Relaying for Interference Management

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Notation

The following table summarizes the notation used in this thesis.

Lower-case bold: $\mathbf{x}$	column vector
Upper-case bold: $\mathbf{X}$	matrix
Calligraphic: $\mathcal{X}$	set
x^n	$(x(1), x(2), \cdots, x(n))$
$\text{Prob}(x)$	probability of an event x
i.i.d.	independent and identically distributed
$\mathcal{N}(0,1)$	Gaussian noise with zero mean and unit variance
Bern($1/2$)	Bernouli distributed random variable with probability $p = 1/2$
$H(X)$	discrete entropy of X
$h(X)$	differential entropy of X
$\text{Corr}(X, Y)$	correlation coefficient between X and Y
$I(X;Y)$	mutual information between X and Y
$\lfloor x \rfloor$	integer part of x
$\lceil x \rceil$	$\lfloor x \rfloor + 1$
$(x)^+$	$\max\{0, x\}$
$C(x)$	$\frac{1}{2}\log(1+x)$, $x > 0$
$C^+(x)$	$\max\{0, C(x)\}$, $x > 0$
$\mathbf{0}_q$	zero vector of length q
$\mathbf{I}_q$	$q \times q$ identity matrix
$\mathbf{X}^T$	transpose of $\mathbf{X}$

Table 1: Notation table

Chapter 1

Introduction

Communications continue to play a vital role in our daily life. This significance of communications has lead to a rapid increase in the number of communication terminals in use. Consequently, this triggered a dramatic increase in the demand on data rates. To cover this increasing demand, research groups have recently put much effort on optimizing network performance and designing new communication systems.

The design of future communication systems is focused on the joint performance enhancement of concurrently active nodes within wireless networks. Such networks suffer from several limitations such as interference and bad connectivity. These limitations are the focus of a large number of investigations by the research community, in particular in recent years. Interference can be a major setback in multi-user networks, which leads to a deterioration in the network performance measured by the achievable rates of communication. Furthermore, connectivity problems such as physical obstructions and bad channel quality can significantly decrease the coverage of a wireless network and thus, have a negative impact on the network performance.

One way to overcome these difficulties is relaying. Relay nodes can be used for managing interference in a wireless network in a controlled way so that the impact of this interference is reduced. Moreover, in wireless networks where connectivity is not always guaranteed, relays can lead to a dramatic improvement in the network performance by providing alternate connectivity paths. Thus, deploying relay nodes in a wireless network can improve the network performance. However, how can we fully exploit the relay benefits in a wireless network? And how can we quantify this performance improvement?

In order to answer these questions, we consider a wireless network consisting of two transmit-receive pairs and one dedicated relay node. Due to the broadcast nature of the wireless channel, the transmit signals interfere with each other and lead to performance deterioration in comparison to interference free communication. The relay is installed in order to enhance the performance of the network by countering this interference. The obtained setup is known as the 2-user interference relay channel (IRC). This elemental network captures the aforementioned limitations and allows us to apply our new ideas on a small scale, thus obtaining insights that can be used as stepping stones towards the understanding of larger networks.

Quantifying the performance improvement obtained by using a relay requires finding tight upper and lower bounds on the network performance. Upper bounds can be derived using information-theoretic tools. Deriving tight lower bounds requires finding an intelligent relaying strategy that can serve the goal of interference management in the IRC. This in turn requires describing the construction of this strategy and measuring its performance in terms of achievable rates. If the upper and lower bounds match, we obtain the capacity of the network. However, since capacity characterization of interference networks is known to be difficult, we resort to an approximative characterization in order to achieve progress on this front by 'climbing the ladder one step at a time'. Namely, we follow an approximative characterization which becomes asymptotically tight at high signal-to-noise ratio. This approximation is supposed to ease the way to 'climbing' the next step, i.e., finding the capacity of the network.

Since our main focus is combating interference, we study an IRC where each transmit signal is received at all receivers with a power which is much larger than the noise power. In this way, we guarantee that the IRC is in the so-called interference-limited regime, and also that the noise does not limit the relay's reception capability. We develop two new relaying strategies for the IRC based on computation at the relay. That is, the relay decodes a linear combination of the transmit signals of the two users. One of the strategies allows each destination to extract the interfering signal and remove it from its received signal, thus reducing the impact of this interference. The other strategy allows the relay to neutralize interference at the destinations on-the-fly, transforming the IRC into two interference free point-to-point channels. The two strategies have the following common property: the main task of the relay is interference cancellation. While this might sound sub-optimal at the first glance, we show that this is an optimal relaying strategy in the IRC.

The proposed strategies have some similarities with network coding. In both proposed strategies, the relay sends a network coded version of the transmit signals to the destinations. However, an important difference between the two exists. In classical network coding, the relay node would construct the network code by itself; it requires knowledge of the source messages or the transmit signals. In our strategies, the network code is constructed by the physical channel. The relay simply decodes the network code it is going to send to the destinations, which does not require knowledge of the source messages or transmit signals at the relay.

The performance of these new strategies is analyzed. It turns out that combining these new relaying strategies with classical strategies characterizes the approximate sum-capacity of the IRC for a wide range of channel parameters.

The major milestones of this characterization can be summarized as follows:

Upper Bounds: We establish new upper bounds on the sum-capacity of the IRC using both novel and classical approaches. As we show later, the given bounds are tight in the characterized regimes. All the upper bounds we provide are given in closed form and require no further optimization.

Lower Bounds: We also establish new sum-capacity lower bounds for the IRC by using novel transmission strategies. The proposed strategies outperform classical strategies that are commonly used in the IRC.

Approximate Capacity: We study the optimality of the given strategies. For a wide range of channel parameters, we show that our strategies are asymptotically optimal, i.e., they provide an asymptotically tight approximation of the sum-capacity in the limit of high signal-to-noise ratio.

Interestingly, while weak reception at the relay might suggest that the benefits of the relay are limited, the study shows that the relay can increase the capacity of the network, even if the channels from the sources to the relay are very weak.

Finally, we explore a new relaying option where the relay does not only relay signals to the destinations but also relays signals back to the sources. This can be also interpreted as relay-source feedback. We investigate the benefits of this kind of feedback, and show that it leads to an increase in the network capacity in certain cases.

In the next chapter, we revisit the history of the IRC and introduce some preliminaries that are required in the following chapters.

Chapter 2

History and Preliminaries

The foundations of information theory were laid down by Shannon [Sha48] in his seminal paper where the capacity of the point-to-point (P2P) channel was characterized. Here, the capacity is defined as the highest rate of information flow from the transmitter to the receiver measured in bits per channel use. Since then, various information theoretic models of communication networks have been studied and analyzed based on this framework (see [EGK11]). Although the capacity of some networks has been solved, the capacity of many other networks is still an open problem.

The main challenge in larger networks is interference. If several users (transmitter-receiver (Tx-Rx) pairs) want to establish communication over the same medium, their signals will inevitably interfere. Since the interfering signals are codewords, they have some structure which can be exploited in the decoding process at the receivers. Therefore, treating interfering Tx-Rx pairs as isolated P2P channels is in general not optimal. The question is then: how can interference be exploited in order to maximize the achievable rates of information flow? This lead to the study of the interference channel (IC) which is the smallest information theoretic model that captures this effect.

2.1 The Interference Channel

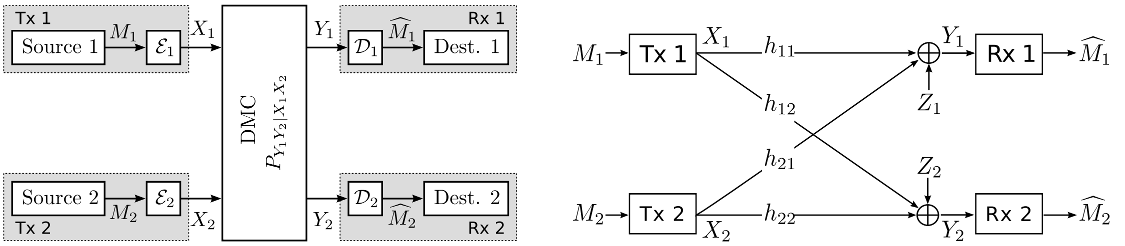
Ahlswede [Ahl71] introduced the interference channel (IC) as an information theoretic model to capture scenarios where simultaneous transmission of dedicated messages by multiple sources to their respective destination takes place on a shared channel. Such a channel is important, for instance, in cellular networks with cell edge users that suffer from interference caused by base stations in neighbouring cells. The phenomenon of interference is not limited to cellular networks and occurs in many other networks such as ad-hoc wireless networks.

The first result on the capacity of the IC appeared in 1975 [Car75] where the capacity region of the Gaussian IC with very-strong interference was characterized. Namely, for a Gaussian IC with signal power P and noise power 1, and where the channel coefficient between Tx j and Rx k is $h_{jk} \in \mathbb{R}$ (Figure 2.1(b)), if the following condition holds

$$h_{jk}^2 \geq h_{jj}^2(1 + h_{kk}^2 P), \quad j,k \in \{1,2\}, j \neq k, \tag{2.1}$$

then the IC is said to be in the very-strong interference regime where the capacity of the IC is equal to the interference free capacity $R_j \leq C(h_{jj}^2 P)$ with $C(x)$ defined as $C(x) = \frac{1}{2}\log(1+x)$. Note that a much lower rate would be achieved were the IC treated as two separate P2P channels. This is an example which shows clearly that exploiting interference can be better than ignoring it. Later on, the capacity of the IC with strong interference, where $h_{jk}^2 \geq h_{jj}$, was characterized [Sat81].

The work on the IC then slowed down for a while, until the last decade when several advances were made on the problem. These advances can be summarized as follows.



(a) A discrete memoryless IC. X_1 and X_2 are the input random variables and Y_1 and Y_2 are the output random variables. The channel has a transition probability $P_{Y_1Y_2|X_1X_2}$.

(b) A normalized Gaussian IC. The output Y_1 for instance is $Y_1 = h_{11}X_1 + h_{21}X_2 + Z_1$. The noises Z_1 and Z_2 are independent $\mathcal{N}(0,1)$ and the transmitters have a power constraint P.

Figure 2.1: In the IC, Tx j wants to send a random message M_j to its receiver Rx j. It does so by encoding the message into a signal $(X_j(1), X_j(2), \cdots, X_j(n))$ and sending this signal over the channel to the receiver to decode it.

Genie-aided Bounds: New bounding techniques such as genie-aided bounds [Kra04, ETW08] were developed. In this approach, the channel is enhanced by giving one of the receivers (or both) some additional information, designed in such a way that facilitates the derivation of a computable upper bound.

Metrics: New metrics for the asymptotic characterization of the capacity of a network were developed. One of these metrics is the degrees-of-freedom (DoF) of the network, which characterizes the scaling of the sum-capacity of a network with respect to the capacity of the P2P channel at high power. Consider for instance a symmetric Gaussian IC (Figure 2.1(b)) where the 'direct channels' are equal $h_{11} = h_{22} = h_d$, the 'cross channels' are also equal $h_{12} = h_{21} = h_c$, the power constraint is P, and the noise power is 1. The DoF of this network (also known as multiplexing gain [HMN05]) is defined as [CJ08]

$$d = \lim_{\mathsf{SNR}\to\infty} \frac{C_\Sigma(\mathsf{SNR})}{\frac{1}{2}\log(\mathsf{SNR})}. \tag{2.2}$$

where C_Σ is the sum capacity of the network, and SNR is the signal-to-noise ratio (here $\mathsf{SNR} = h_d^2 P$). The DoF provide an approximation of the sum-capacity of the network of the form

$$C_\Sigma = \frac{d}{2}\log(\mathsf{SNR}) + o(\log(\mathsf{SNR})). \tag{2.3}$$

Another metric which provides a finer approximation of the sum-capacity is the so-called generalized DoF (GDoF) which was introduced in [ETW08]. The GDoF of the same symmetric IC introduced above is defined as

$$d(\alpha) = \lim_{\mathsf{SNR}\to\infty} \frac{C_\Sigma(\mathsf{SNR},\mathsf{INR})}{\frac{1}{2}\log(\mathsf{SNR})}, \quad \text{where } \alpha = \frac{\log(\mathsf{INR})}{\log(\mathsf{SNR})}, \tag{2.4}$$

the signal-to-noise ratio is $\mathsf{SNR} = h_d^2 P$, and the interference-to-noise ratio is $\mathsf{INR} = h_c^2 P$. This provides an approximation of the sum-capacity of the IC given by

$$C_\Sigma(\alpha) = \frac{d(\alpha)}{2}\log(\mathsf{SNR}) + o(\log(\mathsf{SNR})), \tag{2.5}$$

where the parameter α defined in (2.4) quantifies the INR relative to the SNR on a logarithmic scale. This approximation is finer than (2.3) which does not distinguish between IC's with different interference powers.

Both metrics, DoF and GDoF, are high SNR metrics. The motivation behind using these metrics is focusing on the impact of interference on a network. In general, the performance of a network in terms of achievable rates is hindered by noise and by interference. While at low SNR the network is noise-limited, it is interference-limited at high SNR. From this point of view, a high SNR analysis aims to marginalize the impact of noise in order to obtain a clear view on the impact of interference.

Interference Alignment: Another tool which has been developed recently is interference alignment [Jaf11]. In interference alignment, the transmitters send their signals in such a way that the interference signals 'cast overlapping shadows' at the receiver. That is, the interference signals are aligned in a subspace of the receiver's signal space. Thus, the receiver is able to obtain an interference free signal by simply ignoring this interference subspace. Interference alignment can be accomplished over spacial dimensions (in MIMO networks) [GJ10, JS08, MAMK08], temporal or frequency dimensions [CJ08, CJW10], or signal scale (level) dimensions [BPT10].

The development of these techniques, metrics, and tools has lead to new results in the field of network information theory. For instance, these developments lead to characterizing the DoF of the IC (along with many other networks) [Jaf11], the GDoF and the capacity of the IC within 1 bit [ETW08], the sum-capacity of the IC with noisy interference [AV09, SKC09, MK09], the sum-capacity of the Z-IC (where $h_{12} = 0$) [Sas04] and the IC with mixed interference [WT08], the capacity of classes of IC's with more than 2-users [CS11, SKC08], etc. Out of these results, the GDoF of the IC will be needed in the following chapters. This GDoF is stated in the following lemma.

Lemma 2.1 (Etkin *et al.* [ETW08])**.** *The GDoF of the IC is given by*

$$d(\alpha) = \min\left\{2\max\{1-\alpha,\alpha\}, 2\max\{1,\alpha\}-\alpha, 2\right\}. \tag{2.6}$$

2.1.1 The Interference Channel with General Cooperation

Interference channels with cooperation have also been studied recently. Two variants have been considered, IC's with non-causal cooperation or the so-called cognitive IC [RTD12, HJ09], and IC's with causal cooperation [YT11a, PV11b].

A general IC with causal cooperation is shown in Figure 2.2. In the IC with causal cooperation, Rx 1 for instance sends a feedback signal $\bar{X}_1$ which is received by the remaining nodes. This establishes causal cooperation between Rx 1 and all the remaining nodes. In this case, each node constructs its transmit signal based on all available information. For instance, at time instant i, Tx 1 sends $X_1(i)$ which is constructed from the message M_1 and from the observations of $\bar{Y}_1$ up to time instant i, i.e., $X_1(i) = \mathcal{E}_1(M_1, \bar{Y}_1^{i-1})$ where $\bar{Y}_1^{i-1} = (\bar{Y}_1(1), \cdots, \bar{Y}_1(i-1))$ and $\mathcal{E}_1$ is the encoder at Tx 1. Similarly, Rx 1 feeds back $\bar{X}_1(i) = \bar{\mathcal{E}}_1(Y_1^{i-1})$. Based on this feedback information, the transmission strategy can be improved compared to the IC without cooperation.

For the purpose of this thesis, the IC with cooperation is important since we will use it in the sequel for deriving upper bounds for the network under consideration (the interference relay channel). In addition to genie-aided bounding techniques (introduced

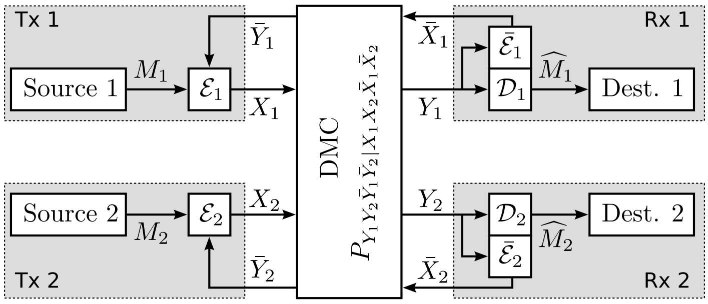


Figure 2.2: An interference channel with general cooperation.

in Section 2.1), another bounding technique that has been used to derive upper bounds for general networks is based on this IC with general cooperation as follows.

Cooperation bounds: For deriving an upper bound for the capacity of a given network, one can enhance the network by allowing some nodes to cooperate. Allowing some nodes to cooperate leads (in some cases) to an IC with general cooperation similar to the one shown in Figure 2.2. For instance, in a 3-user IC, if we let Tx 3, Rx 2, and Rx 3 cooperate to form a super node $\overline{\text{Rx 2}}$ which consists of the set of nodes {Tx 3, Rx 2, Rx 3}, then the overall cooperative network can be modeled as a 2-user IC with uni-directional cooperation from $\overline{\text{Rx 2}}$ to Rx 1 due to the physical channel between Tx 3 $\in$ {Tx 3, Rx 2, Rx 3} and Rx 1. Similar to this example, this cooperation approach has been used to develop upper bounds for different interference networks (see [CJ09] for instance). In the next chapters, we are going to use this approach to develop some upper bounds. Therefore, we introduce the following lemma.

Lemma 2.2 ([Tun12, YT11b]). *The achievable sum-rate in a memoryless IC with general cooperation is upper bounded by*

$$R_\Sigma \leq I(X_1; Y_1, \bar{Y}_2 | Y_2, X_2, \bar{X}_1, \bar{X}_2) + I(X_1, X_2, \bar{X}_1; Y_2 | \bar{X}_2), \tag{2.7}$$

$$R_\Sigma \leq I(X_2; Y_2, \bar{Y}_1 | Y_1, X_1, \bar{X}_1, \bar{X}_2) + I(X_1, X_2, \bar{X}_2; Y_1 | \bar{X}_1). \tag{2.8}$$

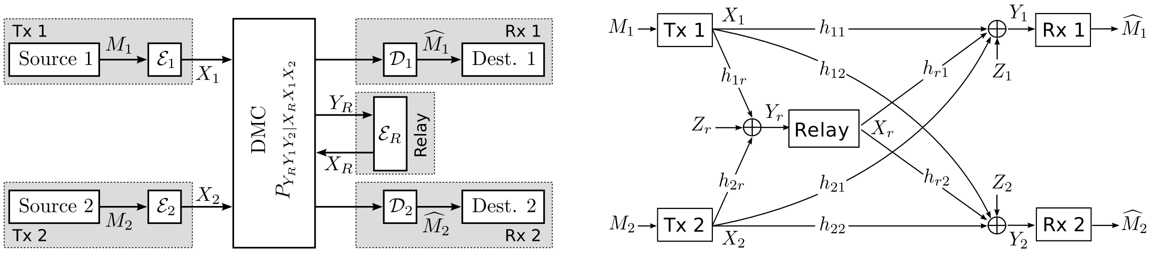
for some distribution $P_{X_1 X_2 \bar{X}_1 \bar{X}_2}$.

2.2 The Interference Relay Channel

The channel parameters in the IC depend on the environment in which the IC is deployed. In the most extreme case, the direct link between each transmitter and its respective receiver might be very weak or even absent due to low coverage or large obstructing objects. In these cases, simply increasing the power at the transmitters will not resolve the problem. A possible solution is to use dedicated relay stations to enable communication.

Relaying has been first studied in conjunction with a P2P channel [CEG79]. Although the capacity has been characterized in some cases, the capacity of the general case remains an open problem. The relay in this case operates as a helper for the transmitter to increase the capacity of the channel. Installing a relay in an IC leads to the so-called interference relay channel (IRC) shown in Figure 2.3(a).

Relaying in interference networks however has a different flavor. A relay can be used not only for increasing coverage and helping the transmitter, but also for interference



(a) A discrete memoryless interference relay channel with input variables X_1, X_2 and X_r, output variables Y_1, Y_2, and Y_r, and a transition probability $P_{Y_1 Y_2 Y_r|X_1 X_2 X_r}$.

(b) A Gaussian interference relay channel. The channel output Y_1 for instance is given by $Y_1 = h_{11}X_1 + h_{21}X_2 + h_{r1}X_r + Z_1$. The noises Z_1, Z_2, Z_r are $\mathcal{N}(0,1)$ and all nodes have power P.

Figure 2.3: The interference relay channel is an IC with a relay dedicated to support the communication between the nodes of the IC.

management. Although the optimal relaying function is in general not known, several attempts have been made to approach the capacity of the IRC. In the past work on the IRC, several relaying varieties have been studied such as decode-forward where the relay decodes the transmit signals and forwards them to the receivers, and compress-forward where the relay compresses its observation and sends the compression index to the receivers. Additionally, several capacity upper bounds have been developed. For instance, in [TY11] new capacity upper bounds were developed for the Gaussian IC with a potent relay, i.e., a relay that has no power constraint. These bounds have been compared with the achievable rates of compress-forward. The capacities of the strong interference regime and a sub-regime of the weak interference regime (of the IRC with a potent relay) were characterized. Since an IRC with a potent relay is more capable than an IRC with a power constraint at the relay, the capacity of the former is an upper bound for the capacity of the latter. Therefore, the results of [TY11] can be regarded as upper bounds for the IRC. In [MDG12], a transmit strategy for the IRC that uses block-Markov encoding at the sources and decode-forward at the relay was proposed. In the same paper, the benefits of interference forwarding were shown. In interference forwarding, the relay attempts to increase the interference at the receivers in such a way that facilitates interference decoding and then cancellation. Furthermore, the authors of [MDG12] have derived a capacity upper bound which improves upon the classical cut-set bounds [CT06], and have studied the strong interference regime of the IRC. As a result, the capacity of a strong interference regime for the discrete memoryless IRC was characterized under some degradedness conditions where decode-forward achieves capacity [MDG12]. In [SE07a], a transmit strategy for the Gaussian IRC (Figure 2.3(b)) based on decode-forward and Carleial's rate-splitting [Car78] was studied. The performance of this scheme was analyzed for the case when the source-relay links are strong, and thus, decoding at the relay does not limit the achievable rates. The IC with a cognitive relay which serves as a performance upper bound for the IRC has also been studied [SE07b, SVJS08, RTDG11, RTD11]. In [SSE11a, SES11, SSE11b, TY12], the IC with an out-of-band relay where the channels from the transmitters to the relay are orthogonal to the remaining channels was considered.

This network, the IRC, will be the main focus of this thesis.

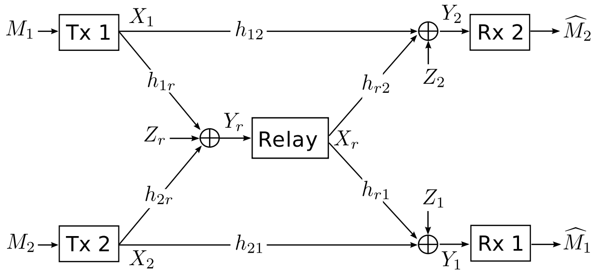


Figure 2.4: A Gaussian butterfly network. The locations of Rx 1 and 2 have been switched for clarity.

2.3 The Butterfly Network

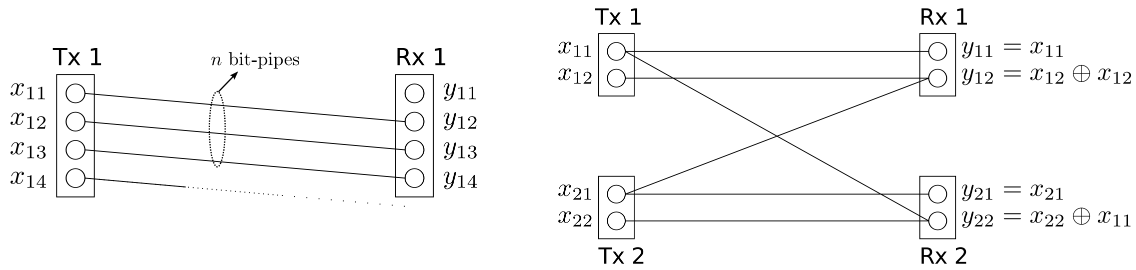
The wireless butterfly network (BFN) is a special case of the IRC which has missing direct links between each transmitter and its respective receiver. Despite being a special case of the IRC, studying the BFN is important in its own sake since the BFN magnifies the impact of relaying in comparison to the IRC. This is true since in the IRC, desired signals can reach each destination from both the respective transmitter and the relay. In the BFN however, this is not the case; desired signals reach each destination only from the relay. Consequently, the BFN provides a more distinct and visible picture of the impact of relaying.

The wireless BFN is not to be confused with the infamous classical butterfly network with multi-cast messages which was used by Ahlswede *et al.* in [ACLY00] to demonstrate the capabilities of network coding. The BFN in [ACLY00] is a wired network where signals do not interfere with each other. In our case, the BFN is a wireless one where simultaneous transmissions interfere additively at the receivers as shown in Figure 2.4.

A wireless BFN with half-duplex nodes was studied by Avestimehr *et al.* in [AH09]. The authors of [AH09] exploited network coding ideas in order to design transmission strategies that were shown to be optimal for the linear-deterministic approximation of the Gaussian BFN at high SNR (see Section 2.4 for an introduction on the deterministic approximation) and to achieve capacity to within 1.95 bits per channel use at any finite SNR. In this thesis, we give special attention to the BFN in the aim of obtaining a better understanding of the IRC.

2.4 The Deterministic Model

The deterministic channel model has been first introduced by El-Gamal and Costa in [EGC82] where the capacity of a class of deterministic IC's has be characterized. Recently, Avestimehr *et al.* [ADT11] have developed a linear-deterministic (LD) approximation of Gaussian networks which significantly simplifies the study of interference networks. The LD model is a model where transmit and received signals are binary vectors, and the impact of noise Gaussian is modeled as clipping a number (determined by the signal-to-noise ratio) of least significant bits of the received binary vector. Namely, a Gaussian P2P channel with a signal-to-noise ratio SNR is modeled by an LD-P2P channel with $n = \left\lceil \frac{1}{2}\log(\mathsf{SNR}) \right\rceil$ bit-pipes as shown in Figure 2.5(a). In this figure, the vertical bar denotes a binary vector, each circle denotes a component of this binary vector, and each line is a bit-pipe which connects an input bit with an



(a) A linear-deterministic approximation of a P2P channel with signal-to-noise ratio SNR. This example has $n = \lceil \frac{1}{2}\log(\mathsf{SNR}) \rceil = 3$ bit pipes; the last bit x_{14} is 'sunk' in noise.

(b) A symmetric linear-deterministic IC with direct channels $n_{11} = n_{22} = 2$ and cross channels $n_{12} = n_{21} = 1$. The interference is modeled by a mod2 sum of interfering bits.

Figure 2.5: The linear-deterministic model focuses on the interaction of signals, while modeling noise as a clipping of the least significant bits.

output bit. In the following chapter, we are going to call each of these circles a 'signal level'. From this figure, it can be clearly seen that the LD-P2P channel capacity is n bits per channel use which is a good high-SNR approximation of the capacity of the corresponding Gaussian P2P channel given by $\frac{1}{2}\log(1+\mathsf{SNR})$.

In multi-user networks, the LD model focuses on the interaction of different signals over a linear-deterministic channel. For instance, the Gaussian IC shown in Figure 2.1(b) is approximated by an LD-IC with input-output equations

$$\mathbf{y}_1 = \mathbf{S}^{q-n_{11}}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_{21}}\mathbf{x}_2, \tag{2.9}$$

$$\mathbf{y}_2 = \mathbf{S}^{q-n_{22}}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_{12}}\mathbf{x}_1, \tag{2.10}$$

where for $j,k \in \{1,2\}$ and $j \neq k$, $n_{jk} = \left\lceil \frac{1}{2}\log(h_{jk}^2 P) \right\rceil$, $q = \max_{jk} n_{jk}$, and $\mathbf{x}_j$ and $\mathbf{y}_j$ are the input and output binary vectors of length q, respectively. Here, the matrix $\mathbf{S}$ is a $q \times q$ down-shift matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{q-1}^T & 0 \\ \mathbf{I}_{q-1} & \mathbf{0}_{q-1} \end{bmatrix}, \tag{2.11}$$

which models the clipping of the least-significant bits of a binary vector by noise. This LD-IC is depicted graphically in Figure 2.5(b). Note that the relative strength of different channels is modeled by a different number of bit-pipes. The weaker the channel (small n_{jk}), the more the bits which will be lost through the channel. In the following chapters, figures similar to Figure 2.5 will be used to approximate different networks. The capacity of the LD-IC was given in [BT08]. In the sequel, we are going to need the sum-capacity of the symmetric LD-IC which is repeated in the following lemma.

Lemma 2.3 ([BT08]). *The sum-capacity of the LD-IC with direct channels $n_{11} = n_{22} = n_d$ and cross channels $n_{12} = n_{21} = n_c$ is given by*

$$C_\Sigma = \min\{2\max\{n_d - n_c, n_c\}, 2\max\{n_d, n_c\} - n_c, 2n_d\}. \tag{2.12}$$

and is achieved using a deterministic version of the Han-Kobayashi scheme [HK81].

Since we study LD models in the following chapters, we are going to need the following simple lemmas. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two binary random vectors[1] from $\mathbb{F}_2^q$, for some integer q. Additionally, let $n_1, n_2 \leq q$. Then we have the following lemmata.

[1]With some abuse of notation, we will use lower-case bold letters to denote random vectors.

Lemma 2.4. *$H(\mathbf{S}^{q-n_1}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_2}\mathbf{x}_2) \leq m$, where $m = \max\{n_1, n_2\}$.*

Proof. The topmost $q-m$ bits of $\mathbf{u} = \mathbf{S}^{q-n_1}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_2}\mathbf{x}_2$ are zeros due to the down-shift. Denote the topmost $q-m$ components of $\mathbf{u}$ by $\mathbf{v}$, and the remaining m components by $\mathbf{w} = (w_1, \cdots, w_m)$. Thus

$$
\begin{aligned}
H(\mathbf{u}) &= H(\mathbf{v}, \mathbf{w}) && (2.13)\\
&\overset{(a)}{=} H(\mathbf{w}) && (2.14)\\
&\overset{(b)}{=} \sum_{i=1}^{m} H(w_i | w_1, \cdots, w_{i-1}) && (2.15)\\
&\overset{(c)}{\leq} \sum_{i=1}^{m} H(w_i) && (2.16)\\
&\overset{(d)}{\leq} m, && (2.17)
\end{aligned}
$$

where (a) follows since $\mathbf{v}$ is deterministic, (b) follows by using the chain rule, (c) follows since conditioning does not increase entropy, and (d) follows since the Bern(1/2) distribution maximizes the binary entropy [CT06]. ☐

Lemma 2.5. $H(\mathbf{S}^{q-n_1}\mathbf{x}_1 | \mathbf{S}^{q-n_2}\mathbf{x}_1) \leq (n_1 - n_2)^+$

Proof. If $n_2 \geq n_1$, then all the bits of $\mathbf{S}^{q-n_1}\mathbf{x}_1$ are included in the condition $\mathbf{S}^{q-n_2}\mathbf{x}_1$, thus the entropy is zero. Otherwise, we can write

$$
\begin{aligned}
H(\mathbf{S}^{q-n_1}\mathbf{x}_1 | \mathbf{S}^{q-n_2}\mathbf{x}_2) &= H(\mathbf{S}^{q-n_1}\mathbf{x}_1 \oplus \mathbf{U}^{n_1-n_2}\mathbf{S}^{q-n_2}\mathbf{x}_2 | \mathbf{S}^{q-n_2}\mathbf{x}_2) && (2.18)\\
&\leq H(\mathbf{S}^{q-n_1}\mathbf{x}_1 \oplus \mathbf{U}^{n_1-n_2}\mathbf{S}^{q-n_2}\mathbf{x}_2) && (2.19)
\end{aligned}
$$

where $\mathbf{U}$ is an up-shift matrix. Denote $\mathbf{S}^{q-n_1}\mathbf{x}_1 \oplus \mathbf{U}^{n_1-n_2}\mathbf{S}^{q-n_2}\mathbf{x}_2$ by $\mathbf{u}$. Then the vector $\mathbf{u} = (u_1, \cdots, u_q)$ has only $m = n_1 - n_2$ unknown components since the remaining components are all zero. The rest of the proof is the same as steps (b), (c), and (d) in the proof of Lemma 2.4. ☐

2.5 Lattice Codes

Translating results from the language of the LD model briefly explained above to the language of Gaussian channel usually requires using lattice codes. For this reason, and due to their desirable linear structure, lattice codes [Loe97] have witnessed increasing usage in wireless networks [SJV+08, BPT10]. Besides being capacity achieving in the P2P channel [EZ04], they can also be used for computation, i.e., decoding linear combinations of codewords [NG11], which makes them perfect candidates for relaying scenarios. The following lattice preliminaries will be required in the sequel. For more details on this subject, the reader is referred to [NG11].

An n-dimensional lattice Λ is a subset of $\mathbb{R}^n$ such that

$$\lambda_1, \lambda_2 \in \Lambda \Rightarrow \lambda_1 + \lambda_2 \in \Lambda, \tag{2.20}$$

i.e., it is an additive subgroup of $\mathbb{R}^n$. The fundamental Voronoi region $\mathcal{V}(\Lambda)$ of Λ is the set of all points in $\mathbb{R}^n$ whose distance to the origin is smaller that that to any other

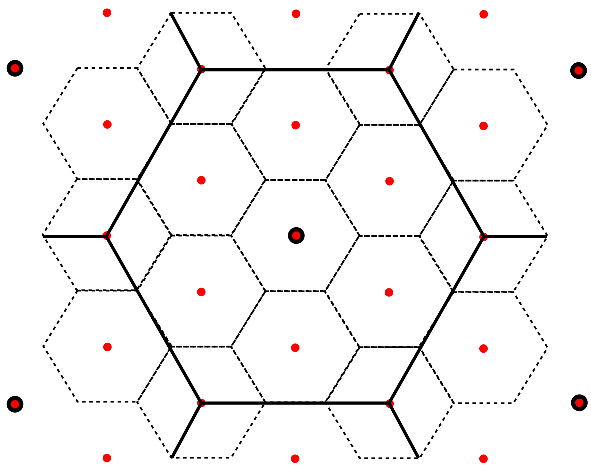


Figure 2.6: An example of a two dimensional nested-lattice consisting of a coarse lattice (bold) nested in a fine lattice (dashed). The small dots are points in $\mathbb{R}^2$ which belong to the fine lattice, while the big dots are those that belong to the coarse lattice.

$\lambda \in \Lambda$. Thus, by quantizing points in $\mathbb{R}^n$ to their closest lattice point, all points in $\mathcal{V}(\Lambda)$ are mapped to the zero vector.

In this work, we need nested-lattice codes. Two lattices are required for nested-lattice codes, a coarse lattice Λ^c which is 'nested' in a fine lattice Λ^f, i.e., $\Lambda^c \subset \Lambda^f$ (see Figure 2.6). We denote a nested-lattice code using a fine lattice Λ^f and a coarse lattice Λ^c by the pair (Λ^f, Λ^c). The codewords are chosen as the fine lattice points $\lambda \in \Lambda^f$ that lie in $\mathcal{V}(\Lambda^c)$, i.e., the nested-lattice codewords are $\lambda \in \Lambda^f \cap \mathcal{V}(\Lambda^c)$. The number of fine lattice points that lie in $\mathcal{V}(\Lambda^c)$ is given by the ratio of the volume of $\mathcal{V}(\Lambda^c)$ to $\mathcal{V}(\Lambda^f)$, which consequently determines the rate R of the code. The power of the code is determined by the second moment of the $\mathcal{V}(\Lambda^c)$. It is shown that a nested-lattice code achieves the capacity of the point-to-point AWGN channel [EZ04].

In the sequel, we are going to need the following result from [NWS07]. Suppose that two nodes A and B, with messages m_A and m_B, respectively, where both messages have rate R, are given. The two nodes use the same nested-lattice codebook (Λ^f, Λ^c) with second moment P, to encode their messages into codewords λ_A and λ_B, respectively, of length n. The nodes then construct their transmit signals x_A^n and x_B^n as

$$x_A^n = (\lambda_A - d_A) \bmod \Lambda^c \quad (2.21)$$
$$x_B^n = (\lambda_B - d_B) \bmod \Lambda^c, \quad (2.22)$$

where d_A and d_B are n-dimensional dither vectors [NG11] uniformly distributed over $\mathcal{V}(\Lambda^c)$, known at the relay and nodes A and B[2]. A relay node receives

$$y_r^n = x_A^n + x_B^n + z_r^n \quad (2.23)$$

where z_r^n is an $\mathcal{N}(0, \sigma^2)$. The following lemmata hold.

Lemma 2.6 ([NWS07]). *The relay can decode the sum* $(\lambda_A + \lambda_B) \bmod \Lambda^c$ *from* y_r^n *reliably as long as* $R \leq C^+\left(\frac{P}{\sigma^2} - \frac{1}{2}\right)$.

[2]In the sequel, we will always use λ as a nested-lattice codeword and d as a random dither. Both λ and d are vectors of length n which can be inferred from the context.

Lemma 2.7 ([NWS07])**.** *If node A knows $(\lambda_A + \lambda_B) \bmod \Lambda^c$ and λ_A, then it can extract λ_B and hence also m_B.*

These lemmata indicate that nested-lattice codes have the property of linearity. That is, the sum of two nested-lattice codes $(\bmod \Lambda^c)$ is also a nested-lattice code, and hence can be decoded. This is in contrast to random codes which do not possess this property. This property make nested-lattice codes an appropriate choice for network coding [ACLY00] over wireless networks.

Chapter 3

Thesis Objectives

Various metrics can be used to quantify the gain that can be achieved by deploying a relay in a Gaussian IC. One such metric is the DoF. In [CJ09], it was shown that relaying, among other factors, can not increase the DoF of an X-network [MAMK08, JS08], which is an interference network where each transmitter has two independent messages, each of which is dedicated to one receiver. As the IC is a special case of the X-channel, relaying also does not increase its DoF. This is a relatively pessimistic result; it basically says that the relay does not have an impact on the asymptotic performance of the channel at high SNR. However, we know that relaying can indeed increase the capacity of the network. In order to quantify this increase, we have to resort to other metrics than the DoF. One can indeed study the capacity of the IRC directly and quantify the gain in bits per channel use. However, recalling that even the capacity of the IC without relay and the capacity of the P2P relay channel without interference are both unknown, aiming for the capacity of the IRC can be quite difficult. Thus, it is reasonable to aim for a metric which is coarser than capacity, but also finer than DoF. A suitable such metric is the GDoF, whose characterization can be seen as an intermediate step between the DoF characterization and the capacity characterization. A GDoF characterization thus progresses our knowledge on the IRC one step further than DoF, and paves the road towards the ultimate goal which is the capacity of the network.

3.1 Problem Statement

In order to characterize the GDoF of the IRC, one has to identify the optimal relaying strategy[1]. For this relaying strategy, the achievable rate has to be quantified leading to a GDoF lower bound. Then this lower bound has to be compared with GDoF upper bounds. If the bounds coincide, then the GDoF characterization is obtained.

Thus the main problem that is studied in this thesis can be summarized as follows:

Identifying optimal relaying strategies for the IRC.

This problem is three-fold, and requires answering several questions that can only be answered in conjunction with one another. While identifying efficient relaying strategies requires proposing coding schemes and describing their construction, judging their optimality can only be done by quantifying their achievable rates and comparing with tight upper bounds. On the other hand, tight upper bound can only be identified by comparing them with the achievable rates of optimal strategies. One can divide the problem in general into three steps:

1. Deriving capacity upper bounds.

[1]Optimal here refers to GDoF-optimality in the Gaussian network, and sum-capacity optimality in the linear-deterministic one.

2. Deriving capacity lower bounds by proposing transmit strategies and calculating their achievable rates.

3. Comparing upper and lower bounds.

If the upper and lower bounds coincide, then the transmit strategy can be classified as being optimal. Otherwise, one or both bounds have to be tightened.

3.2 Motivation

By observing past results on the IRC (see Section 2.2), one can note that despite all preceeding efforts, the optimal relaying strategy is still open. The past results focused on improving common classical strategies, such as using rate splitting with decode-forward [SE07a], or using generalized compress-forward [TY11], etc. A unified strategy which can be optimal for a wide range of paramters is not known.

Solving this problem would ease the way to characterizing the capacity of the IRC, or its approximate capacity within a gap of a constant number of bits. Knowing this capacity, we can know the impact of installing a relay node in a wireless interference network which suffers from coverage problems, bad channel quality, and high interference. Assume that some specified target rates are required to be achieved in this network. Then, knowing the capacity of the IRC allows us to adjust some parameters at the relay in order to achieve this target rate, such as the relay power and the relay location. Without knowing the capacity, one can still achieve some improvement by installing a relay node, but reliable communication can not be guaranteed. Furthermore, knowing the capacity we could decide whether installing a relay node is a good solution for a given problem or otherwise, where different cooperative strategies would be sought.

From a theoretical point of view, identifying optimal relaying strategies for the IRC might help us to develop new coding strategies for networks whose GDoF is still to be characterized, such as IC's with multiple relays or IRC's with more than 2 users.

The partially connected IRC which we also study in this thesis, the so-called wireless butterfly network (BFN) is studied with the following motivation. Unlike the IC where information can only flow from the transmitter to its desired receiver through the direct channel joining the two nodes, in the IRC information can also flow through the relay. From this point of view, the optimal scheme for the IRC is expected to be a mixture of the optimal scheme for the IC and optimal relaying strategies. In order to separate these two elements, we can consider special cases of the IRC. As our main goal is to study the usefulness of a relay for interference management and its impact on the network capacity, it would be helpful to focus on an IRC where the effect of relaying is magnified in comparison to other effects in the network. To do this, we can study the BFN which is an IRC whose direct channels are zero, and therefore, information can only flow via the relay. We expect that studying the BFN will ease the way to understanding the fully-connected IRC beyond the GDoF characterization.

The linear-deterministic approximation of the Gaussian IRC is also studied with the following motivation. First, notice the resemblance between Lemmas 2.1 and 2.3 on pages 7 and 11, respectively. This resemblance suggests that in order to study the GDoF of a network, one can start with the simpler linear-deterministic approximation of the network, and then use the obtained insights to derive the GDoF of the Gaussian network. Therefore, we study the linear-deterministic approximation of the IRC with the aim of finding optimal relaying strategies for the Gaussian IRC, which is indeed accomplished in this thesis.

3.3 Contribution and Organization

In this thesis, we show that a relaying strategy which combines rate-splitting and nested-lattice coding [Loe97,EZ04] at the sources, compute-forward [NG11] at the relay, and successive decoding at the destinations can achieve the GDoF of IRC for a wide range of channel parameters. Unlike the classical relaying strategies, in this strategy named compute-forward (CF) for short, the relay computes a function of the transmit signals and forwards it to the receivers. This is similar to network coding [ACLY00] except that the relay does not have to construct the network code, but the physical channel takes care of this. In other words, the relay decodes a network code which is constructed by the channel. The receivers decode the relay signal and use it for interference cancelation. This reduces the impact of interference from the undesired transmitter, and allows the transmitter to send at higher rates.

We study this relaying strategy for the IRC in Chapter 4. We characterize the sum-capacity of the linear-deterministic IRC and the GDoF of the Gaussian IRC when the source-relay channel is weaker than the cross channel. For this characterization, some new sum-capacity upper bounds for the IRC are developed, and also our novel CF strategy is described. The performance of this new strategy is compared with some classical strategies. In order to simplify the exposition, we restrict ourselves to the study of the symmetric IRC. We briefly comment on the asymmetric case in the conclusion (Chapter 6).

In Chapter 5, the wireless full-duplex BFN is studied. In the half-duplex BFN studied in [AH09], the half-duplex constraint allows orthogonal transmissions by Tx 1, Tx 2, and the relay (in different time slots) in such a way that interference is avoided. In the full-duplex case which we consider, all nodes can send at the same time and hence their signals interfere. We study the capacity of the full-duplex BFN. As expected the study of this network lead to the development of a new relaying strategy for the IRC. The performance of this new strategy, named 'cooperative interference neutralizaton' (CN), is studied in this chapter. In this strategy, each transmitter sends two signals: a 'present' signal destined to the receivers, and a 'future' one destined to the relay. The purpose of the future signal is feeding the relay with the signal to be decoded at the receivers in the next transmission, which allows the relay to operate as a cognitive relay [SVJS08]. The relay performs computation of a linear function of the transmit signals, and forwards a signal designed for interference neutralization. Again, the task of the relay in the CN strategy is also interference cancellation just as in the CF strategy above. As a result of Chapter 5, the capacity region of the linear-deterministic BFN is characterized. The proposed strategy is also extended to the Gaussian BFN.

In the same network, we study the impact of another form of relaying from the relay to the sources (in addition to the destinations) on the capacity of the network. This relaying can be interpreted also as relay-source feedback. We assume that there exist a link from the relay to the sources which establishes a bi-directional relay channel between the two transmitters and the relay as in [RW05, KDMT08, WNPS10, AAT09]. It is shown that this type of feedback provides a net gain in the sum-capacity of the network.

The results of the thesis are finally summarized in Chapter 6.

Chapter 4

The Interference Relay Channel

In this chapter, we study the sum-capacity of the interference relay channel (Figure 2.3(a) on page 9). We propose a novel transmission scheme, and show that it is optimal under some conditions. We start by analyzing the linear-deterministic approximation of the IRC, for which we characterize the sum-capacity. Then, we extend our results for the Gaussian IRC where we approximate the capacity by using the generalized degrees-of-freedom framework. Before we proceed with the analysis of the sum-capacity, let us start with a formal definition of the problem at hand.

