MÉTODOS DE ANÁLISE FUNCIONAL APLICADOS A EQUAÇÕES DIFERENCIAIS

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NOTA EXPLICATIVA

A idéia do presente curso é apresentar aos alunos de Pós-Graduação duas linhas de pesquisa em Análise, atualmente em desenvolvimento no Brasil. Ambas são objeto de estudo nas melhores escolas matemáticas e têm indubitavelmente uma posição central em Análise na atualidade.

O texto é composto de duas partes, uma sobre alguns métodos de Análise Funcional Não-Linear e aplicações a problemas de contorno para equações elíticas, de autoria do Professor David Goldstein Costa. A segunda parte versa sobre operadores integrais de Fourier e propagação de singulareidades, a qual foi redigida pelos Professores Fernando A. Figueiredo Cardoso da Silva e José Ruidival dos Santos Filho.

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ÍNDICE

ALGUNS MÉTODOS DE ANÁLISE FUNCIONAL NÃO-LINEAR E APLICAÇÕES

PROPAGATION OF SINGULARITIES OF SOLUTIONS OF PSEUDODIFFERENTIAL EQUATIONS

ALGUNS MÉTODOS DE

ANÁLISE FUNCIONAL NÃO - LINEAR

E APLICAÇÕES

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I. INTRODUÇÃO

O objetivo destas notas é apresentar alguns métodos de Análise Funcional Não-Linear com vistas à resolução de equações diferenciais parciais elíticas não lineares.

Um desses métodos tem a sua origem na topologia e consiste em determinar quando um homeomorfismo local $\psi:X \to Y$ é um homeomorfismo global. Estaremos interessados na situação em que $X = Y$ é um espaço de Banach. Neste caso, ψ é global se (e somente se) é uma aplicação própria (isto é, a imagem inversa de qualquer compacto é novamente um compacto).

Outro método que desempenhará um papel central nessas notas refere-se a uma versão global do método clássico de Lyapunov-Schmidt. Essencialmente, trata-se de reduzir uma dada equação $T(u) = 0$ num espaço de Banach E de dimensão infinita a um sistema de duas equações, uma das quais em dimensão finita, de maneira que a outra equação (em dimensão infinita) tenha solução única. Então, o estudo da equação original fica reduzido ao estudo de uma equação equivalente em dimensão finita.

Esses métodos serão utilizados no estudo do problema de Dirichlet para a equação

$$(*) \qquad -\Delta u = g(u) + f \quad \text{em} \quad \Omega,$$

onde $\Omega \subset \mathbb{R}^N$ é um domínio limitado, $g:\mathbb{R} \to \mathbb{R}$ é uma função contínua e $f:\Omega \to \mathbb{R}$ é uma funçao dada. Mais precisamente, dada $f \in L^2(\Omega)$, estaremos interessados no problema de existência de soluções fracas $u \in H_0^1(\Omega)$ (1) da equação acima, isto é, funções $u \in H_0^1(\Omega)$ que satisfazem

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)dx = \int_\Omega g(u(x))v(x)dx + \int_\Omega f(x)v(x)dx, \quad \forall\, v \in C_0^\infty(\Omega).$$

Esta formulação generalizada do problema de Dirichlet possibilita um estudo da equação (*) no contexto da teoria dos espaços de Hilbert.

Na Seção II recordamos alguns fatos básicos sobre a teoria linear do problema de Dirichlet e enunciamos os resultados principais a serem demonstrados nas Seções V e VI. As demonstrações seguem as referências lá indicadas. A Seção III trata de resultados abstratos de invertibilidade global para aplicações entre espaços de Banach, resultados esses que, em conjunção com o método de Lyapunov-Schmidt (apresentado na Seção IV), serão utilizados no estudo de (*).

Vale observar que esses métodos são aplicáveis a problemas mais gerais do que (*), por exemplo, envolvendo outros operadores elíticos além do Laplaciano bem como não-linearidades g dependentes também da variável espacial $x \in \Omega$. A nossa escolha de

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(1) $H_0^1(\Omega)$ é o espaço usual de Sobolev obtido por completamento do espaço $C_0^\infty(\Omega) = \{u:\Omega \to \mathbf{R} \mid u \text{ é } C^\infty \text{ e suporte } (u)\subset \Omega\}$ na norma $(\int_\Omega |u|^2dx + \int_\Omega |\nabla u|^2dx)^{1/2}$. Nestas notas consideraremos sempre $H_0^1(\Omega)$ munido da norma $(\int_\Omega|\nabla u|^2dx)^{1/2}$, a qual, pela desigualdade de Poincaré, é equivalente à norma usual (V. [1]).

(*) foi ditada por uma questão de simplicidade de exposição. De fa to, procuramos nos restringir a dois métodos da Análise Funcional Não-Linear: o método baseado em teoremas de invertibilidade global para aplicações entre espaços de Banach[1] e uma versão global do método clássico de Lyapunov-Schmidt. Outros métodos e suas aplica ções podem ser encontrados, por exemplo, em Figueiredo [14] (mé todos de iteração monotônica e do grau de Leray-Schauder) e Castro [11] (método variacional). De uma maneira geral, duas referências básicas (além da já mencionada [9]) de Análise Funcional Não-Linear que o leitor interessado poderá consultar são [23], de L. Nirenberg, e [26], de J. Schwartz.

Finalmente, gostaríamos de expressar os nossos agradeci mentos à Comissão Organizadora, pela oportunidade oferecida, ao Pro fessor Djairo G. de Figueiredo, por suas sugestões e comentários, e ao Sr. Adelio G. do Amaral, pelo seu esmerado trabalho de datilo grafia.

Brasília, maio de 1981

David Goldstein Costa
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(1) Cf. Ambrosetti-Prodi [9] onde é feita uma apresentação bas tante abrangente deste método.

II. FORMULAÇÃO DO PROBLEMA. ENUNCIADOS DOS TEOREMAS PRINCIPAIS.

Vamos considerar o seguinte problema de Dirichlet não-linear

$$(*)\qquad \begin{cases} -\Delta u = g(u) + f & \text{em } \Omega \\ u = 0 & \text{em } \partial\Omega \end{cases}$$

onde $\Omega \subset \mathbb{R}^N$ é um domínio limitado com fronteira $\partial\Omega$, $g : \mathbb{R} \to \mathbb{R}$ é uma função contínua (representando a não-linearidade) e f é uma função dada em $L^2(\Omega)$.

Estaremos interessados no seguinte

Problema. Dada $f \in L^2(\Omega)$, determinar se existe uma solução fraca de (*), isto é, uma função $u \in H_0^1(\Omega)$ tal que

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx = \int_\Omega g(u(x))v(x)\,dx + \int_\Omega f(x)v(x)\,dx, \quad \forall\, v \in H_0^1(\Omega)$$

Inicialmente, vamos recordar alguns resultados básicos sobre o problema de Dirichlet linear

$$(1)\qquad \begin{cases} -\Delta u = \rho & \text{em } \Omega \\ u = 0 & \text{em } \partial\Omega \end{cases}$$

Proposição II.1. **Dado $\rho \in L^2(\Omega)$, existe uma única solução fraca $u = K\rho \in H_o^1(\Omega)$ do problema (1). O operador solução $K: L^2(\Omega) \to H_o^1(\Omega)$ é linear e limitado.**

A demonstração deste resultado decorre da aplicação do Teorema de Representação de Riesz ao funcional linear contínuo

$$H_o^1(\Omega) \ni v \mapsto \int_\Omega \rho(x)v(x)dx,$$

e $K\rho \in H_o^1(\Omega)$ resulta definido por

$$(2) \qquad (K\rho, v)_{H_o^1} = (\rho, v)_{L^2}, \qquad \forall\, v \in H_o^1(\Omega).$$

Agora, como a inclusão $H_o^1(\Omega) \subset L^2(\Omega)$ é compacta, o operador K considerado como um operador de $L^2(\Omega)$ em $L^2(\Omega)$ é compacto. Além disso, em vista da relação (2), ele resulta simétrico de $L^2(\Omega)$ em $L^2(\Omega)$ e positivo. Portanto, o Teorema de Representação Espectral de Hilbert-Schmidt implica o seguinte resultado sobre os autovalores e autofunções (soluções fracas $0 \neq u \in H_o^1(\Omega)$) do problema

$$(3) \qquad \begin{cases} -\Delta u = \lambda u & \text{em } \Omega \\ u = 0 & \text{em } \partial\Omega \end{cases}$$

Proposição II.2. **Os autovalores de (3) formam uma sequência**

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots,$$

com $\lim \lambda_n = +\infty$, e todos os autoespaços são de dimensão finita. Além disso, se

$$\phi_1, \phi_2, \ldots, \phi_n, \ldots$$

são autofunções correspondentes, ortonormalizadas, então elas constituem um sistema ortonormal completo em $L^2(\Omega)$.

Observações.

1. Os $\frac{1}{\lambda_k}$ são os autovalores e os ϕ_k são autofunções correspondentes (ortonormalizadas) do operador $K: L^2(\Omega) \to H_0^1(\Omega) \subset L^2(\Omega)$, isto é, $\lambda_k K\phi_k = \phi_k$. Portanto, $\|K\| = \frac{1}{\lambda_1}$.

2. O conjunto $\{\lambda_k\}$ chama-se o **espectro** do operador $-\Delta$ em Ω com condição de Dirichlet e será designado por $\sigma(-\Delta)$.

Uma vez feitas essas considerações sobre a teoria linear, retornamos ao nosso problema (*) onde vamos sempre supor que a não-linearidade $g:\mathbb{R} \to \mathbb{R}$ verifica a hipótese

(h_1) g é de classe C^1 com g' limitada, digamos

$$\overline{g'(\mathbb{R})} = [a,b].$$

Neste caso, se g "não interage" com o espectro $\sigma(-\Delta)$, pode-se mostrar que o problema (*) é sempre unicamente solúvel. De fato (cf. [3, 4, 13, 20] p.e.), temos o seguinte

Teorema 1 (Existência e Unicidade). **Suponha que g verifica (h_1) com $[a,b] \cap \sigma(-\Delta) = \emptyset$, digamos, $\lambda_n < a \leq g'(s) \leq b < \lambda_{n+1}$ $\forall s \in \mathbb{R}$. Então, para cada $f \in L^2(\Omega)$, o problema (*) possui uma única solu-**

ção fraca $u \in H_0^1(\Omega)$.

Mais geralmente, em situações nas quais a não linearidade "interage" com o espectro de $-\Delta$, isto é, $[a,b] \cap \sigma(-\Delta) \neq \phi$, faremos a hipótese adicional de que g é "assintoticamente linear" no sentido de que

(h_2) Existem $\gamma \in \mathbb{R}$ e $M \geq 0$ tais que $|g(s) - \gamma s| \leq M$ $\forall s \in \mathbb{R}$(1)

Temos dois casos a considerar: o caso **não-ressonante**, quando $\gamma \notin \sigma(-\Delta)$, e o caso **ressonante**, quando $\gamma \in \sigma(-\Delta)$.

No caso não-ressonante, como é de se esperar, o problema (*) é novamente sempre solúvel (cf. [2, 13, 20, 25], p.e.).

Teorema 2 (Existência no Caso Não-Ressonante). **Se $[a,b] \cap \sigma(-\Delta) \neq \phi$ e g verifica (h_2) com $\gamma \notin \sigma(-\Delta)$ então, para cada $f \in L^2(\Omega)$, o problema (*) possui uma solução fraca $u \in H_0^1(\Omega)$ (não necessariamente única).**

O caso ressonante é mais delicado e elimina a possibilidade de existência de solução para toda $f \in L^2(\Omega)$. De fato, mostra-se que existe um subconjunto próprio $\Sigma \subsetneq L^2(\Omega)$, que é limitado nas "direções" ϕ_j's do autoespaço $\ker(-\Delta-\gamma)$, tal que (*) é solúvel se e somente se $f \in \Sigma$. Além disso, pode-se formular condições, conhecidas como condições do **tipo Landesman-Lazer**, que são

(1) Então, $\gamma = \lim_{s\to\pm\infty} \frac{g(s)}{s} \in \overline{g'(\mathbb{R})} = [a,b]$

suficientes para que f pertença a Σ.

A seguir, por uma questão de simplicidade de notação, enunciaremos apenas o resultado de existência para o caso de ressonância no primeiro autovalor λ_1, como considerado originalmente por Landesman e Lazer (cf. [21]). O caso geral de ressonância num autovalor qualquer $\lambda_n \geq \lambda_1$ (cf. [5, 6, 7]) será enunciado e demonstrado na seção VI (Teoremas 3 e 4).

Teorema 3*. Suponha que $\lambda_1 \in [a,b]$ e g verifica (h_2) com $\gamma = \lambda_1$. Suponha ainda que existem os limites

$$h(\pm\infty) = \lim_{s\to\pm\infty} h(s) = \lim_{s\to\pm\infty} [g(s) - \lambda_1 s]$$

e $h(+\infty) < h(s) < h(-\infty)$, $\forall\, s \in \mathbb{R}$. Então, dada $f \in L^2(\Omega)$, o problema (*) é solúvel se e somente se

$$\text{(L.-L.)} \quad -h(-\infty)\int_\Omega \phi_1(x)dx < \int_\Omega f(x)\phi_1(x)dx < -h(+\infty)\int_\Omega \phi_1(x)dx^{(1)}.$$

Resultados sobre multiplicidade de soluções no caso ressonante podem ser obtidos com os métodos destas notas (cf. [5, 6, 7]) mas não serão aqui abordados (cf. [3, 12, 16, 17, 18, 19, 24], onde estes e outros métodos são utilizados). No tocante a multiplicidade, consideraremos apenas uma situação não-ressonante bastante interessante, que foi abordada inicialmente por Ambrosetti

(1) Como se sabe, o primeiro autovalor λ_1 é simples e uma autofunção correspondente não muda de sinal em Ω. Aqui, estamos escolhendo $\phi_1 > 0$ em Ω.

e Prodi ([8]), na qual a não linearidade g "cruza" somente o primeiro autovalor λ_1. A demonstração do resultado de Ambrosetti-Prodi, com os métodos apresentados nestas notas, segue as referências [10, 23] (V. também [14], onde são usados os métodos de iteração monotônica e da teoria do grau de Leray-Schauder).

Teorema 5 (Ambrosetti-Prodi). **Suponha que** $[a,b] \cap \sigma(-\Delta) = \{\lambda_1\}$, **com** $0 < a = \lim_{s\to-\infty} g'(s) < b = \lim_{s\to\infty} g'(s)$. **Dada** $f \in L^\infty(\Omega) \subset L^2(\Omega)$, **decomponha** $f = t\phi_1 + f_2$, **onde** $\int_\Omega f_2\phi_1 dx = 0$. **Então, existe um número real** $\bar{t} = \bar{t}(f_2)$ **tal que o problema (*) possui**

(i) **pelo menos uma solução, se** $t = \bar{t}$;

(ii) **pelo menos duas soluções, se** $t < \bar{t}$;

(iii) **nenhuma solução, se** $t > \bar{t}$.

Observações.

1) Nas referências [8, 10], f é dada em $C^\mu(\bar{\Omega})$ (de onde segue-se que as soluções são clássicas) e supõe-se que $g \in C^2$ é estritamente convexa. Neste caso, obtém-se uma informação precisa sobre o número de soluções (substitua "pelo menos" por "exatamente", no enunciado acima).

2) É importante ressaltar que, em todos os resultados enunciados acima, se $\partial\Omega$ é suficientemente regular (no nosso caso, C^3 é suficiente) e f é dada em $C^\mu(\bar{\Omega})$, então as soluções são clássicas, isto é, pertencem ao espaço $C_o^{2+\mu}(\bar{\Omega})$ das funções em

$C^{2+\mu}(\overline{\Omega})$ que se anulam em $\partial\Omega$. Este fato é consequência da teoria de regularidade de Schauder para equações elíticas (cf. [15]). Nestas notas estaremos sempre supondo que $\partial\Omega$ é suficientemente regular.

III. RESULTADOS DE INVERTIBILIDADE GLOBAL PARA APLICAÇÕES ENTRE ESPAÇOS DE BANACH.

Seja E um espaço de Banach e $\Psi:E \to E$ um homeomorfismo local. Estaremos interessados em saber sob que condições Ψ é um homeomorfismo global.

Inicialmente, observamos que entre os exemplos mais típicos de homeomorfismos globais estão as aplicações Ψ que são perturbações da identidade por uma contração. Em outras palavras, temos o seguinte

Teorema III.1. **Seja $C:E \to E$ uma contração (i.e., para algum $0 \leq \alpha < 1$, $\|C(u) - C(v)\| \leq \alpha\|u-v\|$, $\forall\ u,v \in E$). Então $\Psi = I-C:E \to E$ é um homeomorfismo global.**

Demonstração. Ψ é obviamente contínua (de fato, Lipschitziana com constante $(1+\alpha)$). Por outro lado, o Princípio da Contração de Banach implica que, para todo $f \in E$, existe um único $u \in E$ tal que $\Psi(u) = f$ (basta considerar a contração $u \mapsto C(u) + f$). Logo, Ψ é uma bijeção. Finalmente, a desigualdade

$$\|\Psi(u) - \Psi(v)\| \geq \|u-v\| - \|C(u) - C(v)\| \geq (1-\alpha)\|u-v\|$$

mostra que a inversa Ψ^{-1} é contínua (de fato, Lipschitziana com constante $\frac{1}{(1-\alpha)}$).

■

Seja agora $\Psi : E \to E$ um homeomorfismo local. Se Ψ é um homeomorfismo global então, para cada compacto $A \subset E$, $\Psi^{-1}(A)$ é novamente compacto. Noutras palavras, Ψ é uma aplicação **própria**. O fato notável é que a recíproca é verdadeira, isto é, temos o seguinte (cf. [9, 22]).

Teorema III.2. Seja $\Psi : E \to E$ um homeomorfismo local. Então, Ψ é global se (e somente se) Ψ é uma aplicação própria.

Demonstração. Queremos mostrar que um homeomorfismo local, próprio, é um homeomorfismo global. De início, observe que um homeomorfismo local é uma aplicação aberta: em particular, $\Psi(E)$ é aberto. Também, sendo Ψ própria, não é difícil ver que ela resulta uma aplicação fechada. Logo, $\Psi(E) = E$, isto é, Ψ é sobrejetiva. A demonstração estará completa se mostrarmos que Ψ é injetiva.

A idéia de mostrar que Ψ é injetiva baseia-se no chamado do método do prolongamento analítico. Faremos aqui apenas um esboço da demonstração (cf. [9, 22]). O que ocorre é que Ψ é uma aplicação de recobrimento (por ser homeomorfismo local, próprio) e, como tal, possui a "propriedade de levantamento único de caminhos", isto é, dado qualquer caminho(1) $\beta : [0,1] \to E$ e um ponto $u \in E$ com $\Psi(u) = \beta(0)$, existe um único caminho $\alpha : [0,1] \to E$ tal que $\alpha(0) = u$ e $\Psi(\alpha(t)) = \beta(t)$, $\forall\, t \in [0,1]$ (α é o "levantamento" de β no ponto $u \in \Psi^{-1}(\beta(0))$). Além disso, α depende continuamente de β

(1) Por definição, os caminhos são sempre contínuos.

e $u \in \Psi^{-1}(\beta(0))$. Resulta daí que uma homotopia entre caminhos β_o, β_1 pode ser levantada de maneira única, uma vez fixado um levantamento α_o de β_o. Em particular, se β é um caminho fechado (logo, homotópico a um constante pois E é simplesmente conexo) então, qualquer levantamento α de β é também um caminho fechado.

Agora é fácil verificar que Ψ é injetiva. Com efeito, sejam $u_1, u_2 \in E$ com $\Psi(u_1) = \Psi(u_2) = v$ e considere um caminho qualquer $\alpha:[0,1] \to E$ ligando u_1 a u_2, isto é, $\alpha(0) = u_1$, $\alpha(1) = u_2$. Então, α é o levantamento de $\beta = \Psi \circ \alpha$ no ponto u_1. Como β é um caminho fechado ($\beta(0) = \beta(1) = v$), o caminho α resulta também fechado, como vimos acima. Portanto, $u_1 = u_2$.

■

Observações:

1. Mais geralmente, o teorema acima continua válido para um hoemomorfismo local entre espaços topológicos (de Hausdorff) X e Y, desde que X seja conexo por caminhos e Y seja conexo por caminhos e simplesmente conexo (cf. [22]).

2. Como veremos mais adiante, as aplicações Ψ consideradas na utilização do Teorema III.2 serão de classe C^1 e a condição de homeomorfismo local será obtida através do Teorema da Função Inversa. Neste caso, uma tal Ψ (difeomorfismo local, próprio) resultará um difeomorfismo global.

