\mu\}$ as above, if $d^s(\Gamma) + d^u(\Gamma) < 1$, then

$$\lim_{\mu_o \to 0} \frac{m(B(\varphi_\mu) \cap [0,\mu_o])}{\mu_o} = 0.$$

The following two pictures indicate examples of cycles as above, the first one involving two fixed saddles as basic sets and the second one involving a horseshoe and a saddle.

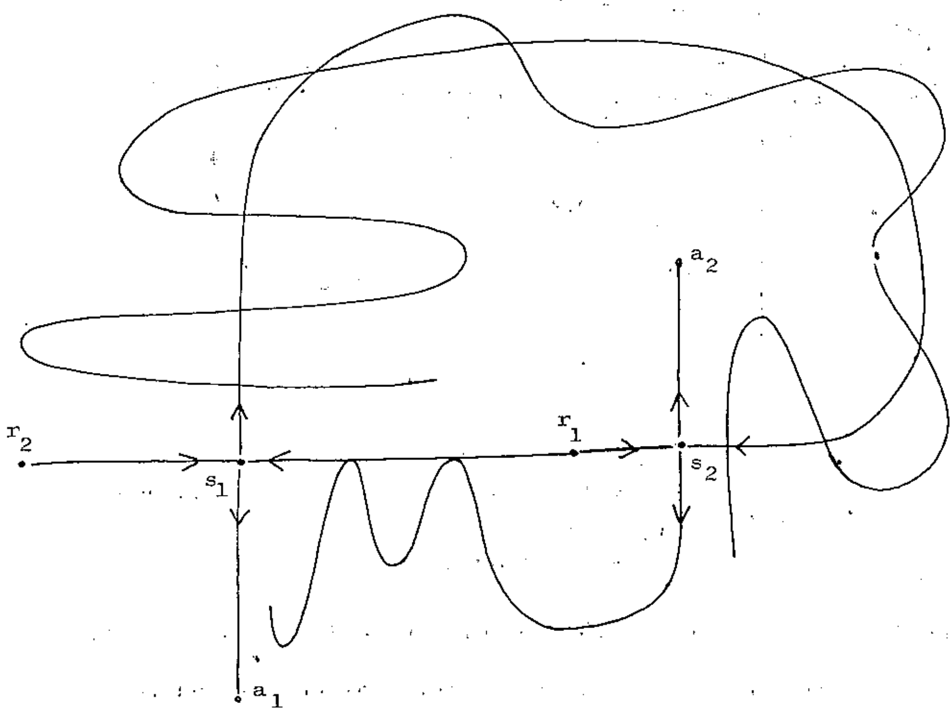



Fig. 1 | $s_1, s_2$ = fixed saddles
$r_1, r_2$ = fixed sources
$a_1, a_2$ = fixed sinks

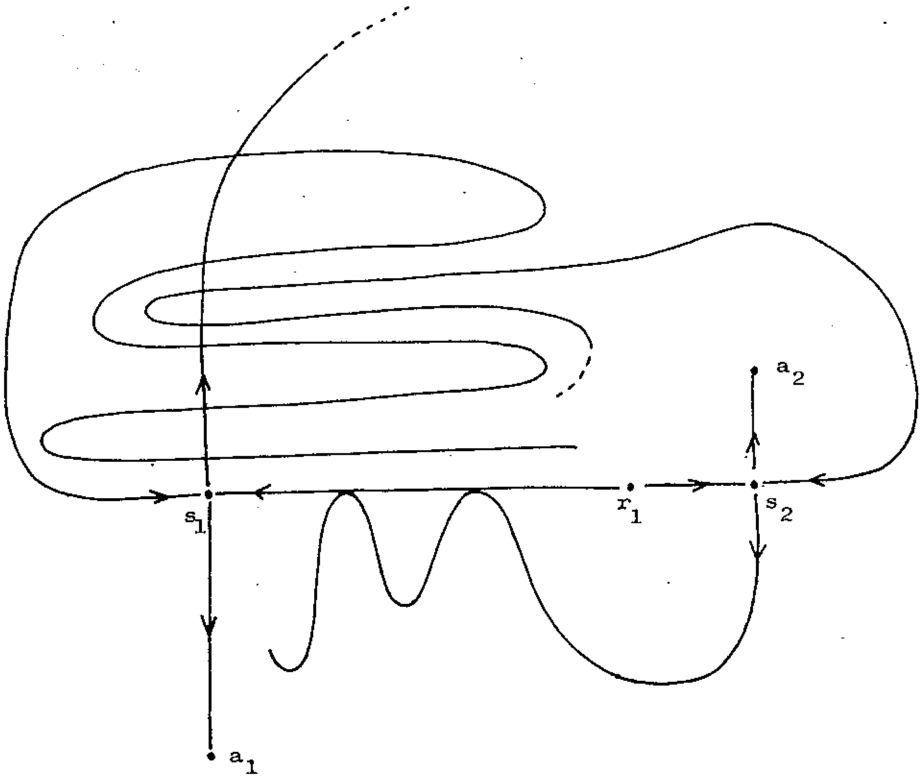



Fig. 2 | $s_2$ = fixed saddle
$s_1$ = fixed saddle part of a horseshoe
$r_1$ = fixed source
$a_1, a_2$ = fixed sinks

(We leave to the reader to complete the main features of Fig. 2)

Again, we suggest a strong correlation between the condition $d^s(\Gamma) + d^u(\Gamma) < 1$ and the hyperbolicity of $\varphi_\mu$, for $\mu$ small.