4.1 System Model of the Interference Relay Channel (IRC)

Consider the IRC shown in Figure 4.1. In this network, transmitter j (Tx j) has a message m_j which it needs to send to receiver j (Rx j), $j \in \{1, 2\}$. This message is a realization of a random variable M_j which is uniformly distributed over the message set

$$\mathcal{M}_j = \{1, \cdots, 2^{\lfloor nR_j \rfloor}\}. \tag{4.1}$$

For reliable communication, the message has to be properly encoded. Thus, the transmitter encodes m_j into an n-symbol codeword $x_j^n = f_j(m_j)$, where the ith component of x_j^n, i.e., $x_j(i)$, $i \in \{1, \cdots, n\}$, is a realization of a real valued random variable X_j. This random variable must satisfy a power constraint given by

$$\mathbb{E}[X_j^2] \leq P_j, \quad j \in \{1, 2\}. \tag{4.2}$$

Then this codeword is transmitted through the channel. At time instant i, the relay receives

$$y_r(i) = h_{1r}x_1(i) + h_{2r}x_2(i) + z_r(i). \tag{4.3}$$

Here, $z_r(i)$ is a realization of an i.i.d. $\mathcal{N}(0, \sigma_r^2)$ noise, and $h_{jr} \in \mathbb{R}$ is the channel coefficient from transmitter j to the relay. Recall that the task of the relay is to support the transmitters in the communication. In general, the relay can send any processed version of what it receives. Thus, keeping in mind that the relay is causal, the relay constructs $x_r(i)$ using an encoding function f_{ri} as follows

$$x_r(i) = f_{ri}(y_r^{i-1}). \tag{4.4}$$

In general, $x_r(i)$ can be modeled by a random variable X_r, which must also satisfy a power constraint given by

$$\mathbb{E}[X_r^2] \leq P_r. \tag{4.5}$$

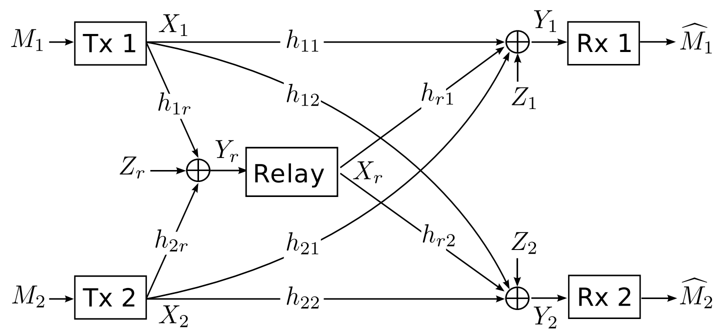


Figure 4.1: The 2-user Gaussian interference relay channel (G-IRC).

As a result, Rx j receives

$$y_j(i) = h_{jj}x_j(i) + h_{kj}x_k(i) + h_{rj}x_r(i) + z_j(i), \quad j \neq k, \tag{4.6}$$

at time instant i, where h_{jj}, h_{kj}, and h_{rj} represent the real valued channel coefficients from Tx j, Tx k, and the relay to Rx j, respectively, as shown in Figure 4.1, and $z_j(i)$ is a realization of an i.i.d. $\mathcal{N}(0, \sigma_j^2)$ noise.

In order to decode m_j, Rx j collects n received symbols which correspond to a complete length of a codeword. Then, it uses a decoder g_j to decode y_j^n to $\hat{m}_j$, i.e., $\hat{m}_j = g_j(y_j^n)$. A choice of message sets $\mathcal{M}_j$, encoding functions, and decoding functions induces an error probability $\mathbb{P}_j$ given by

$$\mathbb{P}_j = \text{Prob}\left(\hat{m}_j \neq m_j\right). \tag{4.7}$$

The overall transmission scheme can be thus characterized by the tuple $(n, R_1, R_2, \mathbb{P}_e)$ referred to as a code, where $\mathbb{P}_e = \max_j \mathbb{P}_j$.

We say that a rate pair (R_1, R_2) is achievable if there exists an $(n, R_1, R_2, \mathbb{P}_e)$ code for which $\mathbb{P}_e \to 0$ as $n \to \infty$ [CT06]. That is, for this rate pair, reliable communication can be guaranteed by using long codewords. The set of all achievable rate pairs is known as the capacity region, and henceforth denoted $\mathcal{C}$, and the sum-capacity C_Σ is defined as the maximum achievable sum-rate, i.e.,

$$C_\Sigma = \max_{(R_1,R_2)\in\mathcal{C}} R_\Sigma. \tag{4.8}$$

where $R_\Sigma = R_1 + R_2$. This quantity, C_Σ, is the main focus of the rest of the chapter.

4.1.1 The Symmetric Gaussian IRC (G-IRC)

The general Gaussian IRC can always be normalized in such a way that $P_1 = P_2 = P_r = P$ and $\sigma_1^2 = \sigma_2^2 = \sigma_r^2 = 1$ (see channel normalization in [Car78]). Even after this normalization, the number of parameters in the general Gaussian IRC is large. Thus, we resort to a symmetric setup for simplicity of exposition. This simplification reduces the number of parameters, while preserving the main features of the IRC. In the symmetric scenario we have

$$h_{11} = h_{22} = h_d \qquad \text{(}\textbf{d}\text{irect channel)}, \tag{4.9}$$

$$h_{12} = h_{21} = h_c \qquad \text{(}\textbf{c}\text{ross channel)}, \tag{4.10}$$

$$h_{r1} = h_{r2} = h_r \qquad \text{(}\textbf{r}\text{elay-destination channel)}, \tag{4.11}$$

$$h_{1r} = h_{2r} = h_s \qquad \text{(}\textbf{s}\text{ource-relay channel)}. \tag{4.12}$$

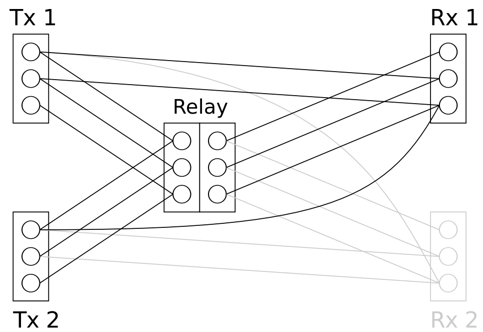


Figure 4.2: A linear-deterministic IRC with $(n_d, n_c, n_r, n_s) = (2, 1, 3, 3)$. Only Rx 1 is emphasized for clarity. Here, Rx 1 observes a bit from the relay on the topmost level, the sum of a bit from the relay and one from Tx 1 on the second level, and the sum of three bits from the transmitters and the relay at the lowermost level.

The powers of the desired signal, the interfering signal, and the relay signal as observed at the receivers of the symmetric G-IRC are specified by the quantities h_d, h_c, and h_r, and the power of the received signal at the relay is specified by h_s. This leaves us with four parameters which completely characterize the network.

4.1.2 The linear-deterministic IRC (LD-IRC)

We introduce here a special case for the IRC described in the previous section. Namely, the linear-deterministic (LD) channel (see Section 2.4) that is by now customarily used to approximate a Gaussian noise network at high SNR. In the LD-IRC, the channel strength is modeled by a positive integer quantity, which specifies the number of 'bit-pipes' between a transmitter and a receiver. For instance, the channel from Tx 1 to Rx 1 which is characterized by h_d in the Gaussian case, is modeled by n_d bit-pipes in the LD case, where $n_d = \lceil (1/2) \log(h_d^2 P) \rceil$. All the remaining channels can be modeled similarly. Thus

$$n_\ell = \left\lceil \frac{1}{2} \log(h_\ell^2 P) \right\rceil, \quad \ell \in \{d, c, r, s\}. \tag{4.13}$$

These integer quantities specify the number of bit-pipes between the transmitters and the receivers of the LD-IRC as shown in Figure 4.2. Each transmitter can send one bit on each such bit-pipe. The receivers receive a mod2 sum of several bits indicated by the bit-pipes arriving at this receiver. Similarly at the relay. Hence, the input-output relations of the LD-IRC can be written as

$$\mathbf{y}_r = \mathbf{S}^{q-n_s}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_s}\mathbf{x}_2, \tag{4.14}$$
$$\mathbf{y}_1 = \mathbf{S}^{q-n_d}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_c}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r, \tag{4.15}$$
$$\mathbf{y}_2 = \mathbf{S}^{q-n_c}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_d}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r, \tag{4.16}$$

where the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_r \in \mathbb{F}_2^q$ are the transmit signals of Tx 1 and 2 and the relay, respectively, and $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_r \in \mathbb{F}_2^q$ are the received signals at Rx 1 and 2 and the relay, respectively. Here q is defined as $q = \max\{n_d, n_c, n_r, n_s\}$ and $\mathbf{S}$ is the $q \times q$ down-shift matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{q-1}^T & 0 \\ \mathbf{I}_q & \mathbf{0}_{q-1} \end{bmatrix}. \tag{4.17}$$

Since the LD model is a high SNR approximation, the noise is neglected. This is done in order to focus on the interaction of different signals (interference) in the absence of noise, which provides a good approximation of the network in the so-called 'interference-limited regime'. Similar to the Gaussian case, the LD-IRC is characterized by four quantities, n_d, n_c, n_r, and n_s. These quantities capture the interaction between the signals taking their different power levels into account.

In the following sections, we will start with analyzing the sum-capacity of the LD-IRC which is easier to analyze due to its deterministic nature, and then we will extend the results to the Gaussian case using the intuitions obtained from the analysis of the LD-IRC.

4.2 On the Sum-capacity of the LD-IRC

The sum-capacity of the LD-IRC is studied in this section. We derive sum-capacity bounds that characterize the sum-capacity of the network as long as $n_s \leq n_c$, i.e., the source-relay channel is weaker than the cross-channel. The main result can be summarized in the following theorem.

Theorem 4.1. *The sum-capacity of the LD-IRC with $n_s \leq n_c$ is given by*

$$C_\Sigma = \min \left\{ \begin{array}{c} 2\max\{n_d, n_r\} \\ 2\max\{n_d, n_s\} \\ \max\{n_d, n_c, n_r\} + \max\{n_d, n_c\} - n_c \\ 2\max\{n_d, n_c\} - n_c + n_s \\ 2\max\{n_c, n_r, n_d - n_c\} \\ 2\max\{n_c, n_d + n_s - n_c\} \end{array} \right\}. \tag{4.18}$$

Recall that the sum-capacity of the LD-IC is given by (see Lemma 2.3 on page 11)

$$C_\Sigma = \min \left\{ \begin{array}{c} 2n_d \\ 2\max\{n_d, n_c\} - n_c \\ 2\max\{n_c, n_d - n_c\} \end{array} \right\}. \tag{4.19}$$

By comparing (4.18) and (4.19), the sum-capacity gain due to the relay is obvious. More precisely, the first two lines in (4.18) are larger than the first line in (4.19), the third and fourth lines in (4.18) are larger than the second line in (4.19), and the last two lines in (4.18) are larger than the last one in (4.19). An example showing the sum-capacity gain is given in Figure 4.3. Interestingly, in this example even though n_s is very weak compared to the other channels, the relay still increases the sum-capacity compared to the IC.

The converse and the achievability of Theorem 4.1 are given in Sections 4.2.1 and 4.2.2, respectively.

4.2.1 Sum-capacity Upper Bounds

Since the IRC has been studied previously, several upper bounds for its capacity exist (cf. [MDG12, TY11, CJ09]). For the purpose of this section however, only the cut-set bounds [CT06] are needed since they are necessary for the pursued sum-capacity characterization. We start by giving the cut-set bounds for the LD-IRC.

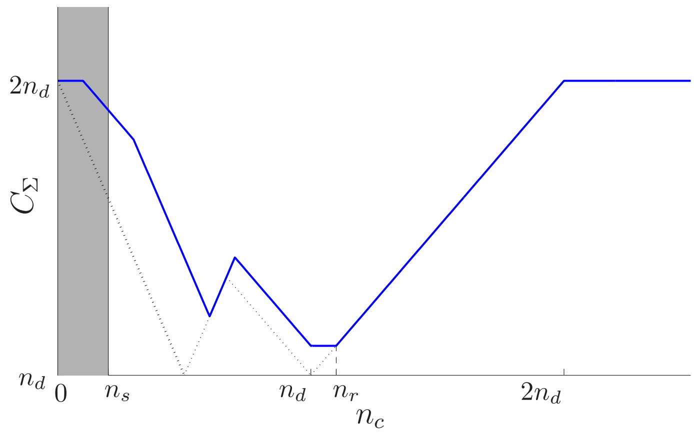


Figure 4.3: The sum-capacity of the LD-IRC with $n_r = 1.1n_d$, $n_s = 0.2n_d$ versus the interference parameter n_c. The sum-capacity of the LD-IC is also shown (dotted) for comparison. The shaded area where $n_c < n_s$ is not characterized.

4.2.1.1 Cut-set Bounds

The cut-set bound [CT06] can be summarized as follows. For a general network consisting of a set of nodes $\mathcal{T}$ with independent messages at each node, an achievable rate tuple must satisfy

$$R(\mathcal{S} \to \mathcal{S}^c) \leq I(X(\mathcal{S}); Y(\mathcal{S}^c)|X(\mathcal{S}^c)), \tag{4.20}$$

for some joint distribution on the inputs, where $\mathcal{S} \subset \mathcal{T}$, $\mathcal{S}^c$ is the complement of $\mathcal{S}$, and $R(\mathcal{S} \to \mathcal{S}^c)$ indicates the sum of the rates from the source nodes in $\mathcal{S}$ to the destination nodes in $\mathcal{S}^c$. The cut-set bound can be applied to the LD-IRC to obtain the following bounds.

Lemma 4.1 (Maric *et al.* [MDG12])**.** *The achievable rates in the LD-IRC are bounded by the region*

$$R_1 \leq \min\{I(\mathbf{x}_1, \mathbf{x}_r; \mathbf{y}_1|\mathbf{x}_2), I(\mathbf{x}_1; \mathbf{y}_1, \mathbf{y}_r|\mathbf{x}_2, \mathbf{x}_r)\} \tag{4.21}$$
$$R_2 \leq \min\{I(\mathbf{x}_2, \mathbf{x}_r; \mathbf{y}_2|\mathbf{x}_1), I(\mathbf{x}_2; \mathbf{y}_2, \mathbf{y}_r|\mathbf{x}_1, \mathbf{x}_r)\} \tag{4.22}$$
$$R_1 + R_2 \leq \min\{I(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_r; \mathbf{y}_1, \mathbf{y}_2), I(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_r|\mathbf{x}_r)\}, \tag{4.23}$$

maximized over all distributions of the binary vectors $\mathbf{x}_1$*,* $\mathbf{x}_2$*, and* $\mathbf{x}_r$ *with* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *independent.*

The individual rate constraints provided by the cut-set bounds, i.e., (4.21) and (4.22), are needed to obtain the first two lines in (4.18) given in the following theorem.

Theorem 4.2. *The sum-capacity of the LD-IRC is upper bounded by*

$$C_\Sigma \leq 2\min\{\max\{n_d, n_r\}, \max\{n_d, n_s\}\}. \tag{4.24}$$

Proof. Consider the first term in the cut-set bounds (4.21). This can be bounded as follows

$$\begin{aligned}R_1 &\le I(\mathbf{x}_1, \mathbf{x}_r; \mathbf{y}_1|\mathbf{x}_2) && (4.25)\\ &= H(\mathbf{y}_1|\mathbf{x}_2) - H(\mathbf{y}_1|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_r) && (4.26)\\ &\overset{(a)}{=} H(\mathbf{S}^{q-n_d}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r|\mathbf{x}_2) && (4.27)\\ &\overset{(b)}{\le} H(\mathbf{S}^{q-n_d}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r) && (4.28)\\ &\overset{(c)}{\le} \max\{n_d, n_r\}, && (4.29)\end{aligned}$$

where (a) follows since $\mathbf{y}_1$ is a deterministic function of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, (b) follows since conditioning does not increase entropy, and (c) follows from Lemma 2.4 (page 12). Using similar steps, the second term in (4.21) can be bounded as follows

$$\begin{aligned}R_1 &\le I(\mathbf{x}_1; \mathbf{y}_1, \mathbf{y}_r|\mathbf{x}_2, \mathbf{x}_r) && (4.30)\\ &= H(\mathbf{y}_1, \mathbf{y}_r|\mathbf{x}_2, \mathbf{x}_r) - H(\mathbf{y}_1, \mathbf{y}_r|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_r) && (4.31)\\ &= H(\mathbf{S}^{q-n_d}\mathbf{x}_1, \mathbf{S}^{q-n_s}\mathbf{x}_1|\mathbf{x}_2, \mathbf{x}_r) && (4.32)\\ &\le H(\mathbf{S}^{q-n_d}\mathbf{x}_1, \mathbf{S}^{q-n_s}\mathbf{x}_1) && (4.33)\\ &\le \max\{n_d, n_s\}. && (4.34)\end{aligned}$$

Similarly, we can get the same bounds for R_2, which proves the statement of the theorem. □

For the purpose of this work, more bounds are required in addition to the cut-set bounds in Theorem 4.2. In what follows, we introduce some new upper bounds that are instrumental for the characterization of the sum-capacity of the LD-IRC.

4.2.1.2 New Upper Bounds

In this subsection, we derive new upper bounds for the sum-capacity of the LD-IRC using genie-aided approaches and cooperation approaches. The following notation is used

$$\mathbf{d}_j^n = \mathbf{S}^{q-n_d}\mathbf{x}_j^n, \quad \mathbf{c}_j^n = \mathbf{S}^{q-n_c}\mathbf{x}_j^n, \quad \mathbf{s}_j^n = \mathbf{S}^{q-n_s}\mathbf{x}_j^n. \tag{4.35}$$

The upper bounds we derive next are general, in the sense that they do not only apply for the case where $n_s \le n_c$. They hold for arbitrary n_d, n_c, n_r, and n_s.

Theorem 4.3. *The sum-capacity of the LD-IRC is upper bounded by*

$$C_\Sigma \le \max\{n_d, n_c, n_r\} + \max\{n_d, n_c, n_s\} - n_c. \tag{4.36}$$

Proof. Consider an enhanced (genie-aided) LD-IRC where a genie gives $\mathbf{x}_1^n$, $\mathbf{y}_r^n$, and $\mathbf{c}_2^n$ to Rx 2 as side information. This enhanced channel has a larger capacity than the original LD-IRC, and hence, the capacity of the former serves as an upper bound for the capacity of the latter. Starting with Fano's inequality [CT06], we have

$$\begin{aligned}n(R_\Sigma - \varepsilon_n) &\le I(\mathbf{x}_1^n; \mathbf{y}_1^n) + I(\mathbf{x}_2^n; \mathbf{y}_2^n, \mathbf{x}_1^n, \mathbf{y}_r^n, \mathbf{c}_2^n) && (4.37)\\ &\overset{(a)}{=} I(\mathbf{x}_1^n; \mathbf{y}_1^n) + I(\mathbf{x}_2^n; \mathbf{y}_2^n, \mathbf{y}_r^n, \mathbf{c}_2^n|\mathbf{x}_1^n) && (4.38)\\ &= I(\mathbf{x}_1^n; \mathbf{y}_1^n) + H(\mathbf{y}_2^n, \mathbf{y}_r^n, \mathbf{c}_2^n|\mathbf{x}_1^n) - H(\mathbf{y}_2^n, \mathbf{y}_r^n, \mathbf{c}_2^n|\mathbf{x}_1^n, \mathbf{x}_2^n), && (4.39)\end{aligned}$$

where $\varepsilon_n \to 0$ as $n \to \infty$, and where (a) follows since $\mathbf{x}_1^n$ and $\mathbf{x}_2^n$ are independent. Now let us consider each term in (4.39) separately. The last term vanishes since

$$H(\mathbf{y}_2^n, \mathbf{y}_r^n, \mathbf{c}_2^n|\mathbf{x}_1^n, \mathbf{x}_2^n) = H(\mathbf{y}_r^n, \mathbf{c}_2^n|\mathbf{x}_1^n, \mathbf{x}_2^n) + H(\mathbf{y}_2^n|\mathbf{x}_1^n, \mathbf{x}_2^n, \mathbf{y}_r^n, \mathbf{c}_2^n) = 0. \tag{4.40}$$

Using the chain rule of entropy, the second term in (4.39) can be rewritten as shown below

$$H(\mathbf{y}_2^n, \mathbf{y}_r^n, \mathbf{c}_2^n|\mathbf{x}_1^n) = \sum_{i=1}^{n} H(\mathbf{y}_2(i), \mathbf{y}_r(i), \mathbf{c}_2(i)|\mathbf{x}_1^n, \mathbf{y}_2^{i-1}, \mathbf{y}_r^{i-1}, \mathbf{c}_2^{i-1}). \tag{4.41}$$

Note that given $\mathbf{y}_r^{i-1}$, $\mathbf{x}_r^i$ can be constructed (see (4.4)). Thus, the contribution of $\mathbf{x}_r^i$ can be subtracted from $\mathbf{y}_2^i$. Moreover, given $\mathbf{x}_1^n$, its contribution can be subtracted from $\mathbf{y}_2^i$ and $\mathbf{y}_r^i$. Thus,

$$\begin{aligned}
H(\mathbf{y}_2^n, \mathbf{y}_r^n, \mathbf{s}_2^n|\mathbf{x}_1^n) &= \sum_{i=1}^{n} H(\mathbf{d}_2(i), \mathbf{s}_2(i), \mathbf{c}_2(i)|\mathbf{x}_1^n, \mathbf{d}_2^{i-1}, \mathbf{s}_2^{i-1}, \mathbf{c}_2^{i-1}) && (4.42)\\
&= \sum_{i=1}^{n} \big[H(\mathbf{c}_2(i)|\mathbf{x}_1^n, \mathbf{d}_2^{i-1}, \mathbf{s}_2^{i-1}, \mathbf{c}_2^{i-1}) + H(\mathbf{d}_2(i), \mathbf{s}_2(i)|\mathbf{x}_1^n, \mathbf{d}_2^{i-1}, \mathbf{s}_2^{i-1}, \mathbf{c}_2^{i})\big]\\
&\overset{(b)}{\le} \sum_{i=1}^{n} \big[H(\mathbf{c}_2(i)|\mathbf{x}_1^n, \mathbf{d}_2^{i-1}, \mathbf{s}_2^{i-1}, \mathbf{c}_2^{i-1}) + H(\mathbf{d}_2(i), \mathbf{s}_2(i)|\mathbf{c}_2(i))\big] && (4.43)
\end{aligned}$$

where $\mathbf{d}_2^n$ and $\mathbf{s}_2^n$ are defined in (4.35) and (b) follows since conditioning does not increase entropy. Further, the first term in (4.39) can be bounded as follows

$$\begin{aligned}
I(\mathbf{x}_1^n; \mathbf{y}_1^n) &= H(\mathbf{y}_1^n) - H(\mathbf{y}_1^n|\mathbf{x}_1^n) && (4.44)\\
&= \sum_{i=1}^{n} \big[H(\mathbf{y}_1(i)|\mathbf{y}_1^{i-1}) - H(\mathbf{y}_1(i)|\mathbf{x}_1^n, \mathbf{y}_1^{i-1})\big] && (4.45)\\
&\overset{(c)}{\le} \sum_{i=1}^{n} \big[H(\mathbf{y}_1(i)) - H(\mathbf{y}_1(i)|\mathbf{x}_1^n, \mathbf{y}_1^{i-1}, \mathbf{s}_2^{i-1}, \mathbf{d}_2^{i-1})\big] && (4.46)\\
&= \sum_{i=1}^{n} \big[H(\mathbf{y}_1(i)) - H(\mathbf{c}_2(i)|\mathbf{x}_1^n, \mathbf{c}_2^{i-1}, \mathbf{s}_2^{i-1}, \mathbf{d}_2^{i-1})\big] && (4.47)
\end{aligned}$$

where (c) follows since conditioning does not increase entropy. Observing that the second term in (4.47) is equivalent to the first term in (4.43), we now substitute (4.40), (4.43), and (4.47) in (4.39) to obtain

$$n(R_\Sigma - \varepsilon_n) \le \sum_{i=1}^{n} H(\mathbf{y}_1(i)) + \sum_{i=1}^{n} H(\mathbf{d}_2(i), \mathbf{s}_2(i)|\mathbf{c}_2(i)). \tag{4.48}$$

From Lemma 2.4 (page 12), we get

$$\sum_{i=1}^{n} H(\mathbf{y}_1(i)) \le n \max\{n_d, n_c, n_r\}. \tag{4.49}$$

On the other hand, assume that $n_d \ge n_s$, then

$$\begin{aligned}
H(\mathbf{d}_2(i), \mathbf{s}_2(i)|\mathbf{c}_2(i)) &= H(\mathbf{d}_2(i)|\mathbf{c}_2(i)) + H(\mathbf{s}_2(i)|\mathbf{c}_2(i), \mathbf{d}_2(i)) && (4.50)\\
&= H(\mathbf{d}_2(i)|\mathbf{c}_2(i)) && (4.51)\\
&\le (n_d - n_c)^+ && (4.52)
\end{aligned}$$

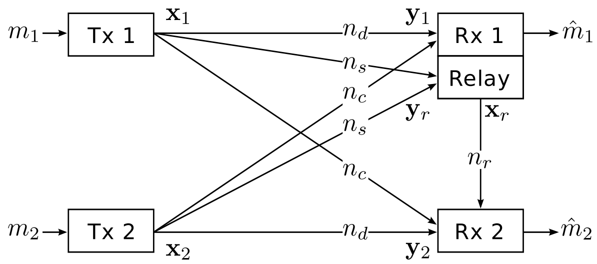


Figure 4.4: A cooperative IC resulting from the cooperation between the relay and Rx 1.

by Lemma 2.5 (page 12), and if $n_s \geq n_d$, then similarly we have

$$H(\mathbf{d}_2(i), \mathbf{s}_2(i)|\mathbf{c}_2(i)) \leq (n_s - n_c)^+. \tag{4.53}$$

Thus, we can write

$$H(\mathbf{d}_2(i), \mathbf{s}_2(i)|\mathbf{c}_2(i)) \leq \max\{(n_d - n_c)^+, (n_s - n_c)^+\} = \max\{n_d, n_s, n_c\} - n_c. \tag{4.54}$$

Substituting (4.49) and (4.54) in (4.48), we obtain

$$n(R_\Sigma - \varepsilon_n) \leq n\left[\max\{n_d, n_c, n_r\} + \max\{n_d, n_s, n_c\} - n_c\right]. \tag{4.55}$$

By dividing by n and letting $n \to \infty$, we get our desired upper bound. □

Notice that if we set $n_s = n_r = 0$, then this bound reduces to the one in the second line in (4.19) which is known as the Z-bound of the IC (or the one sided IC) [ETW08].

Remark 4.1. *If we specialize the upper bound for the X-channel with relays given in [CJ09] to the LD-IRC, we obtain a bound which is a special case of the one in Theorem 4.3. Namely, the two bounds become the same if $n_c \geq n_d$.*

Alternatively, Theorem 4.3 can be shown using a cooperative approach[1], i.e., by allowing some nodes to fully cooperate thus forming a 'super-node'. Then, we treat the resulting network as an IC with general cooperation [Tun12, YT11b]. To prove Theorem 4.3 using this cooperative approach, let the relay and Rx 1 fully cooperate. The resulting network has four nodes as shown in Figure 4.4. In this case, the super-node consisting of the relay and Rx 1 receives $(\mathbf{y}_1, \mathbf{y}_r)$. If we substitute $\bar{Y}_1 = \bar{Y}_2 = \bar{X}_2 = \emptyset$, $\bar{X}_1 = \mathbf{x}_r$, and $Y_1 = (\mathbf{y}_1, \mathbf{y}_r)$ in the bound (2.7) in Lemma 2.2 (page 8), we can write

$$\begin{aligned} R_\Sigma &\leq I(\mathbf{x}_1; \mathbf{y}_1, \mathbf{y}_r|\mathbf{y}_2, \mathbf{x}_2, \mathbf{x}_r) + I(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_r; \mathbf{y}_2) \qquad (4.56)\\ &= H(\mathbf{y}_1, \mathbf{y}_r|\mathbf{y}_2, \mathbf{x}_2, \mathbf{x}_r) - H(\mathbf{y}_1, \mathbf{y}_r|\mathbf{y}_2, \mathbf{x}_2, \mathbf{x}_r, \mathbf{x}_1) + H(\mathbf{y}_2) - H(\mathbf{y}_2|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_r)\\ &\leq H(\mathbf{y}_1, \mathbf{y}_r|\mathbf{y}_2, \mathbf{x}_2, \mathbf{x}_r) + H(\mathbf{y}_2). \qquad (4.57) \end{aligned}$$

where the last step follows since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_r$ are deterministic functions of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_r$. Now, this bound can be evaluated using the same steps as in (4.49) and (4.54) to obtain

$$R_\Sigma \leq \max\{n_d, n_c, n_r\} + \max\{n_d, n_s, n_c\} - n_c. \tag{4.58}$$

In the next theorem, we state another upper bound for the sum-capacity of the LD-IRC and we prove it using the cooperative approach.

[1]The cooperative approach was used in [CJ09] to obtain an upper bound for the X-channel with relays.

Theorem 4.4. *The sum-capacity of the LD-IRC is upper bounded by*

$$C_\Sigma \le 2\max\{n_d, n_c\} - n_c + n_s. \tag{4.59}$$

Proof. Let the relay and Rx 1 fully cooperate to form an IC with uni-directional cooperation between the super-node (relay,Rx 1) and Rx 2 (cf. Figure 4.4). Using the bound (2.8) in Lemma 2.2 on page 8, we can write

$$\begin{aligned}
R_\Sigma &\le I(\mathbf{x}_2;\mathbf{y}_2|\mathbf{y}_1,\mathbf{y}_r,\mathbf{x}_1,\mathbf{x}_r) + I(\mathbf{x}_1,\mathbf{x}_2;\mathbf{y}_1,\mathbf{y}_r|\mathbf{x}_r) && (4.60)\\
&= H(\mathbf{y}_2|\mathbf{y}_1,\mathbf{y}_r,\mathbf{x}_1,\mathbf{x}_r) - H(\mathbf{y}_2|\mathbf{y}_1,\mathbf{y}_r,\mathbf{x}_1,\mathbf{x}_r,\mathbf{x}_2) + H(\mathbf{y}_1,\mathbf{y}_r|\mathbf{x}_r) && (4.61)\\
&\quad - H(\mathbf{y}_1,\mathbf{y}_r|\mathbf{x}_r,\mathbf{x}_1,\mathbf{x}_2) && (4.62)\\
&= H(\mathbf{y}_2|\mathbf{y}_1,\mathbf{y}_r,\mathbf{x}_1,\mathbf{x}_r) + H(\mathbf{y}_1,\mathbf{y}_r|\mathbf{x}_r) && (4.63)
\end{aligned}$$

where the last step follows since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_r$ are deterministic functions of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_r$. We proceed as follows

$$\begin{aligned}
R_\Sigma &\le H(\mathbf{y}_2|\mathbf{y}_1,\mathbf{y}_r,\mathbf{x}_1,\mathbf{x}_r) + H(\mathbf{y}_r|\mathbf{x}_r) + H(\mathbf{y}_1|\mathbf{x}_r,\mathbf{y}_r) && (4.64)\\
&\overset{(a)}{\le} H(\mathbf{d}_2|\mathbf{c}_2,\mathbf{y}_r,\mathbf{x}_1,\mathbf{x}_r) + H(\mathbf{s}_1\oplus\mathbf{s}_2|\mathbf{x}_r) + H(\mathbf{d}_1\oplus\mathbf{c}_2|\mathbf{x}_r,\mathbf{y}_r) && (4.65)\\
&\overset{(b)}{\le} H(\mathbf{d}_2|\mathbf{c}_2) + H(\mathbf{s}_1\oplus\mathbf{s}_2) + H(\mathbf{d}_1\oplus\mathbf{c}_2) && (4.66)\\
&\overset{(c)}{\le} (n_d-n_c)^+ + n_s + \max\{n_d,n_c\} && (4.67)\\
&= n_s + 2\max\{n_d,n_c\} - n_c && (4.68)
\end{aligned}$$

where (a) follows since $H(X|Y) = H(X - f(Y)|Y)$ for some function f, (b) follows since conditioning does not increase entropy, and (c) from Lemmata 2.4 and 2.5 on page 8. ☐

Remark 4.2. *The same bound can be derived using a genie-aided approach where we give $\mathbf{y}_r^n$ to Rx 1 and $(\mathbf{y}_r^n, m_1)$ to Rx 2 as side information [CS12].*

The next theorem presents another sum-capacity upper bound which is inspired from the weak interference upper bound of the IC in [ETW08], but appropriately adapted for the IRC. It uses the novel idea of giving $\mathbf{x}_r(1)$ (only the first instant) to the receivers as side information. Note that $\mathbf{x}_r(1)$ is independent of the messages m_1 and m_2 due to the causality of the relay. This fact is used in the proof of the Theorem 4.5. But before we proceed to this new theorem, we need to present the following lemma.

Lemma 4.2. *The following statement holds for an LD-IRC*

$$H(\mathbf{c}_1^n|\mathbf{x}_r(1)) - H(\mathbf{y}_2^n|\mathbf{x}_r(1),\mathbf{c}_2^n,\mathbf{x}_2^n) = 0 \quad \text{if } n_c \ge n_s. \tag{4.69}$$

Proof. Let us first write

$$\begin{aligned}
H(\mathbf{y}_2^n|\mathbf{x}_r(1),\mathbf{c}_2^n,\mathbf{x}_2^n) &= H(\mathbf{y}_2^n|\mathbf{x}_r(1),\mathbf{x}_2^n) && (4.70)\\
&= \sum_{i=1}^{n} H(\mathbf{y}_2(i)|\mathbf{x}_r(1),\mathbf{x}_2^n,\mathbf{y}_2^{i-1}). && (4.71)
\end{aligned}$$

Now, we use the given information in the condition $(\mathbf{x}_r(1), \mathbf{x}_2^n, \mathbf{y}_2^{i-1})$ to start a recursion which produces $(\mathbf{x}_r(2), \cdots, \mathbf{x}_r(i))$ as follows. First, we extract $(\mathbf{x}_r(1), \mathbf{x}_2(1), \mathbf{y}_2(1))$ from $(\mathbf{x}_r(1), \mathbf{x}_2^n, \mathbf{y}_2^{i-1})$. Then we can use this information to calculate $\mathbf{c}_1(1)$ since

$$\mathbf{c}_1(1) = \mathbf{y}_2(1) \oplus \mathbf{S}^{q-n_d}\mathbf{x}_2(1) \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r(1). \tag{4.72}$$

Then, since $n_c \geq n_s$, we can extract $\mathbf{s}_1(1) = \mathbf{S}^{q-n_s}\mathbf{x}_1(1)$ from $\mathbf{c}_1(1)$. Finally, $\mathbf{x}_r(2)$ can be obtained from $\mathbf{s}_1(1)$ and $\mathbf{x}_2(1)$ since $\mathbf{x}_r(2)$ is a function of $\mathbf{s}_1(1) \oplus \mathbf{s}_2(1)$. At this stage, we know $(\mathbf{x}_r(2), \mathbf{x}_2(2), \mathbf{y}_2(2))$. This information is used to construct $\mathbf{x}_r(3)$. This recursion is continued until $\mathbf{x}_r(i)$ is produced. Thus, we can write

$$\begin{aligned}
H(\mathbf{y}_2^n|\mathbf{x}_r(1), \mathbf{c}_2^n, \mathbf{x}_2^n) &= \sum_{i=1}^{n} H(\mathbf{y}_2(i)|\mathbf{x}_r^i, \mathbf{x}_2^n, \mathbf{y}_2^{i-1}) && (4.73)\\
&\overset{(a)}{=} \sum_{i=1}^{n} H(\mathbf{c}_1(i)|\mathbf{x}_r^i, \mathbf{x}_2^n, \mathbf{c}_1^{i-1}) && (4.74)\\
&\overset{(b)}{=} \sum_{i=1}^{n} H(\mathbf{c}_1(i)|\mathbf{x}_r(1), \mathbf{x}_2^n, \mathbf{c}_1^{i-1}) && (4.75)\\
&= H(\mathbf{c}_1^n|\mathbf{x}_r(1), \mathbf{x}_2^n) && (4.76)\\
&\overset{(c)}{=} H(\mathbf{c}_1^n|\mathbf{x}_r(1)). && (4.77)
\end{aligned}$$

where (a) follows since $H(X|Y) = H(X - f(Y)|Y)$ for some function f, (b) follows since $(\mathbf{x}_r(2), \cdots, \mathbf{x}_r(i))$ can be constructed from $(\mathbf{x}_r(1), \mathbf{x}_2^n, \mathbf{c}_1^{i-1})$ as shown above, and (c) follows since $\mathbf{x}_1^n$, $\mathbf{x}_2^n$, and $\mathbf{x}_r(1)$ are mutually independent. This completes the proof. □

Now we can derive our new upper bound where Lemma 4.2 is exploited.

Theorem 4.5. *The achievable sum-rate of the LD-IRC is upper bounded by*

$$C_\Sigma \leq 2\max\{n_c, n_r, n_d - \max\{n_c, n_s\}\} + 2\max\{n_c, n_s\} - 2n_c. \tag{4.78}$$

Note that for small n_r and n_s, this bound has the same behavior as sum-capacity of the IC with weak interference [ETW08, MK09, SKC09, AV09]. In fact, it will be shown in the next section that treating the LD-IRC as an IC is optimal in some cases.

Proof. First, assume that $n_c \geq n_s$. A genie gives the signals $(\mathbf{c}_1^n, \mathbf{x}_r(1))$ and $(\mathbf{c}_2^n, \mathbf{x}_r(1))$ to receivers 1 and 2, respectively, where $\mathbf{c}_1^n$ and $\mathbf{c}_2^n$ are as defined in (4.35). Then, by using Fano's inequality, we can write

$$\begin{aligned}
n(R_\Sigma - \varepsilon_n) &\leq I(\mathbf{x}_1^n; \mathbf{y}_1^n, \mathbf{c}_1^n, \mathbf{x}_r(1)) + I(\mathbf{x}_2^n; \mathbf{y}_2^n, \mathbf{c}_2^n, \mathbf{x}_r(1)) && (4.79)\\
&\overset{(a)}{=} I(\mathbf{x}_1^n; \mathbf{c}_1^n|\mathbf{x}_r(1)) + I(\mathbf{x}_1^n; \mathbf{y}_1^n|\mathbf{x}_r(1), \mathbf{c}_1^n) + I(\mathbf{x}_2^n; \mathbf{c}_2^n|\mathbf{x}_r(1)) && (4.80)\\
&\quad + I(\mathbf{x}_2^n; \mathbf{y}_2^n|\mathbf{x}_r(1), \mathbf{c}_2^n) && (4.81)\\
&= H(\mathbf{c}_1^n|\mathbf{x}_r(1)) - H(\mathbf{c}_1^n|\mathbf{x}_r(1), \mathbf{x}_1^n) + H(\mathbf{y}_1^n|\mathbf{x}_r(1), \mathbf{c}_1^n) && (4.82)\\
&\quad - H(\mathbf{y}_1^n|\mathbf{x}_r(1), \mathbf{c}_1^n, \mathbf{x}_1^n) + H(\mathbf{c}_2^n|\mathbf{x}_r(1)) - H(\mathbf{c}_2^n|\mathbf{x}_r(1), \mathbf{x}_2^n) && (4.83)\\
&\quad + H(\mathbf{y}_2^n|\mathbf{x}_r(1), \mathbf{c}_2^n) - H(\mathbf{y}_2^n|\mathbf{x}_r(1), \mathbf{c}_2^n, \mathbf{x}_2^n), && (4.84)
\end{aligned}$$

with $\varepsilon_n \to 0$ as $n \to \infty$, where (a) follows from the independence between $\mathbf{x}_1^n$ and $\mathbf{x}_2^n$ on the one hand, and $\mathbf{x}_r(1)$ on the other hand (due to causality (4.4)). Using Lemma 4.2, and since $n_c \geq n_s$ by assumption, we obtain

$$H(\mathbf{c}_1^n|\mathbf{x}_r(1)) - H(\mathbf{y}_2^n|\mathbf{x}_r(1), \mathbf{c}_2^n, \mathbf{x}_2^n) = 0 \tag{4.85}$$
$$H(\mathbf{c}_2^n|\mathbf{x}_r(1)) - H(\mathbf{y}_1^n|\mathbf{x}_r(1), \mathbf{c}_1^n, \mathbf{x}_1^n) = 0. \tag{4.86}$$

Additionally, since $\mathbf{c}_1^n$ and $\mathbf{c}_2^n$ are deterministic functions of $\mathbf{x}_1^n$ and $\mathbf{x}_2^n$, respectively, then

$$H(\mathbf{c}_1^n|\mathbf{x}_r(1), \mathbf{x}_1^n) = H(\mathbf{c}_2^n|\mathbf{x}_r(1), \mathbf{x}_2^n) = 0. \tag{4.87}$$

Thus, we can write our upper bound as

$$n(R_\Sigma - \varepsilon_n) \leq H(\mathbf{y}_1^n|\mathbf{x}_r(1), \mathbf{c}_1^n) + H(\mathbf{y}_2^n|\mathbf{x}_r(1), \mathbf{c}_2^n). \tag{4.88}$$

Consider the first term. This can be bounded as follows

$$H(\mathbf{y}_1^n|\mathbf{x}_r(1), \mathbf{c}_1^n) \leq H(\mathbf{y}_1^n|\mathbf{c}_1^n) \tag{4.89}$$
$$= \sum_{i=1}^{n} H(\mathbf{y}_1(i)|\mathbf{c}_1^n, \mathbf{y}_1^{i-1}) \tag{4.90}$$
$$\leq \sum_{i=1}^{n} H(\mathbf{y}_1(i)|\mathbf{c}_1(i)), \tag{4.91}$$

where the last step follows since conditioning does not increase entropy. Knowing $\mathbf{c}_1(i)$, the contribution of the first n_c components of $\mathbf{x}_1(i)$ in $\mathbf{y}_1(i)$ can be subtracted, leaving $(n_d - n_c)^+$ components unknown. As a result, $\mathbf{y}_1(i)$ has $\max\{n_c, n_r, n_d - n_c\}$ unknown components, whose entropy can be maximized by the Bern(1/2) distribution to obtain

$$H(\mathbf{y}_1^n|\mathbf{c}_1^n, \mathbf{x}_r(1)) \leq n \max\{n_c, n_r, n_d - n_c\} \tag{4.92}$$

and similarly

$$H(\mathbf{y}_2^n|\mathbf{c}_2^n, \mathbf{x}_r(1)) \leq n \max\{n_c, n_r, n_d - n_c\}. \tag{4.93}$$

Substituting (4.92) and (4.93) in (4.88), dividing by n, and then letting $n \to \infty$, we get

$$R_\Sigma \leq 2 \max\{n_c, n_r, n_d - n_c\}. \tag{4.94}$$

Now we consider the case where $n_c \leq n_s$. In this case, let us enhance the receivers by adding $n_s - n_c$ levels to each of them (which is equivalent to reducing the noise power in the corresponding Gaussian IRC). That is, we obtain an enhanced LD-IRC with a direct channel $\bar{n}_d$, a cross channel $\bar{n}_c$, and a relay-destination channel $\bar{n}_r$ given by[2]

$$\bar{n}_d = n_d + n_s - n_c, \quad \bar{n}_c = n_s, \quad \bar{n}_r = n_r + n_s - n_c, \tag{4.95}$$

[2]The over-bar is used to distinguish the parameters of the enhanced channel from those of the original one.

and a source-relay channel $\bar{n}_s = n_s$. Since this channel is clearly more capable than the original one, its capacity provides us with an upper bound. Now notice that the obtained channel has $\bar{n}_c = \bar{n}_s$ which satisfies the inequality $\bar{n}_c \geq \bar{n}_s$, and thus, the bound we derived above in (4.94) applies. Thus, the enhanced channel has an upper bound

$$R_\Sigma \leq 2\max\{\bar{n}_c, \bar{n}_r, \bar{n}_d - \bar{n}_c\} \tag{4.96}$$
$$= 2\max\{n_s, n_r + n_s - n_c, n_d - n_c\} \tag{4.97}$$
$$= 2\max\{n_c, n_r, n_d - n_s\} + 2(n_s - n_c). \tag{4.98}$$

This is also an upper bound for the original LD-IRC with $n_c \leq n_s$. By combining (4.94) and (4.98), we obtain the statement of the theorem. □

Finally, the last upper bound that we need to complete the proof of converse of Theorem 4.1 is presented in the following theorem.