Observamos, agora, que uma classe importante de aplicações próprias pode ser obtida através da seguinte

Proposição III.3. **Seja W um espaço de Banach. Se $L: W \to W$ é um isomorfismo (isto é, linear contínua com inversa contínua) e $N: W \to W$ é uma aplicação contínua, uniformemente limitada, e compacta[(1)], então $\Psi = L - N$ é uma aplicação própria.**

Demonstração. Seja $A \subset W$ um conjunto compacto e $\{w_n\}$ uma sucessão tal que $\Psi(w_n) = Lw_n - N(w_n) = z_n \in A$. Queremos mostrar que, para alguma subsucessão $\{w_{n_j}\}$ de $\{w_n\}$, temos $w_{n_j} \to w_o$ com $\Psi(w_o) = z_o \in A$. Com efeito, podemos já de início supor que $\Psi(w_n) = z_n$ é convergente para um ponto de A,

$$\Psi(w_n) = z_n \to z_o \in A.$$

Por outro lado, como $\{N(w_n)\}$ é um conjunto limitado, o conjunto $\{Lw_n\} = \{z_n + N(w_n)\}$ e, portanto, $\{w_n\}$ resulta um conjunto limitado. Daí, segue-se da compacidade da aplicação N que

$$N(w_{n_j}) \to z$$

para alguma subsucessão $\{w_{n_j}\}$ e $z \in W$. Portanto,

$$Lw_{n_j} = \Psi(w_{n_j}) + N(w_{n_j}) \to z_o + z, \quad \text{e}$$

$$w_{n_j} \to L^{-1}(z_o + z) \equiv w_o.$$

Aplicando Ψ obtemos $\Psi(w_{n_j}) \to \Psi(w_o)$, logo $\Psi(w_o) = z_o \in A$.

■

Finalmente, além do Teorema III.2, vamos apresentar mais um resultado de "monodromia" que fornece condições suficientes pa

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(1) Isto é, conjuntos limitados são transformados por N em conjuntos relativamente compactos.

ra que um difeomorfismo (C^1) local seja um difeomorfismo global. Trata-se, essencialmente, de uma versão global do Teorema da Função Inversa.

Teorema III.4 (Hadamard). **Sejam E um espaço de Banach e $\Psi : E \to E$ uma aplicação de classe C^1 tal que $\Psi'(u)$ é inversível (isto é, um isomorfismo) para todo $u \in E$. Se, para algum $\delta > 0$, temos**

$$\|\Psi'(u) \cdot v\| \geq \delta \|v\|$$

$\forall\, u,v \in E$ (equivalentemente, $\|[\Psi'(u)]^{-1}\| \leq \frac{1}{\delta}$, $\forall\, u \in E$), então Ψ é um difeomorfismo global.

Demonstração. V. [22, Proposição 13 do Capítulo 4] ou [26, Teorema 1.22].

IV. O MÉTODO DE LYAPUNOV-SCHMIDT.

Suponha que queremos estudar uma determinada equação

$$T(u) = 0$$

num espaço de Banach E de dimensão infinita. Se o espaço E decompõe-se numa soma direta $E = V \oplus W$ e $P:E \to V$ (respectivamente, $Q:E \to W$) designa a projeção sobre V ao longo de W (respectivamente, a projeção sobre W ao longo de V), a equação acima é equivalente ao sistema acoplado

$$\begin{cases} \Phi(v,w) = 0 \\ \\ \Psi(v,w) = 0 \end{cases}, \quad v \in V,\ w \in W,$$

onde $\Phi(v,w) = PT(v+w)$ e $\Psi(v,w) = QT(v+w)$.

Uma variante global do método clássico de Lyapunov-Schmidt a qual será utilizada nessas notas consiste em escolher convenientemente os espaços V e W, de maneira que:

(i) V seja de dimensão finita,

(ii) para cada $v \in V$, a equação $\Psi(v,w) = 0$ possua uma única solução $w = w(v)$.

Neste caso, o estudo da equação original fica reduzido ao estudo da equação em dimensão finita

$$\Gamma(v) = 0, \quad v \in V,$$

onde $\Gamma : V \to V$ é dado por $\Gamma(v) = \Phi(v, w(v)) = PT(v+w(v))$.

Este método será utilizado na resolução variacional (soluções fracas em $H_o^1(\Omega)$) do problema

$$(*) \qquad \begin{cases} -\Delta u = g(u) + f & \text{em} \quad \Omega \\ u = 0 & \text{em} \quad \partial\Omega \end{cases}$$

enunciado na Seção II, tanto no caso ressonante (Teoremas 3*, 3 e 4), como no caso não-ressonante mas com a não-linearidade g interagindo com o espectro $\sigma(-\Delta)$, isto é, $\sigma(-\Delta) \cap [a,b] \neq \emptyset$ (Teoremas 2 e 5).

Inicialmente, lembramos que estamos sempre supondo g de classe C^1 com g' limitada, $\overline{g'(\mathbb{R})} = [a,b]$ (hipótese (h_1)). Portanto, o **operador de Niemytskii** G **associado à função** g, $u(x) \overset{G}{\mapsto} g(u(x))$, está definido para todo $u \in L^2(\Omega)$, isto é, $G : L^2(\Omega) \to L^2(\Omega)$.

Lema IV.1. $u \in H_o^1(\Omega)$ **é solução fraca de (*) se e somente se** u **é solução da equação**

$$(**) \qquad u = KG(u) + Kf,$$

onde $K : L^2(\Omega) \to H_o^1(\Omega)$ **é o operador definido na Seção II (Proposição II.1).**

Demonstração. Por definição, $u \in H_o^1(\Omega)$ ser uma solução fraca de

(*) significa

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)dx = \int_\Omega g(u(x))v(x)dx + \int_\Omega f(x)v(x)dx, \quad \forall\ v \in H_o^1(\Omega),$$

isto é,

$$(\dagger) \qquad (u,v)_{H_o^1} = (G(u),\ v)_{L^2} + (f,v)_{L^2}, \quad \forall\ v \in H_o^1(\Omega)$$

Por outro lado, $K: L^2(\Omega) \to H_o^1(\Omega)$ está definido por

$$(K\rho,\ v)_{H_o^1} = (\rho,v)_{L^2}, \quad \forall\ v \in H_o^1(\Omega), \quad \forall\ \rho \in L^2(\Omega)$$

Portanto, a equação (†) acima é equivalente a

$$(u,v)_{H_o^1} = (KG(u),\ v)_{H_o^1} + (Kf,\ v)_{H_o^1}$$

isto é,

$$u = KG(u) + Kf, \quad u \in H_o^1(\Omega).$$

■

Observação IV.1. Note que, fixado um número qualquer $\gamma \in \mathbf{R}$ e pondo $h(s) = g(s) - \gamma s$, a equação (**) é equivalente a

$$(I - \gamma K)u = KH(u) + Kf, \quad u \in H_o^1(\Omega),$$

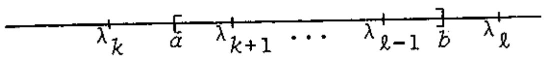
isto é,

$$(***) \qquad \tilde{L}_\gamma u = \tilde{H}(u) + \tilde{f}, \quad u \in H_o^1(\Omega),$$

onde $\tilde{f} = Kf \in H_o^1(\Omega)$, $\tilde{L}_\gamma = I - \gamma K\Big|_{H_o^1} : H_o^1(\Omega) \to H_o^1(\Omega)$ e

$\tilde{H}=KH_1 : H_o^1(\Omega) \to H_o^1(\Omega)$ é a composição do operador $K:L^2(\Omega) \to H_o^1(\Omega)$ com a restrição a $H_o^1(\Omega)$, $H_1 : H_o^1(\Omega) \to L^2(\Omega)$, do operador de Niemytskii H associado à função h.

Sejam, agora, $\lambda_k \in \sigma(-\Delta)$ o maior autovalor tal que $\lambda_k < a$ (caso $\lambda_1 < a$) e $\lambda_\ell \in \sigma(-\Delta)$ o menor autovalor tal que $\lambda_\ell > b$:



Escolhemos $V \subset H_o^1(\Omega)$ como o subespaço (de dimensão finita) gerado pelas autofunções $\phi_{k+1}, \ldots, \phi_{\ell-1}$ (V. Proposição II.2) e $W = V^\perp$ = espaço ortogonal a V em $H_o^1(\Omega)$. Então $E \equiv H_o^1(\Omega) = V \oplus W$ e temos o seguinte

Lema IV.2. A equação (***) é equivalente ao sistema

$$(1) \quad \begin{cases} \Phi(v,w) \equiv \tilde{L}_\gamma v - P\tilde{H}(v+w) - P\tilde{f} = 0 \\ \Psi(v,w) \equiv \tilde{L}_\gamma w - Q\tilde{H}(v+w) - Q\tilde{f} = 0 \end{cases}, \quad v \in V, \quad w \in W,$$

onde $P:E \to V$ e $Q = I-P:E \to W$ são as projeções ortogonais sobre V e W, respectivamente.

Demonstração. Escrevendo $u = v+w = Pu + Qu$, basta verificarmos que $P\tilde{L}_\gamma u = \tilde{L}_\gamma v$ e $Q\tilde{L}_\gamma u = \tilde{L}_\gamma w$, isto é, que V e W são invariantes pelo operador $\tilde{L}_\gamma = I - \gamma K|_E$. Como $K\phi_j = \lambda_j^{-1}\phi_j$ (V. Observação após Proposição II.2) é claro que $V = \langle\phi_{k+1}, \ldots, \phi_{\ell-1}\rangle$ é invariante por $\tilde{L}_\gamma$ e, portanto, $W = V^\perp$ é também invariante por

$\tilde{L}_\gamma$.

■

Para futura referência, é importante observar a esta altura que o espaço $W = V^\perp$ é, de fato, o fecho em E do espaço gerado pelas ϕ_j's, $j \neq k+1, \ldots, \ell-1$:

$$W = \overline{\langle \phi_j \mid j \neq k+1, \ldots, \ell-1 \rangle}^{E}$$

Isto é consequência do

Lema IV.3. **$\{\phi_j\}$ é um sistema ortogonal completo em E.**

Demonstração. Sabemos, da Proposição II.2, que $\{\phi_j\} \subset E$ é um sistema ortonormal completo em $L^2(\Omega)$, isto é, $(\phi_i, \phi_j)_{L^2} = \delta_{ij}$ e se $\theta \in L^2(\Omega)$ é tal que $(\phi_j, \theta)_{L^2} = 0$ $\forall j$ então $\theta = 0$. Portanto, como $\phi_i = \lambda_i K \phi_i$ (e lembrando que $K: L^2 \to E$ é definido por $(K\rho, v)_E = (\rho, v)_{L^2}$, $\forall v \in E$, $\forall \rho \in L^2$), obtemos

$$(\phi_i, \phi_j)_E = \lambda_i (K\phi_i, \phi_j)_E = \lambda_i (\phi_i, \phi_j)_{L^2} = \lambda_i \delta_{ij},$$

e, se $\theta \in E$ é tal que $(\phi_j, \theta)_E = 0$ $\forall j$ então $0 = (\phi_j, \theta)_E = \lambda_j (\phi_j, \theta)_{L^2}$ $\forall j$, isto é, $\theta = 0$.

■

Observação IV.2. Não vamos nos preocupar em tornar os ϕ_j's unitários em $E = H_0^1(\Omega)$ ($\|\phi_j\|_E = \lambda_j^{1/2}$). Basta ficarmos atentos ao fato de que ao escrevermos um elemento $u \in E$ como

$$u = \sum_{j=1}^{\infty} \alpha_j \phi_j,$$

devemos ter $\sum_{j=1}^{\infty} \alpha_j^2\lambda_j$ $(= \|u\|_E^2) < \infty$. Observe, também, que a projeção ortogonal $P:E \to V$ é a restrição $P_o|_E$ da projeção ortogonal $P_o:L^2 \to V$; logo, $Q = I-P:E \to W$ é a restrição $Q_o|_E$ da projeção $Q_o = I-P_o$. Em particular, $PKf = KP_of$ e $QKf = KQ_of$.

V. O CASO NÃO-RESSONANTE

Vamos começar com a situação mais simples do Teorema 1 em que g não interage em absoluto com o espectro $\sigma(-\Delta)$, isto é, $[a,b] \cap \sigma(-\Delta) = \emptyset$. Neste caso, não há necessidade de usar o método de Lyapunov-Schmidt descrito acima. Utilizaremos diretamente o Teorema III.1.

Demonstração do Teorema 1. Estamos supondo que, para algum n,

$$\lambda_n < a \leq g'(s) \leq b < \lambda_{n+1}, \quad \forall\, s \in \mathbb{R}.$$

(Se $b < \lambda_1$, tomamos $\lambda_0 = a - (\lambda_1 - b)$).

Conforme vimos na seção anterior (Observação IV.1), $u \in H_0^1(\Omega)$ é uma solução fraca de (*) se e somente se u é solução da equação

$$(***) \qquad \tilde{L}_\gamma u = \tilde{H}(u) + \tilde{f}$$

onde $\gamma \in \mathbb{R}$ é um número fixado, $\tilde{f} = Kf \in H_0^1(\Omega)$, $\tilde{L}_\gamma = I - \gamma K|_{H_0^1}$, $\tilde{H} = KH_1$, $H_1 = H|_{H_0^1}$, $H: L^2(\Omega) \to L^2(\Omega)$ é o operador de Niemytskii associado à função $h(s) = g(s) - \gamma s$ e $K: L^2(\Omega) \to H_0^1(\Omega)$. Agora, observe que $u \in H_0^1(\Omega)$ é solução de (***) se e somente se $u \in L^2(\Omega)$ é solução da equação

$$L_\gamma u = K_0 H(u) + \tilde{f},$$

onde $L_\gamma = I - \gamma K_o : L^2(\Omega) \to L^2(\Omega)$ e $K_o : L^2(\Omega) \to L^2(\Omega)$ é a composição de K com a inclusão de $H^1_o(\Omega)$ em $L^2(\Omega)$. (Com efeito, se $u \in L^2(\Omega)$ é solução de $L_\gamma u = K_o H(u) + \tilde{f}$ então $u = \gamma K_o u + K_o H(u) + \tilde{f} \in H^1_o(\Omega)$).

Lema V.1. Se $\gamma \notin \sigma(-\Delta)$ então $L_\gamma : L^2(\Omega) \to L^2(\Omega)$ é um isomorfismo. Além disso, $L_\gamma^{-1} K_o : L^2(\Omega) \to L^2(\Omega)$ tem norma $\leq \max_j |\gamma - \lambda_j|^{-1} = 1/\text{dist}(\gamma, \sigma(-\Delta))$.

Demonstração. Dado $u = \sum \alpha_j \phi_j \in L^2(\Omega)$ (logo, $\sum \alpha_j^2 = \|u\|^2_{L^2} < \infty$),

$$L_\gamma u = u - \gamma K_o u = \sum \alpha_j \left(1 - \frac{\gamma}{\lambda_j}\right)\phi_j,$$

de onde vemos que L_γ é uma bijeção ($\gamma \neq \lambda_j \ \forall j$) com inverso L_γ^{-1} dado por

$$L_\gamma^{-1}\left(\sum \beta_j \phi_j\right) = \sum \beta_j \left(1 - \frac{\gamma}{\lambda_j}\right)^{-1}\phi_j$$

Agora, calculando

$$L_\gamma^{-1} K_o\left(\sum \alpha_j \phi_j\right) = L_\gamma^{-1}\left(\sum \alpha_j \lambda_j^{-1} \phi_j\right) = \sum \alpha_j (\lambda_j - \gamma)^{-1}\phi_j$$

e tomando normas, vem

$$\|L_\gamma^{-1} K_o u\|^2_{L^2} = \sum \alpha_j^2 (\lambda_j - \gamma)^{-2} \leq m^2 \sum \alpha_j^2 = m^2 \|u\|^2_{L^2},$$

onde $m = \max_j |\lambda_j - \gamma|^{-1} = 1/\text{dist}(\gamma, \sigma(-\Delta))$, isto é, $\|L^{-1} K_o\| \leq m$.

■

Por outro lado, como $|h'(s)| \leq \max\{|a-\gamma|, |b-\gamma|\}$, se escolhermos $\gamma = (a+b)/2$, obtemos

$$\|H(u) - H(v)\|_{L^2} \leq \frac{b-a}{2} \|u-v\|_{L^2}$$

e

$$\|L_\gamma^{-1} K_o\| \leq \max\{\frac{1}{\gamma - \lambda_n}, \frac{1}{\lambda_{n+1} - \gamma}\} < \frac{2}{b-a} .$$

Portanto, $C = L_\gamma^{-1} K_o H: L^2(\Omega) \to L^2(\Omega)$ é uma contração e o Teorema III.1 implica que a equação

$$u = L_\gamma^{-1} K_o H(u) + L_\gamma^{-1} \tilde{f},$$

ou,

$$L_\gamma u = K_o H(u) + \tilde{f}$$

tem uma única solução $u \in L^2(\Omega)$, a qual, como já observamos inicialmente, é (a única) solução fraca $u \in H_o^1(\Omega)$ de (***). A demonstração do Teorema 1 está completa.

■

Suponhamos, agora, que temos a situação $[a,b] \cap \sigma(-\Delta) \neq \phi$, mais precisamente,

$$[a,b] \cap \sigma(-\Delta) = \{\lambda_{k+1}, \ldots, \lambda_{\ell-1}\}.$$

E, além disso, g verifica a hipótese (h_2), isto é,

$$|h(s)| = |g(s) - \gamma s| \leq M, \quad \forall\, s \in \mathbb{R},$$

com $\gamma \notin \sigma(-\Delta)$. Novamente, queremos resolver a equação

$$(***) \qquad \tilde{L}_\gamma u = \tilde{H}(u) + \tilde{f}^{(1)}, \qquad u \in H_o^1(\Omega) \equiv E,$$

ou equivalentemente (cf. Lema IV.2), o sistema

$$\begin{cases} (1) \quad \tilde{L}_\gamma v - P\tilde{H}(v+w) = P\tilde{f} \\ (2) \quad \tilde{L}_\gamma w - Q\tilde{H}(v+w) = Q\tilde{f} \end{cases}, \quad v \in V,\ w \in W,$$

onde $P:E \to V$ e $Q = I-P:E \to W$ são as projeções ortogonais sobre $V = \langle \phi_{k+1}, \ldots, \phi_{\ell-1} \rangle$ e $W = \overline{\langle \phi_j \mid j \neq k+1, \ldots, \ell-1 \rangle}^{E}$, respectivamente.

Inicialmente, vamos mostrar que, para cada $v \in V$, a equação (2) possui uma única solução $w = w(v) = w(v, Q\tilde{f})$, a qual depende continuamente de v. Para isso, utilizaremos o Teorema III.2 e os vários lemas abaixo.

Lema V.2. Dado $v \in V$, seja $\Psi_v : W \to W$ definida por $\Psi_v(w) = \tilde{L}_\gamma w - Q\tilde{H}(v+w)$. Então, Ψ_v é uma aplicação própria.

Demonstração. De maneira inteiramente análoga à do Lema V.1, vê-se que $\tilde{L}_\gamma : W \to W$ é um isomorfismo. Por outro lado, como a inclusão $E \equiv H_o^1(\Omega) \hookrightarrow L^2(\Omega)$ é compacta, segue-se que a aplicação $H_1 = H|_E : E \to L^2(\Omega)$ é compacta. Além disso, H_1 é uniformemente limitada em vista da hipótese (h_2). Portanto, $N = Q\tilde{H}(v+\cdot) = QKH_1(v+\cdot)$

---

(1) Isto é, escolhemos $\gamma \in \mathbb{R}$ (cf. Observação IV.1) dado pela hipótese (h_2).

é uma aplicação contínua, uniformemente limitada, compacta, de sorte que a Proposição III.3 fornece o resultado.

■

Lema V.3. Ψ_v **é uma aplicação de classe** C^1.

Demonstração. Como $\tilde{L}_\gamma$, Q e K são lineares, basta mostrarmos que $H_1 : E \to L^2(\Omega)$ é de classe C^1. O candidato natural para $H_1'(u)$ é a aplicação linear (contínua) $E \ni v(x) \mapsto h'(u(x))v(x) \in L^2$. Vamos calcular

$$\delta(u,v) = \|H_1(u+v) - H_1(u) - h'(u)v\|_{L^2}, \quad u,v \in E.$$

Temos

$$\delta(u,v) = \left(\int_\Omega |h(u(x) + v(x)) - h(u(x)) - h'(u(x))v(x)|^2 dx\right)^{1/2},$$

ou ainda, como

$$h(u(x) + v(x)) - h(u(x)) = v(x) \int_0^1 h'(u(x) + \tau v(x))d\tau,$$

$$\delta(u,v) = \left(\int_\Omega |v(x)z(x)|^2 dx\right)^{1/2},$$

onde

$$(3) \qquad z(x) = \int_0^1 \left[h'(u(x) + \tau v(x)) - h'(u(x))\right]d\tau.$$

Portanto, pela desigualdade de Hölder,

$$(4) \qquad \delta(u,v) = \|vz\|_{L^2} \leq \|v\|_{L^r}\|z\|_{L^s},$$

onde fixamos r e s com $2 < r < \frac{2N}{N-2}$ e $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$. E, em vista da inclusão contínua (de fato, compacta) $E \subset L^r(\Omega)$ (1), obtemos

$$\delta(u,v) \leq C\|v\|_E \|z\|_{L^s}.$$

Se mostrarmos que $\|z\|_{L^s} \to 0$ quando $\|v\|_E \to 0$, então $\delta(u,v)/\|v\|_E \to 0$ quando $\|v\|_E \to 0$, e H_1 resulta diferenciável (Fréchet) no ponto u com $H_1'(u)\cdot v = h'(u)v$. Ora, $\|v_j\|_E \to 0$ implica (para uma subsucessão) $v_j(x) \to 0$ em quase toda parte, logo $z_j(x) \to 0$ em quase toda parte (V. (3)). Como $|z_j(x)| \leq 2 \sup |h'| \in L^s(\Omega)$, o teorema da convergência dominada de Lebesgue fornece $\|z_j\|_{L^s} \to 0$.