<u>Conjecture</u>. For a family $\varphi_\mu$ as above if $d^s(\Gamma) + d^u(\Gamma) > 1$, then

$$\liminf_{\mu_o \to 0} \frac{m(B(\varphi_\mu) \cap [0,\mu_o])}{\mu_o} > 0.$$

If we change lim inf by lim sup in the conjecture above, still remains a very interesting question.

We observe that theorem above, as well as the corresponding one in Chapter V for homoclinic tangencies, extend previous resul in [21-A] where $L(\varphi_\mu)$ was supposed to be finite for $\mu < 0$. Also the result concerned lim inf instead of the full limit of the relative measure of the bifurcation set.

# APPENDIX IV

## 4. On the shape of some strange attractors

(following Szewc and Tangerman)

It is well known from numerica examples provided by Hénon around 1976 [8-A] that "strange" attractors in the plane may have the shape of the unstable separatrix of a saddle point with associated homoclinic intersections. This phenomenon has not yet been explanied in a satisfactory way - also it may disappear when the diffeomorphism is perturbed (without destroying the homoclinic intersections). A partial explanation however has been given independently by B. Szewc and F. Tangerman around 1981 but was never published. Their argument, in the simplest form, leads to:

Proposition. Let $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism with a hyperbolic saddle point $p$ such that:

- $W^u(p)$ and $W^s(p)$ have a homoclinic intersection $q$, i.e. $p \neq q$;
- the norm of $\det(d\varphi)$ is everywhere smaller than one;
- $W^u(p)$ remains in a bounded region of $\mathbb{R}^2$.

Then there is a non-empty open set $U \subset \mathbb{R}^2$, such that for each $x \in U$, $\omega(x) \subset \overline{W^u(p)}$, i.e. the distance from $\varphi^n(x)$ to $W^u(p)$ goes to zero for $n$ going to infinite.

Proof. Let $U \subset \mathbb{R}^2$ be a bounded open subset whose boundary consists of segments of $W^u(p)$ and $W^s(p)$. Such $U$ exist: due to the homoclinic intersection $q$ one can form a closed curve consisting of segments of $W^u(p)$ and $W^s(p)$.

Consider $\varphi^n(U)$ for $n \geq 0$. Its boundary consists also of segments of $W^s(p)$ and $W^u(p)$; the segments of $W^s(p)$ become shorter and converge to $p$ for increasing $n$. From this and the boundedness of $W^u(p)$ it follows that the boundary of $\varphi^n(U)$, and hence $\varphi^n(U)$ stays in a bounded part of $\mathbb{R}^2$. This means that on $\bigcup_{n\geq 0} \varphi^n(U)$, $|\det d\varphi| < 1-\varepsilon$ for some $\varepsilon > 0$, and hence that the area of $\varphi^n(U)$ goes to zero as $n$ increases.

This means that for any $x \in U$ and $n > 0$, $n$ big, $\varphi^n(x)$ is near the boundary of $\varphi^n(U)$ and hence near $W^u(p) \cup W^s(p)$. Since the part of the boundary formed by $W^s(p)$ has decreasing length, $\varphi^n(x)$ is close to $W^u(p)$. This means that $\omega(x) \subset \overline{W^u(p)}$.

Note that this proposition explains that certain attractors are <u>contained in</u> $\overline{W^u(p)}$ but not that they fill out all of it. In fact there could only be one periodic attractor in $\overline{W^u(p)}$. In that direction it would be interesting to have non-existence results for periodic attractors.

It is not hard to see that the above proposition applies to the Hénon map $H: \mathbb{R}^2 \to \mathbb{R}^2$ given by $H(x,y) = (1-(1,4).x^2+y, (0,3).x)$; the existence of a homoclinic intersection was proved in [16-A]. In fact both Szewc and Tangerman proved more, namely that one can, in the case of the Hénon map, choose $U$ to be a neighbourhood of $\overline{W^u(p)}$.

Another example can be obtained from the second example in the introduction (the pendulum) by a small perturbation. With a first perturbation we make the diffeomorphism, or even the differential equation attracting towards the $\{E=1\}$ level without perturbing the dynamics inside the $\{E=1\}$ level. With a second perturbation we make transverse homoclinic intersections in all the branches of the separatrices of the saddle point $(\pi,0)$ (and hence destroy $\{E=1\}$ as an invariant set). Although this example is not defined on the plane but on an annulus, the arguments of the proof still work.

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