Theorem 4.6. *The sum-rate of the LD-IRC is upper bounded by*

$$C_\Sigma \leq 2\max\{n_s, n_c, n_d - n_c + n_s\}. \tag{4.99}$$

Proof. Give the genie side information $(\mathbf{c}_1^n, \mathbf{y}_r^n)$ and $(\mathbf{c}_2^n, \mathbf{y}_r^n)$ to receivers 1 and 2, respectively, where $\mathbf{c}_1^n$ and $\mathbf{c}_2^n$ are as defined in (4.35). Then we bound R_1 by using Fano's inequality as follows

$$n(R_1 - \varepsilon_{1n}) \leq I(\mathbf{x}_1^n; \mathbf{y}_1^n, \mathbf{c}_1^n, \mathbf{y}_r^n) \tag{4.100}$$
$$= H(\mathbf{y}_1^n, \mathbf{c}_1^n, \mathbf{y}_r^n) - H(\mathbf{y}_1^n, \mathbf{c}_1^n, \mathbf{y}_r^n|\mathbf{x}_1^n) \tag{4.101}$$
$$= H(\mathbf{c}_1^n, \mathbf{y}_r^n) + H(\mathbf{y}_1^n|\mathbf{c}_1^n, \mathbf{y}_r^n) - H(\mathbf{y}_1^n, \mathbf{c}_1^n, \mathbf{y}_r^n|\mathbf{x}_1^n), \tag{4.102}$$

where $\varepsilon_{1n} \to 0$ as $n \to \infty$. Now consider the third term in (4.102). This can be expressed as follows

$$H(\mathbf{y}_1^n, \mathbf{c}_1^n, \mathbf{y}_r^n|\mathbf{x}_1^n) = H(\mathbf{y}_r^n|\mathbf{x}_1^n) + H(\mathbf{y}_1^n, \mathbf{c}_1^n|\mathbf{x}_1^n, \mathbf{y}_r^n) \tag{4.103}$$
$$\overset{(a)}{=} H(\mathbf{s}_2^n|\mathbf{x}_1^n) + H(\mathbf{c}_2^n|\mathbf{x}_1^n, \mathbf{s}_2^n) \tag{4.104}$$
$$= H(\mathbf{c}_2^n, \mathbf{s}_2^n|\mathbf{x}_1^n) \tag{4.105}$$
$$= H(\mathbf{c}_2^n, \mathbf{s}_2^n) \tag{4.106}$$
$$= H(\mathbf{c}_2^n, \mathbf{y}_r^n) - I(\mathbf{x}_1^n; \mathbf{c}_2^n, \mathbf{y}_r^n), \tag{4.107}$$

where (a) follows since knowing $\mathbf{y}_r^n$ we can construct $\mathbf{x}_r^n$. By substituting (4.107) in (4.102) we obtain

$$n(R_1 - \varepsilon_{1n}) \leq H(\mathbf{c}_1^n, \mathbf{y}_r^n) + H(\mathbf{y}_1^n|\mathbf{c}_1^n, \mathbf{y}_r^n) - H(\mathbf{c}_2^n, \mathbf{y}_r^n) + I(\mathbf{x}_1^n; \mathbf{c}_2^n, \mathbf{y}_r^n). \tag{4.108}$$

Similarly we can bound R_2 by

$$n(R_2 - \varepsilon_{2n}) \leq H(\mathbf{c}_2^n, \mathbf{y}_r^n) + H(\mathbf{y}_2^n|\mathbf{c}_2^n, \mathbf{y}_r^n) - H(\mathbf{c}_1^n, \mathbf{y}_r^n) + I(\mathbf{x}_2^n; \mathbf{c}_1^n, \mathbf{y}_r^n), \tag{4.109}$$

where $\varepsilon_{2n} \to 0$ as $n \to \infty$. Next, we add (4.108) and (4.109) to obtain

$$n(R_\Sigma - \varepsilon_{1n} - \varepsilon_{2n}) \leq H(\mathbf{y}_1^n|\mathbf{c}_1^n, \mathbf{y}_r^n) + I(\mathbf{x}_1^n; \mathbf{c}_2^n, \mathbf{y}_r^n) + H(\mathbf{y}_2^n|\mathbf{c}_2^n, \mathbf{y}_r^n) + I(\mathbf{x}_2^n; \mathbf{c}_1^n, \mathbf{y}_r^n). \tag{4.110}$$

Let us write $H(\mathbf{y}_1^n|\mathbf{c}_1^n,\mathbf{y}_r^n)$ as follows

$$H(\mathbf{y}_1^n|\mathbf{c}_1^n,\mathbf{y}_r^n) = H(\mathbf{d}_1^n \oplus \mathbf{c}_2^n|\mathbf{c}_1^n,\mathbf{y}_r^n)$$

which is possible since $\mathbf{x}_r^n$ can be obtained from $\mathbf{y}_r^n$. Now assume that $n_c \geq n_s$. In this case we get

$$\begin{aligned} H(\mathbf{d}_1^n \oplus \mathbf{c}_2^n|\mathbf{c}_1^n,\mathbf{y}_r^n) &\overset{(b)}{=} H(\mathbf{d}_1^n \oplus \mathbf{c}_2^n|\mathbf{c}_1^n,\mathbf{s}_2^n) && (4.111)\\ &\overset{(c)}{\leq} n\max\{0, n_d - n_c, n_c - n_s\} && (4.112) \end{aligned}$$

where (b) follows since for $n_c \geq n_s$, given $\mathbf{c}_1^n$ we can extract $\mathbf{s}_1^n$, and (c) follows from Lemmata 2.4 and 2.5 on page 12. On the other hand, if $n_c \leq n_s$, then

$$\begin{aligned} H(\mathbf{d}_1^n \oplus \mathbf{c}_2^n|\mathbf{c}_1^n,\mathbf{y}_r^n) &\overset{(d)}{=} H(\mathbf{d}_1^n \oplus \mathbf{c}_2^n|\mathbf{c}_1^n,\mathbf{c}_2^n,\mathbf{y}_r^n) && (4.113)\\ &\leq H(\mathbf{d}_1^n|\mathbf{c}_1^n) && (4.114)\\ &\overset{(e)}{\leq} n\max\{0, n_d - n_c\} && (4.115) \end{aligned}$$

where (d) follows since knowing $\mathbf{c}_1^n$, the contribution of the first n_c bits of $\mathbf{x}_1^n$ in $\mathbf{y}_r^n$ can be subtracted, thus uncovering the first n_c bits of $\mathbf{x}_2^n$ which completely specify $\mathbf{c}_2^n$, and (e) follows from Lemma 2.5. By combining both cases $n_c \geq n_s$ and $n_c \leq n_s$, we can write

$$H(\mathbf{y}_1^n|\mathbf{c}_1^n,\mathbf{y}_r^n) \leq n\max\{0, n_d - n_c, n_c - n_s\}. \quad (4.116)$$

Similarly

$$H(\mathbf{y}_2^n|\mathbf{c}_2^n,\mathbf{y}_r^n) \leq n\max\{0, n_d - n_c, n_c - n_s\}. \quad (4.117)$$

Next, we write

$$\begin{aligned} I(\mathbf{x}_1^n;\mathbf{c}_2^n,\mathbf{y}_r^n) &= I(\mathbf{x}_1^n;\mathbf{c}_2^n) + I(\mathbf{x}_1^n;\mathbf{y}_r^n|\mathbf{c}_2^n) && (4.118)\\ &= I(\mathbf{x}_1^n;\mathbf{y}_r^n|\mathbf{c}_2^n) && (4.119)\\ &\leq I(\mathbf{x}_1^n,\mathbf{x}_2^n;\mathbf{y}_r^n|\mathbf{c}_2^n) && (4.120)\\ &\leq H(\mathbf{y}_r^n) && (4.121)\\ &\leq n[n_s]. && (4.122) \end{aligned}$$

Similarly

$$I(\mathbf{x}_2^n;\mathbf{c}_1^n,\mathbf{y}_r^n) \leq n[n_s]. \quad (4.123)$$

By substituting (4.116), (4.117), (4.122), and (4.123) in (4.110), dividing by n, and letting $n \to \infty$, we obtain the desired upper bound. □

To this end, we have presented the upper bounds that are necessary for the sum-capacity characterization of the LD-IRC with $n_s \leq n_c$. Recall that the presented bounds are general, in the sense that they hold even if $n_s \geq n_c$. Since our main focus here is the case $n_s \leq n_c$, we obtain the converse of Theorem 4.1 by writing the bounds in Theorems 4.2, 4.3, 4.4, 4.5, and 4.6 for this special case. The achievability of the sum-capacity is shown in the next section.

4.2.2 Sum-capacity Lower Bounds

The achievability of Theorem 4.1 is proved by using a novel compute-forward (CF) scheme for the LD-IRC next. CF has been already used in some relaying setups to perform physical layer network-coding [NG11, Naz12, WNPS10, AAT09]. The core idea of this scheme is allowing the relay to decode a function of the transmitted signals $\mathbf{x}_1^n$ and $\mathbf{x}_2^n$, and then to forward this function. This scheme exhibits several differences from traditional schemes such as decode-forward (DF) [CEG79, MDG12] and amplify-forward (AF). In DF , the relay decodes both the source messages m_1 and m_2. Note that this decoding imposes a strict rate constraint on the messages, namely, a multiple-access channel (MAC) type sum-rate constraint resulting from treating the channel between the users and the relay as a MAC. For the LD-IRC, such a rate constraint would look like

$$R_1 + R_2 \leq n_s. \tag{4.124}$$

On the other hand, if we construct the transmit signals $\mathbf{x}_1^n$ and $\mathbf{x}_2^n$ in such a way that their sum is decodable at the relay, then we have the CF rate constraint

$$\max\{R_1, R_2\} \leq n_s \tag{4.125}$$

which is clearly less restricting than the MAC sum-rate constraint above. This gives CF an advantage over DF in the 'decode' process; it remains to design the 'forward' process appropriately which is done in the sequel. On the other hand, one might argue that this makes CF similar to AF (at least for the deterministic case). However, this argument is refuted since CF gives the flexibility of re-allocating the bit levels of the forwarded signal arbitrarily, which is not possible if we use AF . As we shall see next, this re-allocation is necessary in some cases (see Figure 4.6(b)). Compress-forward is another common scheme that is used in relay setups [CEG79, TY11], however, its drawback is the relative complexity of its analysis compared to other schemes.

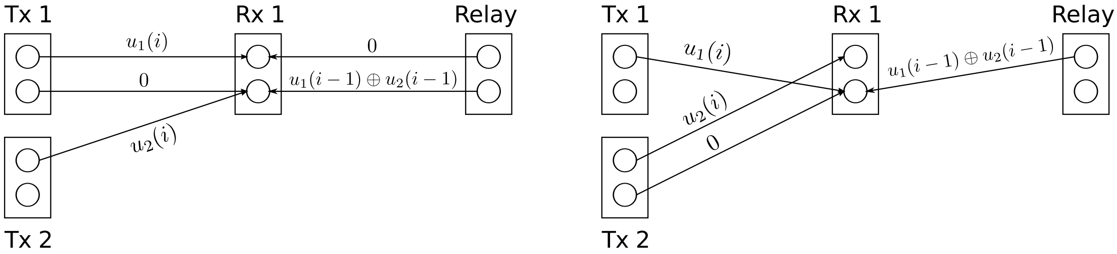
It is important to note the following: *In the LD-IRC, it is not always necessary to incorporate the relay in the transmission process.* For instance, if the channels n_r and n_s are both weak, then incorporating the relay would lead in some cases to unnecessarily binding rate constraints which can be avoided by switching the relay off. In these cases, one would rather operate the IRC as an IC.

4.2.2.1 A Case where the Relay does not Increase Capacity

The sum-rate given by the following expression

$$R_\Sigma = \min\left\{2\max\{n_d - n_c, n_c\}, 2\max\{n_d, n_c\} - n_c, 2n_d\right\} \tag{4.126}$$

is achievable in the LD-IRC (see (4.19)). This is achieved by switching the relay off and operating the IRC as an IC. According to the intuition stated above, there might be cases where this sum-rate is indeed the sum-capacity of the LD-IRC, namely, when the relay channels are weak. Now let us examine the statement of Theorem 4.1 for such a case. The converse of this theorem has been shown in Section 4.2.1, and hence, (4.18) is a valid upper bound. We notice that if in addition to $n_s \leq n_c$, we have $n_r \leq \min\{n_c, n_d\}$, then this upper bound reduces to (4.126). Therefore, the LD-IRC has the same sum-capacity as the corresponding LD-IC obtained by removing the relay,



(a) Weak interference: Rx 1 decodes $u_1(i)$, extracts $u_2(i)$ from $u_1(i) \oplus u_2(i)$ (decoded in channel use $i+1$), subtracts $u_2(i)$, and then decodes $u_1(i-1) \oplus u_2(i-1)$ interference free.

(b) Strong interference: Rx 1 decodes $u_2(i)$, extracts $u_1(i)$ from $u_1(i) \oplus u_2(i)$ (decoded in channel use $i+1$), subtracts $u_1(i)$, and then decodes $u_1(i-1) \oplus u_2(i-1)$ interference free.

Figure 4.5: The CF strategy in an LD-IRC with $n_s = 1$ (the relay has been repositioned for clarity). Due to backward decoding, the relay signal is allowed to interfere with one of the transmit signals while still being decodable, which allows a more efficient use of the bit-pipes.

and consequently, the relay does not increase the sum-capacity of the network in this case.

In what follows, we focus on the remaining cases, where the relay can increase the capacity of the network. In such cases, it is required to find a unified scheme which takes care of both scenarios (strong and weak relay channels). Namely, we need to develop a combination of CF and optimal schemes for the LD-IC, i.e., private and common signaling with successive decoding [ETW08, BT08]. This combination will be constructed in the next paragraphs, but before we proceed to the relatively involved combination of strategies, we describe the CF strategy separately.

4.2.2.2 Compute-forward

The CF strategy works as follows (see Figure 4.5 for an example). Each source sends a signal $\mathbf{u}_j(i)$ with rate R_u in the ith channel use where $i = 1, \cdots, n$ and $j = 1, 2$. That is, user j sends

$$\mathbf{x}_j(i) = \begin{bmatrix} \mathbf{u}_j(i) \\ \mathbf{0}_{q-R_u} \end{bmatrix}. \tag{4.127}$$

The signal $\mathbf{u}_j(i)$ is allocated at the topmost positions of $\mathbf{x}_j(i)$ in order to allow the relay to observe as many bits of $\mathbf{u}_j(i)$ as possible. The relay decodes the sum

$$\bar{\mathbf{u}}(i) = \mathbf{u}_1(i) \oplus \mathbf{u}_2(i), \tag{4.128}$$

which is possible if

$$R_u \leq n_s. \tag{4.129}$$

Then, the relay sends $\bar{\mathbf{u}}(i)$ in the next channel use $i+1$ as follows. The vector $\bar{\mathbf{u}}(i)$ is split into two parts, $\bar{\mathbf{u}}_a(i)$ and $\bar{\mathbf{u}}_b(i)$ of length R_{ua} and R_{ub}, respectively, where $R_u = R_{ua} + R_{ub}$. The purpose of this splitting is to allow different decoding orders at the receivers as we shall see next. The vector $\bar{\mathbf{u}}_a(i)$ is sent on the topmost R_{ua} levels at the relay, and $\bar{\mathbf{u}}_b(i)$ is sent on levels $\{q - n_r + \ell_b - R_{ub} + 1, \cdots, q - n_r + \ell_b\}$ for some integer $\ell_b \leq n_r$. This parameter ℓ_b can be seen as a power allocation parameter for the signal $\bar{\mathbf{u}}_b(i)$. In fact, ℓ_b is the level below which $\bar{\mathbf{u}}_b(i)$ is observed at the receivers, thus, ℓ_b specifies the power at which $\bar{\mathbf{u}}_b(i)$ is received at the receivers relative to the power of noise. Using this construction, we can write

$$\mathbf{x}_r(i+1) = \begin{bmatrix} \bar{\mathbf{u}}_a(i) \\ \mathbf{0}_{n_r-\ell_b-R_{ua}} \\ \bar{\mathbf{u}}_b(i) \\ \mathbf{0}_{\ell_b-R_{ub}} \\ \mathbf{0}_{q-n_r} \end{bmatrix}, \tag{4.130}$$

which requires

$$R_u \leq n_r. \tag{4.131}$$

Note that due to delay, the relay transmits in channel uses $2, \cdots, n+1$. Thus, the overall transmission takes place in $n+1$ channel uses.

The decoding process at the receivers is performed backwards from channel use $n+1$ till 1. In channel use $n+1$, only the relay is active, and the receivers decode $\bar{\mathbf{u}}(n)$ which is possible given (4.131). The receivers then proceed backwards to process the received signal in channel use n. The decoding process at the receivers in time instants $n, \cdots, 1$ depends on whether the LD-IRC has weak interference (Figure 4.5(a)) or strong interference (Figure 4.5(b)). Let us first consider the weak interference case.

4.2.2.2.1 CF with Weak Interference: Consider the decoding process at Rx 1 in time instant n. First, $\bar{\mathbf{u}}_a(n-1)$ is decoded while treating the remaining signals as noise, which requires

$$R_{ua} \leq (n_r - n_d)^+ \tag{4.132}$$

due to the interference from $\mathbf{u}_1(n)$. Next, Rx 1 decodes $\mathbf{u}_1(n)$ bit by bit successively, while simultaneously canceling interference from $\mathbf{u}_2(n)$ (see Figure 4.5(a) for an example with $R_u = 1$). This is possible since Rx 1 knows $\bar{\mathbf{u}}(n) = \mathbf{u}_1(n) \oplus \mathbf{u}_2(n)$ in time instant n since it was decoded in time instant $n+1$. After decoding each bit of $\mathbf{u}_1(n)$, the corresponding bit of $\mathbf{u}_2(n)$ is removed, and this continues till all bits of $\mathbf{u}_1(n)$ are decoded and thus all bits of $\mathbf{u}_2(n)$ are removed. Since the only remaining interference comes from $\bar{\mathbf{u}}_b(n-1)$, then $\mathbf{u}_1(n)$ can be decoded as long as it does not overlap with $\bar{\mathbf{u}}_b(n-1)$ at the receiver, i.e., we need

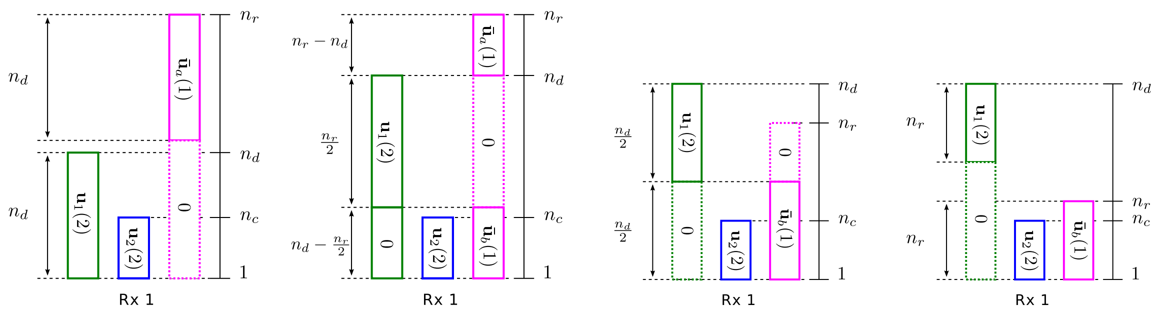
$$R_u \leq n_d - \ell_b, \tag{4.133}$$

with $\ell_b \leq n_d$. Finally, $\bar{\mathbf{u}}_b(n-1)$ is decoded which is possible if

$$R_{ub} \leq \ell_b. \tag{4.134}$$

Now the receiver has obtained $\bar{\mathbf{u}}(n-1)$ and it can proceed to the previous channel use $n-1$. The same process is repeated till channel use 1 is reached. By collecting the rate constraints (4.129), (4.131), (4.132), (4.133), and (4.134), we get the following achievable rate

$$R_u \leq \min\{n_s, n_d - \ell_b, (n_r - n_d)^+ + \ell_b\} \tag{4.135}$$



(a) $n_r \geq 2n_d$. (b) $n_r \geq n_d$. (c) $n_d/2 \leq n_r \leq n_d$. (d) $n_r \leq n_d/2$.

Figure 4.6: Signal level diagrams for CF showing the binary vectors whose sum composes the received signal at Rx 1 at the second channel use.

for some $\ell_b \leq \min\{n_d, n_r\}$. The parameter ℓ_b can be easily optimized to obtain the achievable sum rate

$$R_\Sigma \leq \min\{2n_d, 2n_r, 2n_s, \max\{n_d, n_r\}\}. \tag{4.136}$$

Since source transmission takes place in channel uses $i = 1, \cdots, n$ while the overall transmission requires $n+1$ channel uses, this incurs a rate loss of a fraction $1/(n+1)$ of R_Σ. However, this loss will be neglected since it can be made arbitrarily small by increasing n.

Figure 4.6 shows a signal level diagram for the CF strategy in different scenarios of an LD-IRC with weak interference. The rectangles indicate the shifted vectors $\mathbf{x}_1(i)$, $\mathbf{x}_2(i)$ and $\mathbf{x}_r(i)$ at Rx 1 which observes their sum. The reason for splitting the relay CF signal $\bar{\mathbf{u}}(i)$ into two parts can be seen in Figures 4.6(a) and 4.6(b). The intuition is that the topmost $(n_r - n_d)^+$ levels of the relay signal can be observed at the Rx without any interference. In other words, there is no 'competition' between the relay and the transmitters on these levels. Thus, these levels can be exploited by the relay. On the other hand, the transmitters have to send their CF signals $\mathbf{u}_1(i)$ and $\mathbf{u}_2(i)$ on the their topmost levels so that they are all observed at the relay. In the weak interference regime ($n_c \leq n_d$), the receiver is required to decode $\mathbf{u}_1(i)$. Thus the relay signal has to be designed in such a way that it does not interfere with $\mathbf{u}_1(i)$ at the receiver. Thus, the relay sends a block of zeros on the corresponding levels. Finally, the remaining bits of $\bar{\mathbf{u}}(i)$ (if any) are sent on the remaining levels. Recall that interference from $\mathbf{u}_2(i)$ can be ignored since we subtract it after decoding $\mathbf{u}_1(i)$ and $\bar{\mathbf{u}}(i)$. Given n_s is large, the achievable rate by each user is $R_u = n_r/2$ in this case. Figure 4.6(c) shows a case were $n_r \leq n_d$, in which case each user achieves $R_u = n_d/2$ bits, and Figure 4.6(d) shows a case where $n_r \leq n_d/2$ where each user achieves n_r bits.

4.2.2.2.2 CF with Strong Interference: The strong interference case differs from the weak interference case in that the receiver decodes the interfering CF signal and the relay signal (see Figure 4.5(b) for an example with $R_u = 1$). For instance, Rx 1 decodes $\mathbf{u}_2$ and $\bar{\mathbf{u}}$ as follows. In time instant n, it starts with decoding $\bar{\mathbf{u}}_a(n-1)$, which is possible if

$$R_{ua} \leq (n_r - n_c)^+ \tag{4.137}$$

due to the interference from $\mathbf{u}_2(n)$ which is treated as noise. Next, it decodes $\mathbf{u}_2(n)$ bit by bit. It uses $\mathbf{u}_2(n)$ and $\bar{\mathbf{u}}(n)$ (decoded in time instant $n+1$) to extract its desired signal $\mathbf{u}_1(n)$ and cancel its interference. Using this decoding process, the only remaining interference is that from $\bar{\mathbf{u}}_b(n-1)$. Thus, $\mathbf{u}_2(n)$ can be decoded if

$$R_u \leq n_c - \ell_b, \tag{4.138}$$

for some $\ell_b \leq n_c$. Finally, $\bar{\mathbf{u}}_b(n-1)$ is decoded which requires

$$R_{ub} \leq \ell_b. \tag{4.139}$$

The receiver proceeds with backward decoding till channel use 1 is reached. Collecting the rate constraints (4.129), (4.131), (4.137), (4.138), and (4.139), and maximizing with respect to $\ell_b \leq \min\{n_r, n_c\}$, we get the following achievable sum-rate

$$R_\Sigma \leq \min\{2n_c, 2n_r, 2n_s, \max\{n_c, n_r\}\}. \tag{4.140}$$

Remark 4.3. *Note that in our* CF *strategy, decoding information from the relay signal* $\bar{\mathbf{u}}$ *requires knowledge of* $\mathbf{u}_1$ *or* $\mathbf{u}_2$. *This is in contrast to classical strategies (such as decode-forward and compress-forward) where the receivers can obtain some desired information directly from the relay signal.*

4.2.2.3 Combined Schemes

Now, we are ready to present the capacity achieving scheme corresponding to Theorem 4.1. As mentioned earlier, it consists of a combination of CF, common signaling and private signaling. While common and private signaling are used to recover the sum-capacity of the LD-IC [BT08], CF is used to exploit the relay gain when possible. We will describe the encoding and decoding steps required for this scheme, and then provide the achievable sum-rate in a theorem. We start with the weak interference case.

4.2.2.3.1 Weak Interference Scheme: In the weak interference (WI) case where $n_c \leq n_d$, rate splitting is used at the transmitters in order to split each message into a CF, a common, and a private message. Thus, m_1 for instance is split into

- a CF message m_{1u} with rate $R_u = R_{ua} + R_{ub}$,
- a common (C) message m_{1v} with rate R_v, and
- a private (P) message m_{1w} with rate R_w.

We encode m_{1u}, m_{1v}, and m_{1w} into sequences $\mathbf{u}_1^n$, $\mathbf{v}_1^n$, and $\mathbf{w}_1^n$, where the lengths of $\mathbf{u}_1(i)$, $\mathbf{v}_1(i)$, $\mathbf{w}_1(i)$ are R_u, $\ell_v - \ell_w$ ($\ell_v \geq \ell_w$), and ℓ_w, respectively. Then, Tx 1 sends the CF signal $\mathbf{u}_1(i)$ on the topmost levels, the C signal $\mathbf{v}_1(i)$ on the levels below $q - n_d + \ell_v$, and the P signal $\mathbf{w}_1(i)$ on the levels below $q - n_d + \ell_w$. Thus, at Rx 1, the CF signal, the C signal, and the P signal are received below the levels n_d, ℓ_v, and ℓ_w, respectively, where

$$R_u + \ell_v \leq n_d. \tag{4.141}$$

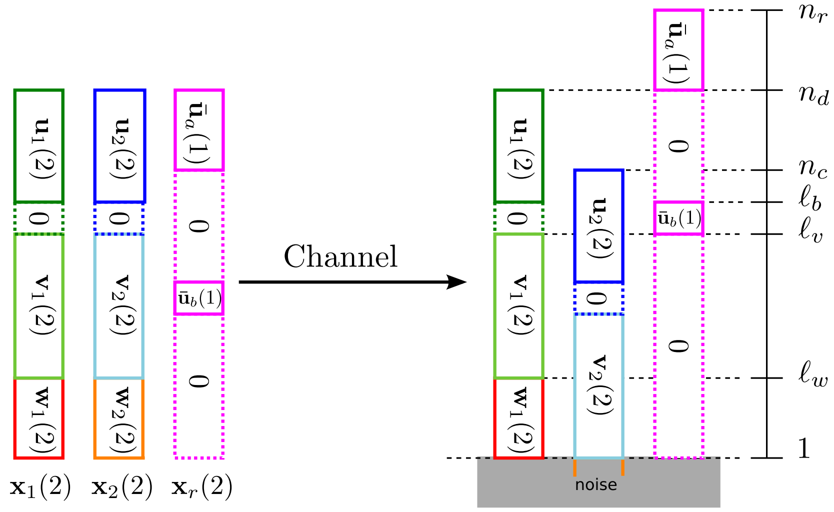


Figure 4.7: A signal level diagram for the WI-CF scheme showing Rx 1 at channel use 2. The relay only decodes the sum of $\mathbf{u}_1$ and $\mathbf{u}_2$. Rx 1 decodes all signals except $\mathbf{w}_2$ which is treated as noise, and $\mathbf{u}_2$ which is extracted from $\mathbf{u}_1$ and $\bar{\mathbf{u}}$.

Time	Tx 1 sends	Relay	
		Decodes	Sends
1	$\mathbf{u}_1(1), \mathbf{v}_1(1), \mathbf{w}_1(1)$	$\bar{\mathbf{u}}(1)$	-
2	$\mathbf{u}_1(2), \mathbf{v}_1(2), \mathbf{w}_1(2)$	$\bar{\mathbf{u}}(2)$	$\mathbf{x}_r(2) = f_r(\bar{\mathbf{u}}(1))$
⋮	⋮	⋮	⋮
n	$\mathbf{u}_1(n), \mathbf{v}_1(n), \mathbf{w}_1(n)$	$\bar{\mathbf{u}}(n)$	$\mathbf{x}_r(n) = f_r(\bar{\mathbf{u}}(n-1))$
n+1	-	-	$\mathbf{x}_r(n+1) = f_r(\bar{\mathbf{u}}(n))$

Table 4.1: The transmit signals of Tx 1 and by the relay in each channel use $i = 1, \cdots, n+1$. Tx 1 transmits in $i = 1, \cdots, n$ where it sends CF, C, and P signals. The relay decodes the sum $\bar{\mathbf{u}}(i) = \mathbf{u}_1(i) \oplus \mathbf{u}_2(i)$, then it maps it to $\mathbf{x}_r(i+1)$ to be forwarded in channel use $i+1$.

The overall construction of $\mathbf{x}_1(i)$ becomes

$$\mathbf{x}_1(i) = \begin{bmatrix} \mathbf{u}_1(i) \\ \mathbf{0}_{n_d - R_u - \ell_v} \\ \mathbf{v}_1(i) \\ \mathbf{w}_1(i) \\ \mathbf{0}_{q-n_d} \end{bmatrix} \tag{4.142}$$

for $i = 1, \cdots, n$ (see Figure 4.7). Tx 1 is silent in time instant $n+1$. A similar splitting and encoding is used by Tx 2.

In our strategy, the relay is only interested in the sum of the CF signal $\bar{\mathbf{u}}(i)$. It uses forward decoding by starting with $i = 1$ till $i = n$ as shown in Table 4.1. In channel use i, the relay decodes $\bar{\mathbf{u}}(i)$. This decoding is possible if the topmost R_u bits of $\mathbf{x}_1(i)$ and $\mathbf{x}_2(i)$ are observed at the relay, i.e.,

$$R_u \leq n_s. \tag{4.143}$$

After decoding $\bar{\mathbf{u}}(i)$, the relay splits it into $\bar{\mathbf{u}}_a(i)$ and $\bar{\mathbf{u}}_b(i)$ of lengths R_{ua} and R_{ub}, respectively, and then sends $\mathbf{x}_r(i+1)$ as in (4.130) in time slot $i+1$, $i = 1, \cdots, n$, for

Time	Step 1	Step 2	Step 3	Step 4
n+1	$\bar{\mathbf{u}}_a(n)$	-	$\bar{\mathbf{u}}_b(n)$	-
n	$\bar{\mathbf{u}}_a(n-1)$	$\mathbf{u}_1(n)$, $\mathbf{u}_2(n) = \bar{\mathbf{u}}(n) \oplus \mathbf{u}_1(n)$	$\bar{\mathbf{u}}_b(n)$	$\mathbf{v}_1(n)$, $\mathbf{v}_2(n)$, $\mathbf{w}_1(n)$
⋮	⋮	⋮	⋮	⋮
2	$\bar{\mathbf{u}}_a(1)$	$\mathbf{u}_1(2)$, $\mathbf{u}_2(2) = \bar{\mathbf{u}}(2) \oplus \mathbf{u}_1(2)$	$\bar{\mathbf{u}}_b(2)$	$\mathbf{v}_1(2)$, $\mathbf{v}_2(2)$, $\mathbf{w}_1(2)$
1	-	$\mathbf{u}_1(1)$, $\mathbf{u}_2(1) = \bar{\mathbf{u}}(1) \oplus \mathbf{u}_1(1)$	-	$\mathbf{v}_1(1)$, $\mathbf{v}_2(1)$, $\mathbf{w}_1(1)$

Table 4.2: Decoding steps at the Rx 1. In step 1, the first part of the relay signal is decoded. Then the desired CF signal is decoded, and the interference from the undesired CF signal is removed. Next, the second part of the relay signal is decoded, and finally, the two common signals and the desired private one are decoded.

some level allocation parameter $\ell_b \leq n_r$. Note that the relay sends nothing in time instant 1. This construction of the relay signal works if

$$R_u \leq n_r. \tag{4.144}$$

Decoding at the receivers proceeds as shown in Table 4.2. The CF signals are decoded as explained in Section 4.2.2.2.1 with the difference that $\mathbf{v}_j(i)$ and $\mathbf{w}_j(i)$ are treated as noise. The rate constraints remain the same (as (4.129), (4.131), (4.132), and (4.133)) except the one for decoding $\bar{\mathbf{u}}_b(i)$ (i.e., (4.134)) which becomes

$$R_{ub} \leq (\ell_b - \ell_v)^+, \tag{4.145}$$

due to the interference from the C and P signals. By combining (4.129), (4.131), (4.132), and (4.145), we obtain the CF rate constraint as follows

$$R_u \leq \min\{n_s, n_r, n_d - \ell_v, n_d - \ell_b, (n_r - n_d)^+ + (\ell_b - \ell_v)^+\} \tag{4.146}$$

for some $\ell_b \leq \min\{n_d, n_r\}$, $\ell_w \leq \ell_v \leq n_d$. Since $(n_r - n_d)^+ + (\ell_b - \ell_v)^+ \leq n_r$, we can simplify this rate constraint to

$$R_u \leq \min\{n_s, n_d - \ell_v, n_d - \ell_b, (n_r - n_d)^+ + (\ell_b - \ell_v)^+\}. \tag{4.147}$$

After decoding the CF signals, their contribution can be removed and only the common and the private signals remain. The resulting received signal after subtracting the CF signals is exactly the same as the received signal in a LD-IC with $\bar{n}_d = \ell_v$ and $\bar{n}_c = (\ell_v - n_d + n_c)^+$. As is already common in the IC, we set the power level of the P signal so that it has the same power level as noise at the undesired receiver [ETW08], i.e., $\ell_w = n_d - n_c$. Plugging these values of $\bar{n}_d$ and $\bar{n}_c$ in the the sum-capacity expression of the LD-IC [BT08, Lemma 4], and using $\ell_w = n_d - n_c$, we get the achievable rate of the C and P message as

$$R_v \leq \min\left\{\frac{\ell_v - n_d + n_c}{2}, (\ell_v - 2n_d + 2n_c)^+\right\} \tag{4.148}$$

$$R_w \leq n_d - n_c. \tag{4.149}$$

Combining (4.147), (4.148), and (4.149), we get the following theorem for the achievable rate of our WI-CF scheme.

Theorem 4.7 (WI-CF). *The sum-rate given by*

$$R_\Sigma = 2R_u + 2R_v + 2R_w, \tag{4.150}$$

where R_u, R_v, and R_w satisfy (4.147), (4.148), *and* (4.149), $\ell_w = n_d - n_c$, *for some $\ell_w \leq \ell_v \leq n_d$ and $\ell_b \leq \min\{n_d, n_r\}$ is achievable.*

Remark 4.4. *Note that if we set $R_u = 0$ and $\ell_v = n_d$, then we recover the achievable sum-capacity of the LD-IC [BT08] with weak interference. Thus, our* WI-CF *scheme is at least as good as that in [BT08] in the weak interference regime.*

The decoding order given in the WI-CF scheme is fixed so that the CF signals are decoded first, and the C and P signals last. While different decoding orders can also be used, we stick to this order because it turns out to be sum-capacity achieving.

4.2.2.3.2 Strong Interference Scheme: In the WI-CF scheme, we have forced the receiver to decode its desired CF signal and then extract the interfering CF signal and use it for interference cancellation. Alternatively, the receiver can also start by decoding the interfering CF signal and then extract its desired CF signal. We call this scheme the strong interference (SI) CF scheme. As private signaling is not needed for achieving capacity in the IC with SI, it is also not needed in the LD-IRC with SI. Thus, we omit the P message part in this scheme. The two remaining messages are the CF message and the C message. Tx 1 sends

$$\mathbf{x}_1(i) = \begin{bmatrix} \mathbf{u}_1(i) \\ \mathbf{0}_{n_c - R_u - \ell_v} \\ \mathbf{v}_1(i) \\ \mathbf{0}_{q-n_c} \end{bmatrix} \tag{4.151}$$

for $i = 1, \cdots, n$, where

$$R_u \leq n_c - \ell_v. \tag{4.152}$$

Notice the ℓ_v denotes the level at which $\mathbf{v}_1(i)$ is observed at Rx 2 (instead of Rx 1 for the WI-CF scheme). The decoding of the CF message is done as in Section 4.2.2.2.2 with the exception that in this case, the C signals are treated as noise. All the CF rate constraints remain the same except the one for decoding $\bar{\mathbf{u}}_b(i)$ which becomes

$$R_{ub} \leq (\ell_b - \ell_v)^+. \tag{4.153}$$

for some $\ell_b \leq \min\{n_c, n_r\}$ and $\ell_v \leq n_c$. Combining this rate constraint with (4.129), (4.131), (4.137), (4.138), and (4.154) we get the following achievable CF rate

$$R_u \leq \min\{n_s, n_c - \ell_v, n_c - \ell_b, (n_r - n_c)^+ + (\ell_b - \ell_v)^+\}. \tag{4.154}$$

After decoding the CF signals, the two C signals are decoded jointly as in [BT08, Lemma 4] achieving

$$R_v \leq \min\left\{\frac{\ell_v}{2}, (\ell_v + n_d - n_c)^+\right\}. \tag{4.155}$$

Combining (4.154) and (4.155), we get the following theorem for the achievable rate of our SI-CF scheme.

Theorem 4.8 (SI-CF Scheme)**.** *The sum-rate given by*

$$R_\Sigma = 2R_u + 2R_v, \tag{4.156}$$

where R_u and R_v satisfy (4.154) *and* (4.155) *for some $\ell_v \leq n_c$ and $\ell_b \leq \min\{n_c, n_r\}$ is achievable.*

Remark 4.5. *Similar to the WI-CF scheme, if we set $R_u = 0$ and $\ell_v = n_c$ in the SI-CF scheme, then we recover the achievable sum-capacity of the LD-IC with SI [BT08].*

The given WI-CF and SI-CF schemes achieve the sum-capacity of the LD-IRC. To show this, one has to carefully choose the parameters ℓ_b, ℓ_v, and ℓ_w, and then compare the achievable rates to the upper bounds.

4.2.3 Sum-Capacity Characterization

Here, we show that the schemes given in Section 4.2.2.3 achieve the sum-capacity of the LD-IRC if $n_s \leq n_c$. In order to achieve the sum-capacity, one has to maximize the expressions (4.150) and (4.156) over ℓ_b and ℓ_v. This maximization is not difficult, but lengthy. In what follows, we give solutions for ℓ_b and ℓ_v that achieve the sum-capacity. The main result of this section is the following theorem.

Theorem 4.9. *The achievable sum-rates given in Theorems 4.7 and 4.8 coincide with the sum-capacity of the LD-IRC with weak interference ($n_c \leq n_d$) and strong interference ($n_c > n_d$), respectively, as long as $n_s \leq n_c$.*

This theorem is proved in the following paragraphs. We also split the analysis here into two parts, weak interference and strong interference.

4.2.3.1 Weak Interference Sum-capacity

Let us first write the sum-capacity expression of Theorem 4.1, where $n_s \leq n_c$, for the WI regime where $n_c \leq n_d$. Since we are considering the WI regime, and since $n_s \leq n_c$, therefore, $n_s \leq n_d$. We get

$$C_\Sigma = \min \left\{ \begin{array}{c} \max\{n_d, n_r\} + n_d - n_c \\ 2n_d - n_c + n_s \\ 2\max\{n_c, n_r, n_d - n_c\} \\ 2\max\{n_c, n_d + n_s - n_c\} \end{array} \right\}. \tag{4.157}$$

Recall that the case $n_r < \min\{n_d, n_c\} = n_c$ has been discussed in Section 4.2.2.1 where it was shown that the relay does not provide any sum-capacity gain. Since our WI-CF scheme is at least as good as the capacity achieving scheme of the LD-IC with WI, then our scheme achieves the upper bound in this case. Thus it remains to consider the other case where $n_s \leq n_c \leq \min\{n_r, n_d\}$. We will consider three WI sub-regimes.

WI-1: If $n_r \leq n_d/2$, it can be easily shown that the sum-capacity reduces to $C_\Sigma = 2n_d - 2n_c$. This sum-capacity can be achieved by the WI-CF scheme since it is also achievable in the LD-IC.

WI-2: Next, consider the case $n_d/2 \leq n_r \leq n_d$. In this case, we can write the sum-capacity expression as follows

$$C_\Sigma = \min \left\{ \begin{array}{c} 2n_d - n_c \\ 2\max\{n_r, n_d - n_c\} \\ 2\max\{n_c, n_d + n_s - n_c\} \end{array} \right\}. \tag{4.158}$$

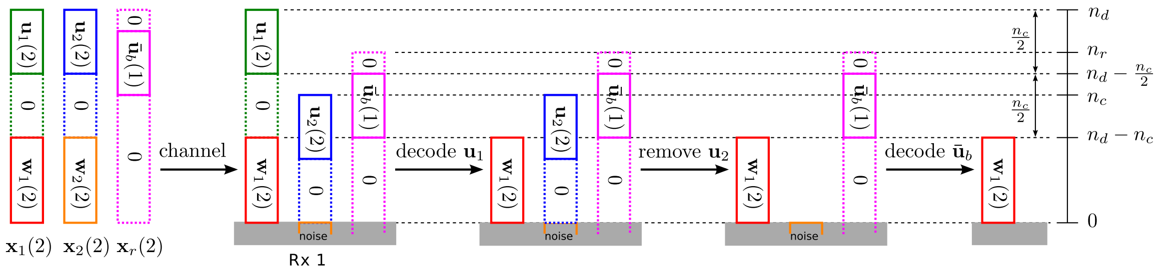


Figure 4.8: A signal level diagram for an LD-IRC where we can achieve $R_\Sigma = 2n_d - n_c$. The achievable rates of the CF message and the P message are $n_c/2$ and $n_d - n_c$, respectively. Notice that the relay sends some zeros on the topmost levels, i.e., it does not send at full power. If we increase the power of $\bar{\mathbf{u}}_b$, then R_u decreases since we have higher noise when decoding $\mathbf{u}_1$ leading to a lower achievable rate.

We will first split this expression into cases where C_Σ matches the sum-capacity of the LD-IC, and other cases where it does not. Namely, we rewrite it as

$$C_\Sigma = \begin{cases} 2n_d - 2n_c & \text{if } n_c \le n_d - n_r \\ \min\{2n_d - n_c, 2n_c\} & \text{if } n_c \ge \frac{n_d+n_s}{2} \\ \min\{2n_d - n_c, 2n_r, 2n_d + 2n_s - 2n_c\} & \text{if } n_d - n_r \le n_c \le \frac{n_d+n_s}{2} \end{cases} \tag{4.159}$$

The first case ($n_c \le n_d - n_r$) matches the sum-capacity of the LD-IC (treating interference as noise) and thus it can be achieved by the WI-CF scheme. The second case also matches the sum-capacity of the LD-IC with $n_c \ge n_d/2$ (see (4.126)), and thus it can be achieved by the WI-CF scheme (note that in this case $n_c \ge (n_d+n_s)/2 \ge \frac{n_d}{2}$).

In the third case ($n_d - n_r \le n_c \le (n_d+n_s)/2$), C_Σ does not match the sum-capacity of the LD-IC, and is equal to

$$C_\Sigma = \begin{cases} 2n_d - n_c & \text{if } 2n_d - 2n_r \le n_c \le 2n_s \\ 2n_r & \text{if } n_c \le \min\{2n_d - 2n_r, n_d + n_s - n_r\} \\ 2n_d + 2n_s - 2n_c & \text{if } n_c \ge \max\{2n_s, n_d + n_s - n_r\} \end{cases} \tag{4.160}$$

The first case in (4.160) can be achieved by setting $\ell_b = (2n_d - n_c)/2$ and $\ell_v = n_d - n_c$. The second case can be achieved by setting $\ell_b = n_r$ and $\ell_v = n_d - n_c$. Finally, the third case in (4.160) can be achieved by setting $\ell_b = n_d + n_s - n_c$ and $\ell_v = n_d - n_c$. Therefore, the sum-capacity in the WI-2 regime is achieved by the WI-CF scheme.

Remark 4.6. *Interestingly, in some cases, the relay does not have to use its full power to achieve the sum-capacity of the LD-IRC, e.g., in the first and third cases above where* $\ell_b < n_r$ *(see Figure 4.8).*

Remark 4.7. *Note that in some cases, the level allocation parameters can be rational. These can be made integer by considering multiple channel uses. For instance, in the first case in* (4.160)*, we need* $\ell_b = (2n_d - n_c)/2$*. In this case, if we consider two channel uses, we get an IRC with levels* $2(n_d, n_c, n_r, n_s)$ *where where* ℓ_b *has to be set to* $2n_d - n_c$ *which is integer.*

The level allocation can be interpreted as follows. First the P signal is set in such a way that it arrives below the noise level at the undesired receiver (as in the IC [ETW08]). Thus, the P signal occupies the lowest $n_d - n_c$ levels at the desired receiver. Now the number of remaining unused levels is $n_d - (n_d - n_c) = n_c$. Our aim is to send the CF signal such that we are still able to perform interference cancellation by proceeding backwards from one channel use to another. Thus, we need to be able to decode the relay signal and the desired CF signal reliably at the destinations. These two signals have to be accommodated at the remaining n_c levels at the receiver, i.e., on levels $\{n_d - n_c + 1, \cdots, n_d\}$. This interval should be optimally divided into two parts, one for the desired CF signal and the other for the relay signal. At best, we can assign $n_c/2$ to each of them as shown in Figure 4.8. In this case, the relay signal is below level $(2n_d - n_c)/2$ at the receiver, and hence we need to set $\ell_b = (2n_d - n_c)/2$. As a result, we can send $n_c/2$ CF bits if $n_c/2 \le n_s$ for the relay to be able to decode the sum of the CF signals. Moreover, we must have $(2n_d - n_c)/2 \le n_r$ which implies $n_c/2 \le n_r - n_d + n_c$. If these conditions hold, then we can achieve $2n_d - n_c$ bits, otherwise, we achieve $2(n_d - n_c + \min\{n_s, n_r - n_d + n_c\}) = \min\{2n_d + 2n_s - 2n_c, 2n_r\}$ (see the three cases in (4.160)).

WI-3: In this case we have $n_d \le n_r$. The sum-capacity in this case is given by

$$C_\Sigma = \min \left\{ \begin{array}{c} n_r + n_d - n_c \\ 2n_d - n_c + n_s \\ 2\max\{n_c, n_d + n_s - n_c\} \end{array} \right\}. \tag{4.161}$$

Let us first rewrite this expression for convenience as

$$C_\Sigma \le \begin{cases} n_d + n_r - n_c, & n_c \le n_d + 2n_s - n_r, \frac{n_d+n_s}{2} \\ 2n_d + 2n_s - 2n_c, & n_d + 2n_s - n_r < n_c \le \frac{n_d+n_s}{2} \\ \min\{2n_c, n_d + n_r - n_c\}, & n_c > \frac{n_d+n_s}{2},\ n_s > n_r - n_d \\ \min\{2n_c, 2n_d + n_s - n_c\}, & n_c > \frac{n_d+n_s}{2},\ n_s \le n_r - n_d \end{cases} \tag{4.162}$$

The first case in (4.162) can be achieved by setting the parameters of the WI-CF scheme to $\ell_b = (3n_d - n_r - n_c)/2$ and $\ell_v = n_d - n_c$, and the second case can be achieved by setting $\ell_b = 2n_d + n_s - n_r - n_c$ and $\ell_v = n_d - n_c$. On the other hand, in the third and last cases in (4.162), the common messages become necessary. Moreover, in these cases we do not need $\bar{\mathbf{u}}_b(i)$, i.e., the relay sends $\bar{\mathbf{u}}(i)$ only on the topmost levels. We set $\ell_b = \ell_v = 2n_d - n_r$ to achieve the third case in (4.162), and we set $\ell_b = \ell_v = n_d - n_s$ to achieve the fourth case.