Finalmente, para verificarmos a continuidade da aplicação $u \to H_1'(u)$, usamos novamente a desigualdade de Hölder e o Teorema de Imersão de Sobolev como acima para obtermos

$$\|H_1'(u+w) - H_1'(u)\| = \sup_{\|v\|_E \leq 1} \|[H_1'(u+w) - H_1'(u)]\cdot v\|_{L^2} \leq C\|h'(u+w) - h'(u)\|_{L^s},$$

e o teorema de Lebesgue para mostrarmos que $\|h'(u+w) - h'(u)\|_{L^s} \to 0$ quando $\|w\|_E \to 0$.

■

Observação V.1. Se tivéssemos considerado o operador de Niemytskii $H: L^2(\Omega) \to L^2(\Omega)$ ao invés de sua restrição H_1 a $H_0^1(\Omega)$, ele **não** resultaria diferenciável (Fréchet), a não ser que fosse linear (cf. [9]).

---

(1) Teorema de Imersão de Sobolev (cf. [1]).

Lema V.4. Para cada $w \in W$, $\Psi_v'(w):W \to W$ é um isomorfismo.

Demonstração. $\Psi_v'(w)$ é a aplicação linear

$$z \to \tilde{L}_\gamma z - QKH_1'(v+w)z, \quad z \in W,$$

onde $\tilde{L}_\gamma : W \to W$ é um isomorfismo e $H_1'(v+w):W \to L^2(\Omega)$ é uma aplicação linear compacta (Isto segue [23, pg. 58] do fato que $H_1 : E \to L^2(\Omega)$ é uma aplicação compacta).

Portanto, $\Psi_v'(w)$ é uma perturbação (linear) compacta de um isomorfismo e, pela Alternativa de Fredholm, $\Psi_v'(w)$ é um isomorfismo se e somente se é injetiva.

Para mostrar que $\Psi_v'(w)$ é injetiva, decompomos $W = \overline{\langle \phi_j \mid j \neq k+1, \ldots, \ell-1 \rangle}^E$ como $W_1 \oplus W_2$, onde

$$W_1 = \langle \phi_j \mid j \leq k \rangle,$$

$$W_2 = \overline{\langle \phi_j \mid j \geq \ell \rangle}^E.$$

(Se $k = 0$, isto é, $\phi_1 \in V$, não há necessidade de decompor). Sejam Q_1, Q_2 as projeções respectivas.

Se $z = z_1 + z_2 = Q_1 z + Q_2 z$ é tal que $\Psi_v'(w)\cdot z = 0$, isto é,

$$\tilde{L}_\gamma(z_1 + z_2) = Q\tilde{H}'(v+w)\cdot(z_1 + z_2),$$

então (projetando sobre W_1 e W_2),

$$\begin{cases} \tilde{L}_\gamma z_1 = Q_1\tilde{H}'(v+w)\cdot(z_1 + z_2) \\ \tilde{L}_\gamma z_2 = Q_2\tilde{H}'(v+w)\cdot(z_1 + z_2) \end{cases}$$

Tomando os produtos internos dessas equações com z_1 e z_2, respectivamente, vem

$$\begin{cases} (\tilde{L}_\gamma z_1, z_1)_E = (\tilde{H}'(v+w)\cdot(z_1+z_2), z_1)_E = (KH_1'(v+w)\cdot(z_1+z_2), z_1)_E \\ (\tilde{L}_\gamma z_2, z_2)_E = (\tilde{H}'(v+w)\cdot(z_1+z_2), z_2)_E = (KH_1'(v+w)\cdot(z_1+z_2), z_2)_E, \end{cases}$$

isto é,

$$\begin{cases} (z_1 - \gamma K z_1, z_1)_E = (H_1'(v+w)\cdot(z_1+z_2), z_1)_{L^2} \\ (z_2 - \gamma K z_2, z_2)_E = (H_1'(v+w)\cdot(z_1+z_2), z_2)_{L^2}, \end{cases}$$

ou ainda, uma vez que $a-\gamma \leq h'(s) \leq b-\gamma$,

$$\begin{cases} \|z_1\|_E^2 - \gamma\|z_1\|_{L^2}^2 \geq \int_\Omega h'(v+w)z_1z_2dx + (a-\gamma)\|z_1\|_{L^2}^2 \\ \\ \|z_2\|_E^2 - \gamma\|z_2\|_{L^2}^2 \leq \int_\Omega h'(v+w)z_1z_2dx + (b-\gamma)\|z_2\|_{L^2}^2. \end{cases}$$

Subtradindo estas equações, obtemos

$$\|z_2\|_E^2 - \|z_1\|_E^2 \leq b\|z_2\|_{L^2}^2 - a\|z_1\|_{L^2}^2,$$

e, em vista do Lema V.5 seguinte,

$$\lambda_\ell\|z_2\|_{L^2}^2 - \lambda_k\|z_1\|_{L^2}^2 \leq b\|z_2\|_{L^2}^2 - a\|z_1\|_{L^2}^2,$$

isto é,

$$(a-\lambda_k)\|z_1\|_{L^2}^2 + (\lambda_\ell - b)\|z_2\|_{L^2}^2 \leq 0$$

Portanto, como $\lambda_k < a < b < \lambda_\ell$, vem $z_1 = z_2 = 0$, isto é, $z = 0$.

■

Lema V.5. **(i)** $\|z_1\|_E^2 \leq \lambda_k \|z_1\|_{L^2}^2, \quad \forall\, z_1 \in W_1$.

(ii) $\|z_2\|_E^2 \geq \lambda_\ell \|z_2\|_{L^2}^2, \quad \forall\, z_2 \in W_2$.

Demonstração.

(i) Se $z_1 \in W_1$ então $z_1 = \sum_{j=1}^{k} \alpha_j \phi_j$, logo

$$\|z_1\|_E^2 = \sum_{j=1}^{k} \alpha_j^2 \lambda_j \leq \lambda_k \sum_{j=1}^{k} \alpha_j^2 = \lambda_k \|z_1\|_{L^2}^2.$$

(ii) De maneira análoga, $z_2 \in W_2$ tem a representação $z_2 = \sum_{j=\ell}^{\infty} \alpha_j \phi_j$ e

$$\|z_2\|_E^2 = \sum_{j=\ell}^{\infty} \alpha_j^2 \lambda_j \geq \lambda_\ell \sum_{j=\ell}^{\infty} \alpha_j^2 = \lambda_\ell \|z_2\|_{L^2}^2.$$

■

Proposição V.6. **Para cada $v \in V$, a equação (2) possui uma única solução $w(v) = w(v, Q\tilde{f})$, a qual depende continuamente de v. Mais ainda, a aplicação $v \mapsto w(v)$ é de classe $C^1(V,W)$ e temos a estimativa**

$$(5) \qquad \|w(v)\|_E \leq c, \quad \forall\, v \in V,$$

onde c é uma constante que depende apenas de $\|f\|_{L^2}$

Demonstração. Em vista dos Lemas V.2, V.4 e do Teorema da Função Inversa, $\Psi_v : W \to W$ é um difeomorfismo local, próprio. Portanto, Ψ_v é um difeomorfismo global (cf. Teorema III.2), e dado $f_2 \in W$, existe um único $w = w(v, f_2)$ tal que

$$\Psi_v(w) = f_2$$

isto é,

$$\tilde{L}_\gamma w - Q\tilde{H}(v+w) = f_2, \tag{6}$$

com a correspondência $f_2 \leftrightarrow w(v, f_2)$ sendo um difeomorfismo. Por outro lado, fixado $f_2 \in W$ (no nosso caso, $f_2 = Q\tilde{f}$), a aplicação $v \to w(v, f_2)$ é de classe $C^1(V,W)$ em vista do Teorema da Função Implícita. Finalmente, como $\tilde{L}_\gamma$ é um isomorfismo e $\tilde{H} = KH_1$ é uniformemente limitada, (6) implica

$$\|w\|_E \leq \text{const.}\|\tilde{L}_\gamma w\|_E \leq \text{const.}(\|Q\tilde{H}(v+w)\|_E + \|Q\tilde{f}\|_E)$$

$$\leq \text{const.}(c_1 + c_2\|f\|_{L^2}).$$

■

Agora, que resolvemos a equação (2) na variável w, substituimos $w = w(v)$ na equação (1) e obtemos a equação

$$\tilde{L}_\gamma v - P\tilde{H}(v+w(v)) = P\tilde{f}, \quad v \in V, \tag{7}$$

isto é,

$$\Theta(v) = P\tilde{f}, \quad v \in V, \tag{8}$$

onde $\Theta: V \to V$ é dado por $\Theta(v) = \tilde{L}_\gamma v - P\tilde{H}(v+w(v))$.

Demonstração do Teorema 2. Queremos mostrar que (8) é sempre solúvel (não necessariamente de maneira única), qualquer que seja $f \in L^2(\Omega)$. Neste ponto é que se evidencia a importância de γ não ser um ponto do espectro $\sigma(-\Delta)$. Com efeito, temos o seguinte

Lema V.7. $\tilde{L}_\gamma : V \to V$ **é inversível.**

Demonstração. Imediata. Basta calcular, com $v = \sum_{j=k+1}^{\ell-1} \alpha_j \phi_j$,

$$\tilde{L}_\gamma v = \sum_{j=k+1}^{\ell-1} \alpha_j \left(1 - \frac{\gamma}{\lambda_j}\right) \phi_j.$$

Observação V.2. Tomando o produto escalar da equação (8) com $\phi_{k+1}, \ldots, \phi_{\ell-1}$, respectivamente, vemos que, em termos das coordenadas α_j de v, ela escreve-se como o seguinte sistema não-linear $n_o \times n_o$ $(n_o = \ell - k - 1)$:

$$(9) \qquad (\lambda_j - \gamma)\alpha_j - \int_\Omega h\left(\sum_{k+1}^{\ell-1} \alpha_i \phi_i + w\left(\sum_{k+1}^{\ell-1} \alpha_i \phi_i\right)\right)\phi_j dx =$$

$$= \int_\Omega f \phi_j dx, \quad j = k+1, \ldots, \ell-1,$$

isto é,

$$D\alpha + F(\alpha) = \beta, \quad \alpha = (\alpha_1, \ldots, \alpha_{n_o}) \in \mathbf{R}^{n_o},$$

onde $\beta \in \mathbf{R}^{n_o}$ é um vetor dado (em termos de f), D é uma matriz diagonal e $F:\mathbf{R}^{n_o} \to \mathbf{R}^{n_o}$ é uma aplicação contínua e limitada (uma vez que $h(s)$ é uma função limitada).

Portanto, a aplicação $D+F:\mathbf{R}^{n_o} \to \mathbf{R}^{n_o}$ (equivalentemente, $\Theta : V \to V$) será sobrejetiva se $T = I+D^{-1}F:\mathbf{R}^{n_o} \to \mathbf{R}^{n_o}$ o for.

Lema V.8. Seja $T:\mathbf{R}^n \to \mathbf{R}^n$ uma aplicação da forma $T = I+N$ onde $N:\mathbf{R}^n \to \mathbf{R}^n$ é contínua e limitada. Então, T é sobrejetiva.

Demonstração. Queremos mostar que, para todo $\delta \in \mathbf{R}^n$, existe $\alpha \in \mathbf{R}^n$ tal que

$$\alpha + N(\alpha) = \delta,$$

isto é,

$$N_\delta(\alpha) \equiv \delta - N(\alpha) = \alpha.$$

Em outras palavras, dado $\delta \in \mathbf{R}^n$, queremos mostrar que a aplicação N_δ definida acima tem um ponto fixo. Ora, seja $r = |\delta| + \sup_\alpha N(\alpha)$ e considere a bola $B(r) \subset \mathbf{R}^n$ de centro na origem e raio r. Então, N_δ transforma a bola $B(r)$ em si mesma e (como N_δ é contínua) o Teorema do Ponto Fixo de Brouwer garante a existência de um ponto fixo $\alpha \in B(r)$.

■

A demonstração do Teorema 2 está completa.

■

Demonstração do Teorema 5.

A seguir, vamos considerar a situação (Ambrosetti-Prodi) em que

$$[a,b] \cap \sigma(-\Delta) = \{\lambda_1\},$$

com $0<a=g'(-\infty) = \lim_{s\to-\infty} g'(s) < b = g'(\infty) = \lim_{s\to\infty} g'(s)$. Observe que a hipótese (h_2) não é satisfeita (para nenhum $\gamma \in \mathbb{R}$): a não linearidade g é assintoticamente linear, mas as "inclinações" $\lim_{s\to\pm\infty} g(s)/s$ são diferentes.

Queremos resolver a equação (**), ou seja, (cf. Observação IV.1), a equação (***) para algum $\gamma \in \mathbb{R}$. Escolhemos $\gamma = \lambda_1$ e, por simplicidade, escrevemos $\tilde{L}_1$ ao invés de $\tilde{L}_{\lambda_1}$:

$$(***)_1 \qquad \tilde{L}_1 u = \tilde{H}(u) + \tilde{f}, \quad u \in H_0^1(\Omega) \equiv E$$

Novamente, temos o sistema equivalente (cf. Lema IV.2)

$$\begin{array}{ll} (11) \\ (12) \end{array} \quad \left\{ \begin{array}{l} -P\tilde{H}(v+w) = P\tilde{f}\ ^{(1)} \\ \tilde{L}_1 w - Q\tilde{H}(v+w) = Q\tilde{f} \end{array} \right. , \quad v \in V, \quad w \in W,$$

onde agora, $V = \langle\phi_1\rangle$ é unidimensional, $W = \overline{\langle\phi_j \mid j \neq 1\rangle}^E$. Observe que, no presente caso, não podemos concluir que a aplicação $\tilde{H} = KH_1 : E \to E$ é uniformemente limitada (pois $h(s) = g(s) - \lambda_1 s$ não é limitada). Logo, a demonstração de que $\Psi_v(w) = \tilde{L}_1 w - Q\tilde{H}(v+w)$ é uma aplicação própria não decorre da Proposição III.3, como no

(1) Note que $\tilde{L}_1 v = (I - \lambda_1 K)v = 0$.

caso anterior. Usaremos o Teorema de Hadamard (Teorema III.4) para mostrar que Ψ_v é um difeomorfismo global.

Lema V.9. Existe $\varepsilon > 0$ tal que $\|\Psi_v'(w)\cdot z\|_E \geq \varepsilon\|z\|_E$, $\forall\, w,z \in W$.

Demonstração. Calculando o produto interno $(\Psi_v'(w)z,\, z)_E$ obtemos

$$(\Psi_v'(w)z,\, z)_E = (\tilde{L}_1 z - Q\tilde{H}'(v+w)z,\, z)_E =$$
$$= \|z\|_E^2 - \lambda_1\|z\|_{L^2}^2 - \int_\Omega h'(v+w)z^2 dx;$$

logo, como $h'(s) = g'(s) - \lambda_1 \leq b - \lambda_1$,

$$(\Psi_v'(w)z,\, z)_E \geq \|z\|_E^2 - \lambda_1\|z\|_{L^2}^2 - (b - \lambda_1)\|z\|_{L^2}^2$$
$$= \|z\|_E^2 - b\|z\|_{L^2}^2.$$

Mas $\|z\|_E^2 \geq \lambda_2\|z\|_{L^2}^2$, $\forall\, z \in W$ (cf. Lema V.5). Portanto,

$$(\Psi_v'(w)z,\, z)_E \geq \varepsilon\|z\|_E^2, \quad \forall\, w,z \in W$$

onde $\varepsilon = 1 - \dfrac{b}{\lambda_2} > 0$. Em particular $\|\Psi_v'(w)z\|_E \geq \varepsilon\|z\|$.

■

Portanto, $\Psi_v : W \to W$ é um difeomorfismo global. Daqui por diante, vamos escrever $v = t\phi_1$ para $v \in V = \langle\phi_1\rangle$ $(t=(v,\phi_1)_E\lambda_1^{-1})$, de sorte que a equação (12) escreve-se como

$$\tilde{L}_1 w - Q\tilde{H}(t\phi_1 + w) = Q\tilde{f} \qquad (12')$$

Proposição V.10. Para cada $t \in \mathbb{R}$, a equação (12') possui uma única solução $w(t) = w(t, Q\tilde{f})$. A aplicação $t \mapsto w(t)$ é de classe $C^1(\mathbb{R}, W)$ e existe uma constante $c = c(f)$ tal que

$$\text{(13)} \qquad \|w(t)\|_E \le c(1 + |t|), \quad \forall\, t \in \mathbb{R}.$$

Demonstração. A demonstração é exatamente aquela da Proposição V.6. Resta apenas obter a estimativa (13).

Tomando o produto interno da equação (12') com w, vem

$$(\tilde{L}_1 w(t), w(t))_E - (\tilde{H}(t\phi_1 + w(t)), w(t))_E = (\tilde{f}, w(t))_E,$$

isto é,

$$\|w(t)\|_E^2 - \lambda_1 \|w(t)\|_{L^2}^2 - (H_1(t\phi_1 + w(t)), w(t))_{L^2} = (f, w(t))_{L^2}.$$

Como $H_1(t\phi_1 + w(t)) = h(t\phi_1 + w(t))$, com $h(s) \doteq g(s) - \lambda_1 s$, e $(\phi_1, w(t))_{L^2} = \lambda_1^{-1}(\phi_1, w(t))_E = 0$, obtemos

$$(H_1(t\phi_1 + w(t)), w(t))_{L^2} = (g(t\phi_1 + w(t)), w(t))_{L^2} - \lambda_1 \|w(t)\|_{L^2}^2,$$

logo

$$\|w(t)\|_E^2 - (g(t\phi_1 + w(t)), w(t))_{L^2} = (f, w(t))_{L^2}$$

Finalmente, usando a desigualdade de Cauchy-Schwarz e a desigualdade de $|g(s)| \le |g(0)| + b|s|$, obtemos

$$\|w(t)\|_E^2 \le \|w(t)\|_{L^2}(\|g(0)\|_{L^2} + b|t| + b\|w(t)\|_{L^2} + \|f\|_{L^2}),$$

ou ainda, uma vez que $\|w(t)\|_{L^2}^2 \le \lambda_1^{-1}\|w(t)\|^2$

$$(1 - \frac{b}{\lambda_2})\|w(t)\|_E^2 \leq \lambda_2^{-1/2}\|w(t)\|_E(\|g(0)\|_{L^2} + \|f\|_{L^2} + b|t|),$$

isto é,

$$\|w(t)\|_E \leq c(1 + |t|).$$

■

Voltamos agora a nossa atenção para a equação (11), onde substituimos w por $w(t)$:

$$(11') \qquad -P\tilde{H}(t\phi_1 + w(t)) = P\tilde{f},$$

ou (tomando o produto interno com ϕ_1),

$$- \int_\Omega h(t\phi_1 + w(t))\phi_1 dx = \int_\Omega f\phi_1 dx,$$

ou ainda,

$$(14) \qquad \lambda_1 t - \int_\Omega g(t\phi_1 + w(t))\phi_1 dx = \int_\Omega f\phi_1 dx$$

(uma vez que $h(s) = g(s) - \lambda_1 s$ e $(w(t), \phi_1)_{L^2} = \lambda_1(w(t), \phi_1)_E = 0$.

Assim sendo, o estudo da nossa equação original $(***)_1$ fica reduzido ao estudo da equação na reta

$$(15) \qquad \theta(t) = \int_\Omega f\phi_1 dx, \quad t \in \mathbb{R},$$

onde $\theta(t)$ é a função (de classe C^1) definida pelo primeiro membro de (14).

Lema V.11. $\theta(t) \to -\infty$ **quando** $|t| \to \infty$.

Demonstração. Vamos considerar o caso $t \to +\infty$. Queremos mostrar que, para toda sucessão $t_n \to \infty$, existe uma subsucessão $t_{n_j} \to \infty$ tal que $\theta(t_{n_j}) \to -\infty$.

Para isso, vamos supor que a estimativa (13) é válida na norma de $C^{1+\alpha}(\overline{\Omega})$ (este será o caso, conforme Lema V.12 a seguir), isto é, $w(t) \in C^{1+\alpha}(\overline{\Omega}) \cap W$ e

$$\|w(t)\|_{C^{1+\alpha}} \leq c_\alpha(1 + |t|), \quad \forall\, t \in \mathbb{R}.$$

Então, a sucessão $\overline{w}(t_n) = w(t_n)/t_n$ é limitada em $C^{1+\alpha}(\overline{\Omega})$, e, como a inclusão $C^{1+\alpha}(\overline{\Omega}) \subset C^1(\overline{\Omega})$ é compacta, existem uma subsucessão $t_{n_j} \to \infty$ e $w_o \in C^1(\overline{\Omega}) \cap W$ tais que

$$\overline{w}(t_{n_j}) = \frac{w(t_{n_j})}{t_{n_j}} \to w_o \quad \text{em} \quad C^1(\overline{\Omega}).$$

Agora, considere

$$\frac{\theta(t_{n_j})}{t_{n_j}} = \lambda_1 - \int_\Omega \frac{g(t_{n_j}\phi_1 + w(t_{n_j}))}{t_{n_j}}\,\phi_1 dx$$

$$= \lambda_1 - \int_\Omega \frac{g(t_{n_j}(\phi_1 + \overline{w}(t_{n_j})))}{t_{n_j}(\phi_1 + \overline{w}(t_{n_j}))}\,(\phi_1 + \overline{w}(t_{n_j}))\phi_1 dx,$$

e decomponha Ω na integral acima como $\Omega = \Omega_+ \cup \Omega_- \cup \Omega_o$, onde $\Omega_\pm = \{x \in \Omega \mid \phi_1(x) + w_o(x) \gtrless 0\}$, $\Omega_o = \{x \in \Omega \mid \phi_1(x) + w_o(x) = 0\}$. Então, passando ao limite quando $t_{n_j} \to \infty$ na expressão acima e usando o teorema da convergência dominada de Lebesgue (tendo em vista que $\lim_{s\to\infty} g(s)/s = b$ e $\lim_{s\to-\infty} g(s)/s = a$ pois $\lim_{s\to\infty} g'(s) = b$ e $\lim_{s\to-\infty} g'(s) = a$), obtemos

$$(16)\qquad \lim_{t_{n_j}\to\infty} \frac{\theta(t_{n_j})}{t_{n_j}} = \lambda_1 - b\int_{\Omega_+} (\phi_1 + w_o)\phi_1 dx - a\int_{\Omega_-} (\phi_1 + w_o)\phi_1 dx$$

Por outro lado, como $w_o \in W$, temos $(w_o, \phi_1)_E = 0$, logo $(w_o, \phi_1)_{L^2} = \lambda_1^{-1}(w_o, \phi_1)_E = 0$. Assim, podemos escrever

$$(17)\qquad 1 = (\phi_1 + w_o, \phi_1)_{L^2} = \int_{\Omega_+} (\phi_1 + w_o)\phi_1 dx + \int_{\Omega_-} (\phi_1 + w_o)\phi_1 dx,$$

isto é,

$$\int_{\Omega_-} (\phi_1 + w_o)\phi_1 dx = 1 - \int_{\Omega_+} (\phi_1 + w_o)\phi_1 dx.$$

Substituindo em (16), obtemos

$$\lim_{t_{n_j}\to\infty} \frac{\theta(t_{n_j})}{t_{n_j}} = \lambda_1 - a - (b-a)\int_{\Omega_+} (\phi_1 + w_o)\phi_1 dx$$
$$\leq \lambda_1 - a - (b-a),$$

pois $\int_{\Omega_+} (\phi_1 + w_o)\phi_1 dx \geq 1$ (V. (17) e lembre-se que $(\phi_1 + w_o)\phi_1 < 0$ em Ω_-). Portanto,

$$\lim_{t_{n_j}\to\infty} \frac{\theta(t_{n_j})}{t_{n_j}} \leq \lambda_1 - b < 0,$$

de sorte que $\theta(t_{n_j}) \to -\infty$ quando $t_{n_j} \to \infty$

De maneira análoga, mostramos que, para toda sucessão

$t_n \to -\infty$, existe uma subsucessão $t_{n_j} \to -\infty$ com $\theta(t_{n_j}) \to -\infty$.

∎

Observação V.3. Note que a função $\theta(t)$ depende de $Q\tilde{f} = QKf$ pois $w(t) = w(t, Q\tilde{f})$. Lembre-se também (Observação IV.2), que $QKf = KQ_o f$ onde $Q_o = I - P_o$ e P_o é a projeção ortogonal $P_o : L^2(\Omega) \to \langle\phi_1\rangle$. Logo, decompondo $f = s\phi_1 + f_2 = P_o f + Q_o f$, temos $\theta(t) = \theta(t, f_2)$.

Em vista do lema anterior, podemos definir

$$\bar{t} = \bar{t}(f_2) = \max_t \theta(t, f_2).$$

É evidente, agora, que a equação (15) é solúvel se e somente se

$$\int_\Omega f\phi_1 dx \leq \bar{t}.$$

Além disso, no caso $\int_\Omega f\phi_1 dx < \bar{t}$, (15) possui no mínimo duas soluções. Portanto, a menos do lema seguinte, a demonstração do Teorema 5 está completa.

∎

Lema V.12. Para cada $t \in \mathbb{R}$, $w(t) \in C^{1+\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ e temos a estimativa

$$\|w(t)\|_{C^{1+\alpha}} \leq c_\alpha(1 + |t|), \quad \forall\, t \in \mathbb{R},$$

onde $c_\alpha = c_\alpha(f)$.

Demonstração. Inicialmente observamos que, pela teoria de regularidade para equações elíticas (cf. [15]), temos $K\rho \in W^{2,p}(\Omega)$

se $\rho \in L^p(\Omega)$, $p > 1$; além disso,

$$\|K\rho\|_{W^{2,p}} \leq c_p \|\rho\|_{L^p} \tag{18}$$

Portanto, se $p > N$, o Teorema de Imersão de Sobolev (cf. [1]) fornece $K\rho \in C^{1+\alpha}(\overline{\Omega})$, $\alpha = 1 - \frac{N}{p}$, logo

$$\|K\rho\|_{C^{1+\alpha}} \leq C\|K\rho\|_{W^{2,p}} \leq C_p'\|\rho\|_{L^p} \tag{19}$$

Por outro lado, $w(t)$ verifica a equação (12'),

$$\tilde{L}_1 w(t) = QKH_1(t\phi_1 + w(t)) + QKf,$$

isto é,

$$w(t) = QKG_1(t\phi_1 + w(t)) + QKf,$$

ou ainda, uma vez que $QK = KQ_o$ (V. Observação IV.2),

$$w(t) = K[Q_o G_1(t\phi_1 + w(t)) + Q_o f] \equiv K\rho. \tag{20}$$

Além disso, em vista da estimativa em $E = H_o^1(\Omega)$ dada pela Proposição V.10 e da imersão de Sobolev $H_o^1(\Omega) \subset L^{p_1}(\Omega)$ (onde $\frac{1}{p_1} = \frac{1}{2} - \frac{1}{N}$ se $N > 2$ e $p_1 > 2$ é arbitrário se $N \leq 2$), obtemos

$$\|w(t)\|_{L^{p_1}} \leq c_1(1 + |t|),$$

logo

$$\|G_1(t\phi_1 + w(t)\|_{L^{p_1}} \leq c_1'(1 + |t|), \tag{21}$$

uma vez que $|g(s)| \leq |g(0)| + b|s|$. Combinando (21), (20) e (18), vem

$$\|w(t)\|_{W^{2,p_1}} \leq C_1(1 + |t|), \tag{22}$$

onde $C_1 = C_1(f)$.

Novamente, em vista da imersão de Sobolev (cf. [1]) $W^{2,p_1}(\Omega) \subset L^{p_2}(\Omega)$ (onde $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2}{N}$ se $N > 2p_1$ e $p_2 > p_1$ é arbitrário se $N \leq 2p_1$), obtemos

$$\|w(t)\|_{L^{p_2}} \leq c_2(1 + |t|),$$

logo (V. (20))

$$\text{(23)} \qquad \|w(t)\|_{W^{2,p_2}} \leq C_2(1 + |t|).$$

Assim, melhoramos a estimativa (22) no espaço $W^{2,p_1}(\Omega)$ para a estimativa (23) no espaço $W^{2,p_2}(\Omega)$ (na pior das hipóteses, $N > 2p_1$, temos $p_2 = p_1 N/(N - 2p_1)$). Continuando com este processo, após um número finito de etapas, obtemos

$$\|w(t)\|_{L^{p_k}} \leq c_k(1 + |t|),$$

logo

$$\|w(t)\|_{W^{2,p_k}} \leq C_k(1 + |t|),$$

com $p_k > N$. Portanto, uma vez mais a imersão de Sobolev ([1]) $W^{2,p_k}(\Omega) \subset C^{1+\alpha}(\overline{\Omega})$, $\alpha = 1 - \frac{N}{p_k}$, fornece

$$\|w(t)\|_{C^{1+\alpha}} \leq c_\alpha(1 + |t|),$$

com $c_\alpha = c_\alpha(f)$.

■

Observação V.4. De fato, note que o processo acima fornece $w(t) \in W^{2,p}(\Omega)$ para todo $p > 1$ e, consequentemente, $w(t) \in C^{1+\mu}(\Omega)$

para todo $0 < \mu < 1$, com

$$\|w(t)\|_{C^{1+\mu}} \leq c_{\mu}(1 + |t|).$$

VI. O CASO RESSONANTE.

Nesta seção vamos novamente considerar a situação $[a,b] \cap \sigma(-\Delta) \neq \phi$, digamos,

$$[a,b] \cap \sigma(-\Delta) = \{\lambda_{k+1}, \ldots, \lambda_{\ell-1}\},$$

sendo que agora vamos supor que g verifica a hipótese (h_2),

$$|h(s)| = |g(s) - \gamma s| \leq M, \quad \forall\ s \in \mathbb{R},$$

com $\gamma \in \sigma(-\Delta)$ (1). Uma vez mais, iremos resolver a equação

$$(***) \qquad \tilde{L}_\gamma u = \tilde{H}(u) + \tilde{f}, \quad u \in H_0^1(\Omega) \equiv E,$$

ou, o sistema

$$\begin{array}{ll} (1) & \left\{ \begin{array}{l} \tilde{L}_\gamma v - P\tilde{H}(v+w) = P\tilde{f} \\ \\ \tilde{L}_\gamma w - Q\tilde{H}(v+w) = Q\tilde{f} \end{array} \right. , \quad v \in V, \quad w \in W \\ (2) & \end{array}$$

onde $P:E \to V$, $Q:E \to W$ são as projeções ortogonais sobre $V = \langle\phi_{k+1}, \ldots, \phi_{\ell-1}\rangle$, $W = \overline{\langle\phi_j \mid j \neq k+1, \ldots, \ell-1\rangle}^{E}$.

De início, vale ressaltar que o fato de γ agora pertencer a $\sigma(-\Delta)$ não perturba em absoluto a análise que fizemos da equação (2) na seção anterior. Afinal, o autoespaço $V_\gamma = N(-\Delta - \gamma)$ está contido em V e a equação (2) é uma equação no espaço $W = V^\perp$.

---

(1) Portanto, γ é um dos λ_j's, $j = k+1, \ldots, \ell-1$.

Assim sendo, para cada $v \in V$, a equação (2) possui uma única solução $w(v) = w(v, Q\tilde{f})$ e a aplicação $v \mapsto w(v)$ é de classe $C^1(V,W)$ e uniformemente limitada (Proposição V.6):

$$(3) \qquad \|w(v)\|_E \leq c, \quad \forall\, v \in V.$$

E o sistema (1), (2) fica sendo equivalente à equação

$$(4) \qquad \Theta(v) = P\tilde{f}, \quad v \in V,$$

onde $\Theta : V \to V$ é dado por $\Theta(v) = \tilde{L}_\gamma v - P\tilde{H}(v + w(v))$.

Passamos agora ao estudo da equação (4). Para isso, vamos considerar os limites

$$\underline{h}(\pm\infty) = \liminf_{s \to \pm\infty} h(s), \quad \overline{h}(\pm\infty) = \limsup_{s \to \pm\infty} h(s)^{(1)}$$

e, para $z \in V_\gamma = N(-\Delta - \gamma)$, colocamos

$$\Omega_\pm(z) = \{x \in \Omega \mid z(x) \gtrless 0\}, \quad \Omega_o(z) = \{x \in \Omega \mid z(x) = 0\}.$$

Além disso, definimos o seguinte subconjunto de $L^2(\Omega)$ (em verdade de $H_o^1(\Omega) = E$),

$$(5) \qquad \Sigma_o = \{f_o \in V \mid \int_\Omega f_o z\, dx < -\int_{\Omega_+(z)} \overline{h}(\infty) z\, dx - \int_{\Omega_-(z)} \underline{h}(-\infty) z\, dx, \quad \forall\, z \in V_\gamma, \ \|z\|_E = 1\},$$

e colocamos $S_o = K(\Sigma_o)$.

---

(1) Lembre-se que estamos supondo $h(s) = g(s) - \gamma s$ limitada. Logo, estes limites existem.

Teorema 3. Dado $\tilde{f} \in E$, decomponha $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 = P\tilde{f} + Q\tilde{f}$.

(i) Para cada $\tilde{f}_2 \in W$, existe um subconjunto próprio (não-vazio) $S = S(\tilde{f}_2) \subset V$ tal que (***) é solúvel se e somente se $\tilde{f}_1 \in S$.

(ii) Suponha $\gamma = \lambda_{k+1}$. Então,

$$S(\tilde{f}_2) \supset S_o, \quad \forall\, \tilde{f}_2 \in W.$$

Observação VI.1. É oportuno recordar que o nosso problema original consiste no problema de existência de soluções fracas $u \in E$ de

$$(*) \qquad \begin{cases} -\Delta u = g(u) + f & \text{em} \quad \Omega \\ u = 0 & \text{em} \quad \partial\Omega, \end{cases}$$

dada $f \in L^2(\Omega)$ (Seção II), ou, equivalentemente (Observação IV.1), de existência de soluções $u \in E$ de (***) com $\tilde{f}$ da forma Kf. Assim, podemos reformular o teorema acima da seguinte maneira:

"Dada $f \in L^2(\Omega)$, decomponha $f = f_1 + f_2 = P_o f + Q_o f$.

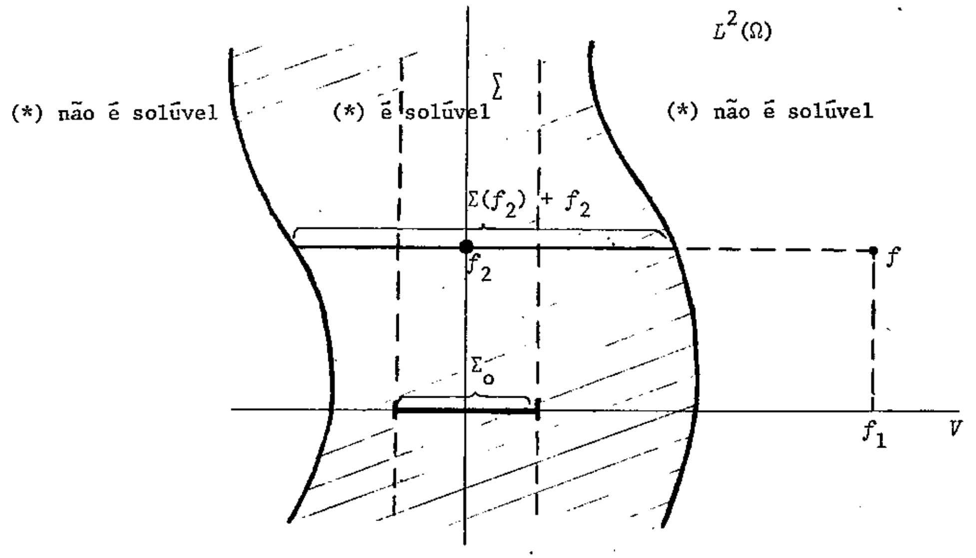
(i) Para toda f_2 existe um subconjunto próprio (não-vazio) $\Sigma(f_2) \subset V$ tal que (*) é solúvel se e somente se $f_1 \in \Sigma(f_2)$, isto é, definindo a "faixa"

$$\Sigma = \bigcup_{f_2} \{f_1 + f_2 \mid f_1 \in \Sigma(f_2)\}$$

em $L^2(\Omega)$, (*) é solúvel se e somente se $f \in \Sigma$.

(ii) Se $\gamma = \lambda_{k+1}$ então $\Sigma(f_2) \supset \Sigma_o$".

A figura abaixo ilustra (a grosso modo) a situação.



Observação VI.2. A condição que define o conjunto Σ_o (V. (5)) é conhecida como **condição do tipo Landesman-Lazer.** Na situação originalmente considerada por Landesman e Lazer ([21]), correspondente a ressonância no primeiro autovalor (isto é, $\gamma = \lambda_1$), o autoespaço V_γ é unidimensional e qualquer autofunção $z \in V_\gamma$ não muda de sinal em Ω, digamos, $V_\gamma = \langle\phi_1\rangle$ com $\phi_1 > 0$ em Ω. Neste caso, a condição de Landesman-Lazer escreve-se

$$(5^*) \qquad -\int_\Omega \underline{h}(-\infty)\phi_1 dx < \int_\Omega f_o \phi_1 dx < -\int_\Omega \overline{h}(\infty)\phi_1 dx.$$

Demonstração do Teorema 3.

(i) Basta definir $S(\tilde{f}_2)$ como a imagem da aplicação $\Theta = \Theta(\cdot,\tilde{f}_2)$. Uma vez que $\tilde{L}_\gamma|_{V_\gamma} = 0$ e $\tilde{H}: H_o^1(\Omega) \to H_o^1(\Omega)$ é uniformemente limitada, o conjunto $S(\tilde{f}_2)$ resulta limitado nas "direções" ϕ_j de V_γ.

(ii) Vamos precisar dos lemas abaixo.

Lema VI.1. Suponha $\gamma = \lambda_{k+1}$. Dadas $\tilde{f}_o = Kf_o$, com $f_o \in \Sigma_o$, e $\tilde{f}_2 \in W$ existe R > 0 tal que

$$(\Theta(v, \tilde{f}_2) - \tilde{f}_o, v)_E > 0$$

para todo $v \in V$ com $\|v\|_E = R$.

Demonstração. Por contradição. Suponha que existe uma sucessão $\{v_n\} \subset V$ com $\|v_n\|_E = n$ e $(\Theta(v_n) - \tilde{f}_o, v_n)_E \leq 0$[1]. Definindo $z_n = \frac{v_n}{n}$ temos $\|z_n\|_E = 1$ e, como V é de dimensão finita, podemos supor que (para uma subsucessão) $z_n \to z_o$, $\|z_o\|_E = 1$. Portanto,

$$\liminf\, (\Theta(v_n), z_n)_E \leq (\tilde{f}_o, z_o)_E,$$

isto é,

$$(6) \qquad \liminf\, \left[n(\tilde{L}_\gamma z_n, z_n)_E - (KH_1(v_n + w(v_n)), z_n)_E\right] \leq (\tilde{f}_o, z_o)_E.$$

Em particular, devemos ter $z_o \in V_\gamma$. Pois, caso contrário, decompondo $V = V_\gamma \oplus V'$, $z_n = z_{n,\gamma} + z_n'$, $z_o = z_{o,\gamma} + z_o'$, $z_o' \neq 0$, obteríamos

$$n(\tilde{L}_\gamma z_n, z_n)_E = n(\tilde{L}_\gamma z_n', z_n')_E = n(\|z_n'\|_E^2 - \lambda_{k+1}\|z_n'\|_{L^2}^2)$$
$$\geq n(\lambda_{k+2} - \lambda_{k+1})\|z_n'\|_{L^2}^2 ,$$

---

(1) Por simplicidade de notação, escreveremos $\Theta(v)$ ao invés de $\Theta(v, \tilde{f}_2)$, isto é, $w(v)$ ao invés de $w(v, \tilde{f}_2)$.

logo $n(\tilde{L}_\gamma z_n, z_n)_E \to \infty$ (uma vez que $\|z_n'\|_{L^2} \to \|z_o'\|_{L^2} \neq 0$), o que é um absurdo em vista de (6). Portanto, $z_o \in V_\gamma$ e

$$(f_o, z_o)_{L^2} = (\tilde{f}_o, z_o)_E \geq \liminf \left[-(KH_1(v_n + w(v_n)), z_n)_E\right]$$

$$= -\limsup\, (H_1(v_n + w(v_n)), z_n)_{L^2},$$

isto é,

$$\int_\Omega f_o z_o dx \geq -\limsup \int_\Omega h(nz_n + w_n) z_n dx,$$

onde escrevemos $w_n = w(v_n)$.

Agora, como $\|w_n\|_E \leq c$ (V. (3)), temos que (para alguma subsucessão e algum $w \in L^2$) $w_n \to w$ em $L^2(\Omega)$ e $w_n(x) \to w(x)$ em quase toda a parte. Logo, decompondo Ω como $\Omega_+(z_o) \cap \Omega_-(z_o) \cap \Omega_o(z_o)$ e usando o lema de Fatou na expressão acima, obtemos

$$\int_\Omega f_o z_o dx \geq -\int_{\Omega_+(z_o)} \overline{h}(\infty) z_o dx - \int_{\Omega_-(z_o)} \underline{h}(-\infty) z_o dx,$$

isto é, uma contradição ao fato de que f_o pertence a Σ_o.