By this point, we have proved Theorem 4.9 for $n_s \le n_c \le n_d$. Next, we consider the strong interference case where $\max\{n_s, n_d\} \le n_c$.

4.2.3.2 Strong Interference Sum-capacity

Now, we consider the LD-IRC with strong interference (SI) $n_c \ge n_d$ and with $n_s \le n_c$. We rewrite the sum-capacity expression in Theorem 4.1 for this case as

$$C_\Sigma = \min \left\{ \begin{array}{c} 2\max\{n_d, n_r\} \\ 2\max\{n_d, n_s\} \\ \max\{n_c, n_r\} \\ n_c + n_s \end{array} \right\}. \tag{4.163}$$

In order to simplify the achievability proof of this sum-capacity, we split the SI regime into two cases.

SI-1: If $n_s \le n_c < n_r$, then we can write C_Σ as

$$C_\Sigma = \min \left\{ \begin{array}{c} 2\max\{n_d, n_s\} \\ n_r \\ n_c + n_s \end{array} \right\}. \tag{4.164}$$

We write this sum-capacity for convenience as

$$C_\Sigma \leq \begin{cases} \min\{n_r, 2n_s\}, & n_d < n_s \leq n_c < n_r \\ \min\{n_c + n_s, n_r, 2n_d\}, & n_s \leq n_d < n_c < n_r \end{cases} \tag{4.165}$$

The first case in (4.165) can be achieved by using the SI-CF scheme in Section 4.2.2.3.2 with $\ell_b = {}^{(2n_c - n_r)}/2$ and $\ell_v = 0$. The second case can be achieved by setting $\ell_b = \ell_v = n_c - n_s$ to achieve $n_c + n_s$ if $n_c \leq \min\{n_r, 2n_d\} - n_s$, and $\ell_b = \ell_v = 2n_c - n_r$ to achieve $\min\{n_r, 2n_d\}$ otherwise.
SI-2: For the other case where $n_r \leq n_c$, the capacity is given by

$$C_\Sigma = \min \left\{ \begin{matrix} 2\max\{n_d, n_r\} \\ 2\max\{n_d, n_s\} \\ n_c \end{matrix} \right\}. \tag{4.166}$$

Let us rewrite this capacity expression as

$$C_\Sigma \leq \begin{cases} \min\{n_c, 2n_d\} & \min\{n_r, n_s\} \leq n_d \\ \min\{n_c, 2n_s\} & n_r \geq n_s \geq n_d \\ \min\{n_c, 2n_r\} & n_s \geq n_r \geq n_d \end{cases} \tag{4.167}$$

Observe that the capacity expression in the first case of (4.167) is achievable by our SI-CF scheme since it is achievable in the LD-IC with strong interference (see Section 4.2.2.1). In the second and third cases, we use our SI-CF scheme while setting $\ell_b = \min\{n_r, {}^{n_c}/2\}$ and $\ell_v = 0$, which achieves the sum-capacity.

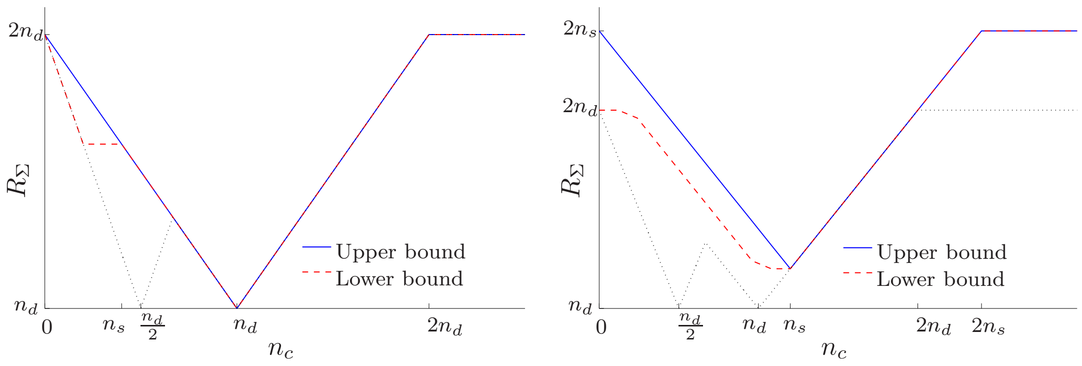
As a result, our WI-CF ans SI-CF schemes achieve the sum-capacity upper bounds of the LD-IRC with $n_s \leq n_c$. This proves Theorem 4.9 and consequently, the proof of achievability of Theorem 4.1 is completed.

A legitimate question that could be asked now is whether our schemes achieve the sum-capacity of the LD-IRC with $n_c < n_s$. The answer to this question turns out to be negative. In Figure 4.9, we plot the achievable sum-rate of our WI-CF and SI-CF schemes versus the upper bounds. Note that the bounds do not coincide if $n_c < n_s$. In fact for this regime ($n_c < n_s$), new schemes such as cooperative interference neutralization (to be introduced in Chapter 5), and new upper bounds are needed. The cooperative interference neutralization scheme will be described in the next chapter for a partially connected IRC.

In the next section, we consider the Gaussian IRC (G-IRC). We translate the upper and lower bounds developed so far to the Gaussian case and characterize the so-called generalized degrees of freedom (GDoF) of the network.

4.3 On the Generalized Degrees of Freedom of the G-IRC

The results we have presented so far for the LD-IRC can be used as stepping stones to develop upper bounds and lower bounds for the sum-capacity of the Gaussian setting. However, in the Gaussian setting, one has to take care of the new properties of the network. For instance, the channel gains are not discrete anymore, but continuous. Additionally, the Gaussian network is noisy and therefore, one should not only construct coding schemes that combat interference, but also ones that combat noise. All these aspects and others will be discussed in the following paragraphs. We start with upper bounds.



(a) $n_r = 0.8n_d$, $n_s = 0.4n_d$ (b) $n_r = 1.2n_d$, $n_s = 1.2n_d$

Figure 4.9: Upper bounds and lower bounds for the sum-capacity of the LD-IRC (all bounds are piecewise-linear). The bounds do not coincide in general if $n_s > n_c$. The sum-capacity of the deterministic IC is shown (dotted) as a benchmark.

4.3.1 Sum-capacity Upper Bounds

Let us start with the cut-set bounds. The expression of the cut-set bounds is the same as given for the LD-IRC in Lemma 4.1 except for one difference: the signals are continuous real-valued random variables instead of binary random vectors. The cut-set bounds for the G-IRC are stated in the following lemma.

Lemma 4.3 ([MDG12])**.** *The achievable rates in the G-IRC are bounded by the region*

$$R_1 \leq \min\{I(X_1, X_r; Y_1|X_2), I(X_1; Y_1, Y_r|X_2, X_r)\} \tag{4.168}$$
$$R_2 \leq \min\{I(X_2, X_r; Y_2|X_1), I(X_2; Y_2, Y_r|X_1, X_r)\} \tag{4.169}$$
$$R_1 + R_2 \leq \min\{I(X_1, X_2, X_r; Y_1, Y_2), I(X_1, X_2; Y_1, Y_2, Y_r|X_r)\}, \tag{4.170}$$

maximized over all distributions of (X_1, X_2, X_r) *satisfying the power constraints* $\mathbb{E}[X_j^2] \leq P$ *for* $j \in \{1, 2, r\}$*, and with* X_1 *and* X_2 *independent.*

The given cut-set bound should be maximized over all such distributions of the triple of input variables (X_1, X_2, X_r). However, it can be noticed that each of the cut-set bounds is maximized by a Gaussian distribution. Then, using a Gaussian distribution for the inputs, evaluating the cut-set bounds, and maximizing them over the set of covariance matrices of the triple (X_1, X_2, X_r) satisfying the power constraints, we obtain the following simple sum-capacity upper bounds.

Theorem 4.10. *The sum-capacity of the G-IRC is upper bounded by*

$$C_\Sigma \leq 2C((|h_d| + |h_r|)^2 P) \tag{4.171}$$
$$C_\Sigma \leq 2C(h_d^2 P + h_s^2 P). \tag{4.172}$$

Proof. Consider the first term in the cut-set bound (4.168), given by

$$R_1 \leq I(X_1, X_r; Y_1|X_2). \tag{4.173}$$

This term has the form of a MISO channel bound, and can be maximized as follows

$$
\begin{aligned}
R_1 &\leq I(X_1, X_r; Y_1|X_2) && (4.174)\\
&= h(Y_1|X_2) - h(Y_1|X_1, X_2, X_r) && (4.175)\\
&\stackrel{(a)}{\leq} h(h_d X_1 + h_r X_r + Z_1) - h(Z_1) && (4.176)\\
&\stackrel{(b)}{\leq} C(h_d^2 P + h_r^2 P + 2h_d h_r \rho_1 P) && (4.177)\\
&\leq C(h_d^2 P + h_r^2 P + 2|h_d||h_r|P) && (4.178)\\
&= C((|h_d| + |h_r|)^2 P), && (4.179)
\end{aligned}
$$

where (a) follows since conditioning does not increase entropy, and (b) follows since the Gaussian distribution $X_1, X_r \sim \mathcal{N}(0, P)$ is a differential entropy maximizer [CT06] where $\rho_1 = \text{Corr}(X_1, X_r) \in [-1, 1]$. Now consider the second term in (4.168) given by

$$R_1 \leq I(X_1; Y_1, Y_r|X_2, X_r). \tag{4.180}$$

Similarly, we can maximize this as follows

$$
\begin{aligned}
R_1 &\leq I(X_1; Y_1, Y_r|X_2, X_r)\\
&= h(Y_1, Y_r|X_2, X_r) - h(Y_1, Y_r|X_1, X_2, X_r)\\
&\leq h(h_d X_1 + Z_1, h_s X_1 + Z_r) - h(Z_1, Z_r)\\
&\leq C(h_d^2 P + h_s^2 P).
\end{aligned}
$$

Similar bounds can be obtained for R_2 which proves the statement of the theorem. □

Many other upper bounds have been previously developed for the G-IRC. For instance, Maric *et al.* [MDG09] tightened the first sum-rate term in the cut-set upper bounds (4.170) by giving enough additional information to Rx 1 so that it is able to construct a less noisy version of the received signal of Rx 2. In this way, it is guaranteed that the enhanced Rx 1 is able to decode m_1 and m_2 and thus an upper bound on C_Σ can be obtained. Other upper bounds were derived by Tian *et al.* [TY11] by using a potent relay approach (a relay with no power constraint). Clearly, any achievable rate pair in the G-IRC is achievable in the IC with a potent relay. Thus, the capacity of the IC with a potent relay serves as an upper bound for the capacity of the G-IRC. However, for the purpose of this document, we only need the cut-set bounds, and our new upper bounds that we present next (a Gaussian version of the bounds in Theorems 4.3, 4.4, 4.5, and 4.6).

It is important at this point to remark that Cadambe *et al.* have shown in [CJ09] that relaying (among other factors) does not increase the DoF of the X-channel. But the IC can be viewed as a special case of the X-channel (with the rate of some messages set to zero). Thus, relaying also does not increase the DoF of the IC. As a result, since the IC has 1 DoF [HMN05], it follows that the IRC also has 1 DoF. From this point of view, the cut-set bounds are clearly not tight at high SNR since they provide a DoF upper bound of 2. Thus, the cut-set bounds are not sufficient for a complete characterization/approximation of the sum-capacity. In the following theorem, we introduce an upper bound which matches the DoF of the G-IRC. Before proceeding, we need to define the following variables

$$S_{sj}^n = h_s X_j^n + Z_r^n, \quad S_{dj}^n = h_d X_j^n + Z_j^n, \quad S_{cj}^n = h_c X_j^n + Z_k^n, \quad j \neq k. \tag{4.181}$$

Theorem 4.11. *The sum-capacity of the G-IRC is upper bounded by*

$$C_\Sigma \leq C\left(P\left(h_d^2 + h_c^2 + h_r^2 + 2|h_r|\sqrt{h_d^2 + h_c^2}\right)\right) + C\left(\frac{h_d^2 P}{h_c^2 P + 1}\right) \tag{4.182}$$

$$+ C\left(\frac{h_s^2 P}{1 + \max\{h_d^2, h_c^2\}P}\right). \tag{4.183}$$

Proof. The proof is based on a genie-aided approach similar to that used in the proof of Theorem 4.3 on page 24, thus, we only highlight the differences here. By constructing a genie-aided channel as in the proof of Theorem 4.3, we can bound R_Σ as follows

$$n(R_\Sigma - \varepsilon_n) \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n, Y_r^n, S_{c2}^n | X_1^n) \tag{4.184}$$

$$= I(X_1^n; Y_1^n) + h(Y_2^n, Y_r^n, S_{c2}^n | X_1^n) - h(Y_2^n, Y_r^n, S_{c2}^n | X_1^n, X_2^n), \tag{4.185}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Then, similar to the proof of Theorem 4.3, the first and second terms in the right hand side of (4.185) can be bounded as

$$I(X_1^n; Y_1^n) \leq \sum_{i=1}^{n} \left[h(Y_1(i)) - h(S_{c2}(i) | X_1^n, S_{c2}^{i-1}, S_{d2}^{i-1}, S_{s2}^{i-1})\right], \tag{4.186}$$

$$h(Y_2^n, Y_r^n, S_{c2}^n | X_1^n) \leq \sum_{i=1}^{n} \left[h(S_{c2}(i) | X_1^n, S_{d2}^{i-1}, S_{s2}^{i-1}, S_{c2}^{i-1}) + h(S_{d2}(i), S_{s2}(i) | S_{c2}(i))\right]. \tag{4.187}$$

Moreover, we can write the third term in (4.185) as

$$h(Y_2^n, Y_r^n, S_{c2}^n | X_1^n, X_2^n) = \sum_{i=1}^{n} \left[h(Z_2(i)) + h(Z_r(i)) + h(Z_1(i))\right], \tag{4.188}$$

since the noises are i.i.d. and are mutually independent of each other and of all the other transmit signals. Plugging (4.186), (4.187), and (4.188) in (4.185), we get

$$n(R_\Sigma - \varepsilon_n) \leq \sum_{i=1}^{n} \left[h(Y_1(i)) - h(Z_1(i)) + h(S_{d2}(i) | S_{c2}(i)) - h(Z_2(i))\right]$$

$$+ \sum_{i=1}^{n} \left[h(S_{s2}(i) | S_{d2}(i), S_{c2}(i)) - h(Z_r(i))\right] \tag{4.189}$$

Now we investigate the terms of (4.189). We have

$$\sum_{i=1}^{n} \left[h(Y_1(i)) - h(Z_1(i))\right] \overset{(a)}{\leq} nC\left(P\left(h_d^2 + h_c^2 + h_r^2 + 2h_r\left(h_d\rho_1 + h_c\rho_2\right)\right)\right) \tag{4.190}$$

$$\overset{(b)}{\leq} nC\left(P\left(h_d^2 + h_c^2 + h_r^2 + 2|h_r|\sqrt{h_d^2 + h_c^2}\right)\right) \tag{4.191}$$

where (a) follows since the Gaussian distribution of (X_1, X_2, X_r) maximizes the differential entropy, with $X_1, X_2, X_r \sim \mathcal{N}(0, P)$, $\rho_1 = \text{Corr}(X_1, X_r)$ and $\rho_1 = \text{Corr}(X_2, X_r)$. Notice that $\rho_1^2 + \rho_2^2 \leq 1$ has to be fulfilled so that the covariance matrix of (X_1, X_2, X_r) is positive semi-definite. Step (b) follows by maximizing over ρ_1, ρ_2 such that $\rho_1^2 + \rho_2^2 \leq 1$. Similarly we can show that

$$\sum_{i=1}^{n} [h(S_{d2}(i)|S_{c2}(i)) - h(Z_2(i))] \overset{(d)}{\leq} nC\left(\frac{h_d^2 P}{1 + h_c^2 P}\right), \tag{4.192}$$

which follows since the Gaussian input X_2 maximizes the conditional differential entropy under a covariance constraint [Tho87], where the expression can be maximized by $X_2 \sim \mathcal{N}(0, P)$. Similarly we can show that

$$\sum_{i=1}^{n} [h(S_{s2}(i)|S_{d2}(i), S_{c2}(i)) - h(Z_r(i))] \leq nC\left(\frac{h_s^2 P}{1 + \max\{h_d^2, h_c^2\}P}\right). \tag{4.193}$$

Substituting (4.191), (4.192), and (4.193) in (4.189), and letting $n \to \infty$ we get the desired bound (4.182). □

This bound reflects the DoF of the G-IRC. Namely, it shows that the sum-capacity of the G-IRC behaves as $\frac{1}{2}\log(P) + o(\log(P))$ which captures the fact that the relay does not increase the DoF of the interference network [CJ09]. The following corollary gives upper bounds that follow from Theorem 4.11.

Corollary 4.1. *In the G-IRC, if $h_d^2 \leq h_s^2$ then the sum-capacity is bounded by*

$$\begin{aligned} C_\Sigma \leq\ & C\left(P\left(h_d^2 + h_c^2 + h_r^2 + 2|h_r|\sqrt{h_d^2 + h_c^2}\right)\right) + C\left(\frac{h_s^2 P}{h_c^2 P + 1}\right) \\ & + C\left(\frac{h_s^2 P}{1 + \max\{h_c^2, h_s^2\}P}\right). \end{aligned} \tag{4.194}$$

Moreover, if $h_c^2 \leq h_s^2$ then

$$\begin{aligned} C_\Sigma \leq\ & C\left(\frac{h_s^2}{h_c^2}P\left(h_d^2 + h_c^2 + h_r^2 + 2|h_r|\sqrt{h_d^2 + h_c^2}\right)\right) + C\left(\frac{h_d^2 P}{h_s^2 P + 1}\right) \\ & + C\left(\frac{h_s^2 P}{1 + \max\{h_d^2, h_s^2\}P}\right). \end{aligned} \tag{4.195}$$

Proof. If $h_d^2 \leq h_s^2$, we can enhance Rx 2 by replacing the noise Z_2 by $\frac{h_d}{h_s} Z_2$ [PV11b]. What we obtain is an equivalent G-IRC where the channels from Tx 1, Tx 2, and the relay to Rx 2 are $\frac{h_c h_s}{h_d}$, h_s, and $\frac{h_r h_s}{h_d}$, respectively, and the noise at Rx 2 is $\mathcal{N}(0, 1)$. Proceeding with the same steps as in Theorem 4.11 we obtain (4.194). The bound in (4.195) is obtained similarly by enhancing Rx 1 where the noise Z_1 is replaced by $\frac{h_c}{h_s} Z_1$, where $h_c^2 \leq h_s^2$. □

It can be noticed that the bounds in Theorem 4.11 and Corollary 4.1 have a similar structure as the bound of the Z-IC (or the one sided IC) [ETW08]. This kind of bounds is useful for characterizing the GDoF for the strong interference scenario, and some sub-regimes of the weak interference scenario (as in the IC) as we shall show in the sequel. As mentioned earlier in Section 4.2, if $h_c^2 \geq h_d^2$, then these bounds become the same as the X-channel bound in [CJ09] specialized to the G-IRC. Otherwise if $h_c^2 < h_d^2$, then our bounds are tighter.

Next, we continue with specializing the bounds derived for the LD-IRC to the G-IRC.

Theorem 4.12. *The sum-capacity of the IRC is upper bounded by*

$$C_\Sigma \leq C(2h_s^2 P) + C(h_d^2 P + h_c^2 P) + C\left(\frac{h_d^2 P}{1+\max\{h_c^2, h_s^2\}P}\right). \tag{4.196}$$

Proof. Similar to the proof of Theorem 4.4 (page 27), we use Tuninetti's cooperation bounds (Lemma 2.2 on page 8) to write

$$\begin{aligned} R_\Sigma &\leq I(X_2;Y_2|Y_1,Y_r,X_1,X_r) + I(X_1,X_2;Y_1,Y_r|X_r) & (4.197)\\ &\leq h(Y_2|Y_1,Y_r,X_1,X_r) - h(Z_2) + h(Y_1,Y_r|X_r) - h(Z_1,Z_r). & (4.198)\end{aligned}$$

Then, we can show that

$$\begin{aligned} h(Y_2|Y_1,Y_r,X_1,X_r) - h(Z_2) &\leq h(S_{d2}|S_{c2},S_{s2}) - h(Z_2) & (4.199)\\ &\leq C\left(\frac{h_d^2 P}{1+\max\{h_c^2,h_s^2\}P}\right) & (4.200)\end{aligned}$$

and

$$\begin{aligned} h(Y_1,Y_r|X_r) - h(Z_1,Z_r) &\leq h(Y_r) + h(Y_1|Y_r,X_r) - h(Z_1,Z_r) & (4.201)\\ &\leq C(2h_s^2P) + C(h_d^2P + h_c^2P) & (4.202)\end{aligned}$$

using a standard information theoretic approach similar to the steps used in the proof of Theorem 4.11. Hence the statement of the theorem is proved. □

Note that this bound is a refinement of the original bound we derived in [CS12, Theorem 3] where it has been derived using a genie-aided approach instead of the cooperation approach that is followed above.

The 'weak interference' upper bound derived in Theorem 4.5 for the LD-IRC, which is inspired from the weak interference upper bound of the IC in [ETW08], is translated to the G-IRC in the following theorem.

Theorem 4.13. *The sum-capacity of the IRC is upper bounded by*

$$C_\Sigma \leq 2C\left((|h_c|+|h_r|)^2P + 4\max\left\{\frac{h_d^2P}{1+h_c^2P}, h_r^2P\right\}\right) + 2C\left(\frac{h_s^2}{h_c^2}\right). \tag{4.203}$$

Proof. The proof uses the same genie-aided approach as in the proof of Theorem 4.5. Before we proceed with the proof, we introduce the following lemma, which is the Gaussian version of Lemma 4.2.

Lemma 4.4. *The following statement holds for a G-IRC*

$$h(h_cX_1^n + U_2^n|X_r(1)) - h(Y_2^n|X_r(1), S_{c2}^n, X_2^n) \leq 0, \tag{4.204}$$

where U_2^n is i.i.d. $\mathcal{N}(0,\sigma^2)$ with $\sigma^2 = \frac{h_c^2}{h_c^2+h_s^2}$, and S_{c2}^n is defined in (4.181).

Proof. We need to show that

$$h(Y_2^n|X_r(1), S_{c2}^n, X_2^n) \geq h(h_cX_1^n + U_2^n|X_r(1)). \tag{4.205}$$

Let $\widetilde{Z}_r^n = Z_r^n - \frac{h_s}{h_c} Z_2^n$. Then we use the following sequence of inequalities

$$
\begin{aligned}
h(Y_2^n|X_r(1), S_{c2}^n, X_2^n) &= h(Y_2^n|X_r(1), Z_1^n, X_2^n) && (4.206)\\
&= h(Y_2^n|X_r(1), X_2^n) && (4.207)\\
&\overset{(a)}{\geq} h(Y_2^n|X_r(1), X_2^n, \widetilde{Z}_r^n) && (4.208)\\
&= \sum_{i=1}^{n} h(Y_2(i)|X_r(1), X_2^n, Y_2^{i-1}, \widetilde{Z}_r^n) && (4.209)\\
&\overset{(b)}{=} \sum_{i=1}^{n} h(Y_2(i)|X_r^i, X_2^n, Y_2^{i-1}, \widetilde{Z}_r^n) && (4.210)\\
&= \sum_{i=1}^{n} h(S_{c1}(i)|X_r^i, X_2^n, S_{c1}^{i-1}, \widetilde{Z}_r^n) && (4.211)\\
&\overset{(b)}{=} \sum_{i=1}^{n} h(S_{c1}(i)|X_r(1), X_2^n, S_{c1}^{i-1}, \widetilde{Z}_r^n) && (4.212)\\
&= h(S_{c1}^n|X_r(1), X_2^n, \widetilde{Z}_r^n) && (4.213)\\
&= h(h_c X_1^n + Z_2^n|X_r(1), \widetilde{Z}_r^n) && (4.214)\\
&\overset{(c)}{=} h(h_c X_1^n + U_2^n|X_r(1)) && (4.215)
\end{aligned}
$$

where (a) follows since conditioning reduces entropy, (b) follows since we can construct $(X_r(2), \cdots, X_r(i))$ from $(X_r(1), X_2^n, Y_2^{i-1}, \widetilde{Z}_r^n)$ or $(X_r(1), X_2^n, h_c X_1^{i-1} + Z_2^{i-1}, \widetilde{Z}_r^n)$ (see details in the proof of Lemma 4.2 on page 27), and (c) follows by using [AV09, Lemma 6] where U_2^n is i.i.d. $\mathcal{N}(0, \sigma^2)$. This concludes the proof of the lemma. □

Now, we can proceed with deriving the upper bound. Let $V_1^n = h_c X_1^n + U_2^n$, $V_2^n = h_c X_2^n + U_1^n$, and let U_1^n and U_2^n be i.i.d. $\mathcal{N}(0, \sigma^2)$ noises independent of all other random variables and mutually independent of each other, with $\sigma^2 = h_c^2/(h_c^2+h_s^2)$. Similar to the the genie-aided approach used in the proof of Theorem 4.5 on page 28, we give $(X_r(1), V_1^n)$ and $(X_r(2), V_2^n)$ as side information to Rx 1 and 2, respectively, to obtain

$$
\begin{aligned}
n(R_\Sigma - \varepsilon_n) \leq\; & h(V_1^n|X_r(1)) - h(U_2^n) + h(Y_1^n|X_r(1), V_1^n) - h(Y_1^n|X_r(1), V_1^n, X_1^n)\\
& + h(V_2^n|X_r(1)) - h(U_1^n) + h(Y_2^n|X_r(1), V_2^n) - h(Y_2^n|X_r(1), V_2^n, X_2^n),
\end{aligned} \tag{4.216}
$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Using Lemma 4.4 above, the following terms are upper bounded by zero

$$
\begin{aligned}
h(V_1^n|X_r(1)) - h(Y_2^n|X_r(1), V_2^n, X_2^n) \leq 0, && (4.217)\\
h(V_2^n|X_r(1)) - h(Y_1^n|X_r(1), V_1^n, X_1^n) \leq 0. && (4.218)
\end{aligned}
$$

Additionally, we have

$$
\begin{aligned}
h(V_1^n|X_{r1}, X_1^n) = h(U_2^n) = \frac{n}{2}\log(2\pi e \sigma^2), && (4.219)\\
h(V_2^n|X_{r1}, X_2^n) = h(U_1^n) = \frac{n}{2}\log(2\pi e \sigma^2), && (4.220)\\
h(Y_1^n|V_1^n, X_{r1}) \leq h(Y_1^n|V_1^n), && (4.221)\\
h(Y_2^n|V_2^n, X_{r1}) \leq h(Y_2^n|V_2^n). && (4.222)
\end{aligned}
$$

Thus we can write

$$n(R_\Sigma - \varepsilon_n) \le h(Y_1^n|V_1^n) + h(Y_2^n|V_2^n) - h(U_1^n) - h(U_2^n) \tag{4.223}$$

$$= \sum_{i=1}^{n} \left[h(Y_1(i)|V_1^n, Y_1^{i-1}) + h(Y_2(i)|V_2^n, Y_2^{i-1}) - \log(2\pi e\sigma^2) \right] \tag{4.224}$$

$$\le \sum_{i=1}^{n} \left[h(Y_1(i)|V_1(i)) + h(Y_2(i)|V_2(i)) - \log(2\pi e\sigma^2) \right], \tag{4.225}$$

where the last step follows since conditioning does not increase entropy. Notice that a jointly Gaussian distribution $p(x_1, x_2, x_r)$ which factors as $p(x_1)p(x_2)p(x_r|x_1, x_2)$ maximizes (4.225). Hence, letting $n \to \infty$, we get

$$R_\Sigma \le \max_{\mathbf{A}\succeq 0} \sum_{\substack{j,k\in\{1,2\}\\ j\neq k}} h(h_d X_{jG} + h_c X_{kG} + h_r X_{rG} + Z_j | h_c X_{jG} + U_k) - \log(2\pi e\sigma^2)$$

where (X_{1G}, X_{2G}, X_{rG}) is a Gaussian random vector with zero mean and covariance matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} P_1 & 0 & \rho_1\sqrt{P_1 P_r} \\ 0 & P_2 & \rho_2\sqrt{P_2 P_r} \\ \rho_1\sqrt{P_1 P_r} & \rho_2\sqrt{P_2 P_r} & P_r \end{bmatrix}, \tag{4.226}$$

and the maximization is over all positive semi-definite matrices $\mathbf{A}$ (denoted $\mathbf{A} \succeq 0$) satisfying $P_1, P_2, P_r \le P$ with correlation coefficients $\rho_1, \rho_2 \in [-1, 1]$. By evaluating this bound, we get

$$R_\Sigma \le \max_{\mathbf{A}\succeq 0} \sum_{j\in\{1,2\}} C(Q_j) - \log(\sigma^2) \tag{4.227}$$

where

$$Q_j = h_c^2 P_j + h_r^2(1-\rho_k^2)P_r + 2h_c h_r \rho_j \sqrt{P_j P_r} + \frac{\sigma^2(h_d\sqrt{P_k} + h_r\rho_k\sqrt{P_r})^2}{\sigma^2 + h_c^2 P_k} \tag{4.228}$$

with $j \neq k$. Finally, we can write

$$Q_j \overset{(d)}{\le} h_c^2 P_j + h_r^2(1-\rho_k^2)P_r + 2h_c h_r \rho_j \sqrt{P_j P_r} + \frac{(h_d\sqrt{P_k} + h_r\rho_k\sqrt{P_r})^2}{1 + h_c^2 P_k} \tag{4.229}$$

$$\overset{(e)}{\le} h_c^2 P_j + h_r^2 P_r + 2|h_c||h_r|\sqrt{P_j P_r} + \frac{\left(|h_d|\sqrt{P_k} + |h_r|\sqrt{P_r}\right)^2}{1 + h_c^2 P_k} \tag{4.230}$$

$$\le (|h_c| + |h_r|)^2 P + 4\max\left\{\frac{h_d^2 P}{1 + h_c^2 P}, h_r^2 P\right\}, \tag{4.231}$$

where (d) follows since $\sigma^2 \le 1$ and (e) follows by maximizing over $\mathbf{A}$. As a result, by substituting (4.231) in (4.227) we obtain the desired bound. □

Note the resemblance between this bound and the sum-capacity of the IC with noisy interference [MK09, SKC09, AV09], particularly when the term $h_d^2 P/(1+h_c^2 P)$ dominates this bound.

Finally, the last upper bound that we derive for the G-IRC is presented in the following theorem. This upper bound is also useful in the weak interference regime.

Theorem 4.14. *The sum-capacity of the G-IRC is upper bounded by*

$$C_\Sigma \leq 2C\left(\left(1-\frac{h_d}{h_c}\right)^2+\frac{h_c^2}{h_s^2}\right)+2C(2h_s^2P). \tag{4.232}$$

Proof. As in the proof of the upper bound for the LD-IRC in Theorem 4.6, a genie gives (S_{c1}^n, Y_r^n) and (S_{c2}^n, Y_r^n) to Rx 1 and Rx 2, respectively. We can bound R_1 as

$$\begin{aligned} n(R_1-\varepsilon_{1n}) &\leq I(X_1^n;Y_1^n,S_{c1}^n,Y_r^n) & (4.233)\\ &= h(S_{c1}^n,Y_r^n)+h(Y_1^n|S_{c1}^n,Y_r^n)-h(Z_2^n)-h(S_{c2}^n,Y_r^n)\\ &\quad+I(X_1^n;S_{c2}^n,Y_r^n) & (4.234)\end{aligned}$$

using the same steps as in the proof of Theorem 4.6, and similarly

$$\begin{aligned} n(R_2-\varepsilon_{2n}) &\leq h(S_{c2}^n,Y_r^n)+h(Y_2^n|S_{c2}^n,Y_r^n)-h(Z_1^n)-h(S_{c1}^n,Y_r^n)\\ &\quad+I(X_2^n;S_{c1}^n,Y_r^n), & (4.235)\end{aligned}$$

where $\varepsilon_{1n},\varepsilon_{2n}\to 0$ as $n\to\infty$. Then

$$\begin{aligned} n(R_\Sigma-\varepsilon_{1n}-\varepsilon_{2n}) &\leq h(Y_1^n|S_{c1}^n,Y_r^n)-h(Z_2^n)+I(X_1^n;S_{c2}^n,Y_r^n) & (4.236)\\ &\quad+h(Y_2^n|S_{c2}^n,Y_r^n)-h(Z_1^n)+I(X_2^n;S_{c1}^n,Y_r^n).\end{aligned}$$

Now, we proceed as in [PV11b] to obtain

$$\begin{aligned} h(Y_1^n|S_{c1}^n,Y_r^n) &\overset{(a)}{\leq} h\left(Z_1^n+\left(1-\frac{h_d}{h_c}\right)Z_2^n-\frac{h_c}{h_s}Z_r^n\,|h_cX_1^n+Z_2^n,Y_r^n\right)\\ &\leq h\left(Z_1^n+\left(1-\frac{h_d}{h_c}\right)Z_2^n-\frac{h_c}{h_s}Z_r^n\right)\\ &= \frac{n}{2}\log\left(1+\left(1-\frac{h_d}{h_c}\right)^2+\frac{h_c^2}{h_s^2}\right)+\frac{n}{2}\log(2\pi e). & (4.237)\end{aligned}$$

where (a) follows since

$$Z_1^n+\left(1-\frac{h_d}{h_c}\right)Z_2^n-\frac{h_c}{h_s}Z_r^n = Y_1^n+\left(1-\frac{h_d}{h_c}\right)(h_cX_1^n+Z_2^n)-\frac{h_c}{h_s}Y_r^n. \tag{4.238}$$

Similarly

$$h(Y_2^n|S_{c2}^n,Y_r^n)\leq\frac{n}{2}\log\left(1+\left(1-\frac{h_d}{h_c}\right)^2+\frac{h_c^2}{h_s^2}\right)+\frac{n}{2}\log(2\pi e). \tag{4.239}$$

Next, we have

$$\begin{aligned} I(X_1^n;S_{c2}^n,Y_r^n) &= I(X_1^n;S_{c2}^n)+I(X_1^n;Y_r^n|S_{c2}^n) & (4.240)\\ &= I(X_1^n;Y_r^n|S_{c2}^n) & (4.241)\\ &= h(Y_r^n|S_{c2}^n)-h(Y_r^n|S_{c2}^n,X_1^n) & (4.242)\\ &\leq h(Y_r^n)-h(Y_r^n|S_{c2}^n,X_1^n,X_2^n) & (4.243)\\ &\leq C(2h_s^2P) & (4.244)\end{aligned}$$

which is the sum-capacity of the multiple-access channel from both transmitters to the relay [Ahl71]. And similarly

$$I(X_2^n; S_{c1}^n, Y_r^n) \leq C(2h_s^2 P). \tag{4.245}$$

By plugging (4.237), (4.239), (4.244), and (4.245) in (4.236), and letting $n \to \infty$, we get the desired upper bound. □

So far, we have presented the cut-set upper bounds and our new upper bounds on the sum-capacity of the G-IRC. These bounds collectively characterize the GDoF of the IRC with $h_s^2 \leq h_c^2$ as we show later on in this section. In order to test the tightness of these upper bounds, we need to compare them with lower bounds obtained by considering some transmission schemes for the IRC.

4.3.2 Sum-capacity Lower Bounds

Various transmission schemes can be employed in the G-IRC. As a benchmark, an IC-type Han-Kobayashi [HK81] (HK) scheme (where the relay is ignored) can be used. By incorporating the relay, a better performance than that of the HK scheme should be achieved. Therefore, a good scheme for the G-IRC should perform at least as good as the HK scheme. We expect that the HK scheme can be very close to optimal if the channels to/from the relay are very weak. Otherwise, if the relay channels are strong, then better performance might be obtained if we use the relay in a decode-forward or compress-forward fashion. Moreover, combinations of these schemes can also be used, leading to numerous possibilities. See for instance schemes that combine cooperative and non-cooperative strategies in the context of the IC with source or destination cooperation in [PV11a, PV11b].

Beside these classical schemes, new ideas can be applied to the G-IRC to construct possibly more capable schemes. Network coding ideas similar to the one used in Section 4.2.2 for the LD-IRC can be extended to the G-IRC using appropriate coding strategies. Note that due to the absence of noise in the LD-IRC, it was sufficient to encode on a symbol-by-symbol basis. The main aim of the constructed code was to combat interference. However, in the G-IRC, one has to take care of both interference and noise. Therefore, channel codes which combat noise (in addition to interference) have to be used. Namely, one can use nested-lattice coding and lattice alignment to establish a cooperation strategy between the relay and the transmitters as we have done for the LD-IRC. We refer the reader to Section 2.5 (page 12) for an introduction on lattice codes.

4.3.2.1 Compute-forward with Nested-lattice Codes

In compute-forward (CF), the relay does not decode the transmitted signals of the users individually, but rather a linear combination thereof. This is rendered possible by using nested-lattice codes. Let us now start by describing the weak interference (WI) variant of our scheme which we call WI-CF .

4.3.2.1.1 Message splitting: Rate splitting is used at the transmitters similar to our scheme for the LD-IRC in Section 4.2.2.3.1. That is, the message m_1 is split into

- a compute-forward (CF) message m_{1u} with rate R_u,

- a common (C) message m_{1v} with rate R_v, and
- a private (P) message m_{1w} with rate R_w.

The CF message refers to the message used for cooperation with the relay. Moreover, the CF message is sub-divided into $K \in \mathbb{N}$ CF sub-messages[3] $m_{1u,k}$, $k = 1, \cdots, K$, with rate $R_{u,k}$ where $R_u = \sum_{k=1}^{K} R_{u,k}$. Thus, the set of messages to be sent from the first transmitter becomes $\{m_{1w}, m_{1v}, m_{1u,1}, m_{1u,2}, \cdots, m_{1u,K}\}$.

4.3.2.1.2 Encoding: The C and P messages do not benefit from the relay, i.e., they are the same as the C and P messages in the IC in [ETW08]. The P message m_{1w} is encoded into w_1^n using a Gaussian random code with power P_w and rate R_w, i.e., W_1 is i.i.d. $\mathcal{N}(0, P_w)$. Similarly, the C message m_{1v} is encoded into $\mathbf{v}_1^n$ using a Gaussian code with power P_v and rate R_v.

In order to facilitate the use of CF, i.e., to allow the relay to decode the sum of the CF signals, the CF messages are encoded using nested-lattice codes that align at the relay (as in Lemma 2.6 on page 13). Each CF message $m_{1u,k}$ is encoded into $u_{1,k}^n$ using a nested-lattice code $(\Lambda_k^f, \Lambda_k^c)$ with rate $R_{u,k}$ and power $P_{u,k}$ (as in 2.21). Thus

$$u_{1,k}^n = (\lambda_{1,k} - d_{1,k}) \bmod \Lambda_k^c \tag{4.246}$$

where $\lambda_{1,k} \in \Lambda_k^f \cap \mathcal{V}(\Lambda_k^c)$ and $d_{1,k}$ is a random dither uniformly distributed over $\mathcal{V}(\Lambda_k^c)$. Moreover, in order to satisfy the power constraint, we set

$$P_w + P_v + \sum_{k=1}^{K} P_{u,k} = P. \tag{4.247}$$

The same encoding is done at Tx 2, where the same nested-lattices $(\Lambda_k^f, \Lambda_k^c)$ are used. Notice that this enables the relay to decode the sum

$$\bar{u}_k = (\lambda_{1,k} + \lambda_{2,k}) \bmod \Lambda_k^c, \tag{4.248}$$

with some rate constraint that we specify next. Tx 1 then sends the sum of all codewords as

$$x_1^n = w_1^n + v_1^n + \sum_{i=1}^{K} u_{1,k}^n. \tag{4.249}$$

This encoding process is repeated in transmission blocks $b \in \{1, \ldots, B-1\}$, each block of length n symbols. The transmitters do not send any messages in block B.

4.3.2.1.3 Relay Processing: Decoding at the relay starts at the end of block $b = 1$. In block 1, the relay receives

$$y_r^n(1) = h_s \sum_{j=1}^{2} \left(w_j^n(1) + v_j^n(1) + \sum_{i=1}^{K} u_{j,k}^n(1) \right) + z_r^n(1), \tag{4.250}$$

[3]Recall that in the LD-IRC we have decoded one of the CF signals at the receiver bit by bit successively while simultaneously canceling interference from other CF signal. In order to mimic this procedure in the G-IRC, we use CF message splitting. This is justified graphically in Section 4.3.3.2.2 on page 66.

where the index in the brackets denotes the block index. It decodes the sum $\bar{u}_k(1) = (\lambda_{1,k}(1) + \lambda_{2,k}(1)) \bmod \Lambda_k^c$, starting with $k = 1$ and ending with $k = K$. Decoding is done successively (successive compute-forward [Naz12]), where at each decoding step, the interference from already decoded signals is removed. Decoding this sum of codewords is possible as long as (see Lemma 2.6 on page 13)

$$R_{u,k} \leq C^+ \left(\frac{h_s^2 P_{u,k}}{1 + 2h_s^2 (\sum_{i=k+1}^{K} P_{u,i} + P_v + P_w)} - \frac{1}{2} \right) \tag{4.251}$$

for all $k \in \{1, \cdots, K\}$. Note that while decoding $\bar{u}_k(1)$, all the signals w_j^n, v_j^n, and $u_{j,l}^n$ for $j = 1, 2$ and $l > k$ are treated as noise.

Remark 4.8. *The channel between the sources and the relay is similar to the doubly dirty MAC [PZEK11], except for the fact that the relay does not need to decode the individual messages but a function thereof. It is thus possible to increase the rate constraints at the relay if we encode the* CF *messages against 'self' interference using the lattice DPC scheme of [PZEK11]. For instance, Tx 1 can encode $u_{1,k}^n$ against the interference caused by $u_{1,l}, n$ for $l > k$, and similarly at Tx 2 which achieves higher rates than* (4.251)*. However, this will not be necessary for the purpose of this chapter.*

Observe that the set of all possible values of $\bar{u}_k(1) \in \mathcal{U}_k$ has size $|\mathcal{U}_k| = 2^{nR_{u,k}}$. The relay maps all the signals $\bar{u}_k(1)$, $k = 1, \cdots, K$, into one message $m_r(1) \in \mathcal{M}_r$, where the message set $\mathcal{M}_r$ has a size which is equal to the size of the Cartesian product of all $\mathcal{U}_k$, i.e.,

$$|\mathcal{M}_r| = |\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_K| = 2^{n \sum_{k=1}^{K} R_{u,k}} = 2^{nR_u}. \tag{4.252}$$

This relay message is then split into $m_{r,a}(1)$ and $m_{r,b}(1)$ with rates $R_{r,a}$ and $R_{r,b}$, respectively, with $R_u = R_{u,a} + R_{u,b}$. These messages are encoded to $\bar{u}_a^n(2)$ and $\bar{u}_b^n(2)$, which are Gaussian codes with power $P_{r,a}$ and $P_{r,b}$, respectively, such that $P_{r,a} + P_{r,b} \leq P$. These codewords are sent in block 2.

The purpose of this splitting is to allow a flexible decoding order at the destinations between the relay signals and the desired CF signals. As we shall see next, we stick to a successive decoding scheme for simplicity. The message $m_{r,b}$ is to be decoded at the destinations after decoding the desired CF messages, while $m_{r,a}$ is decoded before the desired CF messages. This allows us to achieve higher rates than if both $m_{r,a}$ and $m_{r,b}$ are decoded after the desired CF messages in some cases, in particular when $h_r^2 \geq h_d^2$ (for a graphical illustration, see Figure 4.6 on page 35). In this successive decoding framework, this split allows the two decoding orders, the first being the case where the relay signal is decoded before the desired CF signals at the destinations (by setting $P_{r,b} = 0$), and the second being the opposite case (by setting $P_{r,a} = 0$). Moreover, this splitting allows transitions between these two extreme cases, where part of the relay signal is decoded before the desired CF signals, and another part afterwards.

Similar processing is done at the relay in blocks $2, \cdots, B-1$. Due to causality, the relay transmission takes place in blocks $2, \cdots, B$, while the relay does not send any signal in block 1.