■

Lema VI.2. Seja $F:\mathbb{R}^n \to \mathbb{R}^n$ contínua e tal que, para algum $r > 0$,

$$F(\alpha)\cdot\alpha > 0$$

na esfera $|\alpha| = r$. Então, existe α_o na bola $|\alpha| < r$ com $F(\alpha_o) = 0$.

Demonstração. A aplicação $\tilde{F}:\mathbf{R}^n \to \mathbf{R}^n$ definida por

$$\tilde{F}(\alpha) = \begin{cases} \alpha - F(\alpha) & \text{se } |\alpha - F(\alpha)| < r \\ r\,\dfrac{\alpha-F(\alpha)}{|\alpha-F(\alpha)|}, & \text{se } |\alpha - F(\alpha)| \geq r \end{cases}$$

é contínua e transforma a bola $|\alpha| \leq r$ em si mesma. Logo, o teorema do ponto fixo de Brouwer implica que $\tilde{F}(\alpha_o) = \alpha_o$ para algum α_o, $|\alpha_o| \leq r$.

Afirmamos que $|\alpha_o| < r$. Com efeito, se $|\alpha_o| = r$ então $|\tilde{F}(\alpha_o)| = r$ e, portanto, por definição de $\tilde{F}$, devemos ter

$$(7) \qquad |\alpha_o - F(\alpha_o)| \geq r.$$

Por outro lado, temos também

$$r^2 = \alpha_o \cdot \alpha_o = \alpha_o \cdot \tilde{F}(\alpha_o) = \alpha_o \cdot \left(r\,\frac{\alpha_o - F(\alpha_o)}{|\alpha_o - F(\alpha_o)|}\right),$$

isto é,

$$r = \frac{|\alpha_o|^2 - \alpha_o \cdot F(\alpha_o)}{|\alpha_o - F(\alpha_o)|} = \frac{r^2 - \alpha_o \cdot F(\alpha_o)}{|\alpha_o - F(\alpha_o)|}$$

e, uma vez que $\alpha_o \cdot F(\alpha_o) > 0$ (em vista da hipótese), obtemos

$$|\alpha_o - F(\alpha_o)| < r,$$

o que contradiz (7). Portanto, devemos ter $|\alpha_o| < r$, isto é, $|\tilde{F}(\alpha_o)| < r$, de onde concluimos (por definição de $\tilde{F}$) que

$$\alpha_o = \tilde{F}(\alpha_o) = \alpha_o - F(\alpha_o),$$

isto é,

$$F(\alpha_o) = 0.$$

■

Agora, combinando os Lemas VI.1 e VI.2(1), resulta que, dadas $\tilde{f}_o = Kf_o \in S_o$ e $\tilde{f}_2 \in W$, existe $v \in V$ tal que

$$\Theta(v, \tilde{f}_2) = \tilde{f}_o,$$

isto é, $S_o \subset S(\tilde{f}_2)$, $\forall\, \tilde{f}_2 \in W$. Isto mostra a parte (ii) do Teorema 3 e conclui a demonstração.

■

A seguir, passamos à demonstração do Teorema 3* enunciado na Seção II. Trata-se de um caso particular do teorema acima em que $\gamma = \lambda_1$, $\underline{h}(-\infty) = \overline{h}(-\infty) = h(-\infty)$, $\underline{h}(\infty) = \overline{h}(\infty) = h(\infty)$ e $h(\infty) < h(s) < h(-\infty)$, $\forall\, s \in \mathbb{R}$. Conforme já observamos antes (V. (5*)), a condição de Landesman-Lazer escreve-se, neste caso,

$$(5^{**}) \qquad -\int_\Omega h(-\infty)\phi_1 dx < \int_\Omega f_o\phi_1 dx < -\int_\Omega h(\infty)\phi_1 dx.$$

(1) Tome F no Lema VI.2 como a aplicação $\Theta(v) - \tilde{f}_o$ escrita em termos das coordenadas de v (V. Observação V.2).

Demonstração do Teorema 3*. Resta apenas mostrar que (5**) é também uma condição necessária para que (***) seja solúvel.

De fato, projetando a equação $\Theta(v) \equiv \tilde{L}_\gamma v - P\tilde{H}(v + w(v)) = P\tilde{f} = Kf_o$ sobre $V_\gamma = \langle\phi_1\rangle$, obtemos

$$-\int_\Omega h(v+w(v))\phi_1 dx = \int_\Omega f_o\phi_1 dx.$$

Logo, como $\phi_1 > 0$ em Ω e $h(\infty) < h(s) < h(-\infty)$, $\forall\, s \in \mathbf{R}$,

$$-\int_\Omega h(-\infty)\phi_1 dx < \int_\Omega f_o\phi_1 dx < -\int_\Omega h(\infty)\phi_1 dx.$$

■

Observação VI.3. Em [7] supõe-se que $[a,b]$ contém apenas um ponto do espectro $\sigma(-\Delta)$ $(\lambda_{k+1} = \ldots = \lambda_{\ell-1})$, isto é, que $V = V_\gamma$. Assim, o Teorema 3 demonstrado acima constitui uma pequena extensão do resultado correspondente lá apresentado. Naquela situação, entretanto, é possível melhorar a conclusão do item (ii) que passa a ser $S(\tilde{f}_2) \supset S_o \cup S'_o$, $\forall\, \tilde{f}_2 \in W$, onde

$$S'_o = K(\Sigma'_o),$$

$$\Sigma'_o = \{f_o \in V \mid \int_\Omega f_o z dx > -\int_{\Omega_+(z)} \underline{h}(\infty) z dx - \int_{\Omega_-(z)} \overline{h}(-\infty) z dx,\ \ \forall\, z \in V_\gamma,\ \|z\|_E = 1\}.$$

Isto é, temos o seguinte

Teorema 4. **Suponha** $[a,b] \cap \sigma(-\Delta) = \{\gamma\}$. **Dado** $\tilde{f} \in E$, **decomponha**

$\tilde{f} = \tilde{f}_1 + \tilde{f}_2 = P\tilde{f} + Q\tilde{f}$. **Então:**

(i) Para cada $\tilde{f}_2 \in W$, **existe um subconjunto próprio (não--vazio)** $S = S(\tilde{f}_2)$ V **tal que (***) é solúvel se e somente se** $\tilde{f}_1 \in S$;

(ii) $S(\tilde{f}_2) \supset S_o \cup S'_o, \quad \forall\, \tilde{f}_2 \in W$.

Demonstração. Basta verificar o item (ii). Para isso, utilizamos os lemas abaixo, cujas demonstrações são inteiramente análogas às dos Lemas VI.1 e VI.2.

Lema VI.1'. **Dadas** $\tilde{f}'_o = Kf'_o$, **com** $f'_o \in \Sigma'_o$, **e** $\tilde{f}_2 \in W$ **existe** $R' > 0$ **tal que**

$$(\Theta(v, \tilde{f}_2) - \tilde{f}'_o, v)_E < 0$$

para todo $v \in V$ **com** $\|v\|_E = R'$.

Lema VI.2'. **Seja** $F: \mathbb{R}^n \to \mathbb{R}^n$ **contínua e tal que para algum** $r' > 0$,

$$F(\alpha)\cdot\alpha < 0$$

na esfera $|\alpha| = r'$. **Então, existe** α_o **na bola** $|\alpha| < r'$ **com** $F(\alpha_o) = 0$ **(De fato, basta substituir** F **por** $-F$ **no Lema VI.2).**

■

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PROPAGATION OF SINGULARITIES OF SOLUTIONS OF PSEUDODIFFERENTIAL EQUATIONS

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Introduction

We recall some basic facts in the theory of linear partial differential equations in $n \geq 2$ variables, with constant coefficients, the details of which we refer to Hörmander, [18]. Let $P(D)$ be such an operator and let $\Omega \subset \mathbb{R}^n$ be an open set. We consider the following notion:

Definition 1. The set Ω *is said to be strongly* P-*convex if for every compact set* $K \subset \Omega$ *there is a compact set* $K' \subset \Omega$ *such that*

(1) *for all* $\mu \in \mathcal{E}'(\Omega)$, supp $P(-D)\mu \subset K$, *then* supp $\mu \subset K'$

and

(2) *for all* $\mu \in \mathcal{E}'(\Omega)$, sing supp $P(-D)\mu \subset K$, *then* sing supp $\mu \subset K'$.

One can prove, [18]:

Theorem 1. The operator $P(D):\mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$ *is surjective if and only if,* Ω *is strongly* P-*convex.*

Condition (1) (resp. (2)) means that if u is a distribution in a fixed neighborhood of the boundary $\partial\Omega$ of Ω, satisfying $P(-D)u=0$ (resp. $P(-D)u \in C^\infty$), and if $u=0$ (resp. $u \in C^\infty$) in an unspecified neighborhood of $\partial\Omega$, then the same is true in a fixed neighborhood of $\partial\Omega$. These phenomena are known, in general, as the unique continuation property of the support (resp. singular support).

Results such as Theorem 1 are much more difficult to establish in the case of variables, say C^∞, coefficients, [9]. The delimitation of the supports of solutions (if one prefers, the prop agation of zeros) is closely related to the so-called *uniqueness in the Cauchy problem*. A startling contribution in this direction, was made by A. Calderon, [6]. As the reader no doubt knows, the importance of Calderon's work lies not only in its conclusions concerning the Cauchy problem but also in that it contained one of the first and most spectacular applications of the calculus of pseudodifferential operators. In the same way, the delimitation of the singular support of solutions, that is the so-called *propagation of singularities*, has been also the subject of much study recently. It leads at once to the existence, mod. C^{∞}, of solutions of the transpose equation, usually in relatively compact open subsets. But as was shown in Malgrange's thesis, [29], these informations can be used to establish the existence of global solutions, mod. C^{∞}. There, to a large extent, lies the reason for continued interest in the propagation of singularities.

Chapter 1 contains a brief introduction to the theory of Fourier integral Operators. In Chapter 2 we give some results of propagation of singularities and related open problems. Our aim in writting this set of notes was not to present a comprehensive treatment of the subject — this would take us too far afield. Rather, we take the route of presenting the basic ideas in a clear enough fashion so that the eventual readers (hopefully, there will be many!), interested in the details, will find their way through the more specialized literature, extensively listed in the bibliography, in a relatively painless manner.

Chapter 1: Fourier Integral Operators

§1.1 Introduction

Pseudodifferential operators have been developed as a tool for the study of elliptic differential equations. Suitably extended versions are also applicable to hypoelliptic equations but their value is rather limited in genuinely non-elliptic problems. Many operators arising in the solution of differential equations are not pseudolocal, that is, do not preserve the singular support of distributions. For instance, if L is the wave operator

$$L = \frac{\partial^2}{\partial t^2} - \Sigma \frac{\partial^2}{\partial x_i^2} ,$$

the operator E mapping the Cauchy data u and $\partial u/\partial t$, at time $t=0$, to their values at time T is not pseudolocal. Fourier integral operators which are no longer pseudolocal, were introduced in order to study hyperbolic equations. Their calculus can be traced back to the work of Lax [27], Ludwig [28], Maslov [30] and was brought into a final form by Hörmander [20]. They (as well as the less general pseudodifferential operators) are intended to make it possible to handle differential operators with variable coefficients roughly as one would do with differential operators with constant coefficients using the Fourier transformation.

Fourier integral operators (more precisely associated local canonical transformations), together with pseudodifferential operators, can also be used to reduce (at least locally in the cotangent bundle) many partial differential operators P to particularly "simple" forms, such as

$$\partial/\partial x_1 \ , \ \partial/\partial x_1 + i\partial/\partial x_2 \ , \ \partial/\partial x_1 + ix_1\partial/\partial x_2 \ ,$$

a little in the same way as one may always choose local coordinates in which a given real vector field is $\partial/\partial x_1$. Transforming back the results for these simple operators, one can then obtain results for the original P. This procedure, following an idea of Egorov, has been used by Egorov [11], Nirenberg-Trèves [33] and Hörmander [24], in the study of subellipticity and local solvability for general pseudodifferential operators with complex principal symbol p, and by Duistermaat and Hörmander, [9], [22], in the special case that p is real or the Poisson bracket $\{p,\overline{p}\}=0$, when $p=0$, for (semi-)global regularity and existence theorems. The work of Sjöstrand, [38], Sato-Kawai-Kashiwara, [37], Boutet de Monvel, [5], and Trèves, [44], also show that the same procedure is very fruitful in the study of general overdetermined systems.

It is true that some results obtained by Egorov's procedure can also be proved without Fourier integral operators; it is wellknown, for instance, that the energy integral method leads to existence theorems, [21]. However, as was noted by Lax, [27], (a similar device is used in quantum theory, in connection with the "WKB method"), the asymptotic expansions of geometrical optics can be used to construct solutions at least locally and approximately, the solution operators (parametrices) being Fourier integral operators defined by a canonical relation different from the identity, making them very much different from pseudodifferential operators. For instance, let $P=P(x,D)$ be a differential operator of order m, with variable C^{∞} coefficients in an open set $X\subset\mathbb{R}^n$, written in the habitual multi-index notation:

$$P(x,D)=\sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha \ , \ D=(D_1,\dots,D_n) \ , \ D_j=-i\partial/\partial x_j \ ,$$

with principal symbol

$$p(x,\xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \ , \ \xi\in\mathbb{R}^n.$$

We assume that P is strictly hyperbolic with respect to the planes x_n=constant, which means that the characteristic equation

$$p(x,\xi)=0 \ , \ \xi=(\xi',\xi_n) \ ,$$

has m real (distinct) solutions ξ_n, say

$$\tau_1(x,\xi')<\dots<\tau_m(x,\xi') , \text{ when } 0\neq\xi'\in\mathbb{R}^{n-1}.$$

For each of these roots we determine (under suitable uniformity assumptions on p and for x_n-y_n sufficiently small) a solution $\varphi_j(x,y_n,\theta)$ of the differential equation

$$\partial\varphi_j(x,y_n,\theta)/\partial x_n-\tau_j(x,\partial\varphi_j(x,y_n,\theta)/\partial x')=0$$

such that

(1.1.1) $\quad \varphi_j(x,y_n,\theta)=\langle x',\theta\rangle$ when $x_n=y_n$, $\theta\in\mathbb{R}^{n-1}\setminus\{0\}$.

It is clear that (the phase functions) $\varphi_1,\ldots,\varphi_m$ are all the solutions of the (eikonal) equation $p(x,\partial\varphi/\partial x)=0$ satisfying (1.1.1). Now a solution of the Cauchy problem

(1.1.2) $\quad Pu=0\ ,\ D_n^j u=0 \text{ for } j<m-1\ ,\ D_n^{m-1}u=f\in C_o^\infty(\mathbb{R}^{n-1}),\text{when } x_n=y_n,$

can, for x_n sufficiently close to y_n, be written in the form

(1.1.3) $$u(x)=\sum_1^m (2\pi)^{1-n}\int e^{i\varphi_j(x,y_n,\theta)} a_j(x,y_n,\theta)\hat{f}(\theta)d\theta$$

where (the amplitude) $a_j\in S^{1-m}(X\times\mathbb{R}^n)$, and, as usual, $S^\mu(X\times\mathbb{R}^n)$ denotes the set of symbols $a(x,\xi)$ of order μ, that is, $a(x,\xi)\in C^\infty(X\times\mathbb{R}^n)$ and

(1.1.4) $\quad |D_x^\alpha D_\xi^\beta a(x,\xi)|\leq C_{\alpha,\beta,K}(1+|\xi|)^{\mu-|\beta|}\ ,\ x\in K\subset\subset X.$

The construction of the amplitudes a_j is made by successive approximation, with an ordinary differential equation (the transport equations) being solved at each step. Actually one obtains only an approximate solution of (1.1.2) in the sense that Pu will be an integral operator with C^∞ kernel acting on f. Thus (1.1.3) defines a parametrix, and Lax was able to conclude that for x_n-y_n positive and sufficiently small, the singular support in $\mathbb{R}^n\times\mathbb{R}^n$ of a parametrix consists of points (x,y) such that x lies on a bicharacteristics starting at y. Ludwig, [28], proved that the singular support is contained in the union of the bicharacteristics globally by solving the Cauchy problem in steps using a variant of (1.1.3) each time.

As a generalization of pseudodifferential operators as well as of the terms in (1.1.3) with x_n and y_n considered as parameters, it is natural to consider operators of the form

$$(1.1.5) \quad Af(x)=(2\pi)^{-n}\int e^{iS(x,\theta)}a(x,\theta)\hat{f}(\theta)d\theta \ , \ f\in C_0^\infty(\mathbb{R}^n) \ , \ x\in\mathbb{R}^n \ ,$$

where S is real valued and homogeneous of degree one with respect to θ , $a\in S^\mu$ for some μ. If we introduce the definition of the Fourier transform in (1.1.5) we obtain formally

$$(1.1.6) \quad Af(x)=(2\pi)^{-n}\iint e^{i\varphi(x,y,\theta)}a(x,y,\theta)f(y)dyd\theta \ , \ f\in C_0^\infty \ ,$$

where $\varphi(x,y,\theta)=S(x,\theta)-\langle y,\theta\rangle$ and a is actually independent of y. If we drop these special features of φ and of a, we shall have a much more symmetric setup. This is perhaps even more clear if, still formally, we take the scalar product with a function g,

$$(1.1.7) \quad \langle Af,g\rangle=(2\pi)^{-n}\iiint e^{i\varphi(x,y,\theta)}a(x,y,\theta)g(x)f(y)dxdyd\theta, f,g\in C_0^\infty.$$

This means that the distribution kernel of A, which we also denote by A, should be defined by

$$(1.1.8) \quad \langle A,f\rangle=(2\pi)^{-n}\iiint e^{i\varphi(x,y,\theta)}a(x,y,\theta)f(x,y)dxdyd\theta \ , \ f\in C_0^\infty.$$

§1.2 Wave front sets and some geometrical background

The *wave front set* of a distribution is a refinement of the notion of *singular support* of a distribution.

Let $u\in\mathcal{D}'(X)$, X an open set in R^n. According to the

Paley-Wiener theorem u is C^∞ in a neighborhood of x_o, in other words, $x_o \notin$ sing. supp u, if and only if for some $\psi \in C^\infty$ with $\psi(x_o) \neq 0$:

$$\langle e^{-i\tau\langle \cdot,\xi\rangle}\psi, u\rangle = O(\tau^{-N}) \quad \text{for} \quad \tau\to\infty, \tag{1.2.1}$$

uniformly in $|\xi|=1$, for all N.

It turns out that it is very fruitful not only to localize with respect to x but also with respect to ξ. This leads to the following definition of the *wave front set*, WF(u), of a distribution u:

For each (x_o,ξ^o), $\xi^o \neq 0$, we have $(x_o,\xi^o) \notin WF(u)$ if and only if (1.2.1) is valid for some $\psi \in C_o^\infty$, with $\psi(x_o) \neq 0$, and uniformly for all ξ in a neighborhood of ξ^o.

It is not difficult to see that

$$\text{sing supp } u = \bigcap \{x, \psi(x)=0\} \tag{1.2.2}$$

the intersection being taken over all $\psi \in C^\infty(X)$ with $\psi u \in C^\infty(X)$. Replacing the function ψ by zero-order pseudodifferential operators we obtain the following equivalent definition of WF(u):

$$WF(u) = \bigcap \text{char } (A) \quad (\text{char } (A) \text{ is the characteristic set of } A) \tag{1.2.3}$$

the intersection being taken over all $A \in L^o(X)$ with $Au \in C^\infty(X)$, that is: $(x_o,\xi^o) \notin WF(u)$ if and only if $Au \in C^\infty$ for some $A \in L^o(X)$ with a principal symbol $a(x,\xi)$ which is invertible in a conic neighborhood of (x_o,ξ^o). A subset $\Gamma \subset X\times\mathbb{R}^n\setminus\{0\}$ is called a cone if $(x,\xi) \in \Gamma$ implies $(x,\tau\xi) \in \Gamma$ for all $\tau>0$. The definition (1.2.3) has the advantage of being invariant with respect to change of variables and thus lends itself to defining WF(u) when X is a C^∞ manifold.

Proposition 1.2.1. WF(u) *is a closed cone in* $T^*(X)\setminus\{0\}$ *and* sing supp $u=\pi WF(u)$ *where* π *is the cotangent bundle projection*: $T^*(X)\longrightarrow X$.

Proposition 1.2.2. For all pseudodifferential operators A *we have*:

(1.2.4) $$WF(Au)\subset WF(u)\subset WF(Au)\cup \mathrm{char}(A).$$

The second part, extending the regularity theorem for elliptic operators, is obvious; the first part improves the pseudolocal property of pseudodifferential operators.

Let us recall the Poisson bracket of two smooth functions on $T^*(X)$:

(1.2.5) $$\{f,g\} = \sum_1^n \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}.$$

The map $g\longrightarrow\{f,g\}$ is a vector field (derivation), called the Hamiltonian field of f, and denoted by H_f: thus

(1.2.6) $$H_f = \sum_1^n \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j}.$$

It is easily checked that we have:

(1.2.7) $$H_{\{f,g\}}=[H_f,H_g],$$

from which we get

(1.2.8) $$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0 \qquad \text{(Jacobi's identity)}$$

In particular $C^\infty(T^*X)$ with the Poisson bracket is a Lie algebra. In fact, it is a graded Lie algebra if we restrict ourselves to homogeneous functions in ξ-variable. The Poisson bracket may be invariantly defined in terms of the *canonical* 2-form $\omega=\sum_1^n d\xi_j\wedge dx_j$, which is closed, non-degenerate, antisymmetric (i.e. symplectic) and whose only homogeneous primitive is the canonical 1-form $\Sigma\xi_j dx_j$. For each $x\in X$, the form ω_x defines an isomorphism $\tilde{\omega}_x$ of T_xX onto T_x^*X. Let f be a C^∞ function in X, df_x its differential at x. The Hamiltonian field of f at x is the tangent vector to X, at x,

$$H_f(x)=\tilde{\omega}_x^{-1}(df_x).$$

Let v be any smooth vector field in X. We have

$$-\omega(H_f,v)=\langle df,v\rangle=vf.$$

Putting $v=H_g$, where g is another C^∞ function in X, we obtain

$$\{f,g\}=H_f g=\omega(H_f,H_g).$$

It follows that any canonical map $\Phi:T^*(X)\longrightarrow T^*(X')$ (i.e. Φ takes the canonical form $\Sigma d\xi_j\wedge dx_j$ of T^*X into $\Sigma d\xi_j'\wedge dx_j'$) is compatible with the formation of Poisson brackets. We have $H_f f=\{f,f\}=0$, hence f is constant along any integral curve of the Hamiltonian H_f and, consequently, if such a curve has one point in $f^{-1}(0)$, it lies entirely in $f^{-1}(0)$.

Let A be a pseudodifferential operator with homogeneous principal symbol a. Then the *bicharacteristic strips* of A are the integral curves of H_a lying in the characteristic set $a^{-1}(0)$. They satisfy the Hamilton-Jacobi equations:

$$(1.2.9)\qquad \begin{cases} \dot{x}= a_\xi \\ \\ \dot{\xi}=-a_x \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{dx_j}{dt}= \dfrac{\partial a}{\partial \xi_j} \\ \\ \dfrac{d\xi_j}{dt}=-\dfrac{\partial a}{\partial x_j} \end{cases} \qquad j=1,\ldots,n$$

and their projection in the base space X, when they exist (for instance if $a_\xi \neq 0$), is called the *bicharacteristic curve* of A.

§1.3 Fourier integral distributions

Our task now is to give a meaning to integrals of the type (1.1.6) and (1.1.8) (see Hörmander [20]).

A *Fourier integral distribution* is a distribution. A which is defined by an integral of the form

$$(1.3.1)\qquad \langle A,u\rangle=\iint e^{i\varphi(x,\theta)}a(x,\theta)u(x)\,dx\,d\theta \ , \ u\in C_o^\infty(X).$$

Here $\theta=(\theta_1,\ldots,\theta_N)$ are auxiliary variables, called the *frequency variables*. The *phase function* φ is a real valued function in $C^\infty(X\times\mathbb{R}^N\setminus\{0\})$, positively homogeneous of degree one with respect to θ and $d\varphi\neq 0$ everywhere. Moreover it is assumed that φ is *non degenerate*, that is, the differentials of the functions $\partial\varphi/\partial\theta_j$, $j=1,\ldots,N$, are linearly independent in

$$(1.3.2)\qquad C_\varphi=\{(x,\theta)\ ,\ X\times\mathbb{R}^N\setminus\{0\}\ :\ \varphi'_\theta(x,\theta)=0\}.$$

For the *amplitude function* a we suppose that it belongs to the symbol class $S^\mu(X\times\mathbb{R}^N)$ for some μ. If $\mu<-N$ the integral is absolutely convergent and defines a continuous function A. Since φ

has no critical points we can find a first order differential operator

$$L=\Sigma a_j \partial/\partial\theta_j+\Sigma b_j \partial/\partial x_j+c$$

with $a_j\in S^o(X\times\mathbb{R}^N)$ and b_j, $c\in S^{-1}(X\times\mathbb{R}^N)$ such that ${}^tLe^{i\varphi}=e^{i\varphi}$ if ${}^tL$ is the adjoint of L. Now successive integrations by parts in (1.3.1) will reduce the growth of the integrand at infinity until it becomes integrable. This gives a precise definition of the distribution $A\in\mathcal{D}'(X)$. The fact that φ is non-degenerate implies that the set C_φ is an n-dimensional conic C^∞ submanifold of $X\times\mathbb{R}^N\setminus\{0\}$ and that the mapping

(1.3.3) $$C_\varphi\ni(x,\theta)\longmapsto(x,\varphi'_x(x,\theta))\in T^*(X)\setminus\{0\}$$

is an immersion: $C_\varphi\longrightarrow T^*(X)\setminus\{0\}$. The image, denoted Λ_φ, is an n-dimensional conic C^∞ submanifold. The restriction of the form $\Sigma\xi_j dx_j$ to Λ_φ is $\varphi'_x dx=d\varphi-\varphi'_\theta d\theta=0$ since $\varphi'_\theta=0$ on C_φ and so $\varphi=\langle\theta,\varphi'_\theta\rangle=0$ on C_φ, by Euler's identity. In view of the homogeneity this is equivalent to the vanishing on Λ_φ of its differential which is the canonical (symplectic) form $\omega=\Sigma d\xi_j\wedge dx_j$. Thus Λ_φ is a manifold of maximal dimension on which ω vanishes. Such manifolds are called *Lagrangian*. Conversely it can be shown that every coni Lagrangian submanifold Λ of $T^*(X)\setminus\{0\}$ is locally equal to Λ_φ for some non-degenerate phase function φ.

Using the *method of stationary phase* to investigate the asymptotic behavior of integrals of the form (1.3.1) we obtain

(1.3.4) $$WF(A)\subset\Lambda\ .$$

As an example we consider the distribution kernel

(1.3.5) $$(x,y)\longmapsto(2\pi)^{-n}\int e^{i\langle x-y,\theta\rangle}a(x,\theta)\,d\theta,$$

of a pseudodifferential operator A associated to a symbol $a(x,\theta)$; its wave front set lies in $\{(x,y;\xi,\eta);x=y,\xi=-\eta\}$ which is the normal bundle of the diagonal in $X\times X$.

The leading term of this asymptotic expansion gives rise to an "invariantly" defined *principal symbol* of A (being a *density* of order 1/2 with values in a complex line bundle on Λ, called the *Maslov line bundle*). Suppose now that Λ is an arbitrary closed conic Lagrangian manifold in $T^*(X)\setminus\{0\}$. Then a global Fourier integral A of order m, defined by Λ, is a locally finite sum of distributions A_j as in (1.3.1) with $\varphi=\varphi_j$, $a=a_j$, the Λ_{φ_j} forming a locally finite system of open cones in Λ, and with

$$a_j\in S^{m+n/4-N_j/2}$$

(the number $-N_j/2$ in the growth order is necessary to get an order m of A which is independent of the number of frequency variable used, and the number $n/4$ is introduced in order to obtain additivity of the orders when Fourier integral operators, to be defined later, are multiplied); its symbol is defined as the sum of the symbols of each of the integrals. We can therefore define a space $I^m(X,\Lambda)$ of distributions with wave front sets in Λ which locally can be written in the form (1.3.1) with $a\in S^{m+n/4-N/2}$, $n=\dim X$, and φ defining a part of Λ according to (1.3.3). With the elements in $I^m(X,\Lambda)$ one can, as for pseudodifferential operators, associate principal symbols on Λ, which are symbols of order $m+n/4$ modulo symbols of order $m+n/4-1$, with value in the Maslov line bundle. (This agrees with the standard notion of principal symbol in the case of kernels of pseudodifferential operators).

In particular, if σ is the principal symbol of a compactly supported $A \in I^m(X,\Lambda)$, then $|\sigma|^2=\sigma.\overline{\sigma}$ is a density of order 1 on Λ, and

$$\int_X |A|^2 = \tau^{2m}\left(\int_\Lambda |\sigma|^2 + 0(\tau^{-1})\right) \quad \text{as} \quad \tau\to\infty.$$

This shows that, since σ is smooth, the *energy* of A in a domain U in x-space is asymptotically proportional to the n-dimensional volume of $\pi^{-1}(U)$ in Λ, where π denotes the projection $\Lambda \ni (x,\xi) \longmapsto x \in X$. For instance, if U shrinks as a ball with decreasing radius to the point x_o then the ratio between the energy in U and the volume in U (asymptotically as $\tau\to\infty$, let us say for $m=0$) tends to $+\infty$ if and only if x_o is a singular value of π. Such points are called *caustic points* for Λ (because there the light "burns"). There is a close connection between the theory of catastrophies of Thom, [42], and the appearance of caustics.

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^\ell$ be open sets (the results have an obvious extension to manifolds if one works throughout with densities of order 1/2). Then by the kernel theorem of L. Schwartz we can identify $\mathcal{D}'(X\times Y)$ with the space of continuous linear operators $A: C_o^\infty(Y) \longrightarrow \mathcal{D}'(X)$ by means of the formula

(1.3.6) $$\langle Av,u\rangle = \langle K, u\otimes v\rangle, \quad u \in C_o^\infty(X), \; v \in C_o^\infty(Y).$$

From the calculus of wave front sets it follows that if WF(K) does not contain points of the form $(x,y;\xi,0)$, $\xi\neq 0$ or $(x,y,0,\eta)$, $\eta\neq 0$, then A maps, in fact, $C_o^\infty(Y)$ into $C^\infty(X)$, can be extended to a continuous linear mapping: $\mathcal{E}'(Y)\longrightarrow\mathcal{D}'(X)$ and

$$WF(Au)\subset WF'(K)\circ WF(u). \tag{1.3.7}$$

Here $WF'(K)=\{(x,y;\xi,-\eta)\in T^*(X\times Y);(x,y;\xi,\eta)\in WF(K)\}$ and we have identified $T^*(X\times Y)$ with $T^*(X)\times T^*(Y)$ and regarded $WF'(K)$ as a relation between $T^*(X)$ and $T^*(Y)$ acting on $T^*(Y)$. Furthermore A can be extended to a continuous linear mapping: $\mathcal{D}'(Y)\longrightarrow\mathcal{D}'(X)$ if, in addition, the projection of supp K onto X is a proper mapping. Now a *Fourier integral operator of order* m defined by the closed conic Lagrangian submanifold Λ of $T^*(X)\setminus\{0\}\times T^*(Y)\setminus\{0\}$ is an operator $A:C_o^\infty(Y)\longrightarrow\mathcal{D}'(X)$ such that its kernel $K\in I^m(X\times Y,\Lambda)$. That Λ is Lagrangian means that $\omega_{T*(X\times Y)}=\omega_{T*(X)}\oplus\omega_{T*(Y)}$ vanishes on Λ.

Because of (1.3.4) and (1.3.7) ones prefers to work with the relation $C=\Lambda'$ and get that $\omega_{T*(X)}-\omega_{T*(Y)}$ vanishes on C. If C is the graph of a mapping $\Phi:T^*(Y)\longrightarrow T^*(X)$ then this condition means that $\Phi^*.\omega_{T*(X)}=\omega_{T*(Y)}$, that is Φ is a canonical transformation: $T^*(Y)\longrightarrow T^*(X)$. Because C is conic, Φ is homogeneous of degree one. For a general conic Lagrangian manifold Λ, the relation C is therefore called a *homogeneous canonical relation from* $T^*(Y)$ *to* $T^*(X)$. The operator A is called a Fourier integral operator of order m defined by the canonical relation C and one uses the notation $A\in I^m(X,Y;C)$.

Pseudodifferential operators in X are defined as Fourier integral operators with $Y=X$ and

(1.3.8) $\Lambda'_\varphi \subset$ diagonal in $T^*(X)\setminus\{0\}\times T^*(X)\setminus\{0\}=\Delta$

$=$graph of the identity: $T^*(X)\setminus\{0\}\longrightarrow T^*(X)\setminus\{0\}$,

in other words, $I^m(X;X;\Delta)=L^m(X)=$space of pseudodifferential operators of order m in X.

If we have three manifolds X, Y, Z and homogeneous canonical relations C_1 and C_2 from $T^*(Y)$ to $T^*(X)$ and from $T^*(Z)$ to $T^*(Y)$ one can prove that the composition $A_1\circ A_2$ of properly supported operators $A_1\in I^{m_1}(X,Y;C_1)$ and $A_2\in I^{m_2}(Y\times Z;C_2)$ is in

$$I^{m_1+m_2}(X\times Z;C_1\circ C_2),$$

if the appropriate transversality and other conditions are fulfilled which guarantee that $C_1\circ C_2$ is a manifold. Furthermore the principal symbol of $A_1\circ A_2$ is equal to the product of the principal symbols of A_1 and A_2. Assume that C is the graph of a canonical diffeomorphism: $T^*(Y)\longrightarrow T^*(X)$ (in particular dim X=dim Y) and let $A\in I^o(X,Y;C)$; then the adjoint $A^*\in I^o(X,Y;C^*)$, where $C^*=C^{-1}$ (the graph of the inverse diffeomorphism). One has $A^*A\in I^o(Y,Y;\Delta)$, i.e.,A^*A is a pseudodifferential operator of order zero in Y. If P and Q are two pseudodifferential operators in X and Y respectively, with principal symbols p and q, such that PA=AQ, then if $(x,\xi)=F(y,\eta)$ is the equation defining C, one obtains:

(1.3.9) $$p(F(y,\eta))=q(y,\eta),$$

if the principal symbol of A is not zero (i.e if A is elliptic) at $(F(y,\eta),(y,-\eta))$. Conversely, (1.3.9) implies that PA-AQ is of lower order. We can therefore successively construct the symbol of Q, for a given P, so that PA-AQ is of order $-\infty$, provided that the wave front set of P is concentrated near a point where A is elliptic. This is the argument of Egorov alluded to at the introduction, which often allows one to pass from one operator to another with principal symbol modified by a homogeneous canonical transformation.

Finally we observe that if $A\in I^{\mu}(X,Y;C)$ and P is a linear partial differential operator of order m, with principal symbols σ and p respectively, then $PA\in I^{m+\mu}(X,Y;C)$ with principal symbol $\tilde{p}.\sigma$, where $\tilde{p}$ is the lifting from $T^*(X)$ to C by the projection $C\longrightarrow T^*(X)$. So if we want (as in wave mechanics) that PA=0 asymptotically, we need $\tilde{p}\ \sigma=0$, that is $\tilde{p}=0$ *on* C if we take $\sigma\neq 0$. Then PA is actually a Fourier integral distribution defined by C of order $m+\mu-1$, and its principal symbol is equal to

$$\frac{1}{i}\mathcal{L}_{H_{\tilde{p}}}\sigma+\tilde{c}\ a. \tag{1.3.10}$$

Here $\mathcal{L}$ denotes the Lie derivative, [10], (of densities of order 1/2) and $c\in S^{m-1}(T^*(X)\setminus\{0\})$ is another invariantly defined function on $T^*(X)$ called the *subprincipal symbol* of P. We recall that the principal symbol p of P is itself invariantly defined on $T^*(X)$ (see (1.3.9)). Demanding that (1.3.10) be equals to zero means solving an ordinary linear differential equation for σ along the bicharacteristic strips. In particular σ is determined along the whole strip if it is given at one of its points. One says that σ *propagates* along the bicharacteristic strips. Regarding C together with its bicharacteristic strips as the *geometrical* optics of the asymptotic solution A, then the statement is that the geometrical optics describes the propagation of the high-frequency asymptotic waves. The *light rays* are identified with the bicharacteristic curves.

Chapter 2: Propagation of Singularities

§2.1 Pseudodifferential operators of real principal type

Let $P: C^\infty(X) \longrightarrow C^\infty(X)$ be a properly supported pseudodifferential operator of order m in a C^∞ manifold X of dimension n. Assume also that P is of real principal type, that is, its principal symbol p is a real-valued function and H_p and the radial vector field

$$\xi . d_\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}$$

are linearly independent in char(P). In this section we will prove the following result due to Hörmander, [9]:

Theorem 2.1.1. Let u *be a distribution in* X, *then* $WF(u) \setminus WF(Pu)$ *is invariant under the action of the Hamiltonian field* H_p. *Furthermore if* $(x_o, \xi^o) \in \text{char}(P)$ *there is* $w \in \mathcal{D}'(X)$ *such that* $WF(w) \setminus WF(Pw)$, *is equal, near* (x_o, ξ^o), *to the "positive cone" generated by* Γ.

Here Γ is the bicharacteristic of p through $(x_o, {}^{-o})$.

Remark 2.1.0. It is obvious that $WF(u) \setminus WF(Pu)$ is contained in char(P). It follows also from Theorem 2.1.1 that if a point x_o

is in sing supp u \ sing supp Pu then the same is true for the bicharacteristic curve of p_m through x_o, near x_o.

Remark 2.1.1. There are three essentially different proofs of the above theorem. The first one makes use of canonical transformations to reduce (microlocally) our operator P to D_{x_n} in $\mathbb{R}^n$, for which the theorem can be easily proved. The second one uses energy inequalities for hyperbolic operators. This proof gives a more precise way of getting solutions with prescribed singularities, namely in H^s-spaces. We will present a third proof, due to Hörmander, [19] and sketched in Nirenberg, [35]; it is based on a reduction of the operator to a simpler one (although not as simple as in the first proof). The results obtained are not as sharp as in the second proof.

Remark 2.1.2. The reader can see Helffer, [17], for an application of these three different proofs in a related context. More precisely, one needs each of the above methods of proofs in the study of propagation of singularities for boundary value problems.

Proof of Theorem 2.1.1. We may suppose that X is an open set of $\mathbb{R}^n$ and that $Pu \in C^\infty(X)$ since both conclusions in the theorem are microlocal. We also assume that $\frac{\partial p}{\partial \xi_n} \neq 0$ on Γ.

The basic steps to get *Theorem 2.1.1* are now given: Let $(\bar{x};\bar{\xi})$ be a point in Γ which is not in the WF(u), then there exists a zero order pseudodifferential operator B, such that

(2.1.3) $P(Bu) \in C^\infty(\mathcal{O})$ *where* $\mathcal{O}$ *is a connected open neighborhood* *of* x_o *and* $\bar{x}$, $b(x,\xi)\equiv 1$ *in a conic neighborhood of* Γ, b *being a symbol of* B

and

(2.1.4) Bu *is a* C^∞*-function at* $\bar{x}$.

The next step is to prove the existence of a (m-1) - order and first-order pseudodifferential operators Q and $D_{x_n} - \lambda(x,D_{x'})$ respectively $(x'=(x_1,\ldots,x_{n-1}))$, such that

(2.1.5) $P(Bu)=(D_{x_n}-\lambda(x,D_{x'}))(Q(Bu))$ *where* Q *is an elliptic pseudodifferential operator of order one*

and

(2.1.6) *the Cauchy Problem for the operator* $D_{x_n}-\lambda(x,D_{x'})$ *is well posed relatively to* $\{x\in\Theta \mid x_n=\bar{x}_n\}$. *In particular, the solution is* C^∞ *for* C^∞ *data.*

Therefore if one uses (2.1.3) to (2.1.6) and observes that in the proofs of these facts, the factorization of P can be made independent of the construction of the operator B, one obtains easily the first conclusion of the theorem. The second result follows from (2.1.3) to (2.1.6) and

(2.1.7) *there exists a distribution* $w\in\mathcal{D}'(\Theta')$, Θ' *being an open neighborhood of* x_o, *such that* $(D_{x_n}-\lambda(x,D_{x'}))w\in C^\infty(\Theta')$. *Furthermore* $WF(w)$ *is equal to the positive cone generated by the bicharacteristic strip of* $\xi_n-\lambda_1(x,\xi')$ *passing* *through* (x_o,ξ^o).