4.3.2.1.4 Decoding at the destinations: The receivers use backward decoding, i.e., the receivers wait until the end of block B where decoding starts. We explain the decoding process at Rx 1 only since decoding at Rx 2 is similar. At the end of block B, Rx 1 receives only

$$y_1^n(B) = h_r(\bar{u}_a^n(B) + \bar{u}_b^n(B)) + z_1^n, \tag{4.253}$$

since the transmitters do not send in this block. Then, $m_{r,a}(B-1)$ and $m_{r,b}(B-1)$ are decoded successively in this order, which is possible reliably if

$$R_{r,a} \leq C\left(\frac{h_r^2 P_{r,a}}{1 + h_r^2 P_{r,b}}\right) \tag{4.254}$$

$$R_{r,b} \leq C(h_r^2 P_{r,b}). \tag{4.255}$$

Now, Rx 1 has obtained $m_r(B-1)$, and hence it can construct $(\bar{u}_1(B-1), \cdots, \bar{u}_K(B-1))$. Now consider block $B-1$. The received signal at Rx 1 is given by

$$y_1^n(B-1) = h_d x_1^n(B-1) + h_c x_2^n(B-1) + h_r x_r^n(B-1) + z_1^n \tag{4.256}$$

$$\begin{aligned} &= h_d w_1^n(B-1) + h_d v_1^n(B-1) + h_d \sum_{k=1}^{K} u_{1,k}^n(B-1) \\ &\quad + h_c w_2^n(B-1) + h_c v_2^n(B-1) + h_c \sum_{k=1}^{K} u_{2,k}^n(B-1) \\ &\quad + h_r \bar{u}_a^n(B-1) + h_r \bar{u}_b^n(B-1) + z_1^n. \end{aligned} \tag{4.257}$$

Rx 1 decodes the messages in this order:

$$m_{r,a} \to m_{1u,1} \to m_{1u,2} \to \cdots \to m_{1u,K} \to m_{r,b} \to (m_{1v}, m_{2v}) \to m_{1w}.$$

The message $m_{r,a}(B-2)$ (corresponding to the signal $\bar{u}_a^n(B-2)$) is first decoded while treating the other signals as noise, which leads to the rate constraint

$$R_{r,a} \leq C\left(\frac{h_r^2 P_{r,a}}{1 + h_d^2 P + h_c^2 P + h_r^2 P_{r,b}}\right). \tag{4.258}$$

Next, Rx 1 removes the contribution of $\bar{u}_a^n(B-1)$, and decodes $m_{1u,1}(B-1)$ while treating the other signals as noise. Thus, we have the following rate constraint

$$R_{u,1} \leq C\left(\frac{h_d^2 P_{u,1}}{1 + h_d^2(\sum_{i=2}^{K} P_{u,i} + P_v + P_w) + h_c^2 P + h_r^2 P_{r,b}}\right). \tag{4.259}$$

The next step is to perform interference cancellation. Since Rx 1 now knows both $m_{1u,1}(B-1)$ and $\bar{u}_1(B-1)$ (decoded in block B), then, it can extract $m_{2u,1}(B-1)$ (see Lemma 2.7 on page 14). It thus removes its contribution, $h_c u_{2,1}^n(B-1)$, from $y_1^n(B-1)$. After canceling the interference of the first interfering CF signal, the second desired CF signal is decoded while treating the other signals as noise. Next, the second interfering CF signal is canceled and so on. This continues until all CF messages are decoded, leading to the rate constraint given by

$$R_{u,k} \leq C\left(\frac{h_d^2 P_{u,k}}{1 + (h_d^2 + h_c^2)(\sum_{i=k+1}^{K} P_{u,i} + P_v + P_w) + h_c^2 P_{u,k} + h_r^2 P_{r,b}}\right). \tag{4.260}$$

At this stage, after removing the contribution of the decoded signals, the received signal at Rx 1 can be reduced to

$$\tilde{y_1}^n(B-1) = h_d w_1^n(B-1) + h_d v_1^n(B-1) \\ + h_c w_2^n(B-1) + h_c v_2^n(B-1) + h_r \bar{u}_b^n(B-1) + z_1^n. \quad (4.261)$$

Now, $m_{r,b}(B-2)$ (corresponding to the signal $\bar{u}_b^n(B-2)$) is decoded, with the rate constraint

$$R_{r,b} \leq C\left(\frac{h_r^2 P_{r,b}}{1 + (h_d^2 + h_c^2)(P_v + P_w)}\right). \quad (4.262)$$

The remaining C and P messages are treated as in the IC [ETW08], i.e., $m_{1v}(B-1)$ and $m_{2v}(B-1)$ are decoded jointly, and m_{1w} afterwards leading to the rate constraints

$$R_v \leq C\left(\frac{\min\{h_d^2, h_c^2\}P_v}{1 + (h_d^2 + h_c^2)P_w}\right) \quad (4.263)$$

$$2R_v \leq C\left(\frac{h_d^2 P_v + h_c^2 P_v}{1 + (h_d^2 + h_c^2)P_w}\right) \quad (4.264)$$

$$R_w \leq C\left(\frac{h_d^2 P_w}{1 + h_c^2 P_w}\right). \quad (4.265)$$

Notice that the rate constraints (4.258) and (4.262) are more binding than (4.254) and (4.255), and hence the latter two will be ignored. Now since we have $R_{r,a} + R_{r,b} = R_u$, we can write

$$R_u \leq C\left(\frac{h_r^2 P_{r,a}}{1 + h_d^2 P + h_c^2 P + h_r^2 P_{r,b}}\right) + C\left(\frac{h_r^2 P_{r,b}}{1 + (h_d^2 + h_c^2)(P_v + P_w)}\right). \quad (4.266)$$

As long as (4.266) is satisfied, then there exists a split of m_r into $m_{r,a}$ and $m_{r,b}$ which achieves a total CF rate as (4.266). This split is namely corresponding to setting $R_{r,a}$ to be equal to the first term in (4.266) and $R_{r,b}$ to the second one.

Decoding proceeds backwards till block 1 is reached where $m_{1u,k}(1)$ $(k = 1, \cdots, K)$, $m_{1v}(1)$ and $m_{1w}(1)$ are decoded, and as a by-product $m_{2u,k}(1)$ and $m_{2v}(1)$ are also obtained. This leads to the following sum-capacity lower bound.

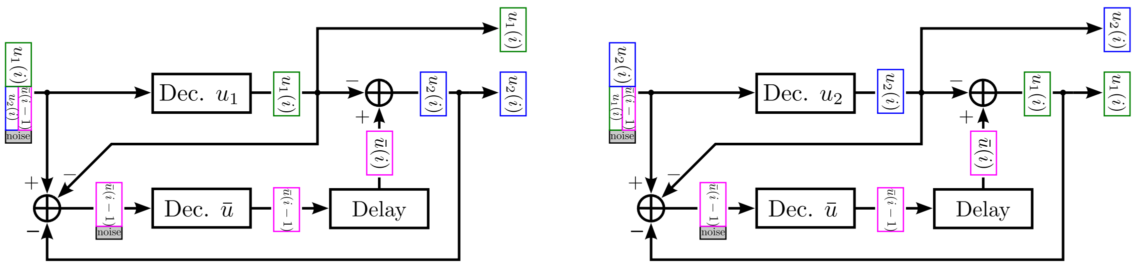
Theorem 4.15 (WI-CF). *The sum-rate given by $R_\Sigma = 2(R_w + R_v + R_u)$ is achievable, where the private message rate R_w satisfies* (4.265)*, the common message rate R_v satisfies* (4.263) *and* (4.264)*, and the CF message rate satisfies*

$$R_u \leq \sum_{k=1}^{K} C^+\left(\frac{h_s^2 P_{u,k}}{1 + 2h_s^2(\sum_{i=k+1}^{K} P_{u,i} + P_v + P_w)} - \frac{1}{2}\right), \quad (4.267)$$

$$R_u \leq \sum_{k=1}^{K} C\left(\frac{h_d^2 P_{u,k}}{1 + (h_d^2 + h_c^2)(\sum_{i=k+1}^{K} P_{u,i} + P_v + P_w) + h_c^2 P_{u,k} + h_r^2 P_{r,b}}\right), \quad (4.268)$$

$$R_u \leq C\left(\frac{h_r^2 P_{r,a}}{1 + h_r^2 P_{r,b} + h_d^2 P + h_c^2 P}\right) + C\left(\frac{h_r^2 P_{r,b}}{1 + (h_d^2 + h_c^2)(P_v + P_w)}\right). \quad (4.269)$$

for some power allocation $P_w + P_v + \sum_{k=1}^{K} P_{u,k} = P$, $P_{r,a} + P_{r,b} \leq P$, and $K \in \mathbb{N}$.



(a) WI-CF . (b) SI-CF .

Figure 4.10: Two decoder structures for the CF signals. One can decode the desired CF signal and then extract the undesired one for interference cancellation, or vice versa.

Proof. By collecting the bounds (4.251), (4.258)-(4.260) and (4.262)-(4.265) we obtain the desired achievable rate. □

Remark 4.9. *Since we have used the same strategy for encoding and decoding the* P *and* C *messages as for the IC in [ETW08], this results in our scheme being at least as good as that in [ETW08]. Namely, by switching the relay off ($P_{r,a} = P_{r,b} = 0$), and setting the powers of the* CF *signals to zero ($P_{u,k} = 0$), our scheme reduces to that of [ETW08].*

Remark 4.10. *As we shall see next, there are cases where the GDoF of the IRC is achieved without using the full power at the relay. For this reason, we have preserved the inequality in the power constraint $P_{r,a} + P_{r,b} \leq P$.*

Note that we have sticked with a specific decoding order in our WI-CF scheme. Different decoding orders can also be used. However, this order turns out to be GDoF achieving and thus we restrict our attention to it (recall that this order achieves the sum-capacity of the LD-IRC (Section 4.2)).

So far, we have forced the receiver to decode its desired CF messages, and then use the cooperation information $m_{r,a}$ and $m_{r,b}$ to extract the CF signal interference and cancel it. Alternatively, the receiver can start by decoding the interfering CF signal, and then, given $m_{r,a}$ and $m_{r,b}$, extract its desired CF messages (Figure 4.10). This gives the following alternative achievable rate.

Theorem 4.16 (SI-CF). *The sum-rate given by $R_\Sigma = 2(R_v + R_u)$ is achievable where the common message rate R_v satisfies*

$$R_v \leq C\left(\min\{h_d^2, h_c^2\} P_v\right) \tag{4.270}$$

$$2R_v \leq C\left(h_d^2 P_v + h_c^2 P_v\right) \tag{4.271}$$

and the CF *message rate R_u satisfies*

$$R_u \leq \sum_{k=1}^{K} C^+\left(\frac{h_s^2 P_{u,k}}{1 + 2h_s^2(\sum_{i=k+1}^{K} P_{u,i} + P_v)} - \frac{1}{2}\right) \tag{4.272}$$

$$R_u \leq \sum_{k=1}^{K} C\left(\frac{h_c^2 P_{u,k}}{1 + (h_d^2 + h_c^2)(\sum_{i=k+1}^{K} P_{u,i} + P_v) + h_d^2 P_{u,k} + h_r^2 P_{r,b}}\right) \tag{4.273}$$

$$R_u \leq C\left(\frac{h_r^2 P_{r,a}}{1 + h_r^2 P_{r,b} + h_d^2 P + h_c^2 P}\right) + C\left(\frac{h_r^2 P_{r,b}}{1 + h_d^2 P_v + h_c^2 P_v}\right), \tag{4.274}$$

for some power allocation $P_v + \sum_{k=1}^{K} P_{u,k} = P$, $P_{r,a} + P_{r,b} \leq P$, *and* $K \in \mathbb{N}$.

Proof. The achievability of these rates can be verified by using the same scheme as that in Theorem 4.15, with decoding the CF interference first instead of the CF desired messages. □

Notice that we did not include a private message in the SI-CF scheme. The reason is that this scheme is useful for characterizing the GDoF of the G-IRC in the strong interference (SI) regime ($h_c^2 \geq h_d^2$). Recall that the sum-capacity of the IC with SI can be achieved without using private messages [Car75, Sat81]. Similarly, we show next that the GDoF of the G-IRC with SI and with $h_s^2 \leq h_c^2$ is achieved without private messages.

To examine the performance of our schemes, one has to carefully choose K and the power allocations, plug in the rate constraints, and compare to the upper bounds. This is done in the next section where we discuss the GDoF of the network.

4.3.3 Generalized Degrees-of-Freedom (GDoF) Characterization

Due to the complicated nature of the given problem, which combines the IC and the relay channels, both of which have unknown capacity in general [EGK11], we resort to an approximative characterization of the sum-capacity. An approximation of the sum-capacity is provided by the DoF of the network, which is known from previous results to be 1 [CJ09]. While the DoF provides interesting insights into the behavior of the system, the GDoF [ETW08] is a much more powerful metric, as it is richer and captures a large variety of scenarios. Next, we show that the bounds we provided in Theorems 4.10, 4.11, 4.12, 4.13, and 4.14 and Corollary 4.1 are GDoF-optimal as long as $h_s^2 \leq h_c^2$. Let us first define the GDoF of the G-IRC.

Definition 4.1. *Let the following variables represent the strength of the different channels (similar to [ETW08])*

$$\alpha = \frac{\log(h_c^2 P)}{\log(h_d^2 P)}, \quad \beta = \frac{\log(h_r^2 P)}{\log(h_d^2 P)}, \quad \gamma = \frac{\log(h_s^2 P)}{\log(h_d^2 P)}, \tag{4.275}$$

and define the GDoF $d(\alpha, \beta, \gamma)$, *or simply* d *as*

$$d = \lim_{h_d^2 P \to \infty} \frac{C_\Sigma(h_d^2 P, \alpha, \beta, \gamma)}{\frac{1}{2}\log(h_d^2 P)}. \tag{4.276}$$

The main result of this section is given in the following theorem.

Theorem 4.17. *The GDoF of the G-IRC with* $\gamma \leq \alpha$ *is given by*

$$d = \min \left\{ \begin{array}{c} 2\max\{1, \beta\} \\ 2\max\{1, \gamma\} \\ \max\{1, \alpha, \beta\} + \max\{1, \alpha\} - \alpha \\ 2\max\{1, \alpha\} - \alpha + \gamma \\ 2\max\{\alpha, \beta, 1 - \alpha\} \\ 2\max\{\alpha, 1 + \gamma - \alpha\} \end{array} \right\}. \tag{4.277}$$

Note that the GDoF expression of the theorem is the same as the sum-capacity expression in Theorem 4.1 with replacing n_d, n_c, n_r, and n_s by 1, α, β, and γ, respectively. The converse and achievability of this Theorem are given in the next paragraphs.

4.3.3.1 GDoF Upper Bounds

In order to prove this theorem, we start by transforming the sum-capacity upper bounds to GDoF upper bounds as follows. Let us start with Theorem 4.10,

$$C_\Sigma \leq 2C((|h_d| + |h_r|)^2 P) \tag{4.278}$$
$$C_\Sigma \leq 2C(h_d^2 P + h_s^2 P). \tag{4.279}$$

For the first bound, we write

$$C_\Sigma \leq 2C((|h_d| + |h_r|)^2 P) \tag{4.280}$$
$$\leq 2C(4\max\{h_d^2, h_r^2\}P) \tag{4.281}$$
$$= 2C(4\max\{h_d^2 P, (h_d^2 P)^\beta\}). \tag{4.282}$$

where in the last step we used the definition of β. Thus

$$d = \lim_{h_d^2 P \to \infty} \frac{C_\Sigma}{\frac{1}{2}\log(h_d^2 P)} \leq 2\max\{1, \beta\}. \tag{4.283}$$

Similarly, the second bound yields

$$d \leq 2\max\{1, \gamma\}. \tag{4.284}$$

Using similar steps, the bounds in Theorem 4.11 and Corollary 4.1 can be combined into the following GDoF bound

$$d \leq \max\{1, \alpha, \beta\} + \max\{1, \alpha, \gamma\} - \alpha. \tag{4.285}$$

The bounds in Theorems 4.12, 4.13, and 4.14 translate to

$$d \leq \max\{1, \alpha\} + \max\{0, 1-\alpha\} + \gamma \tag{4.286}$$
$$d \leq 2\max\{\alpha, \beta, 1-\alpha\} + 2\max\{0, \gamma - \alpha\}, \tag{4.287}$$
$$d \leq 2\max\{0, \alpha - \gamma, 1-\alpha\} + 2\gamma. \tag{4.288}$$

By combining (4.283)-(4.288), we obtain the converse of Theorem 4.17.

4.3.3.2 GDoF Lower Bounds

Before we prove the achievability of Theorem 4.17, we recall the GDoF of the IC [ETW08] given by (see Lemma 2.1 on page 7)

$$d = \min\{2\max\{1-\alpha, \alpha\}, 2\max\{1, \alpha\} - \alpha, 2\}. \tag{4.289}$$

This GDoF is achievable using the Han-Kobayashi (HK) transmission scheme [HK81] with a fixed decoding order [ETW08]. This scheme can be used to achieve the same GDoF in the G-IRC. It turns out that this is GDoF-optimal in the IRC in some cases. For example, if $\gamma \leq \alpha$ and $\beta \leq \min\{1, \alpha\}$, then the GDoF in Theorem 4.17 becomes the same as (4.289) (see Figure 4.11). As a result, if

$$\gamma \leq \alpha \text{ and } \beta \leq \min\{1, \alpha\}, \tag{4.290}$$

then the relay can be switched off without any impact on the GDoF of the G-IRC. Alternatively, we can use WI-CF and the SI-CF while setting the power of the CF

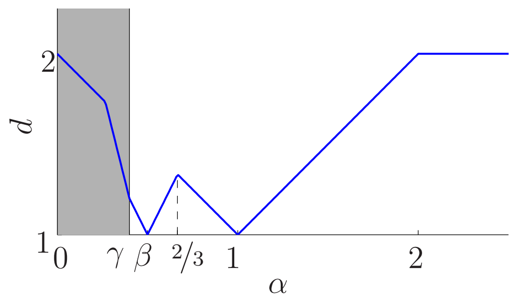


Figure 4.11: The GDoF of the G-IRC for $\beta = 0.5$ and $\gamma = 0.4$. For $\alpha \geq \beta$, the GDoF is the same as that of the IC.

signals to zero which also achieves (4.289). As we shall see next, there are more cases with this property. Consequently, the achievability of Theorem 4.17 is established for $\beta \leq \min\{1, \alpha\}$.

It remains to show that the GDoF in 4.17 is achievable for

$$\gamma \leq \alpha, \quad \text{and} \quad \beta > \min\{1, \alpha\}. \tag{4.291}$$

To show this, we use our WI-CF and SI-CF schemes. We express the rate constraints of these schemes as GDoF constraints using the following definitions.

Definition 4.2. *Define δ_w, δ_v, and $\delta_{u,k}$, $k = 1, \cdots, K$, as*

$$\delta_w = \frac{\log(h_d^2 P_w)}{\log(h_d^2 P)}, \quad \delta_v = \frac{\log(h_d^2(P_v + P_w))}{\log(h_d^2 P)}, \quad \delta_{u,k} = \frac{\log(h_d^2(\sum_{i=k}^K P_{u,i} + P_v + P_w))}{\log(h_d^2 P)}. \tag{4.292}$$

Define α_w and γ_w similarly as δ_w with replacing h_d^2 in the numerator of (4.292) by h_c^2 and h_s^2, respectively, and similarly define α_v, $\alpha_{u,k}$, γ_v, and $\gamma_{u,k}$. Also define β_a and β_b as

$$\beta_a = \frac{\log(h_r^2(P_a + P_b))}{\log(h_d^2 P)}, \quad \beta_b = \frac{\log(h_r^2 P_b)}{\log(h_d^2 P)}. \tag{4.293}$$

With this definition, $\delta_{u,k}$, $\alpha_{u,k}$, and $\gamma_{u,k}$ for instance represent the cumulative received power level at Rx 1, Rx 2, and the relay, respectively, of the signals w_1^n, v_1^n, $u_{1,K}^n$, $u_{1,K-1}^n, \cdots, u_{1,k}^n$, relative to $h_d^2 P$ on a logarithmic scale. Similar interpretation holds for β_a and β_b.

Definition 4.3. *For $m \in \{u, v, w\}$, define d_m as*

$$d_m = \lim_{h_d^2 P \to \infty} \frac{R_m}{\frac{1}{2}\log(h_d^2 P)}. \tag{4.294}$$

In other words, d_u for instance is the GDoF achieved by the CF messages only. To get the achievable GDoF per user, we add $d_u + d_v + d_w$, and to get the sum GDoF, we multiply the per user GDoF by 2, i.e., $d = 2(d_u + d_v + d_w)$.

4.3.3.2.1 Weak Interference: For the WI case ($\alpha \leq 1$), we need only to consider $\gamma \leq \alpha \leq \min\{1, \beta\}$ (cf. (4.291)). Let us express the rate constraints of the WI-CF scheme as GDoF constraints. We start with (4.267). We write this rate constraint first using our notation in Definition 4.2. Here comes the first stage of power allocation. The relay should decode the sum of the CF signals, so their powers have to be stronger than noise, i.e., $h_s^2 P_{u,K} > 1$. On the other hand, the C and P signals do not have to be decoded at the relay, thus their powers can be below the noise power. We obtain

$$R_u \leq \sum_{k=1}^{K} C^+ \left(\frac{h_s^2 P_{u,k}}{1 + 2h_s^2 (\sum_{i=k+1}^{K} P_{u,i} + P_v + P_w)} - \frac{1}{2} \right) \tag{4.295}$$

$$= \sum_{k=1}^{K-1} \left[\frac{1}{2} \log \left(\frac{\frac{1}{2} + h_s^2 (P_{u,k} + P_{u,k+1} + \cdots + P_{u,K} + P_v + P_w)}{1 + 2h_s^2 (\sum_{i=k+1}^{K} P_{u,i} + P_v + P_w)} \right) \right]^+ + \left[\frac{1}{2} \log \left(\frac{\frac{1}{2} + h_s^2 (P_{u,K} + P_v + P_w)}{1 + 2h_s^2 (P_v + P_w)} \right) \right]^+ \tag{4.296}$$

$$= \sum_{k=1}^{K-1} \left[\frac{1}{2} \log \left(\frac{\frac{1}{2} + (h_d^2 P)^{\gamma_{u,k}}}{1 + 2(h_d^2 P)^{\gamma_{u,k+1}}} \right) \right]^+ + \left[\frac{1}{2} \log \left(\frac{\frac{1}{2} + (h_d^2 P)^{\gamma_{u,K}}}{1 + 2(h_d^2 P)^{\gamma_v}} \right) \right]^+ \tag{4.297}$$

By dividing both sides by $\frac{1}{2}\log(h_d^2 P)$ and taking the limit as $h_d^2 P \to \infty$, we get the GDoF constraint

$$d_u \leq \sum_{k=1}^{K-1} (\gamma_{u,k} - \gamma_{u,k+1})^+ + (\gamma_{u,K} - \gamma_v^+)^+ \tag{4.298}$$

$$= \gamma_{u,1} - \gamma_v^+. \tag{4.299}$$

Similarly, one can obtain the following GDoF constraints from (4.268) and (4.269)

$$d_u \leq \sum_{k=1}^{K-1} (\delta_{u,k} - \max\{\delta_{u,k+1}, \alpha_{u,k}, \beta_b\})^+ + (\delta_{u,K} - \max\{\delta_v, \alpha_{u,K}, \beta_b\})^+, \tag{4.300}$$

$$d_u \leq (\beta_a - 1)^+ + (\beta_b - \delta_v^+)^+. \tag{4.301}$$

Let us analyze the first GDoF constraint (4.300). Each receiver needs to decode its desired CF signals before the relay signal $\bar{u}_b^n$. Moreover, the kth desired CF signal should be decoded without interference from the kth undesired CF signal (this is to be removed after the kth desired CF signal is decoded). Thus, we have to set

$$\alpha_{u,k} \leq \delta_{u,k+1}, \quad k = 1, \cdots, K-1 \tag{4.302}$$

$$\alpha_{u,K} \leq \max\{\delta_v, \beta_b\} \tag{4.303}$$

$$\beta_b \leq \delta_{u,K}. \tag{4.304}$$

Consider the second GDoF constraint (4.301). Since the desired C signal has to be decoded at the desired destination, then we have to set

$$\delta_v > 0. \tag{4.305}$$

Moreover, since $P_w + P_v + \sum_{k=1}^{K} P_{u,k} = 1$, we have $\delta_{u,1} = 1$ and $\gamma_{u,1} = \gamma$. As a result, the achievable GDoF of the CF message becomes

$$d_u \leq \min\{\gamma - \gamma_v^+, 1 - \max\{\delta_v, \beta_b\}, (\beta_a - 1)^+ + (\beta_b - \delta_v)^+\}. \tag{4.306}$$

Note the resemblance between this GDoF expression and the achievable CF rate in the LD-IRC given in (4.147). Similarly, the rate constraints of the C and P messages in the WI-CF scheme can be expressed as the following GDoF constraints

$$d_v \leq \min\left\{\alpha_v - \delta_w, \frac{\delta_v - \delta_w}{2}\right\}, \tag{4.307}$$

$$d_w \leq \delta_w, \tag{4.308}$$

where we set $\delta_w \geq 0$ and $\alpha_w \leq 0$. As in the IC [ETW08], we set $\delta_w = 1 - \alpha$ so that the interfering private signal is received at the noise level at the undesired receiver. We get

$$d_v \leq \min\left\{\alpha_v - 1 + \alpha, \frac{\delta_v - 1 + \alpha}{2}\right\}, \tag{4.309}$$

$$d_w \leq 1 - \alpha. \tag{4.310}$$

The remaining free parameters correspond to the C signals (δ_v, α_v, and γ_v), and to the relay signals (β_a and β_b). This leads to the following achievable GDoF as a corollary of Theorem 4.15.

Corollary 4.2. *The achievable GDoF corresponding to the WI-CF scheme in Theorem 4.15) is given by $d = 2(d_u + d_v + d_w)$ where d_u, d_v, and d_w satisfy* (4.306) *and* (4.309) *with $\delta_v \in [1-\alpha, 1]$, $\beta_b \in [0, \min\{1, \beta\}]$, and $\beta_a \in [\beta_b, \beta]$, and $\delta_v, \alpha_v, \gamma_v$ are related as defined in Definition 4.2.*

Note that by setting $\delta_v = 1$ and $\beta_a = \beta_b = 0$, we get the same achievable rate as in (4.289) for the WI regime. Now, we show that the GDoF in Corollary 4.2 coincide with Theorem 4.17 in the WI regime, i.e., that WI-CF achieves the GDoF of the G-IRC in the WI regime. The GDoF of the G-IRC given in Theorem 4.17 can be written for the WI regime as

$$d \leq \min\left\{\begin{array}{c} \max\{1, \beta\} + 1 - \alpha \\ 2 - \alpha + \gamma \\ 2\max\{\alpha, \beta, 1-\alpha\} \\ 2\max\{\alpha, 1 + \gamma - \alpha\} \end{array}\right\}. \tag{4.311}$$

In order to simplify this WI GDoF bound, we subdivide it into three cases similar to the LD-IRC.

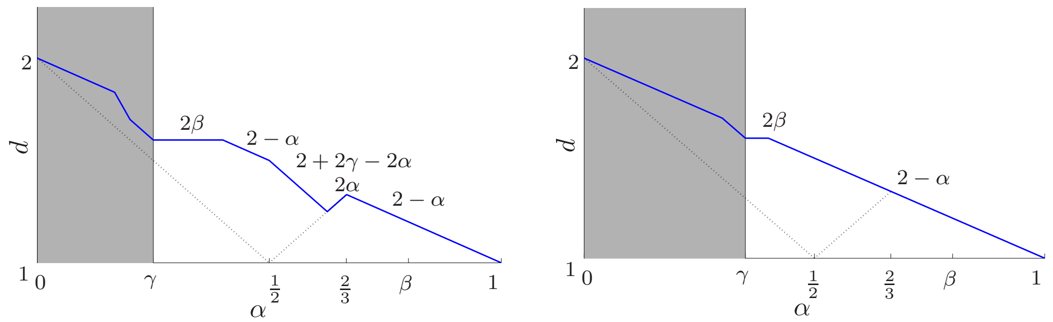
WI-1: In this case, $\beta \leq \frac{1}{2}$ and we need only to consider $\gamma \leq \alpha \leq \beta$ (cf. (4.291)). By evaluating the WI GDoF bounds (4.311) for this case we get

$$d \leq \min\{2 - \alpha, 2\max\{\alpha, 1-\alpha\}\}, \tag{4.312}$$

which is the same as the GDoF of the IC, and hence is achievable by the WI-CF scheme (see Figure 4.11 on page 60). As a result *if $\gamma \leq \alpha \leq \beta \leq \frac{1}{2}$, then the relay can be switched off without any impact on the GDoF of the network.*

WI-2: Now we have $\frac{1}{2} < \beta \leq 1$. The GDoF upper bound becomes

$$d \leq \min\left\{\begin{array}{c} 2 - \alpha \\ 2\max\{\beta, 1-\alpha\} \\ 2\max\{\alpha, 1 + \gamma - \alpha\} \end{array}\right\}. \tag{4.313}$$



(a) $\beta = 0.8$, $\gamma = 0.25$ (b) $\beta = 0.8$, $\gamma = 0.35$

Figure 4.12: The GDoF of the IRC for $\beta = 0.8$ and different values of $\gamma \leq 1$. The GDoF of the IC is also shown (dotted) as a benchmark.

shown in Figure 4.12. For convenience, we express this upper bound as

$$d \leq \begin{cases} 2 - 2\alpha, & \alpha \leq 1 - \beta, \frac{1+\gamma}{2} \\ \min\{2\alpha, 2 - \alpha\}, & 1 - \beta, \frac{1+\gamma}{2} < \alpha \\ \min\{2 - \alpha, 2\beta, 2 + 2\gamma - 2\alpha\}, & 1 - \beta < \alpha \leq \frac{1+\gamma}{2}. \end{cases} \tag{4.314}$$

The GDoF of the first and second cases can be achieved by the WI-CF scheme since it is achievable in the IC (note that in the second case $\alpha \geq 1/2$). Finally, we consider the third case. One can try to maximize the GDoF given in Corollary 4.2 in order to obtain the optimal GDoF. Instead, we consider an easier method. First, we recall the level allocation parameters we used in the LD-IRC. There (Section 4.2.3.1), we used

i) $(\ell_b, \ell_v) = (\frac{2n_d - n_c}{2}, n_d - n_c)$ to achieve $2n_d - n_c$,

ii) $(\ell_b, \ell_v) = (n_r, n_d - n_c)$ to achieve $2n_r$, and

iii) $(\ell_b, \ell_v) = (n_d + n_s - n_c, n_d - n_c)$ to achieve $2n_d + 2n_s - 2n_c$.

This level allocation can be easily translated to a power allocation for the G-IRC. For example, we can translate the level allocation in case (i) given by $(\ell_b, \ell_v) = (\frac{2n_d - n_c}{2}, n_d - n_c)$ to $(\beta_b, \delta_v) = \left(\frac{2-\alpha}{2}, 1 - \alpha\right)$ by replacing n_d by 1 and n_c by α. Thus, we obtained the levels β_b and δ_v. Next, we calculate the power allocation corresponding to these levels. To do this, we write

$$\beta_b = \frac{2 - \alpha}{2} \Rightarrow P_{r,b} = \sqrt{\frac{h_d^4 P}{h_c^2 h_r^4}}, \tag{4.315}$$

$$\ell_v = 1 - \alpha \Rightarrow P_v + P_w = \frac{1}{h_c^2}. \tag{4.316}$$

Observing that $\ell_w = 1 - \alpha \Rightarrow P_w = 1/h_c^2$, this implies that $P_v = 0$ and no C message is needed in this case. Furthermore, we set $P_{ra} = 0$ since $\beta < 1$ in this case, and thus we can not achieve non-zero GDoF for $\bar{u}_a^n$ by decoding it before decoding the desired CF messages. Thus, we need the following power allocation to achieve $d = 2 - \alpha$

$$P_w = \frac{1}{h_c^2}, \quad P_u = P - P_w, \quad P_{r,b} = \sqrt{\frac{h_d^4 P}{h_c^2 h_r^4}} \tag{4.317}$$

and the remaining signals have zero power. The power P_u has to be split between the K CF signals, where K is determined next. To find K, we use conditions (4.302)-(4.304) given by

$$\alpha_{u,k} \leq \delta_{u,k+1}, \quad k = 1, \cdots, K-1 \tag{4.318}$$
$$\alpha_{u,K} \leq \max\{\delta_v, \beta_b\} \tag{4.319}$$
$$\beta_b \leq \delta_{u,K}. \tag{4.320}$$

as follows. If we set

$$P_{u,k} = \left(\frac{h_c^2}{h_d^2}\right)^{k-1} P - \left(\frac{h_c^2}{h_d^2}\right)^{k} P, \quad k = 1, \cdots, K-1 \tag{4.321}$$
$$P_{u,K} = \left(\frac{h_c^2}{h_d^2}\right)^{K-1} P - P_w, \tag{4.322}$$

then we satisfy (4.318) and also $P_u = P - P_w$. Now, we find K such that condition (4.320) is satisfied. We require

$$\beta_b \leq \delta_{u,K} \Rightarrow \sqrt{\frac{h_d^4 P}{h_c^2}} \leq h_d^2 \left(\frac{h_c^2}{h_d^2}\right)^{K-1} P \tag{4.323}$$
$$\Rightarrow K - 1 \leq \frac{\alpha}{2(1-\alpha)}. \tag{4.324}$$

From (4.319), noting that $\delta_v \leq \beta_b$ in this case, we get

$$\beta_b \geq \alpha_{u,K} \Rightarrow K - 1 \geq \frac{\alpha}{2(1-\alpha)} - 1. \tag{4.325}$$

From (4.324) and (4.325), we obtain the desired K

$$K = \left\lceil \frac{\alpha}{2(1-\alpha)} \right\rceil. \tag{4.326}$$

In summary, to achieve $d = 2 - \alpha$ in the third case in (4.314), we set $P_w = 1/h_c^2$, $P_v = P_{r,a} = 0$, $P_{r,b} = \sqrt{\frac{h_d^4 P}{h_c^2 h_r^4}}$, and

$$P_{u,K} = \left(\frac{h_c^2}{h_d^2}\right)^{K-1} P - P_w, \qquad K = \left\lceil \frac{\alpha}{2(1-\alpha)} \right\rceil \tag{4.327}$$
$$P_{u,k} = \left(\frac{h_c^2}{h_d^2}\right)^{k-1} P - \left(\frac{h_c^2}{h_d^2}\right)^{k} P, \quad k = 1, \cdots, K-1. \tag{4.328}$$

Note that if $\alpha \leq 2/3$, then $K = 1$, i.e., no splitting of the CF message is required. Otherwise, splitting the CF message is required. However, if $\alpha > 2/3$, then CF is not needed to achieve $2 - \alpha$ since this GDoF can be also achieved in the IC.

Using similar analysis, we can show that the level allocation in (ii) and (iii) lead to $K = 1$, $P_v = P_{r,a} = 0$, $P_w = 1/h_c^2$, $P_{u,1} = P - P_w$, and $P_{r,b} = P$ to achieve $d = 2\beta$, and $P_{r,b} = \frac{h_d^2 h_s^2 P}{h_c^2 h_r^2}$ to achieve $d = 2 + 2\gamma - 2\alpha$.

It is notable that in this case the relay does not use its full power to achieve the GDoF. This is relatively counterintuitive, since one would expect that we should use all the relay power to be optimal. Additionally, the common messages are not necessary in this regime.

WI-3: In this case, the G-IRC has strong relay-destination channels $\beta > 1$, and hence, part of the relay signal can be sent at higher power and decoded first at the destinations before the desired CF signals. The GDoF in this case is

$$d \leq \min \left\{ \begin{array}{c} 1-\alpha+\beta \\ 2-\alpha+\gamma \\ 2\max\{\alpha, 1-\alpha+\gamma\} \end{array} \right\}. \tag{4.329}$$

Let us first rewrite this expression for convenience as

$$d \leq \begin{cases} 1-\alpha+\beta, & \alpha \leq 1+2\gamma-\beta, \frac{1+\gamma}{2} \\ 2-2\alpha+2\gamma, & 1+2\gamma-\beta < \alpha \leq \frac{1+\gamma}{2} \\ \min\{2\alpha, 1-\alpha+\beta\}, & \alpha > \frac{1+\gamma}{2},\ \gamma > \beta-1 \\ \min\{2\alpha, 2-\alpha+\gamma\}, & \alpha > \frac{1+\gamma}{2},\ \gamma \leq \beta-1 \end{cases} \tag{4.330}$$

Again, using the same analysis as before, we can translate the level allocation of the LD-IRC in the regime WI-3 to a power allocation policy for the G-IRC. In the first two cases in (4.330), we set the WI-CF parameters to $P_w = \frac{1}{h_c^2}$, $P_v = 0$,

$$P_{u,K} = \left(\frac{h_c^2}{h_d^2}\right)^{K-1} P - P_w, \tag{4.331}$$

$$P_{u,k} = \left(\frac{h_c^2}{h_d^2}\right)^{k-1} P - \left(\frac{h_c^2}{h_d^2}\right)^{k} P, \tag{4.332}$$

for $k = 1, \cdots, K-1$, and $P_{r,a} = P - P_{r,b}$, with

$$K = \left\lceil \frac{\beta+\alpha-1}{2(1-\alpha)} \right\rceil, \quad P_{r,b} = \sqrt{\frac{h_d^6 P}{h_r^6 h_c^2}} \tag{4.333}$$

to achieve $d = 1-\alpha+\beta$, and

$$K = \left\lceil \frac{\gamma}{1-\alpha} \right\rceil, \quad P_{r,b} = \frac{h_d^4 h_s^2 P}{h_r^4 h_c^2} \tag{4.334}$$

to achieve $d = 2-2\alpha+2\gamma$.

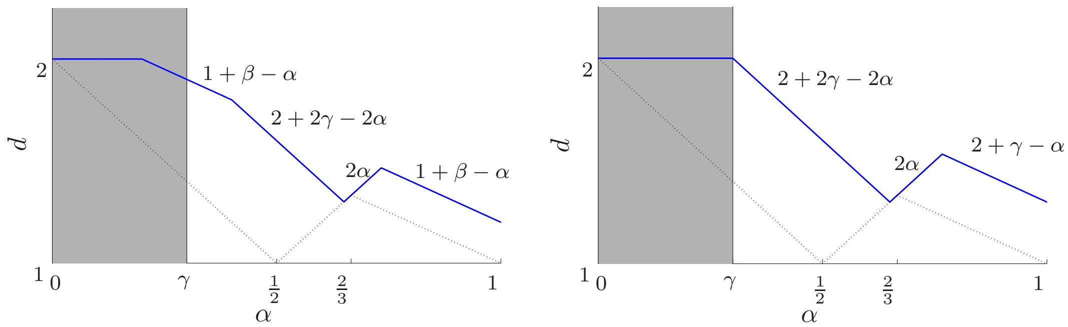
In the third and last cases in (4.330), we need the common messages. Moreover, in these cases we do not need $\bar{u}_b^n$, i.e., the relay does not split its message, but sends it using a Gaussian code with power P. We set $P_w = \frac{1}{h_c^2}$, $P_{r,a} = P$, $P_{r,b} = 0$,

$$P_{u,K} = \left(\frac{h_c^2}{h_d^2}\right)^{K-1} P - P_v - P_w, \tag{4.335}$$

$$P_{u,k} = \left(\frac{h_c^2}{h_d^2}\right)^{k-1} P - \left(\frac{h_c^2}{h_d^2}\right)^{k} P, \tag{4.336}$$

for $k = 1, \cdots, K-1$, with

$$K = \left\lceil \frac{\beta-1}{1-\alpha} \right\rceil, \quad P_v = \frac{h_d^2 P}{h_r^2} - P_w \tag{4.337}$$



(a) $\beta = 1.2$, $\gamma = 0.3$ (b) $\beta = 1.4$, $\gamma = 0.3$

Figure 4.13: The GDoF of the IRC for $\beta = 0.8$ and different values of $\gamma \leq 1$. The GDoF of the IC is also shown (dotted) for comparison.

to achieve $d = \min\{2\alpha, 1 - \alpha + \beta\}$, and

$$K = \left\lceil \frac{\gamma}{1-\alpha} \right\rceil, \quad P_v = \frac{1}{h_s^2} - P_w \tag{4.338}$$

to achieve $d = \min\{2\alpha, 2 - \alpha + \gamma\}$.

To this end, we have shown the achievability of the GDoF of the G-IRC with $h_s^2 \leq h_c^2$ and with WI $h_c^2 \leq h_d^2$. GDoF plots for this case are shown in Figure 4.13. At this point, we discuss the CF rate splitting used in the WI-2 and WI-3 regimes above before we prove the GDoF achievability for the strong interference case.

4.3.3.2.2 Why to split the CF message into K parts? A graphical illustration: In this discussion, we refer to Figures 4.14 and 4.15 which capture the general idea, and can be extended to other cases. Figure 4.14 shows a scenario where CF splitting is not used. In this case, Rx 1 starts by decoding $\bar{u}_a$ achieving $\beta - 1$ GDoF, then it decodes $u_{1,1}$ achieving $d_{u,1} = 1 - \alpha$ GDoF. Next, it performs interference cancellation[4] by removing the contribution of $u_{2,1}$, and decodes $\bar{u}_b$ achieving $2 - \alpha - \beta$ GDoF (note that $(\beta - 1) + (2 - \alpha - \beta) = 1 - \alpha = d_{u,1}$). Finally, the P signal is decoded which achieves $d_w = 1 - \alpha$. The total achieved GDoF is then

$$2(d_u + d_w) = 2\min\{1-\alpha, (\beta - 1) + (2 - \alpha - \beta)\} + 2(1 - \alpha) = 4 - 4\alpha. \tag{4.339}$$

By looking on the scale on the right edge of Figure 4.14, we can see the wasted opportunity to achieve some extra GDoF in the interval $[3 - 2\alpha - \beta, \alpha]$. Namely, after decoding $u_{1,1}$ and removing $u_{2,1}$, another signal, say $u_{1,2}$, can be decoded without decreasing the GDoF of $u_{1,1}$ as long as its power is higher than w_1 (so that it achieves non-zero GDoF) and lower than u_2 (so that it does not decrease the GDoF of u_1). At most, we can choose the power of $u_{1,2}$ such that $\delta_{u,2} = \alpha_{u,1}$ (which explains (4.318)).

Thus, if we split u_j, $j = 1, 2$, into $u_{j,1}$ and $u_{j,2}$ as in Figure 4.15(a) such that $\delta_{u,2} = \alpha_{u,1}$, we can achieve higher GDoF. In this case, after decoding $\bar{u}_a$, the first desired CF signal $u_{1,1}$ can be decoded with a GDoF of $d_{u,1} = 1 - \alpha$ as if the only interference comes from the second interfering CF signal. Once $u_{1,1}$ is decoded, we cancel the interference from $u_{2,1}$ and proceed to decode the $u_{1,2}$, *achieving some extra*

[4] Recall that in block b, Rx 1 knows the sum of the desired and the interfering CF signals from the decoded relay signal in block $b + 1$.

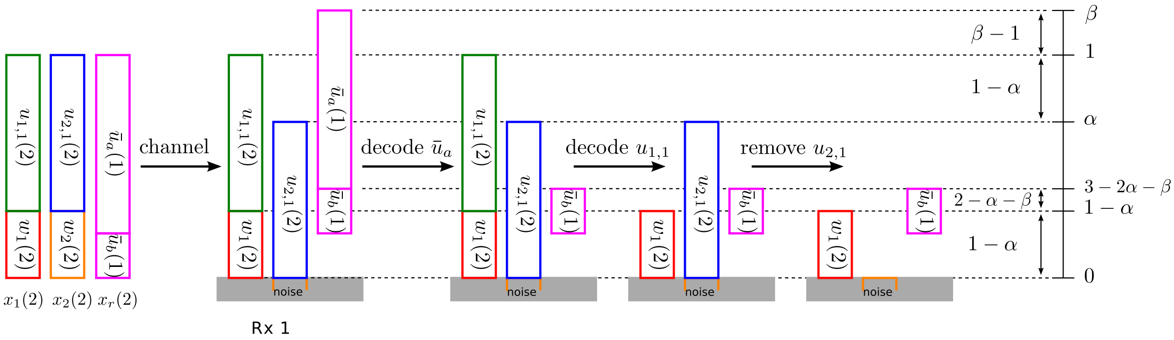


Figure 4.14: Power levels of transmitted and received signals at Rx 1 in the second transmission block. No CF message splitting is used in this example.

non-zero GDoF. In this case, $d_{u,2} = \frac{3\alpha+\beta-3}{2}$ (Figure 4.15(b)). Then, $\bar{u}_b$ is decoded with a GDoF of $\frac{1+\alpha-\beta}{2}$, and finally, w_1 is decoded with a GDoF of $d_w = 1-\alpha$. This achieves a total GDoF of $2(d_u + d_w) = 1 + \beta - \alpha$ which is clearly larger than $4 - 4\alpha$ in this case. This GDoF is optimal for the example in this figure.

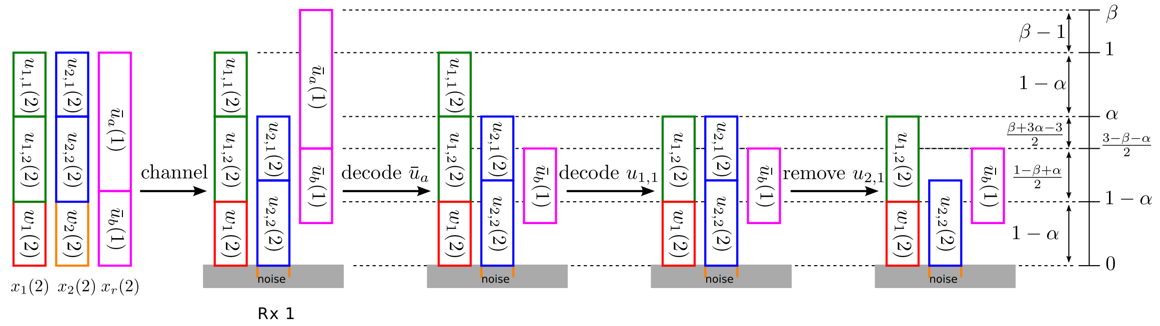
Now we ask ourselves how many such CF messages can we have? In other words, what is the largest number of CF message splits K that we should choose in order to maximize the achievable GDoF? This question was answered in our discussion of the WI-2 case, however, here we explain it graphically. To answer this question, we have first to choose the power of the relay signal $\bar{u}_b$. This is done as follows. As explained previously for the LD-IRC, the WI-CF is based on decoding two observations of the same signal, namely, the desired CF signal and the relay signal (which indicates the sum of the CF signals of the two users). To maximize the GDoF achievable by the CF messages, we try to divide the interval $[1-\alpha, \beta]$ (from the received power level of the private signal to that of the relay signal (see Figure 4.15)) into two equal parts, one which is assigned to the relay signals, and the other assigned to the desired CF signals. Since only the relay signal can be received with power level higher than 1 (since in this case $\beta > 1$) the interval $[1, \beta]$ is allocated to $\bar{u}_a$. This allow us to achieve $d_a = \beta - 1$ and it remains to achieve $d_b = \frac{\beta-(1-\alpha)}{2} - (\beta - 1) = \frac{1+\alpha-\beta}{2}$. Thus, the signal $\bar{u}_b$ must be received with power level of $\frac{3-\alpha-\beta}{2}$ in order to be able to decode it with a GDoF of $d_b = \frac{1+\alpha-\beta}{2}$ while treating the P signal as noise. The power level of the relay signal $\bar{u}_b$ given by $\frac{3-\alpha-\beta}{2}$ is equivalent to

$$\frac{\log(h_r^2 P_{r,b})}{\log(h_d^2 P)} = \frac{\log\left(\sqrt{\frac{h_d^6 P}{h_c^2 h_r^2}}\right)}{\log(h_d^2 P)} = \frac{\log\left(h_r^2\sqrt{\frac{h_d^6 P}{h_c^2 h_r^6}}\right)}{\log(h_d^2 P)} \tag{4.340}$$

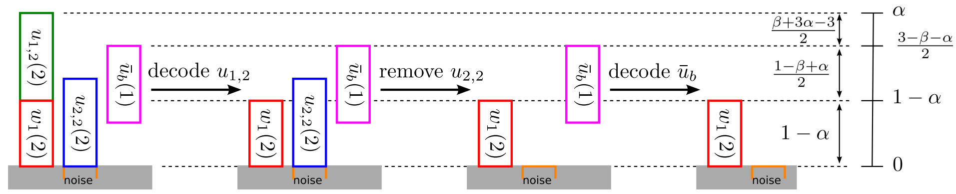
leading to $P_{r,b} = \sqrt{\frac{h_d^6 P}{h_c^2 h_r^6}}$ (see (4.333)). Now, the Kth CF signal can have non-zero GDoF if its power level is larger than the power of the relay signal $\bar{u}_b$ (see Figure 4.15(b) where the power of $u_{1,2}$ is larger than that of $\bar{u}_b$). Thus, it is required that $1-(K-1)(1-\alpha) \geq \frac{3-\beta-\alpha}{2}$, leading to $K - 1 \leq \frac{\beta+\alpha-1}{2(1-\alpha)}$. Therefore, the largest K that we can choose is

$$K = \left\lceil \frac{\beta+\alpha-1}{2(1-\alpha)} \right\rceil, \tag{4.341}$$

as given in (4.333). This guarantees that while decoding the Kth desired CF signal,



(a) Rx 1 decodes $\bar{u}_a$ followed by $u_{1,1}$ which achieves $d_a = \beta - 1$ and $d_{u,1} = 1 - \alpha$, respectively. Then, it removes the contribution of $u_{2,1}$. Note that $u_{1,2}$ does not decrease the GDoF achieved by $u_{1,1}$.