For the existence of a pseudodifferential operator satisfying (2.1.3) and (2.1.4), it is enough to show that there are open connected neighborhoods $\mathcal{O}$ and $\bar{\mathcal{O}}$ of x_o and $\bar{x}$ respectively ($\mathcal{O}$ being also a neighborhood of $\bar{x}$) such that $[P,B]\in L^{-\infty}(\mathcal{O})$ (the class of regularizing pseudodifferential operators in $\mathcal{O}$), $b(x,\xi) = 1$ in a conic neighborhood of $T^*\bar{\mathcal{O}}\cap\Gamma$ and ess supp $b\cap WF(u)\cap T^*(\mathcal{O})=\emptyset$. Making use of the symbolic calculus of $L^{\infty}(X)$ with $b\sim b_o+b_1+\ldots$, we have

$$\begin{aligned}(2.1.8)\quad \textit{symbol of}\quad [P,B]&\sim\sum\frac{1}{\alpha!}\{[\partial_\xi^\alpha(p+p_{m-1}+\ldots).D_x^\alpha(b_o+b_{-1}+\ldots)]-\\ &-[\partial_\xi^\alpha(b_o+b_{-1}+\ldots).D_x^\alpha(p+p_{m-1}+\ldots)]\}\sim\frac{1}{i}H_p b_o+\{\frac{1}{i}H_p b_{-1}+\frac{1}{i}H_{p_{m-1}}b_o+\\ &+\sum_{|\alpha|=2}[(\partial_\xi^\alpha p)(D_x^\alpha b_o)-(\partial_\xi^\alpha b_o)(D_x^\alpha p)]\}+\ldots\ .\end{aligned}$$

The general term of this last asymptotic expansion is of the form

$$\frac{1}{i}H_p b_{-j}+f_j$$

where f_j is a homogeneous function of degree $m-j-1$ depending on p, p_{m-k} $(1\leqslant k\leqslant j)$ and $b_{-k'}$ $(0\leqslant k'\leqslant j-1)$. Therefore we impose

$$H_p b_o = 0 \tag{2.1.9}$$

and

$$H_p b_{-j}+if_j=0,\ j>1, \tag{2.1.10}$$

in order to have $[P,B]\in L^{-\infty}(X)$. Consider

$$V^{\varepsilon}_{(\bar{x};\bar{\xi})}=\{(x;\xi)\in T^*X\setminus\{0\}\ :\ |x-\bar{x}|<\varepsilon,\left|\frac{\xi}{|\xi|}-\frac{\bar{\xi}}{|\bar{\xi}|}\right|<\varepsilon\}$$

with $\varepsilon>0$ small enough such that

$$V^{\varepsilon}_{(\overline{x};\overline{\xi})} \cap WF(u)=\emptyset .$$

Since p is a real-valued function and $\frac{\partial p}{\partial \xi_n} \neq 0$ in $T^*X\setminus\{0\}$, it is possible to solve, inductively , the equations (2.1.9) and (2.1.10) satisfying respectively the initial conditions:

(2.1.11)
$$b_o\Big|_{x_n=\overline{x}_n} = \psi(x';\xi)$$

and

(2.1.12)
$$b_{-j}\Big|_{x_n=\overline{x}_n} = 0 \quad \text{if} \quad j>0,$$

where ψ is a zero-order homogeneous symbol such that

$$\operatorname{supp} \psi \cap \{x \mid x_n=\overline{x}_n\} \subset V^{\varepsilon}_{(\overline{x};\overline{\xi})} \quad \text{and} \quad \psi \equiv 1 \quad \text{in} \quad V^{\varepsilon/2}_{(\overline{x};\overline{\xi})} .$$

It is clear that the solutions b_{-j} are homogeneous functions of order $-j$. Also if $\varepsilon>0$ is small enough there exist solutions b_j, $j\leq 0$, in a relatively compact open connected subset $\mathcal{O}$ of X. This set $\mathcal{O}$ can be taken in such a way that the projection of Γ onto $\mathcal{O}$ contains $\{x_0,\overline{x}\}$. These arguments, as we have mentioned before, are enough to guarantee the existence of B satisfying (2.1.3) and (2.1.4).

We observe that if $\overline{x}$ is sufficiently close to x_0 and $\varepsilon>0$ is small enough, the restriction of Bu to $\{x \in \mathcal{O} \mid x_n = \overline{x}_n\}$ exists and is a C^∞ function. We start now the proof of the assertions (2.1.5), (2.1.6) and (2.1.7). By the implicit function theorem we may choose an open neighborhood $U_o \subset \mathbb{R}^n \times \mathbb{R}^{n-1}$ of the set $\{(x,\xi') \mid \exists\ \xi_n,\ (x,\xi',\xi_n) \in \Gamma \cap T^*\mathcal{O}\}$ and a real-valued C^∞ function $\lambda_1(x,\xi')$, defined in U_o, such that

$$p(x,\xi',\lambda_1(x,\xi'))=0 \quad \forall \quad (x,\xi') \in U_o$$

(since $\frac{\partial p}{\partial \xi_n} \neq 0$ near Γ).

Extending $\xi_n-\lambda_1(x,\xi')$ to an open conic neighborhood V of Γ, we may consider

$$q_{m-1}(x,\xi)=\frac{p(x,\xi)}{\xi_n-\lambda_1(x,\xi')}$$

as a C^∞ - function homogeneous of degree m-1, defined in V. It is easy to see that $q_{m-1}(x,\xi) \neq 0$ in V. We remark that here an open conic neighborhood means "conic away from the origin". We get a complete factorization of P near Γ, by showing:

Lemma 2.1.2. There exist properly supported pseudodifferential operators Q *and* $D_{x_n}-\lambda(x,D_{x'})$ *for which* (2.1.5) *is satisfied in an open conic neighborhood of* Γ.

Proof As before we consider

$$\lambda(x,\xi') \sim (\xi_n-\lambda_1(x,\xi'))+\lambda_o(x,\xi')+\dots$$

and

$$q(x,\xi) \sim q_{m-1}(x,\xi)+q_{m-2}(x,\xi)+\dots \quad .$$

We want that

(2.1.13) symbol of $P\sim\sum\frac{1}{\alpha!}\partial_\xi^\alpha(\tau-\lambda_1-\sum_{i=0}^{-\infty}\lambda_i)\cdot D_x^\alpha(q_{m-1}+q_{m-2}+\dots)$

near Γ, that is

$$p(x,\xi)=(\xi_n-\lambda_1)q_{m-1}, \tag{2.1.14}$$

$$p_{m-1}(x,\xi)=(\xi_n-\lambda_1)q_{m-2}-\lambda_o q_{m-1}+$$

$$+D_{x_n}q_{m-1}-\sum_{i=1}^{n-1}\frac{\partial}{\partial\xi_i}\lambda_1\cdot D_{x_i}q_{m-1},$$

$$\dots \quad .$$

The functions λ_1 and q_{m-1} have been chosen in order that the first equation of (2.1.14) be satisfied. Making $\xi_n=\lambda_1$ in the second equation of (2.1.14) we get

$$p_{m-1}(x\ \xi',\lambda_1(x,\xi'))=[-\lambda_o q_{m-1}+$$

$$+D_{x_n}q_{m-1}-\sum_{i=1}^{n-1}\frac{\partial}{\partial\xi_i}\lambda_1\cdot D_{x_i}q_{m-1}](x,\xi',\lambda_1(x,\xi')).$$

Therefore we must have

$$\lambda_o(x,\xi')=\frac{(-p_{m-1}+D_{x_n}q_{m-1}-\sum_{i=1}^{n-1}\frac{\partial}{\partial\xi_i}\lambda_1\cdot D_{x_i}q_{m-1})(x,\xi',\lambda_1(x,\xi'))}{q_{m-1}(x,\xi',\lambda_1(x,\xi'))} \tag{2.1.15}$$

and

$$q_{m-2}(x,\xi)=\frac{(p_{m-1}+\lambda_o q_{m-1}-D_{x_n}q_{m-1}-\sum_{i=1}^{n-1}\frac{\partial}{\partial\xi_i}\lambda_1\cdot D_{x_i}q_{m-1})(x,\xi)}{\xi_n-\lambda_1(x,\xi')}$$

We observe that the last expression has a unique extension to

$\xi_n-\lambda_1(x,\xi)=0$, by our choice of $\lambda_o(x,\xi')$. As can be easily shown the same method can be applied to determine q_{m-j} and λ_{-k}, $j\geq 3$ and $k\geq 1$, as homogeneous functions near Γ satisfying (2.1.14). We extend these functions to $T^*\mathcal{O}$ by making use of an appropriate cut-off function, in such a way that (2.1.14) remains true in an open conic neighborhood of $\Gamma\cap T^*(\mathcal{O})$. This last conic set can be taken such that it contains supp b (by taking $\varepsilon>0$ small enough in the construction of b). With this last restriction we get pseudodifferential operators Q and $D_{x_n}-\lambda(x,D')$ satisfying (2.1.5).

The statement (2.1.6) follows by using a litlle of Functional Analysis and approximation arguments (see Taylor [41], Chapter IV).

Finally (2.1.7) is implied by W.K.B. method, that is, we can determine a distribution w given by an oscilatory integral $\int e^{i\varphi(x,\theta)}\, a(x,\theta)d\theta$, such that $(D_{x_n}-\lambda(x,D_{x'}))w\in C^\infty$ and WF(w) is equal, near Γ, to the positive cone generated by Γ. The phase function φ has to verify the eikonal equation $D_{x_n}\varphi-\lambda_1(x,(\nabla\varphi)')=0$ and the $a_{\ell-j}$ in $a\sim a_\ell+a_{\ell-1}+\ldots$ are determined by the so-called transport equations (see references given in Chapter I).

§2.2 Pseudodifferential operators admitting radial points

In this section we consider an operator P satisfying the conditions of §2.1 with the difference that we allow now $\nabla_\xi p$ to be the null vector. More precisely, we assume that P admits isolated *radial points* in the following sense:

Definition 2.2.1. A point $(x_o,\xi^o)\in T^*(x)\setminus\{0\}$ *is said to be a radial point of the operator* P *if at* (x_o,ξ^o):

(2.2.1) H_p *is proportional to the radial vector* $\xi.\partial_\xi$.

Observe that *Theorem 2.1.1* is actually microlocal theorem, as can be readily seen by going over the proof. At a radial point (x_o,ξ^o) the bicharacteristic strip of P is the ray

$$\Sigma=\{(x_o,t\xi^o)\ |\ t>0\},$$

which is contained in WF(u) if $(x_o,\xi^o)\in(WF(u)\setminus WF(Pu))$, because wave front sets are conic subsets of $T^*X\setminus\{0\}$. Therefore, at a radial point the statement of *Theorem 2.1.1* is also true.

V. Guillemin and D. Schaeffer, [13], found, generically, microlocal models for this situation (the analogous of D_{x_n} in the context of Theorem 2.1.1). Furthermore they initiated the study of how singularities propagate at these radial points. More specifically, they found out how singularities do propagate at radial points in 2-dimensions and detected all the cases in which singularities do not propagate in n-dimensions. The latter means that there exists a distribution $u\in\mathcal{D}'(X)$ such that $Pu\in C^\infty$ and $WF(u)=\Sigma$. Finally Santos Filho, [36], completed their study.

We shall state the results and make some comments in the 2-dimensional case. We refer the reader to [13] and [36] for the general case.

Proposition 2.2.1. Assume that P *satisfies the hypotheses above with* dim X=2 *and let* (x_o,y_o,ξ^o,η^o) *be a radial point; then under certain generic conditions,* P *is microlocally equivalent to*

(2.2.2) $$L_b^\alpha=\frac{\partial^2}{\partial x^2}+(y+bx^2)\frac{\partial^2}{\partial y^2}+\alpha\frac{\partial}{\partial y}$$

in a conic neighborhood of the ray passing through $(x_o, y_o, \xi^o, \eta^o)$, *for some* $b \in \mathbb{R}$ *and* $\alpha \in \mathbb{C}$.

Remark 2.2.3. The generic conditions imposed on P are invariant by canonical change of variables. In terms of the model L_b^α it simply means that (b,α) is restricted to the set $\Lambda=\{(b,\alpha)\in\mathbb{R}\times\mathbb{C};\ b<\frac{1}{16}, m\mathrm{Re}(\frac{\sqrt{1-16b}}{2})-n\neq 0,$ $\alpha-k(\frac{1}{2}+\frac{\sqrt{1-16b}}{2})-j\neq 0,\ m,n,k,j\in\mathbb{Z},\ n\neq 0,\ k\geq 0\}\cup\{(b,\alpha)\in\mathbb{R}\times\mathbb{C}\ |\ b>\frac{1}{16}\}$.

Consider $u\in\mathcal{D}'(\mathbb{R}^2)$ such that $L_b^\alpha(u)\in C^\infty$ and $(0,0;0,1)\in WF(u)$. Near the singular ray $\Sigma_o=\{(0,0;0,t);t>0\}$ the bicharacteristic flow of L_b^α is singular. We project these bicharacteristics into the plane $\{\eta=1\}$ by the radial map

$$(x,y,\xi,\eta)\longmapsto(x,y,\xi/\eta,1).$$

Along the projection of the null bicharacteristic strip we have

$$y=-\left[\left(\frac{\xi}{\eta}\right)^2+bx^2\right].$$

Therefore we may ignore the y variable. In this way we obtain linear system that governs the projected bicharacteristic strip

$$\begin{bmatrix}\dot{x}\\ \dot{\xi}\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2b & 1\end{bmatrix}\begin{bmatrix}x\\ \xi\end{bmatrix}. \tag{2.2.4}$$

The eigenvalues of this matrix system,

$$\lambda_\pm=\{1\pm\sqrt{1-16b}\}/2,$$

are invariant under the possible changes of variables. This explains in part why parameters are needed in the canonical form (2.2.2).

The qualitative behavior of the solutions of (2.2.4) depends, of course, on the value of b. There are three significant ranges for b. If $b>\frac{1}{16}$, then $\lambda_{\pm}$ are complex conjugates with positive real part; so the projected bicharacteristics are spirals tending to the origin as $t\to-\infty$ and to infinity as $t\to\infty$. If $0<b<\frac{1}{16}$, both eigenvalues are real and positive; in this case also all solutions of (2.2.4) tend to zero as $t\to-\infty$ and to infinity as $t\to+\infty$, but without the spiralling of the previous case. Finally if $b<0$, the eigenvalues are real and of opposite sign, so the projected bicharacteristics have saddle point behavior near the origin. If $b<0$ there are only four projected bicharacteristics that tend asymptotically to Σ, as $t\to\pm\infty$; we call them Σ_{inc} and Σ_{out} according to whether $t\to\infty$ or $t\to-\infty$. We can prove then:

Theorem 2.2.2. Let (b,α) *be a point in* Λ. *Then according to whether* $b>0$ *or* $b<0$

(2.2.5) *there exists* $u\in\mathcal{D}'(\mathbb{R}^2)$ *such that* $L_b^{\alpha}u\in C^{\infty}$ *with*

$$WF(u)=\{(0,0;0,t);t>0\}$$

or

(2.2.6) *for each* $u\in\mathcal{D}'(\mathbb{R}^2)$ *such that*

$$(0,0;0,1)\in(WF(u)\backslash WF(L_b^{\alpha}u)),$$

$WF(u)$ *intersects* Σ_{inc} *or* Σ_{out}.

§2.3 *Hyperbolic pseudodifferential operators with multiple characteristics*

We shall now drop the assumption that the pseudodifferen tial operator P on X, of order m, is of principal type and suppose that its principal symbol p, still real-valued, satisfies the constant multiplicity condition:

$$p=q_1^{r_1}\cdots q_s^{r_s}\ ,\ r_j \text{ positive integers }, \tag{2.3.1}$$

where each q_j , $j=1,\dots,s$ is a symbol of principal type and the characteristic sets $q_j^{-1}(0)$ are disjoint in $T^*(X)\setminus\{0\}$.

Definition 2.3.1 The operator P *is said to verify the Lévi condition* $L_{(x_o,\xi^o)}$ *at the point* $(x_o,\xi^o)\in p^{-1}(0)\subset T^*(X)\setminus\{0\}$, *if for every phase function* $\varphi(x)$, *solution of the equation*

$$q_j(x,\operatorname{grad}\varphi(x))=0 \qquad (\textit{if } j \textit{ is such that } q_j(x_o,\xi^o)=0)$$

in a neighborhood of x_o *with* $\operatorname{grad}\varphi(x_o)=\xi^o$ *and for all amplitude* $a\in C_o^\infty(X)$, *supported in a neighborhood of* x_o *where* $\operatorname{grad}\varphi\neq 0$, *we have*

$$e^{-it\varphi}P(e^{it\varphi}a)=0(t^{m-r_j}) \quad \textit{as} \quad t\to+\infty.$$

The operator P *verifies the Lévi condition* (L) *if* $L_{(x_o,\xi^o)}$ *is satisfied at every point in* $p^{-1}(0)$.

The Lévi condition is necessary and sufficient for the C^∞ Cauchy problem to be well posed for P. It is a condition on the terms of degree $m-(\bar{r}-1)$, where $\bar{r}=\max r_j$; in particular it is always satisfied if P is of principal type. It implies that the transport equations are ordinary differential equations along the bicharacteristic, whose order is precisely the multiplicity of the characteristic which contains it. Chazarain, [7], proved the following result which is an extension of Theorem 2.1.1:

Theorem 2.3.1. Let p *satisfies* (2.3.1) *and* (L). *If* $u \in \mathcal{D}'(X)$ *it, follows that* $WF(u)\backslash WF(Pu)$ *is a subset of* $p^{-1}(0)$, *which is invariant under the bicharacteristic flow (it is understood that on* $q_j^{-1}(0)$ *one considers the flow defined by the Hamiltonian* H_{q_j} *of* q_j).

The proof is analogous to that of Theorem 2.1.1; microlocally P behaves as having principal symbol q^r, q real of principal type. There are however two difficulties. First, one has to show that the Lēvi condition is invariant by canonical transformation. Second if one tries to reduce the operator to a simple model, like D_n^r, as might be expected, one finds an obstruction which is only circumvented by a reduction to a system of the type $D_n I$, I the identity matrix.

We turn now our attention to the case where the pseudo-differential operator P is hyperbolic with double (variable) characteristic, that is:

(2.3.2) $$P=P_1P_2+Q \ , \ P_i \in L_c^{m_i}(X) \ , \ Q \in L^{m_1+m_2-1}(X) \ ,$$

the principal symbol p_i of P_i being a real-valued, principal type, homogeneous functions of degree m_i on $T^*(X)$, i=1,2. We denote by $\Sigma=\{(x,\xi)\in T^*(X): p_1(x,\xi)=p_2(x,\xi)=0\}$ and assume that P satisfies:

(2.3.3) Σ *is involutive*, i.e, $\{p_1,p_2\}=0$ *on* Σ.

(2.3.4) *The subprincipal symbol of* P *vanishes on* Σ.

(2.3.5) *The Hamiltonian vector fields* H_{p_i} *and the cone axis* $\xi.\partial_\xi$

are linearly independent on Σ , i=1,2.

Assumption (2.3.3) says that the vector fields H_{p_1} and H_{p_2} are tangent to Σ and, because of (2.3.5), they form an involutive distribution on Σ.

Proposition 2.3.1. Let $P \in L^m(X)$ *be of the form* (2.3.2) *and satisfy* (2.3.3). *Then* P *satisfies* (2.3.4) *if and only if the principal symbol* q *of* Q *vanishes on* Σ.

Proof Let (x,ξ) denote the standard coordinates in $T^*(X)$, obtained from local coordinates $x_1,\ldots,x_n$, in X, by taking $dx_1,\ldots,dx_n$ as basis for the cotangent vectors. The subprincipal symbol of P (see (1.3.10)), which we will denote by C_p is given by

$$C_p(x,\xi)=p_{m-1}(x,\xi)-\frac{1}{2i}\sum_{j=1}^{n}\frac{\partial^2 p(x,\xi)}{\partial x_j \partial \xi_j}. \tag{2.3.6}$$

The symbolic calculus of pseudodifferential operators gives:

$$p_{m-1}=q+\frac{1}{i}\sum_{j=1}^{n}\frac{\partial p_1}{\partial \xi_j}\frac{\partial p_2}{\partial x_j}$$

and hence,

$$C_p=q+\frac{1}{2i}\{p_1,p_2\},$$

from which Proposition 2.3.1 follows.

Remark 2.3.1. By Theorem 3.6.3 of [10] and condition (2.3.3) we know that for all $(x_o,\xi^o)\in\Sigma$, there exists a C^∞ solution of the non-linear first order partial differential equations

$$(2.3.7)\qquad p_1(x,\operatorname{grad}\varphi(x))=p_2(x,\operatorname{grad}\varphi(x))=0\ ,\ \operatorname{grad}\varphi(x_o)=\xi^o,$$

in a neighborhood of x_o.

The operador P in (2.3.2) then satisfies the Lévi condition at $(x_o,\xi^o)\in\Sigma$, if for such a φ and for all $a\in C_o^\infty(X)$ supported near x_o, we have:

$$(2.3.8)\qquad e^{-it\varphi}P(e^{it\varphi}a)=0(t^{m_1+m_2-2})\quad \text{as}\quad t\to+\infty.$$

Note that outside Σ, P is of principal type which implies that the Lévi condition is always verified there.

Proposition 2.3.2. Let $P\in L^m(X)$ *be of the form* (2.3.2) *and satisfy* (2.3.3). *Then* P *satisfies the Lévi condition for every* $(x,\xi)\in\Sigma$ *if and only if the subprincipal symbol of* P *vanishes on* Σ.

Proof We have (in local coordinates) for $\varphi\in C^\infty(X)$ and $a\in C_o^\infty(X)$, supported in points x where $\operatorname{grad}\varphi(x)\neq 0$,

$$(2.3.9)\qquad e^{-it\varphi}P(e^{it\varphi}a)(x)=t^{m_1+m_2}p(x,\operatorname{grad}\varphi(x))a(x)$$
$$+t^{m_1+m_2-1}(p_{m-1}(x,\operatorname{grad}\varphi(x))a(x))$$
$$+t^{m_1+m_2-1}\Big[\frac{1}{i}\sum_{j=1}^{n}\frac{\partial}{\partial\xi_j}(x,\operatorname{grad}\varphi(x))\frac{\partial a}{\partial x_j}(x)$$
$$+\frac{1}{2i}\sum_{j,k=1}^{n}\frac{\partial^2 p}{\partial\xi_j\partial\xi_k}\ \frac{\partial^2\varphi(x)}{\partial x_j\partial x_k}a(x)\Big]$$
$$+0(t^{m_1+m_2-2}).$$

Let $(x_o,\xi^o)\in\Sigma$ and $\varphi\in C^\infty(X)$ satisfy (2.3.7). Then $p(x,\operatorname{grad}\varphi(x))=0$, near x_o, and

(2.3.10) $\qquad \dfrac{\partial p}{\partial \xi_i}(x,\text{grad}\,\varphi(x))=0$ near x_o , $i=1,\ldots,n$.