(b) Rx 1 then decodes $u_{1,2}$ achieving $d_{u,2} = \frac{\beta+3\alpha-3}{2}$. Finally, it removes the contribution of $u_{2,2}$ and decodes $\bar{u}_b$ followed by w_1 which achieves $d_b = \frac{1-\beta+\alpha}{2}$ and $d_w = 1 - \alpha$, respectively.

Figure 4.15: Power levels of transmitted (with CF message splitting) and received signals at Rx 1 in the second transmission block. At Rx 1, the interval $[1-\alpha, \beta]$ is divided into two parts: the part $[\frac{3-\beta-\alpha}{2}, 1]$ is assigned to the desired CF signals, and the rest to the relay signals. With $K = 2$ we achieve $d_u = d_{u,1} + d_{u,2} = d_a + d_b = \frac{\alpha+\beta-1}{2}$ and $d_w = 1 - \alpha$ for a total of $\frac{1+\beta-\alpha}{2}$ GDoF per user. Notice that although the interference power is larger than the power of $\bar{u}_b$, by splitting the CF message into two parts we were able to decode it with a GDoF of $\frac{\alpha+\beta-1}{2}$ as if there is no CF interference.

the strongest interferer is the relay signal $\bar{u}_b$ since this gives $\alpha - (K-1)(1-\alpha) = 1 + K(1-\alpha) \leq \frac{3-\alpha-\beta}{2}$, which follows from the choice of K.

In summary, K is chosen as the largest integer such that the received power level of the Kth desired CF signal is larger than the power level of the relay signal $\bar{u}_b$ which in turn is larger than the received power level of the Kth interfering CF signal (see Figure 4.15).

Following these guidelines, we proceed with showing the achievability of Theorem 4.17 for the strong interference G-IRC.

4.3.3.2.3 Strong Interference: Similar to the weak interference case, we start by expressing the rate constraints of the SI-CF scheme as GDoF constraints to obtain the following corollary.

Corollary 4.3. *The achievable GDoF corresponding to the SI-CF scheme in Theorem 4.16 is given by* $d = 2(d_u + d_v)$ *where* d_u *and* d_v *satisfy*

$$d_u \leq \min\{\gamma - \gamma_v^+, \alpha - \max\{\alpha_v, \beta_b\}, (\beta_a - \alpha)^+ + (\beta_b - \alpha_v)^+\} \tag{4.342}$$

$$d_v \leq \min\left\{\delta_v, \frac{\alpha_v}{2}\right\}. \tag{4.343}$$

for some $\alpha_v \in [0, \alpha]$, $\beta_b \in [0, \min\{\alpha, \beta\}]$, $\beta_a \in [\beta_b, \beta]$, *and* $\delta_v, \alpha_v, \gamma_v$ *related as given in Definition 4.2.*

The conditions for splitting the CF message for the SI case are as follows

$$\delta_{u,k} \leq \alpha_{u,k+1}, \quad k = 1, \cdots, K-1 \tag{4.344}$$
$$\delta_{u,K} \leq \max\{\alpha_v, \beta_b\} \tag{4.345}$$
$$\beta_b \leq \alpha_{u,K}. \tag{4.346}$$

First, by setting $\alpha_v = 1$ and $\beta_a = \beta_b = 0$, we recover the GDoF of the IC as in (4.289) for the SI regime. Additionally, the GDoF given in this corollary coincides with Theorem 4.17 under the conditions $\alpha > 1$ and $\gamma \leq \alpha$, given by

$$d \leq \min \left\{ \begin{array}{c} 2\max\{1,\beta\} \\ 2\max\{1,\gamma\} \\ \max\{\alpha,\beta\} \\ \gamma+\alpha \end{array} \right\}. \tag{4.347}$$

We only need to discuss $\beta \geq 1$ since if $\beta \leq 1 \leq \alpha$, then the G-IRC has the same GDoF as the IC (cf. (4.290)). In order to simplify the proof, we split SI into two cases.
SI-1: If $\gamma \leq \alpha < \beta$, then the GDoF upper bound becomes

$$d \leq \min \left\{ \begin{array}{c} 2\max\{1,\beta\} \\ 2\max\{1,\gamma\} \\ \beta \\ \gamma+\alpha \end{array} \right\}. \tag{4.348}$$

We rewrite this upper bound for convenience as

$$d \leq \left\{ \begin{array}{ll} \min\{\beta, 2\gamma\}, & 1 < \gamma, \beta \\ \min\{\gamma+\alpha, \beta, 2\}, & \gamma \leq 1 < \beta \end{array} \right. \tag{4.349}$$

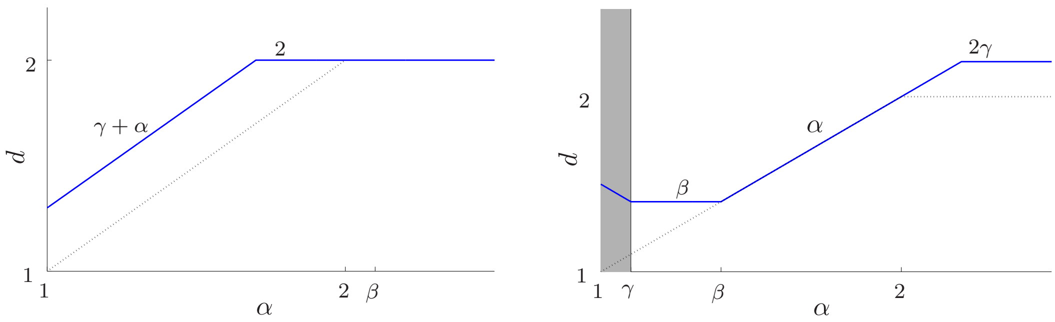
Let us start with the first case in (4.349). Here we need to achieve $\min\{\beta, 2\gamma\}$. We can do this by maximizing the achievable GDoF in Corollary 4.3, or by translating the level allocation of the LD-IRC to power allocation of the G-IRC. Either way, we get $P_v = 0$, $P_{r,b} = \sqrt{\frac{h_c^4 P}{h_r^6}}$, $P_{r,a} = P - P_{r,b}$,

$$P_{u,k} = \left(\frac{h_d^2}{h_c^2}\right)^{k-1} P - \left(\frac{h_d^2}{h_c^2}\right)^{k} P, \quad k = 1, \cdots, K-1, \tag{4.350}$$
$$P_{u,K} = \left(\frac{h_d^2}{h_c^2}\right)^{K-1} P, \tag{4.351}$$

and $K = \left\lceil \frac{\beta}{2(\alpha-1)} \right\rceil$. In the second case, we need to achieve $\min\{\gamma+\alpha, \beta, 2\}$, which is possible by using $P_{r,a} = P$, $P_{r,b} = 0$,

$$P_{u,k} = \left(\frac{h_d^2}{h_c^2}\right)^{k-1} P - \left(\frac{h_d^2}{h_c^2}\right)^{k} P \quad k = 1, \cdots, K-1, \tag{4.352}$$
$$P_{u,K} = \left(\frac{h_d^2}{h_c^2}\right)^{K-1} P - P_v, \tag{4.353}$$

and setting $P_v = \frac{1}{h_s^2}$ and $K = \left\lceil \frac{\gamma}{\alpha-1} \right\rceil$ to achieve $\gamma + \alpha$, and $P_v = \frac{h_c^2 P}{h_r^2}$ and $K = \left\lceil \frac{\beta-\alpha}{\alpha-1} \right\rceil$ to achieve $\min\{\beta, 2\}$.



(a) $\beta = 2.1$, $\gamma = 0.3$ (b) $\beta = 1.4$, $\gamma = 1.1$

Figure 4.16: The GDoF of the IRC with strong interference for two different cases. The GDoF of the IC is also shown (dotted) for comparison. In all cases, the GDoF gain obtained by using the relay is apparent.

SI-2: In this case, $\gamma, \beta \leq \alpha$ and the GDoF expression of Theorem 4.17 becomes

$$d \leq \min \left\{ \begin{array}{c} 2\max\{1,\beta\} \\ 2\max\{1,\gamma\} \\ \alpha \end{array} \right\}. \tag{4.354}$$

which can be rewritten as

$$d \leq \begin{cases} \alpha & \alpha \leq 2 \\ 2, & \alpha > 2,\ \min\{\beta,\gamma\} \leq 1 \\ \min\{\alpha, 2\gamma\}, & \alpha > 2,\ \beta \geq \gamma \geq 1 \\ \min\{\alpha, 2\beta\}, & \alpha > 2,\ \gamma \geq \beta \geq 1 \end{cases} \tag{4.355}$$

Observe that the first and second cases are always achievable in the case of SI since they are achievable in the IC. In the third and fourth cases, we use SI-CF with $P_v = P_{r,a} = 0$, $P_{r,b} = \min\left\{P, \sqrt{\frac{h_c^2 P}{h_r^4}}\right\}$, $P_{u,1} = P$, and $K = 1$, which achieves the upper bound. GDoF plots for the IRC with strong interference are shown in Figure 4.16.

Thus, our SI-CF scheme achieves the GDoF of the G-IRC with SI and as a result, this ends the achievability proof of Theorem 4.17. To this end, we have finished the characterization of the GDoF of the G-IRC with $h_s^2 \leq h_c^2$.

4.4 Comparison with classical schemes

Since the GDoF is a high SNR metric, let us see how does our CF schemes perform at low to moderate SNR in comparison to two classical schemes: decode-forward, and compress-forward.

4.4.1 Classical Schemes

Designing a transmission scheme for the G-IRC involves not only designing a coding strategy at the sources and a decoding strategy at the destinations, but also a relaying strategy at the relay. Therefore, the number of various transmission schemes for the G-IRC can be enormous. Here we only compare our scheme with the two most common schemes, namely, decode-forward and compress-forward. We first give the achievable rates of the decode-forward scheme and the compress-forward scheme.

4.4.1.1 Decode-forward

We illustrate a decode-forward (DF) scheme which is a restricted version of the DF scheme in [SE07a]. Namely, we restrict the decoding order, such that the common messages are decoded before the private message. This scheme achieves the following sum-rate.

Theorem 4.18 (DF [SE07a]). *The sum-rate $R_\Sigma = 2R_w + 2R_v$ is achievable where R_w and R_v satisfy the rate constraints for decoding at the relay:*

$$R_w \leq \min\left\{C(h_s^2 P_w), \frac{1}{2}C(2h_s^2 P_w)\right\} \tag{4.356}$$

$$R_v \leq \min\left\{C(h_s^2 P_v), \frac{1}{2}C(2h_s^2 P_v)\right\} \tag{4.357}$$

$$R_w + R_v \leq \min\left\{C(h_s^2(P_w + P_v)), \frac{1}{2}C(2h_s^2(P_w + P_v))\right\} \tag{4.358}$$

$$2R_w + R_v \leq C(2h_s^2 P_w + h_s^2 P_v) \tag{4.359}$$

$$R_w + 2R_v \leq C(h_s^2 P_w + 2h_s^2 P_v) \tag{4.360}$$

and for decoding at the receivers:

$$R_v \leq C\left(\frac{(h_d\sqrt{\bar{P}_v} + h_r\sqrt{P_{r,v}})^2}{1 + (h_d\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2 + (h_c\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2 + h_c^2 P_w}\right) \tag{4.361}$$

$$R_v \leq C\left(\frac{(h_c\sqrt{\bar{P}_v} + h_r\sqrt{P_{r,v}})^2}{1 + (h_d\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2 + (h_c\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2 + h_c^2 P_w}\right) \tag{4.362}$$

$$2R_c \leq C\left(\frac{(h_d\sqrt{\bar{P}_v} + h_r\sqrt{P_{r,v}})^2 + (h_c\sqrt{\bar{P}_v} + h_r\sqrt{P_{r,v}})^2}{1 + (h_d\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2 + (h_c\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2 + h_c^2 P_w}\right) \tag{4.363}$$

$$R_p \leq C\left(\frac{(h_d\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2}{1 + (h_c\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}})^2 + h_c^2 P_v}\right), \tag{4.364}$$

for all $P_w + P_v + \bar{P}_w + \bar{P}_v \leq P$ and $2P_{r,w} + 2P_{r,v} \leq P$.

At high P, where the system is interference limited, the advantage of CF over DF is obvious. The DF scheme has the problem of the bottleneck at the relay; the achievable rate is always upper bounded by (4.358), i.e., $C(2h_s^2(P_w + P_v)) \leq C(2h_s^2 P)$ which gives a GDoF constraint of $d \leq \gamma$. Thus, it is not possible to achieve a GDoF higher than γ, which in contrast is achievable by CF (e.g. the achievability of 2γ in (4.355)). Thus, CF is clearly superior to DF. Next we briefly describe the DF scheme.

4.4.1.1.1 Transmitter processing: Tx 1 transmits $B-1$ messages $m_1(b)$, $b = 1, \cdots, B-1$, in a window of B blocks, where B is some large integer. In block b, the message of Tx 1 $m_1(b)$ is split into 2 messages, a private (P) message $m_{1,w}(b)$ with rate R_w, to be decoded by Rx 1, and a common (C) message $m_{1,v}(b)$ with rate R_v, to be decoded by both receivers. These messages are then encoded using a Gaussian random code into two independent i.i.d. sequences $w_1^n(b)$ and $v_1^n(b)$, with powers P_w

and P_v, respectively. In each block b, the transmit signal $x_1^n(b)$ is constructed in the following way

$$x_1^n(b) = \sqrt{\frac{\bar{P}_v}{P_v}} u_1^n(b-1) + \sqrt{\frac{\bar{P}_w}{P_w}} w_1^n(b-1) + v_1^n(b) + w_1^n(b). \tag{4.365}$$

Thus, $x_1^n(b)$ is a scaled superposition of private and common codewords from blocks b and $b-1$ (block Markov encoding). At the Tx 2, a similar procedure is done. The power constraint is satisfied if $P_w + P_v + \bar{P}_w + P_v \leq P$.

4.4.1.1.2 Relay processing: The relay decodes in a forward manner starting from block 1 where it decodes $w_1^n(1)$, $v_1^n(1)$, $w_2^n(1)$ and $v_2^n(1)$. The rate constraint for reliable decoding at the relay are given in (4.356)-(4.360). After decoding the P and C messages, the relay sends the sum

$$x_r^n(b) = \sqrt{\frac{P_{r,w}}{P_w}} (w_1^n(b-1) + w_2^n(b-1)) + \sqrt{\frac{P_{r,v}}{P_v}} (v_1^n(b-1) + v_2^n(b-1)) \tag{4.366}$$

in block b. The power constraint at the relay is satisfied if $2P_{r,w} + 2P_{r,v} \leq P$.

4.4.1.1.3 Receiver Processing: As a result, the received signal at Rx 1 in block b can be written as

$$\begin{aligned} y_1^n(b) = &\frac{1}{\sqrt{P_w}} \left[(h_d\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}}) w_1^n(b-1) + (h_c\sqrt{\bar{P}_w} + h_r\sqrt{P_{r,w}}) w_2^n(b-1) \right] \\ &+ \frac{1}{\sqrt{P_v}} \left[(h_d\sqrt{\bar{P}_v} + h_r\sqrt{P_{r,v}}) v_1^n(b-1) + (h_c\sqrt{\bar{P}_v} + h_r\sqrt{P_{r,v}}) v_2^n(b-1) \right] \\ &+ h_d(w_1^n(b) + v_1^n(b)) + h_c(w_2^n(b) + v_2^n(b)) + z_1^n. \end{aligned} \tag{4.367}$$

The receivers use Willems' backward decoding [Wil82] to decode the signals, starting from the last block B. Each receiver decodes both the C messages and its P message. Then the decoding proceeds backwards where in each block b, the known P and C signals at Rx 1 ($w_1^n(b)$, $v_1^n(b)$, and $v_2^n(b)$ decoded in block $b+1$) are subtracted, and then $w_1^n(b-1)$, $v_1^n(b-1)$, and $v_2^n(b-1)$ are decoded. This procedure continues until block 1 is reached. The decoding of P and C messages at each receiver is done in a similar way as in [ETW08]. Each receiver decodes both the C messages while treating the other signals as noise. This is possible with arbitrarily small error probability if the rates of the C message satisfy (4.361)-(4.363). Then, each receiver subtracts both the decoded C signals, and decodes its own P message treating the remaining interference as noise. The error probability can be made arbitrarily small if (4.364) is satisfied. This leads to the achievable rate in Theorem 4.18.

4.4.1.2 Compress-forward

Another cooperative strategy that can be used in relay networks is the compress-forward strategy. In this strategy, the relay compresses its received observation and maps it to an index, then it uses a channel code to send this index to the receivers. This scheme was considered for the relay channel in [CEG79] and for G-IRC in [TY11]. Two variants of this scheme exist, namely, compress-forward with forward decoding (CFF), and compress-forward with backward decoding (CFB). We start with CFF.

Theorem 4.19 (CFF). *CFF achieves $R_\Sigma = 2R_w + 2R_v$ where*

$$R_v \leq \min\{I(V_1; Y_1, \hat{Y}_r|V_2), I(V_1; Y_2, \hat{Y}_r|V_2)\} \tag{4.368}$$

$$2R_v \leq \min\{I(V_1, V_2; Y_1, \hat{Y}_r), I(V_1, V_2; Y_2, \hat{Y}_r)\} \tag{4.369}$$

$$R_w \leq I(W_1; Y_1, \hat{Y}_r|V_1, V_2) \tag{4.370}$$

where $\hat{Y}_r = Y_r + Z_f$, Z_f is a Gaussian noise, independent of all other variables, with variance

$$\sigma_f^2 = \frac{h_s^2(h_c - h_d)^2(P_w + P_v)^2 + (h_d^2 + h_c^2 + 2h_s^2)(P_w + P_v) + 1}{h_r^2 P_r}, \tag{4.371}$$

and $W_1, W_2 \sim \mathcal{N}(0, P_w)$, $V_1, V_2 \sim \mathcal{N}(0, P_v)$, such that $P_w + P_v \leq P$ and $P_r \leq P$.

The compress-and-forward transmit strategy is performed block-wise as for the DF strategy. We introduce rate splitting to the compress-forward strategy, i.e., each transmitter sends a P message and a C message. In block b, Tx j encodes its P and C messages $m_{j,w}(b)$ and $m_{j,v}(b)$ with rates R_w and R_v, respectively, into codewords $w_j^n(b)$ and $v_j^n(b)$, respectively, with i.i.d. components such that $W_j \sim \mathcal{N}(0, P_w)$ and $V_j \sim \mathcal{N}(0, P_v)$, and $P_w + P_v \leq P$. Then the signal $x_j^n(b) = w_j^n(b) + v_j^n(b)$ is sent. At the end of block b, the relay compresses its received signal $y_r^n(b)$ using Wyner-Ziv coding [WZ76] with rate R_r and assigns it to an index $m_r(b)$ and then encodes this index into x_r^n with i.i.d. $X_r \sim \mathcal{N}(0, P_r)$ such that $P_r \leq P$. This codeword x_r^n is sent in the next block $b+1$, and hence denoted $x_r^n(b+1)$.

Now consider blocks 1 and 2. Due to causality, the relay does not send any signal in the first block 1, thus, the received signals at Rx 1 for instance in these blocks are

$$y_1^n(1) = h_d x_1^n(1) + h_c x_2^n(1) + z_1^n(1) \tag{4.372}$$

$$y_1^n(2) = h_d x_1^n(2) + h_c x_2^n(2) + z_1^n(2) + h_r x_r^n(2). \tag{4.373}$$

where $x_r^n(2)$ represents the compressed $y_r^n(1)$. The receiver decodes $x_r^n(2)$ first from $y_1^n(2)$ while treating the other signals as noise. Reliable decoding of $x_r^n(2)$ is possible if

$$R_r \leq \frac{1}{2}\log\left(1 + \frac{h_r^2 P_r}{1 + (h_d^2 + h_c^2)(P_w + P_v)}\right). \tag{4.374}$$

Same is done at the Rx 1. Now, by applying [HM06, Proposition 1], Rx 1 knowing $m_r(1)$ is able to decompress a noisy version of $y_r^n(1)$, and thus is able to obtain the signal given by

$$\mathbf{y}_{f1}^n(1) = \begin{bmatrix} y_1^n(1) \\ y_r^n(1) + z_f^n \end{bmatrix}, \tag{4.375}$$

where z_f^n is a realization of an i.i.d. Gaussian noise Z_f^n representing the compression noise, which is independent of all other variables. Using (4.374), the variance of Z_f can be written as given in equation (4.371). Similar processing is performed by Rx 2. By proceeding forward block by block, the received signal in each block can be written as (4.375). The receivers then proceed by jointly decoding the C messages first from the equivalent received signal (4.375), and then decoding the P message, successively in this order while treating the undesired P signal as noise. The achievable rates are thus bounded by (4.368)-(4.370).

Backward decoding can also be used at the receivers instead of forward decoding. In this case, we can write the achievable rate as given in the next theorem.

Theorem 4.20 (CFB)**.** *CFB achieves* $R_\Sigma = 2R_w + 2R_v$ *where* R_v *and* R_w *are bounded as in* (4.368)-(4.370)*, with* $\hat{Y}_r = Y_r + Z_b$*,* $Z_b \sim \mathcal{N}(0, \sigma_b^2)$*,*

$$\sigma_b^2 = \frac{h_s^2(h_c - h_d)^2(P_w + P_v)^2 + (1 + h_r^2 P_r)(2h_s^2(P_w + P_v) + 1) + (h_d^2 + h_c^2)(P_w + P_v)}{\frac{h_r^2 P_r}{1 + h_c^2 P_w}(1 + (h_d^2 + h_c)^2(P_w + P_v) + h_r^2 P_r)}, \tag{4.376}$$

$W_1, W_2 \sim \mathcal{N}(0, P_w)$*,* $V_1, V_2 \sim \mathcal{N}(0, P_v)$*, and* $P_w + P_v \leq P$ *and* $P_r \leq P$*.*

Decoding in CFB starts from the last blocks B and $B + 1$ and then proceeds backwards. Assume that the decoding of the C and P messages in block $b + 1$ was successful. Then, the received signal at Rx 1 in blocks b and $b + 1$ can be written as

$$y_1^n(b) = h_r x_r^n(b) + h_d x_1^n(b) + h_c x_2^n(b) + z_1^n(b) \tag{4.377}$$
$$y_1^n(b+1) = h_r x_r^n(b+1) + h_c w_2^n(b+1) + z_1^n(b+1), \tag{4.378}$$

where $x_r^n(b+1)$ represents the compressed $y_r^n(b)$, and where the contribution of the decoded C and P signals has been removed. The receiver decodes $x_r^n(b+1)$ first, decompresses $m_r(b)$ into a noisy version of $y_r^n(b)$, and then decodes the common messages and its private message from its equivalent received signal given by [HM06, Proposition 1]

$$\mathbf{y}_{b1}^n(b) = \begin{bmatrix} y_1^n(b) \\ y_r^n(b) + z_b^n \end{bmatrix}, \tag{4.379}$$

where z_b^n is a realization of an i.i.d. Gaussian noise Z_b^n. Notice that in this case, in addition to $w_2^n(b)$, $x_r^n(b)$ is also treated as noise. Same is done at the Rx 2. The resulting constraint for decoding m_r is

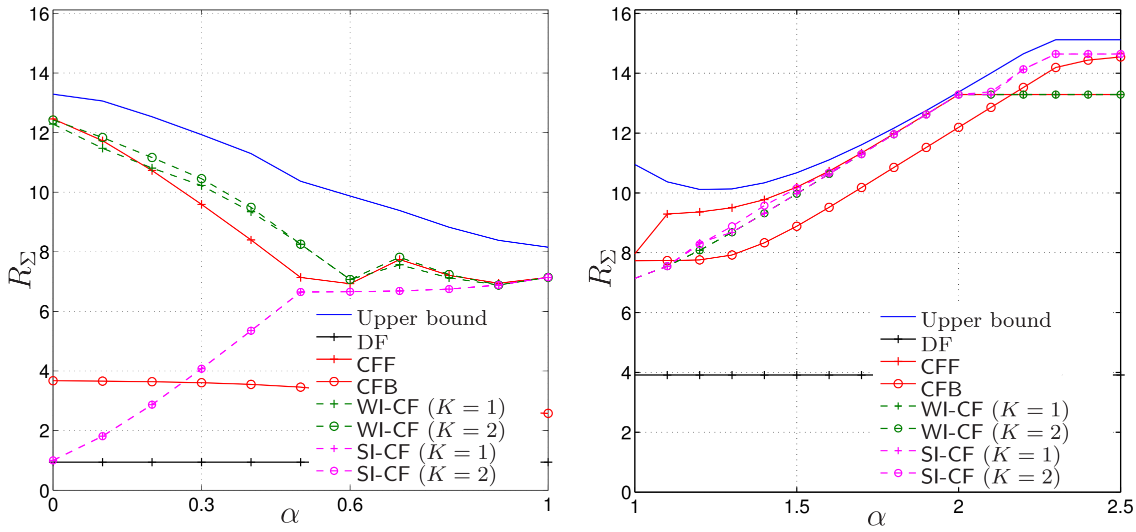
$$R_r \leq \frac{1}{2}\log\left(1 + \frac{h_r^2 P_r}{1 + h_c^2 P_w}\right). \tag{4.380}$$

As a result, using (4.380), the variance of Z_b is σ_b^2 given in (4.376). Then proceeding backward, every block can be written as (4.379), and decoding is reliable if (4.368)-(4.370) hold with $\hat{Y}_r = Y_r + Z_b$.

Although comparing our CF schemes and compress-forward in both its variants is a tedious task, we can notice the following difference between the two. In our schemes, we have forwarding 'selectivity' at the relay, in the sense that the relay can select which part of the received signal to decode and forward. On the other hand, in compress-forward, the relay forwards a compression index of everything it receives. This gives our CF schemes an advantage especially in the cases where it is better not to forward everything at the relay, and thus, more power can be used for forwarding less signals resulting in a higher power efficiency.

4.4.2 Comparison

With the achievable sum-rates of the two classical schemes and that of our CF schemes given, we can proceed to compare their performance in terms of sum-rate. We start with some high P evaluation. Figure 4.17 shows the sum-capacity upper and lower bounds as a function of α for two cases of an G-IRC with $h_d = 1$, $P = 40$dB. In Figure 4.17(a), we show the bounds for weak interference and weak relay channels

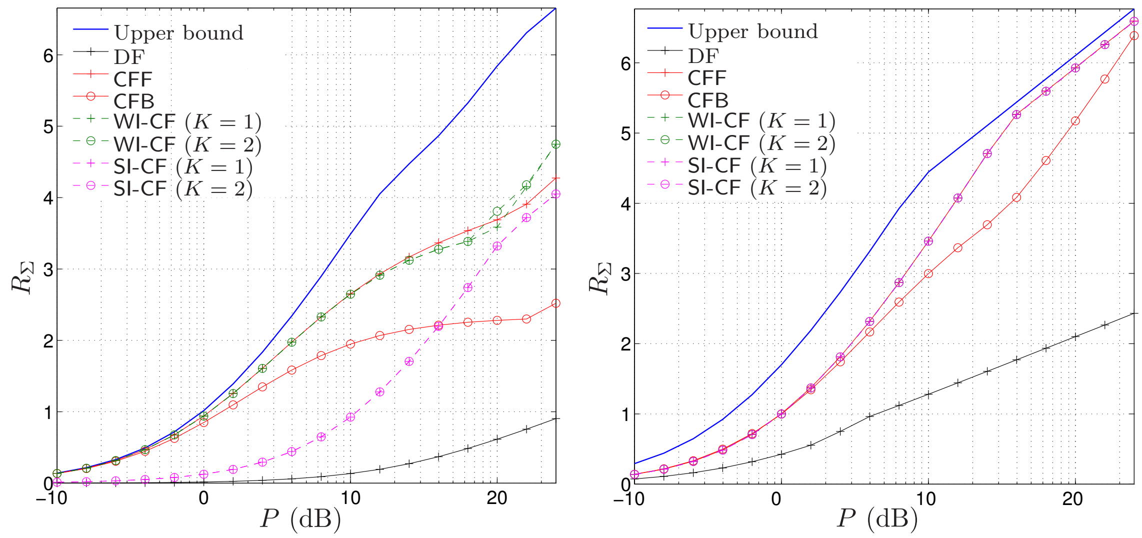


(a) $(h_d, h_r, h_s) = (1, 0.6, 0.025)$, $P = 40$dB (b) $(h_d, h_r, h_s) = (1, 6, 1.6)$, $P = 40$dB

Figure 4.17: The achievable sum-rate for different schemes in comparison with the sum-capacity upper bound as a function of α.

where $(\beta, \gamma) \approx (0.9, 0.2)$. The achievable rates of WI-CF and SI-CF are shown for two cases ($K = 1$ and $K = 2$). WI-CF achieves rates which are close to the upper bound. It can be seen that the achievable rate of the WI-CF scheme increases if we increase K. The SI-CF scheme, which is not designed for the weak interference scenario, achieves lower rates than WI-CF . Here we can see the effect of the relay bottleneck in the DF scheme. DF performs very poorly if h_s is weak. On the other hand, CFB and CFF perform better than DF, and furthermore, CFF achieves rates very close to the upper bound, but still lower than WI-CF . Figure 4.17(b) shows a scenario with strong interference where and strong relay channels $(\beta, \gamma) \approx (1.4, 1.1)$. Here the performance SI-CF dominates the others. We can see that SI-CF achieves rates that are also close to the upper bound, and outperforms the other schemes for a wide range of α. The WI-CF is inferior to SI-CF in this regime, as its achievable sum-rate saturates to the very-strong IC sum-capacity when $\alpha > 2$ (around 13 bits per channel use in this example), while the sum-rate of SI-CF increases further. A similar statement can be made about CFF and CFB. Here, although h_s is stronger than h_d, the DF scheme still suffers from the relay bottleneck.

A comparison of the achievable schemes at low to moderate P is shown in Figure 4.18 as a function of P. In the first plot, Figure 4.18(a), we show the bounds for a G-IRC with weak relay and cross channels. In this example, we can see that the performance is dominated by either WI-CF or CFF. These two schemes have similar performance at low P, and different performance as P increases. In the second plot, Figure 4.18(b), we show an example with weak h_r and strong h_s and h_c. The performance of DF improves, compared to that in Figure 4.18(a), since h_s is larger. However, its performance is still worse than CF. CF and CFF have nearly the same performance in this example.



(a) $(h_d, h_c, h_r, h_s) = (1, 0.3, 0.8, 0.15)$ (b) $(h_d, h_c, h_r, h_s) = (1, 6, 0.5, 1.5)$

Figure 4.18: The achievable sum-rate for different schemes in comparison with the sum-capacity upper bound as a function of P.

4.5 Summary

In this chapter, we have characterized the sum-capacity of the linear-deterministic IRC and also the GDoF of the Gaussian IRC for all cases where the source-relay channel is weaker than the interference (cross) channel. Consequently, we have an approximation of the sum-capacity of the G-IRC of the form

$$C_\Sigma(\alpha, \beta, \gamma) = \frac{d(\alpha, \beta, \gamma)}{2} \log(\mathsf{SNR}) + o(\log(\mathsf{SNR})) \tag{4.381}$$

for half[5] the space of all channel parameters, where d is the derived GDoF, SNR is the ratio of the available transmit power to noise power, and the parameters α, β, and γ quantify the strength of the cross, relay-destination, and source-relay channel, respectively. For achieving this goal, we developed novel sum-capacity upper bounds for the network. We have also developed a novel transmission scheme for the IRC based on compute-forward at the relay. This new optimal scheme achieves a GDoF in the IRC which is higher than that of the IC. Therefore it shows that while a relay does not increase the DoF of the IC, it indeed increases its GDoF. By using numerical examples, we showed that the new scheme outperforms classical decode-forward and compress-forward schemes.

By proposing a generic and optimal transmission strategy for the IRC, we have an answer to the main problem of the thesis stated in Chapter 3. This work sheds light on optimal relaying strategies in interference networks, and methods for interference management in such networks. It turns out that network coding ideas are very important in wireless interference networks. These ideas which can be implemented using lattice codes outperform classical relaying strategies and can have several implementations and significant impact in practical wireless networks where interference is a major setback.

[5]The remaining regime is discussed in Chapter 6.

Chapter 5

The Butterfly Network

In the previous chapter, we have studied the interference relay channel (IRC), and we have characterized its sum-capacity (in the linear-deterministic case) and its GDoF (in the Gaussian case) for the regime where the source-relay channels are weaker than the cross channels. As seen in Figure 4.9, the bounds do not coincide in the remaining regime, and thus some new techniques have to be developed for this case. In order to study new techniques, we resort in this chapter to a special IRC where the relaying potential of the network is magnified.

First, the network is made more dependent on the relay by removing the direct channels. Observe that information in the IRC can flow from transmitters to receivers through the direct channel (between a transmitter and its respective receiver) and through the relay. If the direct channels are removed, then information can only flow from transmitters to receivers via the relay. Consequently, the partially connected IRC where the direct channels have zero capacity is more dependent on the relay, and hence, the impact of relaying can be studied in isolation of the impact of direct channels. The gained insights from this partially connected IRC might lead to a better understanding of the optimal relaying strategy in the fully connected one.

Furthermore, in order to make the relay more capable in this partially connected network, and thus, put more emphasis on the relaying component of the network, we explore another relaying possibility. Observe that the relay in the the classical IRC forwards information only to the destinations. But the relay can also relay information backwards to the sources. This backward relaying can be thought of as feedback from the relay to the sources, and can lead to a relevant performance improvement in the IRC.

In order to study these aspects in this chapter, we consider a full-duplex butterfly network (BFN) where the relay performs both forward and backward relaying (BFN with relay-source feedback).

5.1 The Memoryless IRC with Relay-source Feedback

Since the BFN is a special case of an IRC, and since we are adding an additional relaying option, i.e., relay-source feedback, we will start by defining the IRC with feedback. The reason for doing so is that the general feedback model allows us to easily describe the proposed outer bounds for the relay-source feedback model which is the subject of investigation in Section 5.1.2.

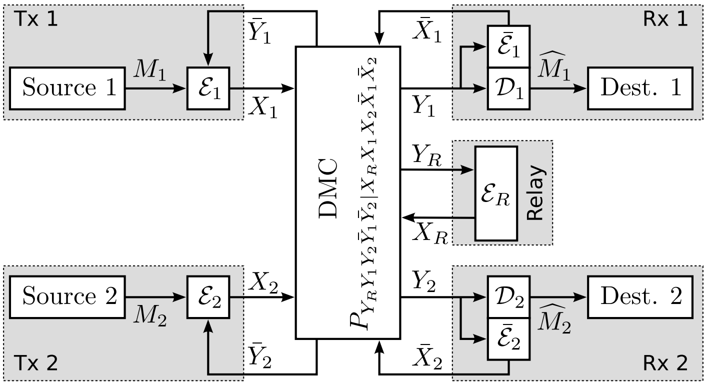


Figure 5.1: The general memoryless Interference Relay Channel with Feedback (IRCF).

5.1.1 Channel Model for the Memoryless IRC with General Feedback

A memoryless IRC with general feedback (IRCF) is shown in Fig. 5.1. All nodes are full-duplex and causal. Tx $j \in \{1, 2\}$ has an independent message $M_j \in \{1, \dots, 2^{\lfloor nR_j \rfloor}\}$, where $n \in \mathbb{N}$ is the code-length and $R_j \in \mathbb{R}_+$ the rate in bits per channel use, to be sent to Rx j. The operations performed at each node can be described in general as follows:

- The relay receives Y_R and sends X_R, where the ith symbol of X_R^n is constructed from Y_R^{i-1} using an encoding function $\mathcal{E}_{R,i}$, i.e., $X_R(i) = \mathcal{E}_{R,i}(Y_R^{i-1})$.
- Tx 1 receives[1] feedback information $\bar{Y}_1$ and sends X_1, where $X_1(i)$ is constructed from the message M_1 and from $\bar{Y}_1^{i-1}$ using an encoding function $\mathcal{E}_{1,i}$, i.e., $X_1(i) = \mathcal{E}_{1,i}(M_1, \bar{Y}_1^{i-1})$. Similarly at Tx 2, i.e., $X_2(i) = \mathcal{E}_{2,i}(M_2, \bar{Y}_2^{i-1})$.
- Rx 1 receives Y_1 and feeds back $\bar{X}_1$, where $\bar{X}_1(i)$ is constructed from Y_1^{i-1} using an encoding function $\bar{\mathcal{E}}_{1,i}$, i.e., $\bar{X}_1(i) = \bar{\mathcal{E}}_{1,i}(Y_1^{i-1})$. After n channel uses, Rx 1 tries to obtain M_1 from Y_1^n using a decoding function $\mathcal{D}_1$, i.e., $\widehat{M_1} = \mathcal{D}_1(Y_1^n)$. An error occurs if $M_1 \neq \widehat{M_1}$. Rx 2 similarly feeds back $\bar{X}_2(i) = \bar{\mathcal{E}}_{2,i}(Y_2^{i-1})$ and decodes $\widehat{M_2} = \mathcal{D}_2(Y_2^n)$. An error occurs if $M_2 \neq \widehat{M_2}$.

The channel has a transition probability $P_{Y_R,Y_1,Y_2,\bar{Y}_1,\bar{Y}_2|X_R,X_1,X_2,\bar{X}_1,\bar{X}_2}$ and is assumed to be memoryless, that is, for all $i \in \mathbb{N}$ the following Markov chain holds

$$
\begin{aligned}
&(W_1, W_2, X_R^{i-1}, X_1^{i-1}, X_2^{i-1}, \bar{X}_1^{i-1}, \bar{X}_2^{i-1}, Y_R^{i-1}, Y_1^{i-1}, Y_2^{i-1}, \bar{Y}_1^{i-1}, \bar{Y}_2^{i-1}) \\
&\to (X_R(i), X_1(i), X_2(i), \bar{X}_1(i), \bar{X}_2(i)) \to (Y_R(i), Y_1(i), Y_2(i), \bar{Y}_1(i), \bar{Y}_2(i)). \qquad (5.1)
\end{aligned}
$$

We use the standard information theoretic definition of a code, probability of error and achievable rates [CT06]. We aim to characterize the capacity defined as the convex

[1] Although source nodes are both transmitters and receivers at the same time (due to feedback), we will still refer to them as transmitters.

closure of the set of non-negative rate pairs (R_1, R_2) such that $\max_{j\in\{1,2\}} \mathbb{P}[M_j \neq \widehat{M}_j] \to 0$ as $n \to \infty$.

This model generalizes various well studied channel models. For instance, it models the classical IC [Car78] (for $\bar{Y}_1 = \bar{Y}_2 = Y_R = X_R = \bar{X}_1 = \bar{X}_2 = \emptyset$), the IC with cooperation [Tun12] (for $Y_R = X_R = \emptyset$), the classical IRC [MDG12] also discussed in the previous chapter (for $\bar{Y}_1 = \bar{Y}_2 = \bar{X}_1 = \bar{X}_2 = \emptyset$), etc.

5.1.2 Upper Bounds for the Memoryless IRC with Relay-source Feedback

The memoryless IRC with relay-source feedback is obtained from the IRCF model in Section 5.1.1 by setting $\bar{X}_1 = \bar{X}_2 = \emptyset$. Next, we derive several upper bounds on the achievable rate pairs for the general memoryless IRC with relay-source feedback. We note that the described techniques apply to the general IRCF and do not require necessarily $\bar{X}_1 = \bar{X}_2 = \emptyset$. We start with the cut-set bound [CT06], and then we adapt upper bounds for the general memoryless IC with cooperation given in [Tun12] to our channel model.

5.1.2.1 Cut-set Bounds

The cut-set bound [CT06] which we introduced previously in (4.20) can be applied to the IRCF to bound R_1 as

$$R_1 \leq I(X_1; Y_R, \bar{Y}_2, Y_1 | X_R, X_2) \tag{5.2}$$
$$R_1 \leq I(X_1, X_2; Y_R, Y_1 | X_R) \tag{5.3}$$
$$R_1 \leq I(X_R, X_1; \bar{Y}_2, Y_1 | X_2) \tag{5.4}$$
$$R_1 \leq I(X_R, X_1, X_2; Y_1), \tag{5.5}$$

for some input distribution P_{X_R,X_1,X_2}. Similarly, we can bound R_2 by replacing the subscripts 1 and 2 with 2 and 1. Also using the cut-set bounds, the sum-rate can be bounded as

$$R_1 + R_2 \leq I(X_1, X_2; Y_R, Y_1, Y_2 | X_R) \tag{5.6}$$
$$R_1 + R_2 \leq I(X_R, X_1, X_2; Y_1, Y_2), \tag{5.7}$$

for some input probability distribution P_{X_R,X_1,X_2}.

5.1.2.2 Cooperation Upper Bounds

As mentioned earlier, the IC with general cooperation is a special case of the IRCF obtained by setting $Y_R = X_R = \emptyset$. An upper bound for the sum-capacity of the IC with general cooperation was given by Tuninetti in [Tun12] (see Lemma 2.2 on page 8). We restate this bound here, given by

$$R_1 + R_2 \leq I(X_1; Y_1, \bar{Y}_2 | Y_2, X_2, \bar{X}_1, \bar{X}_2) + I(X_1, X_2, \bar{X}_1; Y_2 | \bar{X}_2), \tag{5.8}$$
$$R_1 + R_2 \leq I(X_2; Y_2, \bar{Y}_1 | Y_1, X_1, \bar{X}_1, \bar{X}_2) + I(X_1, X_2, \bar{X}_2; Y_1 | \bar{X}_1). \tag{5.9}$$

for some $P_{X_1,X_2,\bar{X}_1,\bar{X}_2}$. In the IRCF, if we let the relay perfectly cooperate with one of the other nodes in the network, then the model again reduces to an IC with general cooperation in which one of the nodes has an enhanced input and output. Since cooperation cannot decrease capacity, any outer bound for the obtained IC with general cooperation is an upper bound to the capacity of the IRCF. For instance, if Tx 1, cooperates with the relay, then in Tuninetti's bounds (5.8) and (5.9), we replace X_1 with (X_1, X_R) and $\bar{Y}_1$ with $(\bar{Y}_1, Y_R)$. Since we do not consider feedback from the receivers, we set $\bar{X}_1 = \bar{X}_2 = \emptyset$ after this substitution. By considering the various cooperation possibilities, we get the following upper bounds:

1. Full cooperation between Tx 1 and the relay leads to an IC with bi-directional cooperation between the transmitters where Tx 1 sends (X_1, X_R) and receives $(\bar{Y}_1, Y_R)$:

$$R_1 + R_2 \leq I(X_1, X_R; Y_1, \bar{Y}_2 | Y_2, X_2) + I(X_1, X_R, X_2; Y_2), \tag{5.10}$$
$$R_1 + R_2 \leq I(X_2; Y_2, \bar{Y}_1, Y_R | Y_1, X_1, X_R) + I(X_1, X_R, X_2; Y_1). \tag{5.11}$$

   Cooperation between Tx 2 and the relay leads to similar bounds.

2. Full cooperation between Rx 1 and the relay leads to an IC with uni-directional cooperation between Rx 1 and Rx 2 and with feedback from Rx 1 to the transmitters, where Rx 1 sends X_R and receives (Y_1, Y_R):

$$R_1 + R_2 \leq I(X_1; Y_1, Y_R, \bar{Y}_2 | Y_2, X_2, X_R) + I(X_1, X_2, X_R; Y_2), \tag{5.12}$$
$$R_1 + R_2 \leq I(X_2; Y_2, \bar{Y}_1 | Y_1, Y_R, X_1, X_R) + I(X_1, X_2; Y_1, Y_R | X_R). \tag{5.13}$$

   Cooperation between Rx 2 and the relay leads to similar bounds.

These upper bounds will be used next to upper bound the capacity region of the BFN with relay-source feedback. As it turns out, these bounds suffice to characterize the capacity region of the symmetric linear-deterministic BFN.