Differentiating (2.3.10), we get

(2.3.11) $$\frac{\partial^2 p}{\partial x_i \partial \xi_i}(x,\text{grad}\,\varphi(x))+\sum_{k=1}^{n} \frac{\partial^2 p}{\partial \xi_i \partial \xi_k}\;\frac{\partial^2 \varphi(x)}{\partial x_i \partial x_k}=0,$$

near x_o.

We get from (2.3.9) and (2.3.11) for $a \in C_o^\infty(X)$, supported near x_o:

(2.3.12) $$e^{-it\varphi}P(e^{it\varphi}a)(x)=t^{m_1+m_2-1}\Big[p_{m-1}(x,\text{grad}\,\varphi(x))a(x) - \frac{1}{2i}\sum_{j=1}^{n}\frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\text{grad}\,\varphi(x))a(x)\Big]+O(t^{m_1+m_2-1}),$$

which implies Proposition 2.3.2.

For operators of the form (2.3.2) satisfying the involutive condition (2.3.3), Ivrii and Petkov, [26], have shown that (2.3.4) is a necessary condition for the Cauchy problem to be well posed.

Let Γ_j (resp. Γ_j^+), $j=1,2$, denote the bicharacteristic (resp. forward bicharacteristic) of P_j through a point $(x_o,\xi^o)\in\Sigma$.

G. Uhlmann, [46], constructed a microlocal parametrix for operators verifying (2.3.2) to (2.3.5), from which one can derive (see [31]) the following result on the propagation of singularities:

Theorem 2.3.2. If $u \in \mathcal{D}'(X)$ *is such that* $(x_o,\xi^o)\notin WF(Pu)$ *and* $\Gamma_j^+ \cap WF(u) \cap V$ *is empty,* $j=1,2$, *for any conic neighborhood* V *of* (x_o,ξ^o), *then* $(x_o,\xi^o)\notin WF(u)$.

If we consider now P, still of the form (2.3.2), with principal symbol $p=p_1^k p_2^\ell$, k, ℓ positive integers and p_1, p_2 satisfying the same conditions (2.3.3) to (2.3.5), it can be shown (see [47]) that the "geometry associated to P" is the same as in the case $k=\ell=1$. For this reason (as in *Theorem 2.3.1* for constant multiplicity), the propagation of singularities is the same as in *Theorem 2.3.2* which corresponds to $k=\ell=1$.

We finish this section by mentioning that similar results of propagation of singularities have been obtained by N. Hanges, [14], for operators (2.3.2) with non-involutive double characteristics. More precisely, we assume now that instead of (2.3.3) we have

(2.3.13) Σ *is non-involutive,* i.e $\{p_1,p_2\}\neq 0$ *on* Σ.

It follows from (2.3.13) that (2.3.5) also holds. We suppose that C_P, the subprincipal symbol of P, avoids certain discrete sets of values, that is:

$$(2.3.14)_j \qquad \frac{(-1)^{j+1} i C_P(x_o,\xi^o)}{\{p_1,p_2\}\ (x_o,\xi^o)} - \frac{1}{2} \neq 0,1,2,\ldots, j=1,2.$$

Theorem 2.3.3 (Hanges). Let P *satisfy* (2.3.2), (2.3.13), $(2.3.14)_j$ *and let* $u \in \mathcal{D}'(X)$ *with* $(x_o,\xi^o) \notin WF(Pu)$. *If* $\Gamma_j \cap WF(u) \cap V$ *is empty,* $j=1,2$, *for any conic neighborhood* V *of* (x_o,ξ^o), *then* $(x_o,\xi^o) \notin WF(u)$.

A microlocal model for the operator P in Theorem 2.3.3 is given by $L=t\partial_t - B(x,D_x)$, where $B \in L^o_c(\mathbb{R}^N)$ is properly supported, $N=n-1$. Hanges, [14], proved Theorem 2.3.3 by constructing explicit microlocal parametrices for L. He obtained also results without assuming $(2.3.14)_j$ and, in fact, without any condition on the lower order terms.

§2.4 *Pseudodifferential operators with complex principal symbols*

In this section we shall study properly supported pseudodifferential operators $P \in L^m(X)$, where X is an n-dimensional C^∞ manifold, assuming that P has homogeneous complex-valued principal symbol p.

First, we shall require that

(2.4.1) $\{p,\overline{p}\}=0$ *when* $p=0$

(2.4.2) H_{Rep} , H_{Imp} *and the cone axis* $\xi.\partial_\xi$ *are linearly independent when* $p=0$

Condition (2.4.1) is necessary (see [33]) in order to have even local existence of solutions for both P and ${}^tP$. In the case that P is a differential operator, (2.4.1) and (2.4.2) together imply the condition ($\mathcal{P}$) which is known to be necessary and sufficient for the local solvability of P. Because of (2.4.2), also d Rep and d Imp are linearly independent at p=0, so the characteristic manifold of P, char (P), is a conic C^∞ submanifold of $T^*(X)\setminus\{0\}$ of codimension two. From (2.4.1) it follows that H_{Rep} and H_{Imp} are tangent to char (P) and that $\{Rep,Imp\}=a\ Rep+b\ Imp$ for some smooth a and b. In view of the Jacobi identity (1.2.8) this implies that

(2.4.3) $[H_{Rep},H_{Imp}]=a\ H_{Rep}+b\ H_{Imp}$, *when* $p=0$.

By the Frobenius theorem we then have, through every characteristic point, a two dimensional manifold contained in the characteristics and tangent to H_{Rep} and H_{Imp}. Therefore these two vector fields define a two dimensional foliation of char (P). In analogy with the real case the leafs of this foliation are called the *bi-characteristic leafs* of the operator P. Duistermaat and Hörmander, [9], then proved the following result which is the analogue of Theorem 2.1.1:

Theorem 2.4.1. If $u\in\mathcal{D}'(X)$, *then* $WF(u)\setminus WF(Pu)$ *is invariant under the bicharacteristic foliation in* char $(P)\setminus WF(Pu)$. *Furthemore, if* $(x_o,\xi^o)\in$ char (P) *there is* $w\in\mathcal{D}'(X)$ *such that* $WF(w)\setminus WF(Pw)$ *is equal, near* (x_o,ξ^o), *to the (two dimensional) bicharacteristic leaf* of P *passing through* (x_o,ξ^o).

The proof of Theorem 2.4.1 can be made parallel to that of Theorem 2.1.1 (see Remark 2.1.1). One can show that in the present situation our operator P is microlocally equivalent to the Cauchy-Riemann operator $\partial/\partial\bar{z}$ in $\mathbb{R}^n$, $z=x_{n-1}+ix_n$. For this, one looks for the existence of a homogeneous canonical transformation $(y,\eta)=\chi(x,\xi)$, from a conic neighborhood U of (x_o,ξ^o) to a conic neighborhood of

$$(0,\eta_o)\in T^*(R^n)\setminus\{0\},$$

such that $\mathrm{Re}p=\eta_{n-1}$ and $\mathrm{Im}p=\eta_n$. However, this requires (Proposition 6.1.3, [9]) that:

(2.4.4) $\{p,\bar{p}\}=2i\{\mathrm{Re}p,\mathrm{Im}p\}=0$ on a conic neighborhood of (x_o,ξ^o).

The necessity of (2.4.4) is obvious: we have $\{\eta_{n-1},\eta_n\}=0$ and be̲cause Poisson brackets are preserved by canonical transformations, we must have $\{\mathrm{Re}p,\mathrm{Im}p\}=0$.

The following lemma shows that (2.4.4) can be obtained if we multiply by a suitable elliptic factor:

Lemma 2.4.1. If $(x_o,\xi^o)\in$ char (P), *one can find a homogeneous* C^∞ *function* a *of degree* 1-m *with* $a(x_o,\xi^o)\neq 0$ *such that* $\{q,\bar{q}\}=0$ *in a conic neighborhood of* (x_o,ξ^o) *if* q=ap.

One proves Theorem 2.4.1 directly for $\partial/\partial\bar{z}$ and then passes to P by means of Fourier integral operators corresponding to χ.

From now on we assume that $X\subset\mathbb{R}^n$ is an open set and that P is a differential operator of principal type, of order m, with analytic coefficients in X.

Consider the Lie algebra, generated by H_{Rep} and H_{Imp} (for the commutation bracket $[H_{\mathrm{Rep}}, H_{\mathrm{Imp}}]$). It is known then (see Nagano [32]) that each point of $T^*(X)\backslash\{0\}$ belongs to one, and only one, subset $\mathcal{M}$ of $T^*(X)\backslash\{0\}$ having the following properties:

(2.4.5) $\mathcal{M}$ *is a connected analytic submanifold of* $T^*(X)\backslash\{0\}$;

(2.4.6) *The tangent space to* $\mathcal{M}$ *at anyone of its points, is exactly equal to the* "freezing" *of* $\mathcal{A}$ *at that point.*

(2.4.7) $\mathcal{M}$ *is maximal for properties* (2.4.5) *and* (2.4.6).

We shall refer to the submanifolds $\mathcal{M}$ as the leaves defined by P. Differently from the case of an integrable algebra, covered by the Theorem of Frobenius, the dimensions of the leaves are not necessarily the same and, in fact, they may vary from one to n. When n=dim X=2, Godin, [12], studied the propagation of singularities of solutions of Pu=f, $f\in\mathcal{D}'(X)$, assuming that P verifies the local solvability condition:

for all $(x_o,\xi^o)\in T^*(X)\setminus\{0\}$, *there exists a conic neighborhood* Γ *of* (x_o,ξ^o) *such that for all complex number* z *which satisfies* $d_\xi \mathrm{Re}zp\neq 0$ *in* Γ, *the function* Imzp *does not change sign when restricted to the bicharacteristic strips of* Rezp *in* Γ.

We observe that because p is analytic and dim X=2, it was not necessary to require that char (P) satisfy "good" symplectic properties, namely (2.4.2). We shall describe Godin's result : let $F(x_o,\xi^o)$ be the leaf of the Nagano foliation through a point $(x_o,\xi^o)\in$ char (P), let $E'(x_o,\xi^o)$ be the cone generated by the connected component of $F(x_o,\xi^o)\cap$ char (P) which contains (x_o,ξ^o) and let $E(x_o,\xi^o)$ be the connected component of $E'(x_o,\xi^o)\cap V$ passing through (x_o,ξ^o), where V is a conic neighborhood of that point. In other words,

$$E(x_o,\xi^o)=\bigcup_{\substack{z\in\mathbb{C}\\ d_\xi(\mathrm{Re}zp)(x_o,\xi^o)\neq 0}} \Gamma_z^o \tag{2.4.8}$$

where Γ_z^o is the greatest closed subinterval containing (x_o,ξ^o) and on which p vanishes, of the bicharacteristic strip Γ_z of Rezp through (x_o,ξ^o).

Theorem 2.4.2. If P *satisfies* $(\mathcal{P})$, *if* $u\in\mathcal{D}'(X)$ *is such that* $WF(Pu)\cap V$ *is empty, then* $(x_o,\xi^o)\in WF(u)$ *implies that* $E(x_o,\xi^o)$ *is contained in* WF(u).

Observe that the propagator $E(x_o,\xi^o)$ is invariant by multiplication by elliptic symbols. We remark that if $(\mathcal{P})$ is not satisfied there exists a point where Theorem 2.4.2 is not true (see Proposition 3.3.5, [21]). One may reduce the proof of Theorem 2.4.2 to the case of the following operator (microlocal model for P):

(2.4.9) $$L=D_t-ix^k g(x,t)D_x+c(x,t)+T \ , \ T\in L^{-1}$$

where k>0 is an integer and $g\not\equiv 0$ is a real analytic function defined in a neighborhood $\Omega=\{(x,t):|x|<M,|t|<M\}$, of the origin in $\mathbb{R}^2$, which verifies, besides ($\mathcal{P}$), the condition

(Q) *for all* $x_o\in(-M,M)$, *the function* $t\longmapsto g(x_o,t)$ *does not vanish identically in any open subinterval contained in* $\{t,|t|<T\}$.

(We remind that ($\mathcal{P}$) and (Q) conveniently formulated, are both necessary and sufficient for the hypoellipticity of a differential operator of principal type with C^∞ coefficients, of any order). Observe that when $g\equiv 0$, Theorem 2.1.1 applies and when k=0, L is *subelliptic* which makes the question of propagation trivial. Furthermore L is subelliptic if $x\neq 0$ and $E(0,t_o,\xi^o,0)$ is the manifold $\tau=x=0$, sgn ξ=sgn ξ^o. To prove Theorem 2.4.2 for L one introduces an extra variable, following a method of Helffer, [16], and then shows an inequality for the new operator, following a method of Sjöstrand, [39].

We return now to the case where dim X=n is arbitrary, P still being an analytic differential operator (or even a classical analytic pseudodifferential operator, [4]) of principal type of order m. Hanges, [15], constructed distributions u such that, Pu is analytic and u has prescribed wave front set (see [2] for related results). More precisely, let $(x_o,\xi^o)\in$ char (P), let $F_{(x_o,\xi^o)}$ be the leaf through (x_o,ξ^o) of the Nagano foliation determined by H_{Rep} and H_{Imp} and let $E_{(x_o,\xi^o)}$ be the cone generated by $F_{(x_o,\xi^o)}$. Since P is of principal type, dim $E(x_o,\xi^o)=k+1$, $k\geq 1$. The question is: when can we find $u\in\mathcal{D}'(X)$ so that Pu is analytic and $WF(u)=E(x_o,\xi^o)$? An obvious necessary condition is

$$(2.4.10)\qquad E(x_o,\xi^o)\subset \text{char }(P).$$

Actually this is the only assumption one needs:

Theorem 2.4.3 (Hanges). Let P *satisfy* (2.4.10). *Then there exists an open conic neighborhood* V *of* (x_o,ξ^o) *and* $u\in\mathcal{D}'(X)$ *such that*

$$WF_a(Pu)\cap V=\emptyset \text{ and } WF(u)\cap V=E(x_o,\xi^o)\cap V=WF_a(u)\cap V.$$

(Here WF_a denotes the *analytic wave front set* to be defined in the next section).

Hanges also conjectured that if u is a distribution such that Pu is analytic, then either $E(x_o,\xi^o)$ is contained in $WF_a(u)$ or is disjoint from $WF_a(u)$. Later he showed that the conjecture is correct when k=1, i.e. the case when P admits a real bicharacteristics through (x_o,ξ^o), an assumption weaker than that p should be real valued which corresponds to Theorem 2.1.1. It is not difficult to see that (2.4.10) implies that $E(x_o,\xi^o)$ is *isotropic* (i.e $E(x_o,\xi^o)\subset E(x_o,\xi^o)^{\perp\omega}$) for the simplectic 2-form ω and hence that dim $E(x_o,\xi^o)=k+1\leq n$. Hange's conjecture has been completely proved by Hanges-Sjöstrand, [40], under the assumption that the projec tion $E(x_o,\xi^o) \longrightarrow X$ has injective differential.

§2.5 *Some open problems*

Let, as before $P \in L^m(X)$, with principal symbol p, X being an n-dimensional C^∞ manifold. We have been interested in the following question:

> *Given* $(x_o,\xi^o) \in T^*X\backslash\{0\}$ *find the biggest connected subset* $C_{(x_o,\xi^o)} \subset T^*X\backslash\{0\}$ *passing through* (x_o,ξ^o), *such that* $C_{(x_o,\xi^o)} \subset (WF(u)\backslash WF(Pu))$ *for all* $u \in \mathcal{D}'(X)$ *with*
>
> $$(x_o,\xi^o) \in (WF(u)\backslash WF(Pu)).$$

We shall call the set $C_{(x_o,\xi^o)}$ above the *propagator* of P passing through (x_o,ξ^o).

As indicated in the constant coefficient case, mentioned in the Introduction, the knowledge of the propagators of P is related to the solvability of the equation Pu=f (see, for instance [9] and [22] where one solves such equations semi-globally, that is, one finds a solution modulo C^∞, in relatively compact open subsets of X). Nirenberg and Treves, [33,34], and later Beals and Fefferman, [3], and Hörmander, [24], initiated and gave a complete answer to the question of finding necessary and sufficient condition for the local solvability of differential operators of principal type. More precisely, they showed that $(\mathcal{P})$ is such a condition. It still remains as a difficult problem to prove similar results for pseudodifferential operators. It was conjectured by Nirenberg and Treves, [33], that the right condition should be

(ψ) *there is no positively homogeneous complex valued function* q *in* $C^{\infty}(T^*X\setminus\{0\})$ *such that* Imqp *changes sign from - to + when one moves in the positive direction on a bicharacteristic strip of* Reqp *on which* $q\neq 0$.

Only recently it was shown that this condition is necessary (see Hörmander, [24]). Nevertheless, the sufficiency of condition (ψ) has no yet been established and, in fact,very litlle is known about it. For this reason, the question of propagation of singularities turns out being important, even from the point of view of local solvability. Besides the work of Godin, mentioned in §2.4, Hörmander,[24], and Dencker, [8], have made some advance in the problem. It is therefore, natural to propose the following problem:

Q1. *Let* P *be a classical pseudodifferential operator in* $\Omega\subset\mathbb{R}^n$, $n\geq 2$, *of principal type, satisfying the condition* (ψ). *Is it possible to obtain a theorem of propagation of singularities for* P?

Remark 2.5.1. If the condition (P) is not satisfied, there are simple examples which show that $C_{(x_o,\xi^o)}$ is $\{(x_o,t\xi_o);t>0\}$ (see, for instance Hörmander, [23]).

In the case of radial points, treated in §2.2, we can pose similar questions when the coefficients are analytic. First we recall that in the analytic context, we have also a notion of analytic wave front set, WF_a, which is motivated by the following: given $u\in\mathcal{D}'(\mathbb{R}^n)$, $x_o\in \text{sing supp}_a(u)$ *if and only if there is a sequence* $\{\phi_N\}\subset C_o^{\infty}(\mathbb{R}^n)$, $\phi_N\equiv 1$ *in a fixed neighborhood of the origin such that*

$$|(\phi_N u)^{\wedge}(\xi)| \leq C^N(1+|\xi|/N)^{-N}.$$

Here sing $\text{supp}_a u$ denotes the analytic singular support of u. By analogy with the C^∞ wave front set, we then define the analytic wave front set as follows:

Definition 2.5.1. Let $u \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n$ *an open subset and*

$$(x_o, \xi^o) \in T^*\Omega \setminus \{0\};$$

we say that $(x_o, \xi^o) \notin WF_a(u)$ *if and only if there exists a neighborhood* U *of* x_o *and a sequence* $u_N \in \mathcal{E}'(\Omega)$ *such that* $u_N = u$ *in* U *and*

$$|\hat{u}_N(\xi)| \leq C^N(1+|\xi|/N)^{-N}$$

when ξ *belongs to some fixed conic open neighborhood of* ξ_o.
Andersson, [1], proved the result analogous to Theorem 2.1.1 when $P(x,D)$ is a differential operator of real principal type with analytic coefficients and WF replaced by WF_a; in other words, the conclusion is that $WF_a(u)$ is invariant under the bicharacteristic foliation.

Q2. *Let* $P(x,D)$ *be a partial differential operator with analytic coefficients and let* $(x_o, \xi^o) \in T^*X \setminus \{0\}$ *be an isolated radial point for* P. *The conclusions obtained in §2.2 still hold?*

Actually one should start by searching the corresponding models in the analytic context.

Theorem 2.3.2 of §2.3 proves, in particular, that for some

$j\in\{1,2\}$, $\Gamma_j=\Gamma_j^+\cup\Gamma_j^-$ is contained (in a neighborhood of $(x_o,\xi^o)\in\Sigma$) in $WF(u)\setminus WF(Pu)$ if $(x_o,\xi^o)\in WF(u)\setminus WF(Pu)$. This leads to the following question:

Q3. *For each* $(x_o,\xi^o)\in\Sigma$ *and* $j\in\{1,2\}$ *there exists* $u_j\in\mathcal{D}'(X)$ *such that* $WF(u_j)\setminus WF(Pu_j)$ *is equal, near* (x_o,ξ^o), *to the cone generated by the integral curve of* H_{p_j} *passing through* (x_o,ξ^o)?

By looking at simple example one can show that the hypotheses (2.3.3) and (2.3.5) are not sufficient in order to obtain the conclusion of *Theorem 2.3.2*. Therefore we pose:

Q4. *Find necessary conditions, weaker than* (2.3.4), *in such a way that the conclusions of Theorem 2.3.2 remain true.*

Finally consider

Q5. *Is Hange's conjecture (see §2.5) true in its full generality?*

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