5.2 The Linear-deterministic Butterfly Network (LD-BFN)

Similar to the previous chapter, we first consider the linear-deterministic (LD) approximation [ADT11] of the BFN. Later on, we discuss the Gaussian BFN. We start with the IRC with a dedicated *out-of-band feedback channel* between the relay and the transmitters. For this reason, we write X_R as (X_r, X_f) where X_r is the in-band relay signal to the receivers and X_f is the out-of-band feedback signal to the sources. Moreover, the input signals of the channels in this case are all binary vectors, thus, we denote them as $\mathbf{y}_r$, $\mathbf{y}_1$, $\mathbf{y}_2$, $\bar{\mathbf{y}}_1$, and $\bar{\mathbf{y}}_2$ which are elements of the binary field of length q. As in the previous chapter, we resort to a symmetric setup for simplicity of exposition. This simplification reduces the number of parameters, and thus leads to complete analytical, clean, and insightful capacity region characterization. Finally, since the partially connected BFN has no direct channels, we set $n_d = 0$ in the LD-IRC with feedback. As a result, the input-output relations of this symmetric LD-BFN with

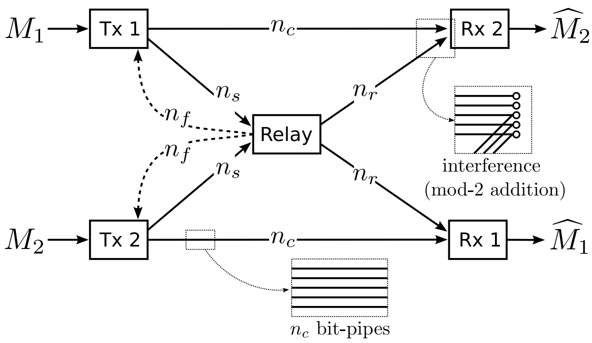


Figure 5.2: The linear-deterministic butterfly network with relay-source feedback. Notice that we have switched the places of receivers 1 and 2 for clarity.

out-of-band relay-source feedback are

$$\mathbf{y}_r = \mathbf{S}^{q-n_s}(\mathbf{x}_1 \oplus \mathbf{x}_2), \tag{5.14}$$
$$\bar{\mathbf{y}}_1 = \mathbf{S}^{q-n_f}\mathbf{x}_f, \tag{5.15}$$
$$\bar{\mathbf{y}}_2 = \mathbf{S}^{q-n_f}\mathbf{x}_f, \tag{5.16}$$
$$\mathbf{y}_1 = \mathbf{S}^{q-n_c}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r, \tag{5.17}$$
$$\mathbf{y}_2 = \mathbf{S}^{q-n_c}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r. \tag{5.18}$$

where $\mathbf{y}_r$ is the channel output at the relay, $\bar{\mathbf{y}}_1$ and $\bar{\mathbf{y}}_2$ are the received feedback signal at the transmitters, $\mathbf{y}_1$ and $\mathbf{y}_2$ are the received signals at the receivers, $q = \max\{n_c, n_r, n_s, n_f\}$, and $\mathbf{S}$ is the $q \times q$ down-shift matrix. Here, n_c denotes the **c**ross channel, n_r denotes the **r**elay-destination channel, n_s denotes the **s**ource-relay channel, and n_f denotes the **f**eedback channel (see Figure 5.2).

The main focus of the rest of this chapter is to determine the capacity region of the network described by (5.14)-(5.18), and thus, identify the optimal relaying strategies in this setup. In the following subsections, we provide matching upper and lower bounds for the LD-BFN with feedback thereby completely characterizing the capacity region. The main result of this section is stated in the following theorem.

Theorem 5.1. *The capacity region of the LD-BFN with relay-source feedback is the set of rate pairs* (R_1, R_2) *which satisfy*

$$0 \le R_1 \le \min\{n_s, n_r + n_f, \max\{n_c, n_r\}\} \tag{5.19}$$
$$0 \le R_2 \le \min\{n_s, n_r + n_f, \max\{n_c, n_r\}\} \tag{5.20}$$
$$R_1 + R_2 \le \max\{n_r, n_c\} + n_c \tag{5.21}$$
$$R_1 + R_2 \le \max\{n_r, n_c\} + (n_s - n_c)^+ \tag{5.22}$$
$$R_1 + R_2 \le n_s + n_c. \tag{5.23}$$

Intuitively, these bounds can be explained as follows. Since communication is only possible via the relay (due to the absence of the direct channel), Tx 1 can not send more bits per channel use than the relay can receive; thus, we have the bound $R_1 \le n_s$ in (5.19). Now assume that the channel to the relay is very strong (say of infinite capacity); in this case, the rate achieved by Tx 1 can not exceed the capacity of the outgoing channels from the relay, i.e., $n_r + n_f$ in (5.19). Finally, the rate R_1 can not exceed the amount of information that can be received by Rx 1, which is given by $\max\{n_c, n_r\}$, and hence the bound $R_1 \le \max\{n_c, n_r\}$ in (5.19). Similar reasoning holds for the bound in (5.20).

Interestingly, the sum-rate bounds in (5.21)-(5.23) do not depend on the feedback parameter n_f. As we shall see in the following sections, given $n_r > n_c$, the region in Theorem 5.1 is the same as for $n_f = 0$, i.e., there is no gain from the availability of a dedicated relay-source feedback channel. In this case, the relay-destination link is so strong that the relay can help the receivers resolve their signals without the need of transmitter cooperation. On the other hand, when $n_r < n_c$, relaying can be improved upon by transmitter cooperation enabled by the presence of feedback. In this case, we can have a 'net-gain' from feedback that is larger than the 'cost' of feedback. We will expand on this idea after we proved the achievability of Theorem 5.1. The proof of the converse is given in the next subsection.

5.2.1 Upper Bounds for the LD-BFN with Feedback

Here we specialize the general bounds given in Section 5.1.2 to the LD-BFN described in Section 5.2. We substitute the output definition in (5.14)-(5.18) in the outer bounds in Section 5.1.2. From the cut-set bound (5.2), if we replace $\mathbf{x}_R$ by $(\mathbf{x}_r, \mathbf{x}_f)$ we get

$$\begin{aligned}
R_1 &\le I(\mathbf{x}_1; \mathbf{y}_r, \bar{\mathbf{y}}_2, \mathbf{y}_1 | \mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_2) && (5.24)\\
&= H(\mathbf{y}_r, \bar{\mathbf{y}}_2, \mathbf{y}_1 | \mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_2) - H(\mathbf{y}_r, \bar{\mathbf{y}}_2, \mathbf{y}_1 | \mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_2, \mathbf{x}_1) && (5.25)\\
&= H(\mathbf{S}^{q-n_s}\mathbf{x}_1 | \mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_2) && (5.26)\\
&\le H(\mathbf{S}^{q-n_s}\mathbf{x}_1) && (5.27)\\
&\le n_s, && (5.28)
\end{aligned}$$

where the last step follows since the Bern(1/2) distribution maximizes the binary entropy [CT06]. Similarly, the cut-set bounds in (5.4) and (5.5) reduce to

$$R_1 \le n_r + n_f, \quad (5.29)$$
$$R_1 \le \max\{n_c, n_r\}, \quad (5.30)$$

respectively. These bounds combined give (5.19). Similarly, the bound in (5.20) for R_2 follows by the symmetry in the network.

The sum-rate cut-set bound in (5.7) becomes

$$\begin{aligned}
R_1 + R_2 &\le I(\mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2) && (5.31)\\
&= H(\mathbf{y}_1, \mathbf{y}_2) && (5.32)\\
&= H(\mathbf{y}_1) + H(\mathbf{y}_2 | \mathbf{y}_1) && (5.33)\\
&\le H(\mathbf{y}_1) + H(\mathbf{y}_2 \oplus \mathbf{y}_1) && (5.34)\\
&\le \max\{n_r, n_c\} + n_c, && (5.35)
\end{aligned}$$

where the last steps follows by using Lemma 2.4 (page 12). This proves (5.21). These are the necessary cut-set upper bounds for our problem. The remaining cut-set bounds are redundant given the cooperation bounds that we derive next, and are thus omitted.

Next, we evaluate the cooperation bounds for the LD-BFN. It turns out that the bound (5.10) for the LD-BFN with feedback is redundant given (5.35). Thus, we omit its derivation. Next, we consider the bound in (5.11), which yields

$$\begin{aligned} R_1 + R_2 &\leq I(\mathbf{x}_2; \mathbf{y}_2, \bar{\mathbf{y}}_1, \mathbf{y}_r | \mathbf{y}_1, \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_r) + I(\mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_2; \mathbf{y}_1) && (5.36)\\ &= H(\mathbf{y}_2, \bar{\mathbf{y}}_1, \mathbf{y}_r | \mathbf{y}_1, \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_f) - H(\mathbf{y}_2, \bar{\mathbf{y}}_1, \mathbf{y}_r | \mathbf{y}_1, \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_2) && (5.37)\\ &\quad + H(\mathbf{y}_1) - H(\mathbf{y}_1 | \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_2) && (5.38)\\ &= H(\mathbf{S}^{q-n_s}\mathbf{x}_2 | \mathbf{S}^{q-n_c}\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_f) + H(\mathbf{S}^{q-n_c}\mathbf{x}_2 \oplus \mathbf{S}^{q-n_r}\mathbf{x}_r) && (5.39)\\ &\leq (n_s - n_c)^+ + \max\{n_c, n_r\}, && (5.40) \end{aligned}$$

by using Lemmas 2.4 and 2.5 (page 12). This proves (5.22). Note that this bound can be tighter than the sum-rate cut-set bound in (5.35). The bound in (5.12) is similar to (5.11). Finally, the bound in (5.13) becomes

$$\begin{aligned} R_1 + R_2 &\leq I(\mathbf{x}_2; \mathbf{y}_2, \bar{\mathbf{y}}_1 | \mathbf{y}_1, \mathbf{y}_r, \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_f) + I(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_r | \mathbf{x}_r, \mathbf{x}_f) && (5.41)\\ &= H(\mathbf{y}_2, \bar{\mathbf{y}}_1 | \mathbf{y}_1, \mathbf{y}_r, \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_f) - H(\mathbf{y}_2, \bar{\mathbf{y}}_1 | \mathbf{y}_1, \mathbf{y}_r, \mathbf{x}_1, \mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_2) && (5.42)\\ &\quad + H(\mathbf{y}_1, \mathbf{y}_r | \mathbf{x}_r, \mathbf{x}_f) - H(\mathbf{y}_1, \mathbf{y}_r | \mathbf{x}_r, \mathbf{x}_f, \mathbf{x}_1, \mathbf{x}_2) && (5.43)\\ &= H(\mathbf{S}^{q-n_c}\mathbf{x}_2, \mathbf{S}^{q-n_s}\mathbf{x}_1 \oplus \mathbf{S}^{q-n_s}\mathbf{x}_2 | \mathbf{x}_r, \mathbf{x}_f) && (5.44)\\ &\leq n_s + n_c. && (5.45) \end{aligned}$$

This bound yields (5.23). This completes the proof of the converse of Theorem 5.1.

5.2.2 Transmission schemes for the LD-BFN with feedback

Here, we prove the achievability of Theorem 5.1. To do this, we develop an achievable rate region based on a combination of transmission schemes. First, we describe the different required coding strategies separately, then we prove the achievability of Theorem 5.1 by using a careful combination of these strategies for different parameter regimes. We start with a novel relaying strategy.

5.2.2.1 Cooperative Interference Neutralization (CN)

We propose a signaling scheme which we call 'cooperative interference neutralization', or CN for short[2]. The main idea of CN is to allow the relay to know some information about future source transmissions, in order to facilitate interference neutralization. Note that if the relay knows the future source transmissions, then at each transmission instant, the relay can be treated as a cognitive relay (IC's with cognitive relay have been studied in [RTD10, SE07b, SVJS08]). In our case, the relay is causal. However, the relay can be given access to future source information as follows. Each transmitter sends two CN signals in the ith channel use, which we call $\mathbf{t}_j(i)$ and $\mathbf{t}_j(i+1)$, each of which has length R_t, where $j \in \{1, 2\}$ is the transmitter index, and where i is the channel use index. $\mathbf{t}_j(i)$ is the CN signal to be decoded by the receiver in the ith channel use, while $\mathbf{t}_j(i+1)$ is to be decoded in the next channel use $i+1$. Therefore, the transmitter sends the present and the future CN signals. The future one, $\mathbf{t}_j(i+1)$ is intended for the relay, and is not decoded at the receivers. The relay attempts to decode $\bar{\mathbf{t}}(i+1) = \mathbf{t}_1(i+1) \oplus \mathbf{t}_2(i+1)$ in the ith channel use which requires $2R_t \leq n_s$. Then, $\bar{\mathbf{t}}(i+1)$ is sent in the next channel use $i+1$, on the same levels where $\mathbf{t}_2(i+1)$ is observed at Rx 1 (note that $\mathbf{t}_2(i+1)$ is interference from the perspective of Rx 1),

[2] In [MDFT08], an interference neutralization strategy has also been used for a deterministic two hop network *without* direct connectivity between source and destination nodes.

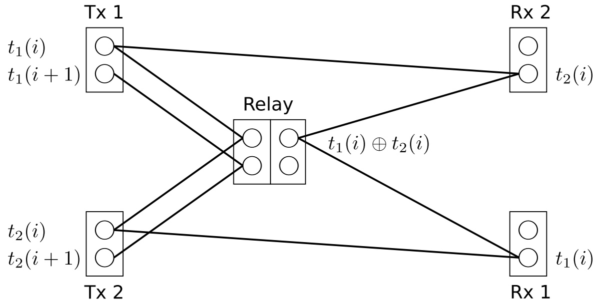


Figure 5.3: A graphical illustration of the CN strategy. Note how the transmitters pass the future CN signals to the relay without disturbing the receivers.

resulting in interference neutralization since $\bar{\mathbf{t}}(i+1) \oplus \mathbf{t}_2(i+1) = \mathbf{t}_1(i+1)$. This allows Rx 1 to decode its desired CN signal in channel use $i+1$ as long as $R_t \leq \min\{n_c, n_r\}$.

The relay signal $\bar{\mathbf{t}} = \mathbf{t}_1 \oplus \mathbf{t}_2$ can be interpreted as a network code [ACLY00]. However, notice that instead of decoding the transmit signals at the relay, and then constructing this network code which is useful for both receivers, the relay directly decodes the desired network code. This leads to a more relaxed rate constraints at the relay. This idea will be extended to Gaussian relay networks in the sequel.

An illustrative example for CN is given in Figure 5.3. Thanks to CN, each receiver decodes its desired signal interference free. By a careful adjustment of the levels of each signal, it is possible to pass the future CN signals to the relay without even disturbing the receivers. From Figure 5.3 we remark that by using CN each transmitter can send R_t bits per channel use over R_t levels at the receiver while using $2R_t$ levels at the relay. Due to this fact, *this strategy is preferable when n_s is larger than n_c.*

To realize the CN strategy we use block Markov encoding. Each transmitter sends n signals in $n+1$ channel uses. Starting with an initialization step, the transmitters send $\mathbf{t}_j(1)$ in channel use $i=0$ while the relay remains silent. Then, each transmitter sends both $\mathbf{t}_j(i)$ and $\mathbf{t}_j(i+1)$ in the ith channel use for $i = 1, \cdots, n-1$ while the relay sends $\mathbf{t}_1(i) \oplus \mathbf{t}_2(i)$. Finally, in the nth channel use, each transmitter sends $\mathbf{t}_j(n)$ only and the relay sends $\mathbf{t}_1(n) \oplus \mathbf{t}_2(n)$. Each receiver decodes its desired CN signal starting from $i=1$ till $i=n$. Thus, assuming that $\mathbf{t}_j$ is a binary vector of length R_t, each transmitter is able to successfully deliver nR_t bits over the span of $n+1$ channel uses for a total rate of $\frac{n}{n+1}R_t$ which approaches R_t for large n.

Remark 5.1. *A strategy similar to the CN strategy was also used in the interference channel with generalized feedback in [YT11b], where the transmitters exchanges bits below the noise floor of the receivers, which are then used in the next slot to zero force the interference. A half-duplex variant of this scheme also appeared in [AH09].*

5.2.2.2 Decode-forward (DF)

Next, we describe the decode-forward (DF) strategy. Although this strategy is well known [CEG79], we describe it briefly and then highlight a possibility to combine DF and CN in an effective way afterwards.

In the ith channel use, $i = 1, \cdots, n$, Tx j sends a vector of length $R_{1v} + R_{2v}$ where the DF signal $\mathbf{v}_j(i)$ of length R_{jv} is zero-padded to length $R_{1v} + R_{2v}$ as follows

$$\text{Tx 1 sends} \begin{bmatrix} \mathbf{v}_1(i) \\ \mathbf{0}_{R_{2v}} \end{bmatrix}, \quad \text{Tx 2 sends} \begin{bmatrix} \mathbf{0}_{R_{1v}} \\ \mathbf{v}_2(i) \end{bmatrix}. \tag{5.46}$$

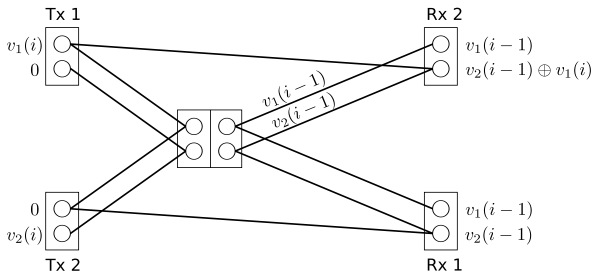


Figure 5.4: A graphical illustration of the decode-forward strategy. Rx 2 starts be removing $v_1(i)$ (known from the decoding process in channel use $i+1$) from its received signal. Then it decodes $v_1(i-1)$ and $v_2(i-1)$. Using this strategy in this setup, each transmitter can send 1 bit per channel use.

The relay decodes both $\mathbf{v}_1(i)$ and $\mathbf{v}_2(i)$ in the ith channel use, which is possible if $R_{1v} + R_{2v} \leq n_s$, and forwards them in channel use $i+1$ (see Figure 5.4).

The receivers start decoding backwards starting from channel use $n+1$ where only the relay is active, and they both decode $\mathbf{v}_1(n)$ and $\mathbf{v}_2(n)$, which requires $R_{1v} + R_{2v} \leq n_r$. Decoding proceeds backwards to the nth channel use. In the nth channel use, the receivers start by removing $\mathbf{v}_j(n)$ from the received signal (which they know from channel use $n+1$), and then they decode both $\mathbf{v}_1(n-1)$ and $\mathbf{v}_2(n-1)$. In this way, the receivers obtain their desired DF signals, R_{1v} bits from Tx 1 and R_{2v} bits from Tx 2. Decoding proceeds backwards till the first channel use is reached. Thus, Tx 1 and Tx 2 achieve R_{1v} and R_{2v} bits per channel use, respectively, for large n.

5.2.2.3 Superposition of DF and CN

At this point, a remark about the DF strategy as compared to the CN strategy in Section 5.2.2.1 is in order. Due to backward decoding, the interference caused by the DF signal, $\mathbf{v}_1(i)$ at Rx 2 for instance, is not harmful since it can be removed as long as the decoding of the DF signals was successful in channel use $i+1$. This is the reason why the relay and the transmitters can send over the same levels at the receivers (as in Figure 5.4). Therefore, *the DF signals $\mathbf{v}_j(i)$ (from the transmitters) should be received 'clean' at the relay but not necessarily so at the receivers.* In fact, the DF signals sent from the transmitters do not have to be received at all at the receivers since they are decoded from the relay signal.

Now consider the CN signals where an opposite statement holds. Since the relay decodes $\mathbf{t}_1(i+1) \oplus \mathbf{t}_2(i+1)$ in channel use i (in a forward fashion), and since the transmitters send 'present' and 'future' CN signals, i.e., $\mathbf{t}_j(i)$ and $\mathbf{t}_j(i+1)$ in the ith channel use, then the contribution of $\mathbf{t}_1(i) \oplus \mathbf{t}_2(i)$ in $\mathbf{y}_r(i)$ can be removed by the relay since it is known from the decoding process in time slot $i-1$. Thus, if the present CN signals overlap with other signals at the relay, the latter can still be decoded by the relay after removing the contribution of the present CN signals. On the other hand, the present CN signal is important at the receivers, since it is the signal that participates in the interference neutralization process. We summarize this statement by saying that *the present CN signal must be received 'clean' at the receivers but not necessarily so at the relay.* Additionally, *the future CN signal must be received 'clean' at the relay, but does not have to be received at all at the receivers.*

Combining these observations, we can construct a hybrid scheme where both CN and DF are used, and where the CN signals and the DF signals overlap at the relay and

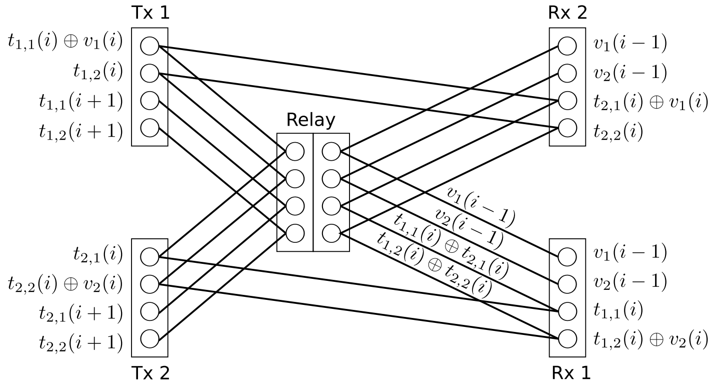


Figure 5.5: A combination of DF and CN. The relay can obtain $v_1(i)$ and $v_2(i)$ in the ith channel use after removing $t_{1,1}(i) \oplus t_{2,1}(i)$ and $t_{1,2}(i) \oplus t_{2,2}(i)$ which it has decoded in channel use $i-1$. Thus, this interference between the CN signal and the DF signals at the relay is not harmful. In the ith channel use, Rx 1 starts by removing $v_2(i)$ (known from the decoding process in channel use $i+1$) from its received signal. Then it decodes $v_1(i-1)$, $v_2(i-1)$, $t_{1,1}(i)$, and $t_{1,2}(i)$. Using this strategy each transmitter can send 3 bit per channel use which achieves the sum-capacity upper bound (cf. Theorem 5.1).

the receivers in a smart way, as illustrated in Figure 5.5. In this figure, Tx 1 allows its present CN signal $\mathbf{t}_1(i) = [t_{1,1}(i), t_{1,2}(i)]^T$ to overlap with the DF signal $v_1(i)$. And thus these signals also overlap at the receivers. Nevertheless, the relay is still able to decode the necessary information and forward it to the receivers which can still recover their desired information. If we do not use this property, i.e., if we send $\mathbf{t}_j$ and $\mathbf{v}_j$ without overlap, then reliable decoding at the relay for instance requires $2R_t + R_{1v} + R_{2v} \leq n_s$. Otherwise, if we allow the signals to overlap, reliable decoding at the relay requires $R_t + \max\{R_t, R_{1v} + R_{2v}\} \leq n_s$. Clearly, this overlap allows a much more efficient exploitation of the bit-pipes.

In the following, we discuss two variants of feedback, a symmetric bi-directional variant, and an asymmetric uni-directional variant.

5.2.2.4 Symmetric Feedback

Here both transmitters use the same strategy. This strategy exploits the feedback channel between the relay and the transmitters to establish cooperation between the transmitters. The transmitters exchange information among each other as in the LD bi-directional relay channel in [NWS07, AAT09]. Tx j, $j \in \{1, 2\}$, sends a feedback (F) signal $\mathbf{s}_j(i)$ of length R_s in the ith channel use. The relay decodes the sum $\mathbf{s}_1(i) \oplus \mathbf{s}_2(i)$ in the ith channel use, which requires $R_s \leq n_s$, and feeds it back to the transmitters in channel use $i+1$. In channel use $i+1$, Tx 1 for instance decodes $\mathbf{s}_1(i) \oplus \mathbf{s}_2(i)$ from the feedback channel, which requires $R_s \leq n_f$, and then it extracts $\mathbf{s}_2(i)$ from this sum using $\mathbf{s}_1(i)$. Then, Tx 1 sends $\mathbf{s}_2(i)$ to Rx 2 using its cross channel in channel use $i+2$, which allows Rx 2 to to decode $\mathbf{s}_2(i)$ if $R_s \leq n_c$. A similar procedure is done at the Tx 2. Figure 5.6 shows an example of this F strategy. Note that both transmitters always send information to the relay which renders some levels at the transmitters always occupied. Thus, the transmitters have to use *other levels* for sending the F signals to the respective receivers. In general, for each F bit, the symmetric F strategy uses 2 levels at the transmitters and 1 level for feedback.

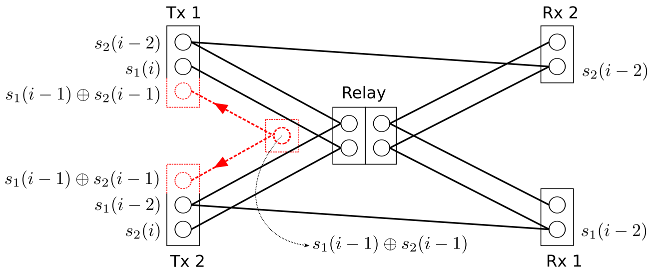


Figure 5.6: Symmetric feedback: Tx 1 and Tx 2 exchange their signals as in a bi-directional relay channel. At the same time, Tx j sends the signal of Tx k acquired via feedback to Rx k. The dotted lines denote the out-of-band feedback channels.

Note that this scheme incurs a delay of 2 channel uses. Each transmitter sends n F signals from the first channel use till channel use $i = n$. The relay feeds these signals back in the channel uses $i = 2, \cdots, n+1$. Finally, Tx 1 and 2 send the F signals to their respective receivers in channel uses $i = 3, \cdots, n+2$. If the F signals $\mathbf{s}_j$ are vectors of length R_s, then each transmitter can successfully deliver nR_s bits in $n+2$ channel uses. Thus the rate that each transmitter can achieve per channel use approaches R_s for large n.

Notice from Figure 5.6 that we have sent the F signals on levels that could have also been used by the relay to send the same amount of bits using CN for instance. As we shall see, this symmetric F strategy does not increase the capacity if $n_c \leq n_r$ which is the case in Figure 5.6. The F strategy would increase the capacity if n_c is larger than n_r, in which case the transmitters would send the F signals to their respective receivers over levels that are not accessible by the relay, thus not disturbing the relay transmission while doing so.

5.2.2.5 Asymmetric Feedback

The symmetric F strategy achieves symmetric rates for the F signals, i.e., the rate achieved by Tx 1 is equal to that of Tx 2. We can also use the F strategy in an asymmetric fashion as follows. Tx 1 sends $\mathbf{s}_1(i)$ to the relay in the ith channel use, the relay decodes this signal and feeds it back to Tx 2 in channel use $i+1$, which sends it to Rx 1 in the channel use $i+2$ *on the same level used by Tx 1*. This causes the signals $\mathbf{s}_1(i)$ and $\mathbf{s}_1(i-2)$ to interfere at the relay. However, the relay can always resolve this interference since it decoded $\mathbf{s}_1(i-2)$ in channel use $i-2$. If the vector $\mathbf{s}_1(i)$ has length R_s, then this strategy achieves the rate point $(R_s, 0)$.

An illustrative example for the asymmetric F strategy is given in Figure 5.7 where Tx 1 can send 1 bit per channel use to Rx 1, achieving the rate pair $(1, 0)$. Note that the same rate pair can be achieved using the symmetric F strategy (Figure 5.6) by setting $\mathbf{s}_2 = 0$. But this would be inefficient since it consumes 2 levels at the relay for reception. The same rate pair can be achieved using the asymmetric F strategy while using only 1 level at the relay as shown in Figure 5.7. This leaves one level at the relay unused, providing more flexibility to combine the F strategy with other strategies. Since our aim is to characterize the capacity region of the LD-BFN with feedback, we are going to need strategies which achieve asymmetric rates efficiently. Both the symmetric and the asymmetric F strategies will be used in the sequel.

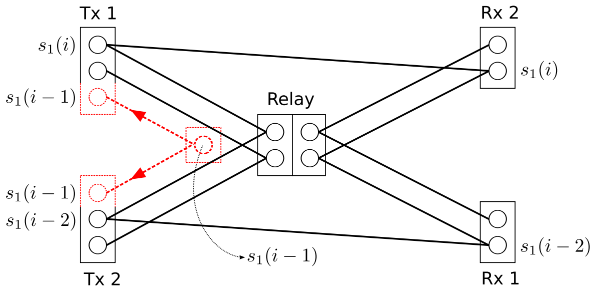


Figure 5.7: Asymmetric feedback: Tx 1 sends the $s_1(i)$ to Rx 1 via the relay and Tx 2. The dotted lines denote the out-of-band feedback channels.

5.2.2.6 Compute-forward

The last strategy we need for achieving the capacity region of the LD-BFN is the compute-forward (CF) strategy. The CF strategy has already been discussed in Section 4.2.2.2 (page 33). To avoid repetition, the reader can refer to the previous chapter for more details. We denote CF signals by $\mathbf{u}_j$ and their rates by R_u. It can be easily seen from Section 4.2.2.2 that the CF strategy can deliver R_u bits for each receiver while using R_u levels at the relay and $2R_u$ levels at the receivers. For this reason, this strategy is preferable when the number of levels at the receivers $\max\{n_c, n_r\}$ is larger than n_s.

In the following sections, we develop capacity achieving schemes for the LD-BFN with feedback which are based on combinations of the four strategies given above.

5.2.3 Capacity Region of the LD-BFN with Feedback

In this section, we prove that the region given in Theorem 5.1 is achievable. We split the proof into two cases. First, we consider the case $n_c \leq n_r$ where feedback does not increase the capacity of the LD-BFN. Then we consider the opposite case.

If $n_c \leq n_r$, then the feedback channel n_f does not have any contribution in the rate constraints in Theorem 5.1, which reduces to

$$0 \leq R_1 \leq \min\{n_s, n_r\} \tag{5.47}$$

$$0 \leq R_2 \leq \min\{n_s, n_r\} \tag{5.48}$$

$$R_1 + R_2 \leq \min\{n_r + n_c, n_r + (n_s - n_c)^+, n_s + n_c\}. \tag{5.49}$$

This region can be achieved without exploiting the feedback link n_f, and thus without using the F strategy. Hence, in this section we only use the strategies that do not exploit feedback, i.e., CF, CN, and DF.

Remark 5.2. *Note that some rate pairs on the boundary of the capacity region in* (5.47)-(5.49) *can have rational components. In this case, this rate pair can be made integer by considering multiple channel uses. For instance, if we want to achieve a rate pair* $(p_1/q_1, p_2/q_2)$, *we use* q_1q_2 *channel uses to achieve a rate pair* (p_1q_2, p_2q_1) *over an LD-BFN with* q_1q_2 *times the number of levels for each channel. This achieves the desired rational rate pair in each channel use.*

5.2.3.1 A Case where Feedback does not Increase Capacity

This regime where $n_c \leq n_r$ is split into two cases. Namely, a case where $n_s \leq n_c$, and another case where $n_s > n_c$. In the first case, CF is better than CN since n_s is small, and vice versa in the second case.

5.2.3.1.1 Case $n_s \leq n_c \leq n_r$: As discussed in Section 5.2.2, in the first case we should not use CN since n_s is small. Since $\max\{n_c, n_r\} \geq n_s$, we use the CF strategy according to the discussion in Section 5.2.2. Moreover, we use DF for achieving asymmetric rate tuples. The combination of CF and DF is sufficient for achieving the capacity region. The following lemma summarizes the result for this regime.

Lemma 5.1. *In the LD-BFN with feedback with $n_s \leq n_c \leq n_r$, the following region is achievable*

$$0 \leq R_1 \leq n_s \tag{5.50}$$

$$0 \leq R_2 \leq n_s \tag{5.51}$$

$$R_1 + R_2 \leq n_r, \tag{5.52}$$

This achievable rate region coincides with the region given in Theorem 5.1. Thus, this lemma characterizes the capacity region of the LD-BFN with feedback with $n_s \leq n_c \leq n_r$. The sum-capacity given by this lemma is not new, in fact, it was given in Theorem 4.1 on page 22. However, this lemma characterizes not only the sum-capacity but also the capacity region. The next paragraphs are devoted for the proof of this lemma.

Proof. Let us construct $\mathbf{x}_1(i)$ in the ith channel use as follows

$$\mathbf{x}_1(i) = \begin{bmatrix} \mathbf{u}_1(i) \\ \mathbf{v}_1(i) \\ \mathbf{0}_{R_{2v}} \\ \mathbf{0}_{n_c - R_u - R_{1v} - R_{2v}} \\ \mathbf{0}_{q-n_c} \end{bmatrix}. \tag{5.53}$$

Here, the signal $\mathbf{v}_1$ is a vector of length R_{1v}, and the signal $\mathbf{u}_1$ is a vector of length R_u. We construct $\mathbf{x}_2(i)$ similarly, with $\mathbf{u}_1(i)$, $\mathbf{v}_1(i)$ and $\mathbf{0}_{R_{2v}}$ replaced with $\mathbf{u}_2(i)$, $\mathbf{0}_{R_{1v}}$, and $\mathbf{v}_2(i)$, respectively.

In the ith channel use, the relay observes the topmost n_s bits of $\mathbf{x}_1(i) \oplus \mathbf{x}_2(i)$. Under the following condition

$$R_u + R_{1v} + R_{2v} \leq n_s, \tag{5.54}$$

the relay is able to observe $\bar{\mathbf{u}}(i) = \mathbf{u}_1(i) \oplus \mathbf{u}_2(i)$, $\mathbf{v}_1(i)$, and $\mathbf{v}_2(i)$ and hence to decode them.

Since in this case $n_r \geq n_c$, the relay can access levels at the receivers above those that can be accessed by the transmitters. Then, the signal $\bar{\mathbf{u}}(i)$ to be forwarded by the relay is split into two parts: $\bar{\mathbf{u}}_a(i)$ and $\bar{\mathbf{u}}_b(i)$ of length R_{ua} and R_{ub}, respectively, where $R_u = R_{ua} + R_{ub}$. The first part $\bar{\mathbf{u}}_a(i)$ is sent such that it arrives on levels above the signals from the transmitters, and the second part $\bar{\mathbf{u}}_b(i)$ is sent below[3]. The relay also sends the DF signals $\mathbf{v}_1(i)$ and $\mathbf{v}_2(i)$. Figure 5.8 shows the transmit signals of the

[3]A similar splitting was used in Section 4.2.2.2.

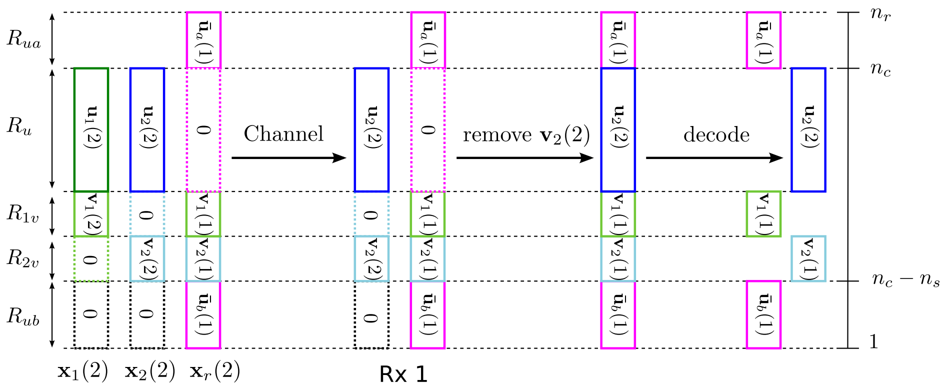


Figure 5.8: A graphical illustration of the transmit signals at channel use 2, the received signal at Rx 1, and the decoding process, using the capacity achieving scheme of the LD-BFN with feedback with $n_s \leq n_c \leq n_r$.

transmitters and the received signal of Rx 1 ($\mathbf{y}_1(i)$). Notice from this figure that the relay forwards $\mathbf{x}_r(i+1)$ in the next channel use where

$$\mathbf{x}_r(i+1) = \begin{bmatrix} \mathbf{0}_{n_r-n_c-R_{ua}} \\ \bar{\mathbf{u}}_a(i) \\ \mathbf{0}_{R_u} \\ \mathbf{v}_1(i) \\ \mathbf{v}_2(i) \\ \bar{\mathbf{u}}_b(i) \\ \mathbf{0}_{n_c-R_{ua}-2R_{ub}-R_{1v}-R_{2v}} \\ \mathbf{0}_{q-n_r} \end{bmatrix}. \tag{5.55}$$

This construction requires

$$R_{ua} + 2R_{ub} + R_{1v} + R_{2v} \leq n_c, \tag{5.56}$$

$$R_{ua} \leq n_r - n_c. \tag{5.57}$$

Rx 1 waits until the end of channel use $n+1$ where only the relay is active, and decodes $\bar{\mathbf{u}}_a(n)$ and $\bar{\mathbf{u}}_b(n)$ thereby recovering $\mathbf{u}_1(n) \oplus \mathbf{u}_2(n)$, and it also decodes $\mathbf{v}_1(n)$ and $\mathbf{v}_2(n)$. Then it proceeds backward to the nth channel use. The received signal at Rx 1 can be written as (see Figure 5.8)

$$\mathbf{y}_1(n) = \begin{bmatrix} \mathbf{0}_{q-n_r} \\ \mathbf{0}_{n_r-n_c-R_{ua}} \\ \bar{\mathbf{u}}_a(n-1) \\ \mathbf{u}_2(n) \\ \mathbf{v}_1(n-1) \\ \mathbf{v}_2(n) \oplus \mathbf{v}_2(n-1) \\ \bar{\mathbf{u}}_b(n-1) \\ \mathbf{0}_{n_c-R_{ua}-2R_{ub}-R_{1v}-R_{2v}} \end{bmatrix}. \tag{5.58}$$

Since Rx 1 knows $\mathbf{v}_2(n)$, it can remove it from the received signal. Then it proceeds with decoding $\bar{\mathbf{u}}_a(n-1)$, $\bar{\mathbf{u}}_b(n-1)$, $\mathbf{u}_2(n)$, $\mathbf{v}_1(n-1)$, and $\mathbf{v}_2(n-1)$. Having $\mathbf{u}_2(n)$ and $\mathbf{u}_1(n) \oplus \mathbf{u}_2(n)$ allows Rx 1 to obtain $\mathbf{u}_1(n)$. Additionally, Rx 1 obtains $\mathbf{v}_1(n-1)$ which is a desired signals. Furthermore, $\mathbf{v}_2(n-1)$ and $\bar{\mathbf{u}}(n-1)$ are obtained which are used in the decoding process in channel use $n-1$. Using this process, Rx 1 was able to recover its desired signals comprising of R_u and R_{1v} bits, respectively. Rx 2 performs similar operations. The receivers proceed backwards till channel use 1 is reached.

The rate achieved by Tx j is $R_j = R_u + R_{jv}$. Collecting the rate constraints (5.54), (5.56), and (5.57), we get the following constraints on the non-negative rates R_{ua}, R_{ub}, R_{1v}, and R_{2v}:

$$R_{ua} + R_{ub} + R_{1v} + R_{2v} \leq n_s \tag{5.59}$$

$$R_{ua} + 2R_{ub} + R_{1v} + R_{2v} \leq n_c \tag{5.60}$$

$$R_{ua} \leq n_r - n_c. \tag{5.61}$$

Using Fourier-Motzkin's elimination [EGK11, Appendix D] we get the achievable region given in Lemma 5.1. □

5.2.3.1.2 Case $n_c \leq \min\{n_s, n_r\}$: Now we consider the case where the cross channel is weaker than both the source-relay channel and the relay-destination channel. Note that this is a regime that was not covered in Chapter 4. Thus, characterizing the capacity of the LD-BFN in this regime can serve as a stepping stone towards the capacity of the LD-IRC with $n_c \leq n_s$.

As we have mentioned earlier, if $n_s \geq n_c$, then we can pass some future information to the relay without the receivers noticing by using the CN strategy. Thus, we use CN in addition to CF and DF. For this case, we have the following lemma.

Lemma 5.2. *The rate region defined by the following rate constraints*

$$0 \leq R_1 \leq \min\{n_s, n_r\} \tag{5.62}$$

$$0 \leq R_2 \leq \min\{n_s, n_r\} \tag{5.63}$$

$$R_1 + R_2 \leq \min\{n_s + n_c, n_r + n_c, n_r + n_s - n_c\}, \tag{5.64}$$

is achievable in the LD-BFN with feedback with $n_c \leq \min\{n_s, n_r\}$.

This achievable region coincides with the region given in Theorem 5.1, and thus characterizes the capacity of the LD-BFN with feedback with $n_c \leq \min\{n_s, n_r\}$. We provide this capacity achieving scheme next.

Proof. At time instant i, Tx 1 sends the following signal

$$\mathbf{x}_1(i) = \begin{bmatrix} \mathbf{0}_{n_c - R_u - R_{1v} - R_{2v} - R_t} \\ \mathbf{u}_1(i) \\ \mathbf{v}_1(i) \\ \mathbf{0}_{R_{2v}} \\ \begin{bmatrix} \mathbf{t}_1(i) \\ \mathbf{0}_{n_s - n_c - R_t} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0}_{n_s - n_c - \bar{R}_{1v} - \bar{R}_{2v}} \\ \bar{\mathbf{v}}_1(i) \\ \mathbf{0}_{\bar{R}_{2v}} \end{bmatrix} \\ \mathbf{t}_1(i+1) \\ \mathbf{0}_{q-n_s} \end{bmatrix}, \tag{5.65}$$

where $\mathbf{t}_1(i+1)$ is the future information passed to the relay. The signals $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{t}_1$, and $\bar{\mathbf{v}}_1$ are vectors of length R_u, R_{1v}, R_t, and $\bar{R}_{1v}$, respectively. Notice that this

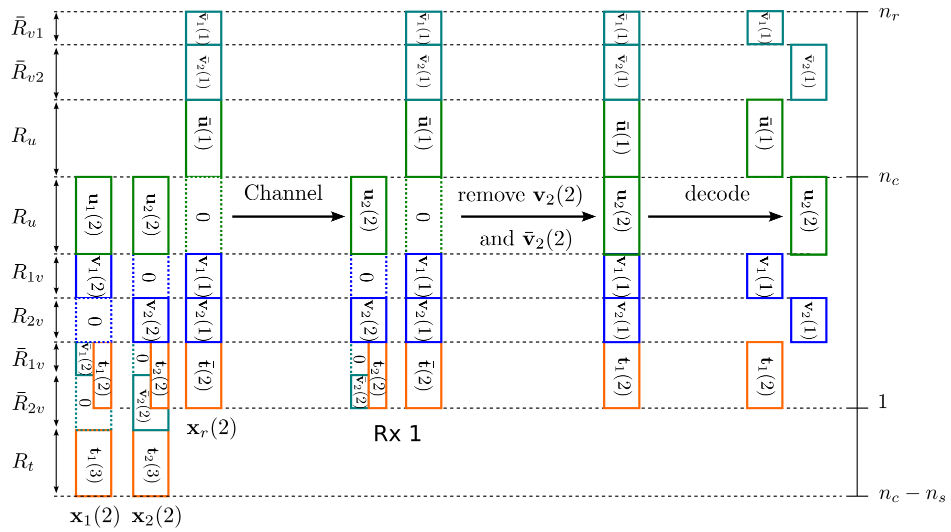


Figure 5.9: The transmit signals, received signal at Rx 1, and decoding process at Rx 1 at channel use 2, for the capacity achieving scheme of the LD-BFN with feedback with $n_c \leq \min\{n_s, n_r\}$. Notice the overlap of $\bar{\mathbf{v}}_2$ and $\mathbf{t}_1$ at Rx 1.

construction requires that

$$R_u + R_{1v} + R_{2v} + R_t \leq n_c \tag{5.66}$$
$$\bar{R}_{1v} + \bar{R}_{2v} \leq n_s - n_c \tag{5.67}$$
$$R_t \leq n_s - n_c. \tag{5.68}$$

Tx 2 uses a similar construction, with zeros instead of $\mathbf{v}_1(i)$ and $\bar{\mathbf{v}}_1(i)$, and with $\mathbf{v}_2(i)$ and $\bar{\mathbf{v}}_2(i)$ instead of $\mathbf{0}_{R_{2v}}$ and $\mathbf{0}_{\bar{R}_{2v}}$, respectively. Using this construction, there can be an overlap between $\mathbf{t}_1(i)$ and $\mathbf{t}_2(i)$ on one hand, and $\bar{\mathbf{v}}_1(i)$ and $\bar{\mathbf{v}}_2(i)$ on the other hand, at the relay and at the receivers (see Figure 5.9). However, this overlap is not harmful (similar to the one discussed in Section 5.2.2.2). The overlapping DF signals are marked with a bar to distinguish them from $\mathbf{v}_1$ and $\mathbf{v}_2$ which do not overlap with any signal at the relay.

The received signal at the relay consists of the topmost n_s bits of $\mathbf{x}_1(i) \oplus \mathbf{x}_2(i)$. Let us write $\mathbf{y}_r(i)$ as follows

$$\mathbf{y}_r(i) = \begin{bmatrix} \mathbf{0}_{q-n_s} \\ \mathbf{0}_{n_c-R_u-R_{1v}-R_{2v}-R_t} \\ \mathbf{u}_1(i) \oplus \mathbf{u}_2(i) \\ \mathbf{v}_1(i) \\ \mathbf{v}_2(i) \\ \begin{bmatrix} \mathbf{t}_1(i) \oplus \mathbf{t}_2(i) \\ \mathbf{0}_{n_s-n_c-R_t} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0}_{n_s-n_c-\bar{R}_{1v}-\bar{R}_{2v}} \\ \bar{\mathbf{v}}_1(i) \\ \bar{\mathbf{v}}_2(i) \end{bmatrix} \\ \mathbf{t}_1(i+1) \oplus \mathbf{t}_2(i+1) \end{bmatrix}. \tag{5.69}$$

In the ith channel use, the relay knows $\bar{\mathbf{t}}(i) = \mathbf{t}_1(i) \oplus \mathbf{t}_2(i)$ from the decoding process in the channel use $i-1$. This allows it to remove $\mathbf{t}_1(i) \oplus \mathbf{t}_2(i)$ from $\mathbf{y}_r(i)$. Then, the relay can decode $\bar{\mathbf{u}}(i) = \mathbf{u}_1(i) \oplus \mathbf{u}_2(i)$, $\mathbf{v}_1(i)$, $\mathbf{v}_2(i)$, $\bar{\mathbf{v}}_1(i)$, $\bar{\mathbf{v}}_2(i)$, and finally $\bar{\mathbf{t}}(i+1) = \mathbf{t}_1(i+1) \oplus \mathbf{t}_2(i+1)$. At the end of channel use i, the relay constructs the following signal

$$\mathbf{x}_r(i+1) = \begin{bmatrix} \mathbf{0}_{n_r-\bar{R}_{1v}-\bar{R}_{2v}-2R_u-R_{1v}-R_{2v}-R_t} \\ \bar{\mathbf{v}}_1(i) \\ \bar{\mathbf{v}}_2(i) \\ \bar{\mathbf{u}}(i) \\ \mathbf{0}_{R_u} \\ \mathbf{v}_1(i) \\ \mathbf{v}_2(i) \\ \bar{\mathbf{t}}(i+1) \\ \mathbf{0}_{q-n_r} \end{bmatrix} \tag{5.70}$$

and sends it in channel use $i+1$. The constituent signals of $\mathbf{x}_r(i+1)$ in (5.70) fit in an interval of size n_r if

$$\bar{R}_{1v} + \bar{R}_{2v} + 2R_u + R_{1v} + R_{2v} + R_t \leq n_r. \tag{5.71}$$

Consider now the processing at Rx 1 (the processing at Rx 2 follows along similar lines). Rx 1 observes the topmost n_c bits of $\mathbf{x}_2(i)$ and the topmost n_r bits of $\mathbf{x}_r(i)$ at the ith channel use (Figure 5.9). It uses backward decoding. Assuming that the decoding process in channel use $i+1$ was successful, Rx 1 can remove the contribution of $\bar{\mathbf{v}}_2(i)$ and $\mathbf{v}_2(i)$ from its received signal. It then proceeds to decode $\bar{\mathbf{v}}_1(i-1)$, $\bar{\mathbf{v}}_2(i-1)$, $\bar{\mathbf{u}}(i-1)$, $\mathbf{u}_2(i)$, $\mathbf{v}_1(i-1)$, $\mathbf{v}_2(i-1)$, and $\mathbf{t}_1(i)$. Decoding then proceeds backwards till $i=1$.

Collecting the rate constraints (5.66)-(5.68) and (5.71), we conclude that the non-negative rates $\bar{R}_{1v}$, $\bar{R}_{2v}$, R_{1v}, R_{2v}, R_u, and R_t can be achieved if they satisfy

$$R_u + R_{1v} + R_{2v} + R_t \leq n_c \tag{5.72}$$

$$\bar{R}_{1v} + \bar{R}_{2v} \leq n_s - n_c \tag{5.73}$$

$$R_t \leq n_s - n_c \tag{5.74}$$

$$\bar{R}_{1v} + \bar{R}_{2v} + 2R_u + R_{1v} + R_{2v} + R_t \leq n_r. \tag{5.75}$$

Using Fourier Motzkin's elimination with $R_j = R_{jv} + \bar{R}_{jv} + R_u + R_t$, we get the achievable rate region given in Lemma 5.2. □

At this point, we have finished the proof of the achievability of Theorem 5.1 for $n_c \leq n_r$ where feedback does not increase capacity. Next, we consider the opposite case where feedback indeed increases capacity.

5.2.3.2 A Case where Feedback Increases Capacity

The statement of Theorem 5.1 for the LD-BFN with $n_c > n_r$ reduces to

$$R_1 \leq \min\{n_s, n_r + n_f, n_c\} \tag{5.76}$$

$$R_2 \leq \min\{n_s, n_r + n_f, n_c\} \tag{5.77}$$

$$R_1 + R_2 \leq n_c + (n_s - n_c)^+. \tag{5.78}$$

In this case, n_f contributes to the outer bounds. If the region defined by these upper bounds is achievable, then feedback has a positive impact on the BFN. This is what we shall prove next. That is, we show that this region is in fact achievable, and hence that relay-source feedback increases the capacity of the network if $n_c > n_r$ when compared to the case $n_f = 0$.

5.2.3.2.1 Case $\max\{n_r, n_s\} < n_c$**:** We start by stating the achievable region described in this subsection in the following lemma.

Lemma 5.3. *The rate region defined by the following inequalities*

$$0 \le R_1 \le \min\{n_s, n_r + n_f\} \tag{5.79}$$

$$0 \le R_2 \le \min\{n_s, n_r + n_f\} \tag{5.80}$$

$$R_1 + R_2 \le n_c, \tag{5.81}$$

is achievable in the LD-BFN with feedback with $\max\{n_r, n_s\} < n_c$.

Notice that this achievable region matches the region given in Theorem 5.1. Therefore, the achievability of this region proves the achievability of Theorem 5.1 for the case $\max\{n_r, n_s\} < n_c$. We prove this lemma in the rest of this subsection.

Since $n_c > n_r$ the transmitters can use the upper $n_c - n_r$ levels at the receivers which are not accessible by the relay to send feedback information to the receivers. Thus, in this case we use the F strategy. Since $\max\{n_c, n_r\} > n_s$ in this case, we also use the CF strategy following the intuition in Section 5.2.2.6. Furthermore, we use DF to achieve asymmetric rate pairs.

Proof. In the ith channel use, Tx 1 sends the following signal (Figure 5.10)

$$\mathbf{x}_1(i) = \begin{bmatrix} \mathbf{u}_1(i) \\ \mathbf{v}_1(i) \\ \mathbf{0}_{R_{2v}} \\ \mathbf{s}_1(i) \\ \mathbf{s}_2(i-2) \\ \bar{\mathbf{s}}_1(i) \\ \bar{\mathbf{s}}_2(i-2) \\ \mathbf{0}_{n_c - R_u - R_{1v} - R_{2v} - R_{1s} - R_{2s} - 2\bar{R}_s} \\ \mathbf{0}_{q-n_c} \end{bmatrix}. \tag{5.82}$$

Here, the signal $\mathbf{s}_j$, $j = 1, 2$, is the signal used to establish the asymmetric F strategy, which is a vector of length R_{js}. The signal $\mathbf{s}_2(i-2)$ is available at Tx 1 via feedback. The signal $\bar{\mathbf{s}}_j$ is the signal used in the symmetric F strategy, and is a vector of length $\bar{R}_s$. We use both symmetric feedback and asymmetric feedback to achieve all points in the closure of the region given in Lemma 5.3. The CF signal $\mathbf{u}_1$ has length R_u, and the DF signal $\mathbf{v}_1$ has length R_{1v}. Tx 2 sends a similar signal (see Figure 5.10). The given construction works if

$$R_u + R_{1v} + R_{2v} + R_{1s} + R_{2s} + 2\bar{R}_s \le n_c. \tag{5.83}$$

The relay observes the topmost n_s bits of $\mathbf{x}_1(i) \oplus \mathbf{x}_2(i)$. We want the relay to be able to observe $\mathbf{u}_1(i) \oplus \mathbf{u}_2(i)$, $\mathbf{v}_1(i)$, $\mathbf{v}_2(i)$, $\mathbf{s}_1(i) \oplus \mathbf{s}_1(i-2)$, $\mathbf{s}_2(i-2) \oplus \mathbf{s}_2(i)$, and $\bar{\mathbf{s}}_1(i) \oplus \bar{\mathbf{s}}_2(i)$. This is possible if we choose

$$R_u + R_{1v} + R_{2v} + R_{1s} + R_{2s} + \bar{R}_s \le n_s. \tag{5.84}$$

Given this condition is satisfied, the relay starts by removing $\mathbf{s}_1(i-2)$ and $\mathbf{s}_2(i-2)$ (known from past decoding) from $\mathbf{y}_r(i)$. Next, it decodes the sum of the CF signals

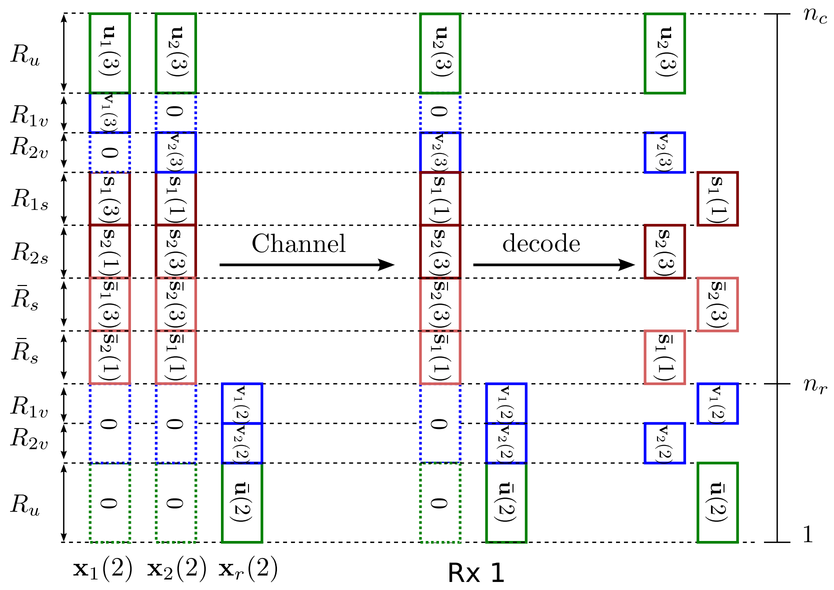


Figure 5.10: A graphical illustration of the transmit signals and the received signals at Rx 1 in channel use 3, for an LD-BFN with feedback with $\max\{n_r, n_s\} \leq n_c$ and $n_f > 0$.

$\bar{\mathbf{u}}(i) = \mathbf{u}_1(i) \oplus \mathbf{u}_2(i)$, the DF signals $\mathbf{v}_1(i)$ and $v_2(i)$, the F signals $\mathbf{s}_1(i)$ and $\mathbf{s}_2(i)$, and $\bar{\mathbf{s}}_1(i) \oplus \bar{\mathbf{s}}_2(i)$. Then it sends the following signals

$$\mathbf{x}_r(i+1) = \begin{bmatrix} \mathbf{0}_{n_r - R_u - R_{1v} - R_{2v}} \\ \mathbf{v}_1(i) \\ \mathbf{v}_2(i) \\ \bar{\mathbf{u}}(i) \\ \mathbf{0}_{q-n_r} \end{bmatrix}, \quad \mathbf{x}_f(i+1) = \begin{bmatrix} \begin{bmatrix} \mathbf{s}_1(i) \\ \mathbf{0}_{(R_{2s}-R_{1s})^+} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{s}_2(i) \\ \mathbf{0}_{(R_{1s}-R_{2s})^+} \end{bmatrix} \\ \bar{\mathbf{s}}_1(i) \oplus \bar{\mathbf{s}}_2(i) \\ \mathbf{0}_{q-\max\{R_{1s}, R_{2s}\}-\bar{R}_s} \end{bmatrix}, \quad (5.85)$$

over the relay-destination channel and the feedback channel, respectively, in channel use $i+1$, which requires

$$R_u + R_{1v} + R_{2v} \leq n_r \quad (5.86)$$

$$R_{1s} + \bar{R}_s \leq n_f \quad (5.87)$$

$$R_{2s} + \bar{R}_s \leq n_f. \quad (5.88)$$

Note that the F signals $\mathbf{s}_1$ and $\mathbf{s}_2(i)$ are network coded at the relay after they are zero padded to the same length. This allows a more efficient use of the feedback channel.

Tx 1 decodes the feedback signal and extracts $\mathbf{s}_2(i)$ and $\bar{\mathbf{s}}_2(i)$. Therefore, in channel use $i+2$, Tx 1 knows the F signals of Tx 2 which justifies the transmission of $\mathbf{s}_2(i-2)$ and $\bar{\mathbf{s}}_2(i-2)$ in $\mathbf{x}_1(i)$ in (5.82). A similar processing is performed at Tx 2.

Assume that

$$2R_u + 2R_{1v} + 2R_{2v} + R_{1s} + R_{2s} + 2\bar{R}_s \leq n_c. \quad (5.89)$$

In this case, Rx 1 for instance is able to observe all the signals sent by Tx 2 and the relay. The received signal $\mathbf{y}_1(i)$ is then

$$\mathbf{y}_1(i) = \begin{bmatrix} \mathbf{0}_{q-n_c} \\ \mathbf{u}_2(i) \\ \mathbf{0}_{R_{1v}} \\ \mathbf{v}_2(i) \\ \mathbf{s}_1(i-2) \\ \mathbf{s}_2(i) \\ \bar{\mathbf{s}}_2(i) \\ \bar{\mathbf{s}}_1(i-2) \\ zv_{n_c-2R_u-2R_{1v}-2R_{2v}-R_{1s}-R_{2s}-2\bar{R}_s} \\ \mathbf{v}_1(i-1) \\ \mathbf{v}_2(i-1) \\ \bar{\mathbf{u}}(i-1) \end{bmatrix}. \tag{5.90}$$

Rx 1 decodes backwards. In each channel use i, it decodes $\mathbf{u}_2(i)$, $\mathbf{s}_1(i-2)$, $\bar{\mathbf{s}}_1(i-2)$, $\mathbf{v}_1(i-1)$, and $\bar{\mathbf{u}}(i-1)$. Then it extracts the desired CF signal from $\bar{\mathbf{u}}(i)$ and $\mathbf{u}_2(i)$. Similar processing is performed by Rx 2. Collecting the bounds (5.83), (5.84), (5.86), (5.87), (5.88), and (5.89), we see that a pair R_j with $R_j = R_u + R_{jv} + R_{js} + \bar{R}_s$ is achievable if

$$R_u + R_{1v} + R_{2v} + R_{1s} + R_{2s} + \bar{R}_s \leq n_s \tag{5.91}$$
$$R_u + R_{1v} + R_{2v} \leq n_r \tag{5.92}$$
$$R_{1s} + \bar{R}_s \leq n_f \tag{5.93}$$
$$R_{2s} + \bar{R}_s \leq n_f \tag{5.94}$$
$$2R_u + 2R_{1v} + 2R_{2v} + R_{1s} + R_{2s} + 2\bar{R}_s \leq n_c. \tag{5.95}$$

Solving this set of linear inequalities using the Fourier Motzkin's elimination, we get the achievable region given in Lemma 5.3. □

5.2.3.2.2 Case $n_r < n_c \leq n_s$: In this case, the relay observes more bits than the receivers since $n_s \geq n_c$. Thus, the transmitters can exploit the additional $n_s - n_c$ bits by using the CN strategy of Section 5.2.2.1. Additionally, we use the F strategy for feedback, and the DF strategy to achieve asymmetric rates. In what follows, we prove the following lemma.

Lemma 5.4. *The region defined by*

$$0 \leq R_1 \leq \min\{n_r + n_f, n_c\} \tag{5.96}$$
$$0 \leq R_2 \leq \min\{n_r + n_f, n_c\} \tag{5.97}$$
$$R_1 + R_2 \leq n_s, \tag{5.98}$$

is achievable in the LD-BFN with feedback with $n_r < n_c \leq n_s$.

This lemma proves Theorem 5.1 for the case $n_r < n_c \leq n_s$ since the achievable region of this lemma matches the region given in the Theorem 5.1.

Proof. In this case, Tx 1 sends a DF signal vector $\mathbf{v}_1(i)$ of length R_{1v} two CN signal vectors $\mathbf{t}_1(i)$ and $\mathbf{t}_1(i+1)$ of length R_t each, two F signal vectors $\mathbf{s}_1(i)$ (asymmetric)

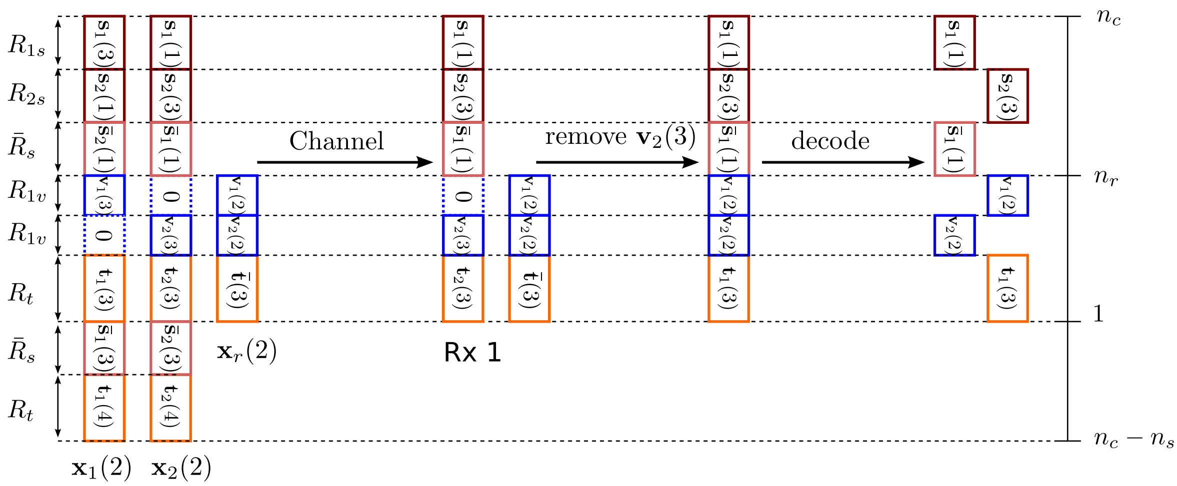


Figure 5.11: The transmit signals and received signal of Rx 1 at channel use 3 for an LD-BFN with feedback with $n_r < n_c \leq n_s$ and $n_f > 0$.

and $\bar{\mathbf{s}}_1(i)$ (symmetric) of length R_{1s} and $\bar{R}_s$, respectively. Additionally, it sends the F signals of Tx 2 (acquired through feedback) $\mathbf{s}_2(i-2)$ and $\bar{\mathbf{s}}_2(i-2)$ of length R_{2s} and $\bar{R}_s$, respectively, as shown if Figure 5.11.

Notice that out of these signals, two do not have to be observed at the receivers, namely $\mathbf{t}_1(i+1)$ and $\bar{\mathbf{s}}_1(i)$. These two signals have to be decoded at the relay to establish the F and the CN strategies. Thus, these signals can be sent below the noise floor of the receivers, i.e., in the lower $n_s - n_c$ levels observed at the relay. Assume that these signals do not fit in this interval of length $n_s - n_c$, i.e., $R_t + \bar{R}_s > n_s - n_c$. In this case, part of these signals is sent below the noise floor, and part above it. For this reason, we split these signals to two parts:

$$\bar{\mathbf{s}}_1(i) = \begin{bmatrix} \bar{\mathbf{s}}_{1a}(i) \\ \bar{\mathbf{s}}_{1b}(i) \end{bmatrix}, \quad \mathbf{t}_1(i) = \begin{bmatrix} \mathbf{t}_{1a}(i) \\ \mathbf{t}_{1b}(i) \end{bmatrix}, \tag{5.99}$$

where $\bar{\mathbf{s}}_{1a}$, $\bar{\mathbf{s}}_{1b}$, $\mathbf{t}_{1a}$, and $\mathbf{t}_{1b}$ have length $\bar{R}_{sa}$, $\bar{R}_{sb}$, R_{ta}, and R_{tb}, respectively, where $\bar{R}_{sa} + \bar{R}_{sb} = \bar{R}_s$ and $R_{ta} + R_{tb} = R_t$ (this split is not shown in Figure 5.11 for clarity). As a result, Tx 1 sends

$$\mathbf{x}_1(i) = \begin{bmatrix} \mathbf{0}_{n_c - R_{1s} - R_{2s} - 2\bar{R}_{sa} - \bar{R}_{sb} - R_{1v} - R_{2v} - 2R_{ta} - R_{tb}} \\ \mathbf{s}_1(i) \\ \mathbf{s}_2(i-2) \\ \bar{\mathbf{s}}_{2a}(i-2) \\ \bar{\mathbf{s}}_{2b}(i-2) \\ \bar{\mathbf{s}}_{1a}(i) \\ \mathbf{t}_{1a}(i+1) \\ \mathbf{v}_1(i) \\ \mathbf{0}_{R_{2v}} \\ \mathbf{t}_{1a}(i) \\ \mathbf{t}_{1b}(i) \\ \bar{\mathbf{s}}_{1b}(i) \\ \mathbf{t}_{1b}(i+1) \\ \mathbf{0}_{n_s - n_c - \bar{R}_{sb} - R_{tb}} \\ \mathbf{0}_{q - n_s} \end{bmatrix}. \tag{5.100}$$

The vectors $\bar{\mathbf{s}}_{1a}(i)$ and $\mathbf{t}_{1a}(i+1)$ are sent above $\mathbf{v}_1(i)$, $\mathbf{t}_{1a}(i)$, and $\mathbf{t}_{1b}(i)$ since that latter signals have to align with the signals sent from the relay (see Sections 5.2.2.1, 5.2.2.4, and 5.2.2.5) and the relay can only access lower levels since $n_r < n_c$ in this case. The transmit signal of Tx 2 is constructed similarly. This construction requires

$$R_{1s} + R_{2s} + 2\bar{R}_{sa} + \bar{R}_{sb} + R_{1v} + R_{2v} + 2R_{ta} + R_{tb} \leq n_c \tag{5.101}$$

$$\bar{R}_{sb} + R_{tb} \leq n_s - n_c. \tag{5.102}$$

The relay receives the topmost n_s bits of $\mathbf{x}_1(i) \oplus \mathbf{x}_2(i)$. We write $\mathbf{y}_r(i)$ as

$$\mathbf{y}_r(i) = \begin{bmatrix} \mathbf{0}_{q-n_s} \\ \mathbf{0}_{n_c-R_{1s}-R_{2s}-2\bar{R}_{sa}-\bar{R}_{sb}-R_{1v}-R_{2v}-2R_{ta}-R_{tb}} \\ \mathbf{s}_1(i) \oplus \mathbf{s}_1(i-2) \\ \mathbf{s}_2(i) \oplus \mathbf{s}_2(i-2) \\ \bar{\mathbf{s}}_{1a}(i-2) \oplus \bar{\mathbf{s}}_{2a}(i-2) \\ \bar{\mathbf{s}}_{1b}(i-2) \oplus \bar{\mathbf{s}}_{2b}(i-2) \\ \bar{\mathbf{s}}_{1a}(i) \oplus \bar{\mathbf{s}}_{2a}(i) \\ \mathbf{t}_{1a}(i+1) \oplus \mathbf{t}_{2a}(i+1) \\ \mathbf{v}_1(i) \\ \mathbf{v}_2(i) \\ \mathbf{t}_{1a}(i) \oplus \mathbf{t}_{2a}(i) \\ \mathbf{t}_{1b}(i) \oplus \mathbf{t}_{2b}(i) \\ \bar{\mathbf{s}}_{1b}(i) \oplus \bar{\mathbf{s}}_{2b}(i) \\ \mathbf{t}_{1b}(i+1) \oplus \mathbf{t}_{2b}(i+1) \\ \mathbf{0}_{n_s-R_{1s}-R_{2s}-2\bar{R}_s-R_{1v}-R_{2v}-2R_t} \end{bmatrix}. \tag{5.103}$$

The relay starts processing this signal by removing the past F signals $\mathbf{s}_j(i-2)$ and $\bar{\mathbf{s}}_j(i-2)$ (decoded in channel use $i-2$) and the present CN signals $\mathbf{t}_{ja}(i)$ and $t_{jb}(i)$ (decoded in channel use $i-1$) from $\mathbf{y}_r(i)$. Then it decodes $\mathbf{s}_1(i)$, $\mathbf{s}_2(i)$, $\bar{\mathbf{s}}_{1a}(i) \oplus \bar{\mathbf{s}}_{2a}(i)$, $\bar{\mathbf{s}}_{1b}(i) \oplus \bar{\mathbf{s}}_{2b}(i)$, $\mathbf{v}_1(i)$, $\mathbf{v}_2(i)$, $\bar{\mathbf{t}}_a(i+1) = \mathbf{t}_{1a}(i+1) \oplus \mathbf{t}_{2a}(i+1)$, and $\bar{\mathbf{t}}_b(i+1) = \mathbf{t}_{1b}(i+1) \oplus \mathbf{t}_{2b}(i+1)$. Then, the relay sends

$$\mathbf{x}_r(i+1) = \begin{bmatrix} \mathbf{0}_{n_r-R_{1v}-R_{2v}-R_{ta}-R_{tb}} \\ \mathbf{v}_1(i) \\ \mathbf{v}_2(i) \\ \bar{\mathbf{t}}_a(i+1) \\ \bar{\mathbf{t}}_b(i+1) \\ \mathbf{0}_{q-n_r} \end{bmatrix}, \quad \mathbf{x}_f(i+1) = \begin{bmatrix} \begin{bmatrix} \mathbf{s}_1(i) \\ \mathbf{0}_{(R_{2s}-R_{1s})^+} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{s}_2(i) \\ \mathbf{0}_{(R_{1s}-R_{2s})^+} \end{bmatrix} \\ \bar{\mathbf{s}}_{1a}(i) \oplus \bar{\mathbf{s}}_{2a}(i) \\ \bar{\mathbf{s}}_{1b}(i) \oplus \bar{\mathbf{s}}_{2b}(i) \\ \mathbf{0}_{n_f-\max\{R_{1s},R_{2s}\}-\bar{R}_{sa}-\bar{R}_{sb}} \\ \mathbf{0}_{q-n_f} \end{bmatrix}. \tag{5.104}$$

to the receivers and to the transmitters, respectively, in channel use $i+1$. The given signals fit in the interval of length n_r and n_f if

$$R_{1v} + R_{2v} + R_{ta} + R_{tb} \leq n_r \tag{5.105}$$

$$R_{1s} + \bar{R}_{sa} + \bar{R}_{sb} \leq n_f \tag{5.106}$$

$$R_{2s} + \bar{R}_{sa} + \bar{R}_{sb} \leq n_f. \tag{5.107}$$

For efficient use of the feedback channel, the relay adds the signals $\mathbf{s}_1(i)$ and $\mathbf{s}_2(i)$ together after zero-padding them to the same length, and feeds the sum back. Tx 1 at channel use $i+1$ subtracts its own F signals from its received feedback signal, and then decodes the F signals of Tx 2, i.e., $\mathbf{s}_2(i)$, $\bar{\mathbf{s}}_{2a}(i)$, and $\bar{\mathbf{s}}_{2b}(i)$. These signals are sent in channel use $i+2$ as seen in (5.100).

In the ith channel use, Rx 1 observes

$$\mathbf{y}_1(i) = \begin{bmatrix} \mathbf{0}_{q-n_c} \\ \mathbf{0}_{n_c-R_{1s}-R_{2s}-2\bar{R}_{sa}-\bar{R}_{sb}-R_{1v}-R_{2v}-2R_{ta}-R_{tb}} \\ \mathbf{s}_1(i-2) \\ \mathbf{s}_2(i) \\ \bar{\mathbf{s}}_{1a}(i-2) \\ \bar{\mathbf{s}}_{1b}(i-2) \\ \bar{\mathbf{s}}_{2a}(i) \\ \mathbf{t}_{2a}(i+1) \\ \mathbf{v}_1(i-1) \\ \mathbf{v}_2(i) \oplus \mathbf{v}_2(i-1) \\ \mathbf{t}_{1a}(i) \\ \mathbf{t}_{1b}(i) \end{bmatrix}. \tag{5.108}$$

Decoding at Rx 1 is done backwards. In the ith channel use, it starts with removing the already known DF signal $\mathbf{v}_2(i)$ (decoded in channel use $i+1$). Then it proceeds with decoding each of $\mathbf{s}_1(i-2)$, $\bar{\mathbf{s}}_{1a}(i-2)$, $\bar{\mathbf{s}}_{1b}(i-2)$, $\mathbf{v}_1(i-1)$, $\mathbf{v}_2(i-1)$, $\mathbf{t}_{1a}(i)$, and $\mathbf{t}_{1b}(i)$. Rx 2 proceeds similarly.

Collecting the bounds (5.101), (5.102), (5.105), (5.106), and (5.107) we get

$$R_{1s} + R_{2s} + 2\bar{R}_{sa} + \bar{R}_{sb} + R_{1v} + R_{2v} + 2R_{ta} + R_{tb} \le n_c \tag{5.109}$$
$$\bar{R}_{sb} + R_{tb} \le n_s - n_c \tag{5.110}$$
$$R_{1v} + R_{2v} + R_{ta} + R_{tb} \le n_r \tag{5.111}$$
$$R_{1s} + \bar{R}_{sa} + \bar{R}_{sb} \le n_f \tag{5.112}$$
$$R_{2s} + \bar{R}_{sa} + \bar{R}_{sb} \le n_f, \tag{5.113}$$

where the rates R_{1s}, R_{2s}, $\bar{R}_{sa}$, $\bar{R}_{sb}$, R_{1v}, R_{2v}, R_{ta}, and R_{tb} are non-negative. Solving this set of linear inequalities using the Fourier Motzkin's elimination with $R_j = R_{js} + \bar{R}_{sa} + \bar{R}_{sb} + R_{jv} + R_{ta} + R_{tb}$ yields the achievable rate region given in Lemma 5.4. □

By this point, we have finished the proof of the achievability of Theorem 5.1, and hence, we have characterized the capacity region of the LD-BFN with relay-source feedback. This also shows that the developed relaying strategies are optimal. The feedback gain is discussed next.

5.2.4 Net Feedback Gain

Relay-source feedback increases the capacity of the BFN with respect to the non-feedback case. However, is this feedback efficient? In other words, is there a *net-gain* when using feedback? In this section, we discuss the net-gain attained by exploiting feedback and we answer the question above in the affirmative.

First, let us define what we mean by net-gain. Let C_0 be the sum-capacity of a BFN without feedback ($n_f = 0$), and let C_{n_f} be the sum-capacity with feedback ($n_f \neq 0$), which is achieved by feeding back $r_f \le n_f$ bits per channel use through the feedback channel. Let η be defined as the ratio

$$\eta = \frac{C_{n_f} - C_0}{r_f}. \tag{5.114}$$

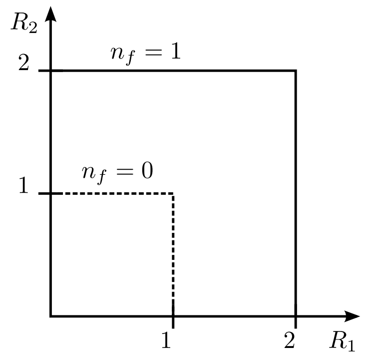


Figure 5.12: The capacity region of the deterministic BFN with $(n_c, n_s, n_r) = (6, 3, 1)$ with $(n_f = 1)$ and without $(n_f = 0)$ feedback.

We say that we have a net-gain if the ratio of the sum-capacity increase to the number of feedback bits is larger than 1, i.e., $\eta > 1$. Otherwise, if $\eta \leq 1$, then we have no net-gain because then $C_{n_f} - C_0 \leq r_f$, i.e., the gain is less than the cost.

Note that if $n_c \leq n_r$, then there is no feedback gain at all, since in this case, the capacity region in Theorem 5.1 is the same as $n_f = 0$. Now, consider for sake of example the case $n_c > n_f$ with a LD-BFN with $(n_c, n_s, n_r) = (6, 3, 1)$. The capacity region of this BFN without feedback is shown in Figure 5.12 where the sum-capacity is $C_0 = 2$ bits per channel use corresponding to the rate pair $(R_1, R_2) = (1, 1)$. This rate pair is achieved by using the CF strategy, where Tx 1 sends $\mathbf{x}_1(i) = [\mathbf{u}_1(i),\ \mathbf{0}_5^T]^T$ and Tx 2 sends $\mathbf{x}_2(i) = [\mathbf{u}_2(i),\ \mathbf{0}_5^T]^T$, and the relay sends $\mathbf{x}_r(i) = [\mathbf{u}_1(i-1) \oplus \mathbf{u}_2(i-1),\ \mathbf{0}_5]^T$.

Now consider the case with $n_f = 1$. In this case, the sum-capacity is $C_1 = 4$ bits per channel use corresponding to the corner point of the capacity region $(R_1, R_2) = (2, 2)$ as shown in Figure 5.12. To achieve this, the transmitters use the same CF strategy used for $n_f = 0$, which achieves $R_1 = R_2 = 1$ bit per channel use. Additionally each transmitter sends a feedback bit $\bar{\mathbf{s}}_j(i)$ to the other transmitter via the relay using the symmetric F strategy. This way, each transmitter acquires the F signal of the other transmitter, which it forwards then to the respective receiver. This F strategy requires feeding back only $r_f = 1$ bit, namely $\bar{\mathbf{s}}_1(i) \oplus \bar{\mathbf{s}}_2(i)$. With this we have $\eta = {}^{(C_1 - C_0)}/_{r_f} = 2$, i.e., a net-gain: *for each feedback bit, we gain 2 bits in the sum-capacity.* The reason for achieving this net gain is the design of the F strategy in which the F signals are network coded, thereby making each fed back bit useful at both transmitter. In other words, each fed back bit is worth two bits, one at each transmitter.

The analysis in this section lead to the development of a new scheme, namely, the cooperative interference neutralization scheme. Next, we extend this idea to the Gaussian BFN (G-BFN) in order to see how this scheme can be applied in the G-BFN, and more generally, in the G-IRC.

5.3 The Gaussian BFN (G-BFN) with Feedback

Figure 5.13 shows a real-valued symmetric G-BFN with feedback. This network is simply a G-IRC (cf. Section 4.1.1 on page 20) with the direct channel set to zero ($h_d = 0$) with an additional component which is the *out-of-band* relay-source feedback channels. We will not repeat the description of the model, but just describe the feedback channel. The feedback signal x_f^n is constructed from the relay received signal y_r^n. Due to causality, the ith symbol $x_f(i)$ (just like $x_r(i)$) is constructed from $(y_r(1), \cdots, y_r(i-1))$. The

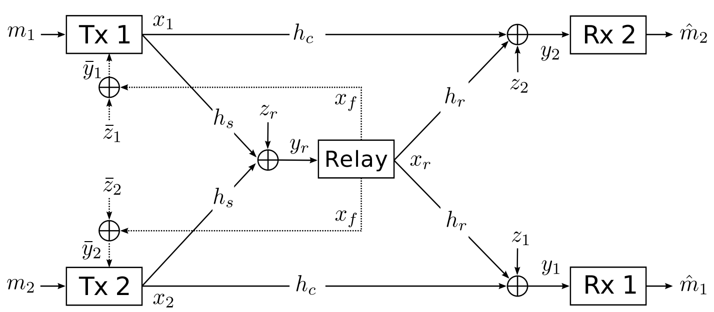


Figure 5.13: The Gaussian butterfly network with relay-source feedback (dotted).

feedback signal has a power constraint P. The received feedback at the transmitters at time instant i can be written as

$$\bar{y}_j(i) = h_f x_f(i) + \bar{z}_j(i), \quad j = 1, 2, \tag{5.115}$$

where h_f is the real valued feedback channel gains, and the noises $\bar{z}_j$ are i.i.d. $\mathcal{N}(0, 1)$.

5.3.1 Transmission Strategies

We describe here the extension of the transmission strategies we described in Section 5.2.2 to the G-BFN. The extension requires some knowledge of nested-lattice codes. For the required preliminaries on nested lattice codes, the reader is referred to Section 2.5, and is referred to [EZ04, NG11, NWS07] for more details.

5.3.1.1 Cooperative Interference Neutralization

The following strategy is a forwarding strategy which does not exploit feedback. In cooperative interference neutralization (CN), the relay sends a signal x_r^n in a way such that the interference at both receivers is removed and 'replaced' with the desired signal. For instance, if Tx 1 sends $t_1^n(b)$ in transmission block b, then the received signal at Rx 2 is $y_2^n = h_c t_1^n(b) + z_2^n(b)$ which contains no desired information. Now if the relay sends $-\frac{h_c}{h_r}(t_1^n(b) + t_2^n(b))$, then y_2^n becomes $y_2^n = -h_c t_2^n(b) + z_2^n(b)$. Notice that the relay replaced the interfering signal $t_1^n(b)$ by the desired signal $t_2^n(b)$.

This process requires giving the relay access to $t_1^n(b)+t_2^n(b)$ before the block at which they are decoded at the receivers. If the relay acquires $t_1^n(b)$ and $t_2^n(b)$ in block $b-1$, then interference neutralization can be performed. To enable this, each transmitter sends $t_j^n(b)$ in addition to $t_j^n(b-1)$ in block $b-1$. This enables the relay to obtain the sum $t_1^n(b) + t_2^n(b)$ in block $b-1$ by decoding both $t_1^n(b)$ and $t_2^n(b)$. However, decoding $t_1^n(b)$ and $t_2^n(b)$ in block $b-1$ leads to several limitations:

(i) Decoding both users' signals at the relay induces a rate constraint of the type

$$R_\Sigma \leq C(2h_s^2 P) = (1/2)\log(P) + o(\log(P)), \tag{5.116}$$

which can be a decisive rate constraint and a bottleneck for the performance.

(ii) If the relay wants to send both users' signals, such as

$$x_r^n(b) = -(h_c/h_r)(t_1^n(b) + t_2^n(b)), \tag{5.117}$$

then the relay power has to be shared between the two signals, e.g., each signal gets a power $P/2$, which leads to rate degradation.

Applying the CN strategy while using nested-lattice codes solves both problems at once. Let the signal $t_j^n(b)$, $j \in \{1,2\}$, be designed using a nested-lattice code with rate R_t as $t_j^n(b) = (\lambda_j(b) - d_j(b)) \bmod \Lambda^c$, where λ_j is a codeword of the nested lattice code with fine lattice Λ^f and coarse lattice Λ^c, and d_j is a random dither. The power of this nested-lattice code is chosen to be P. In blocks $b = 0, \cdots, B$, Tx j sends

$$x_j^n(b) = \alpha_1 t_j^n(b) + \alpha_2 t_j^n(b+1) \tag{5.118}$$

where $t_j^n(0)$ and $t_j^n(B+1)$ are zero. The power constraint is satisfied at Tx 1 and 2 if $\alpha_1^2 + \alpha_2^2 \leq 1$. Note that since the channel is symmetric, the same parameters α_1 and α_2 will be used by both users.

In block b, the relay receives

$$y_r^n(b) = \sum_{j=1}^{2} h_s(\alpha_1 t_j^n(b) + \alpha_2 t_j^n(b+1)) + z_r^n(b). \tag{5.119}$$

Assume that the relay knows $t_1^n(b) + t_2^n(b)$ in block b, then the relay subtracts the contribution of this sum from $y_r^n(b)$, and then decodes the sum $(\lambda_1(b+1) + \lambda_2(b+1)) \bmod \Lambda^c$ [NG11]. This decoding is possible if R_t satisfies

$$R_t \leq C^+\left(h_s^2\alpha_2^2 P - \frac{1}{2}\right). \tag{5.120}$$

Note that this rate constraint is more relaxed than (5.116), which solves problem (i) above. In the next transmission block, the relay reconstructs $h_s(t_1^n(b+1) + t_2^n(b+1))$ from $(\lambda_1(b+1)+\lambda_2(b+1)) \bmod \Lambda^c$ (successive compute-forward [Naz12]) and subtracts it from $y_r^n(b+1)$. Then it decodes $(\lambda_1(b+2) + \lambda_2(b+2)) \bmod \Lambda^c$, and so on.

The decoded $\bmod \Lambda^c$ sum has power P as the original nested-lattice code. The relay simply scales this $\bmod \Lambda^c$ sum and sends it, which requires less power than (5.117). This solves problem (ii) above. The relay sends

$$x_r^n(b+1) = -\frac{h_c\alpha_1}{h_r}\left[(\lambda_1(b+1) + \lambda_2(b+1)) \bmod \Lambda^c\right] \tag{5.121}$$

in block $b+1$ (here, $x_f^n = 0$). The rate constraint at the relay is satisfied if $h_c^2\alpha_1^2/h_r^2 \leq 1$. Note that the relay does not re-encode the decoded signal, it scales it and forwards it for the purpose of interference neutralization.

Rx 1 receives

$$y_1^n(b) = h_c\alpha_1 t_2^n(b) - h_c\alpha_1\left[(\lambda_1(b) + \lambda_2(b)) \bmod \Lambda^c\right] + h_c\alpha_1\tilde{z}_1^n(b), \tag{5.122}$$

where we used the notation

$$\tilde{z}_1^n = (1/h_c\alpha_1)(h_c\alpha_2 t_2^n(b+1) + z_1^n), \tag{5.123}$$

for brevity. This term will be treated as noise in the decoding process. First, Rx 1 divides $y_1^n(b)$ by $h_c\alpha_1$, then it adds $d_2(b)$, and then it calculates the quantization error with respect to Λ^c by using a $\bmod \Lambda^c$ operation [NG11]. After these operations, Rx 1 obtains

$$\begin{aligned}
&\left[\frac{y_1^n(b)}{\alpha_1 h_c} + d_2(b)\right] \bmod \Lambda^c \\
&\qquad = \left[t_2^n(b) - (\lambda_1(b) + \lambda_2(b)) \bmod \Lambda^c + \tilde{z}_1^n(b) + d_2(b)\right] \bmod \Lambda^c && (5.124)\\
&\qquad = \left[\lambda_2(b) - d_2(b) - \lambda_1(b) - \lambda_2(b) + \tilde{z}_1^n(b) + d_2(b)\right] \bmod \Lambda^c && (5.125)\\
&\qquad = \left[-\lambda_1(b) + \tilde{z}_1^n(b)\right] \bmod \Lambda^c, && (5.126)
\end{aligned}$$

where we used the property $(x \bmod \Lambda^c + y) \bmod \Lambda^c = (x + y) \bmod \Lambda^c$ [NG11]. This allows Rx 1 to decode $\lambda_1(b)$ as long as

$$R_t \leq C\left(\frac{\alpha_1^2 h_c^2 P}{1 + \alpha_2^2 h_c^2 P}\right). \tag{5.127}$$

This rate constraint follows since nested-lattice codes achieve the capacity of the AWGN channel [EZ04]. A similar procedure is used by Rx 2.

In summary, in the cooperative interference neutralization strategy, the relay decodes a sum of codewords, which relaxes the rate constraint at the relay compared to classical decoded-forward. Furthermore, it neutralizes interference and allows the receivers to decode the desired signals interference free.

Remark 5.3. *The half-duplex* CN *strategy used in the half-duplex LD-BFN in [AH09] was extended in the same paper to the Gaussian case using amplify-forward. We extend our full-duplex* CN *strategy to the Gaussian case using computation which gives more flexibility especially when* CN *is combined with other strategies, and therefore, performs better than amplify-forward.*

5.3.1.2 Feedback

As mentioned earlier, the feedback strategy exploits the bi-directional relay channel established by the channels h_s and h_f to exchange signals between the two transmitters. The feedback process is exactly the same process used for exchanging information in the bi-directional relay channel [NWS07]. We only briefly describe it here.

The users use a nested-lattice code of power P and rate R_s to construct the feedback signals (F signals) as $s_j^n(b) = (\lambda_j(b) - d_j(b)) \bmod \Lambda^c$ where $j \in \{1, 2\}$, b is the block index denoting the bth transmission block, λ_j is the nested-lattice codeword, and d_j is a random dither. Tx 1 sends

$$x_1^n(b) = \alpha_1 s_1^n(b) + \alpha_2 s_2^n(b-2) \tag{5.128}$$

with $\alpha_1^2 + \alpha_2^2 \leq 1$, i.e., it sends its own F signal and the F signal of Tx 2. The latter is acquired via feedback as follows. The received signal at the relay in block b is

$$y_r^n(b) = \sum_{j=1}^{2} h_s(\alpha_1 s_j^n(b) + \alpha_2 s_j^n(b-2)) + z_r^n(b). \tag{5.129}$$

The relay first removes the interference from $s_1^n(b-2) + s_2^n(b-2)$ (assuming that decoding $(\lambda_1(b-2) + \lambda_2(b-2)) \bmod \Lambda^c$ was successful in block $b-2$ [Naz12]), and then it decodes the sum $(\lambda_1(b) + \lambda_2(b)) \bmod \Lambda^c$. The relay then maps the decoded sum to a Gaussian codeword $x_f^n(b+1)$ with average power P and feeds it back to the transmitters in block $b+1$ (here we use $x_r^n = 0$). Tx 1 gets

$$\bar{y}_1^n(b+1) = h_f x_f^n(b+1) + \bar{z}_1^n(b+1), \tag{5.130}$$

from which it decodes $x_f^n(b+1)$ and acquires $s_2^n(b)$. The acquired F signal of Tx 2 ($s_2^n(b)$) is sent to Rx 2 in block $b+2$ after scaling it with α_2 (see (5.128)). The received signal at Rx 2 is then

$$y_2^n(b) = h_c(\alpha_1 s_1^n(b) + \alpha_2 s_2^n(b-2)) + z_2^n(b), \tag{5.131}$$

from which $s_2^n(b-2)$ is decoded while treating $s_1^n(b)$ as noise. This allows Tx 2 to send its F signal through both the relay and Tx 1. The rate constraints for reliable decoding are then

$$R_s \leq \min\left\{C^+\left(h_s^2\alpha_1^2P-\frac{1}{2}\right), C\left(h_f^2P\right), C\left(\frac{h_c^2\alpha_2^2P}{1+h_c^2\alpha_1^2P}\right)\right\}. \tag{5.132}$$

The first constraint corresponds to decoding at the relay, the second to decoding x_f^n at the transmitters, and the last to decoding $s_j^n(b-2)$ at the receivers.

5.3.1.3 Compute-forward

As the BFN is a special case of the IRC, the compute-forward (CF) strategy of the IRC (namely the SI-CF strategy) can be used in the BFN in a straightforward way. For this reason, we do not describe this strategy here in details, the reader is referred to Section 4.3.2.1 for more on this strategy. The resulting rate constraints when using CF in the BFN are

$$R_u \leq \min\left\{C^+\left(h_s^2\alpha_1^2P-\frac{1}{2}\right), C\left(\frac{h_c^2\alpha_1^2P}{1+h_r^2\alpha_2^2P}\right), C\left(h_r^2\alpha_2^2P\right)\right\}, \tag{5.133}$$

where $\alpha_1, \alpha_2 \leq 1$. The first constraint corresponds to decoding at the relay, and the second and third ones corresponds to decoding at the receivers (successively).

Next, we construct combinations of these three schemes and examine their performance in the G-BFN in terms of sum-rate.

5.3.2 Achievable Rates in the G-BFN with Feedback

We construct two combinations of the three strategies. Namely, one combination of CN and F, and another combination of CF and F. Intuitively, we expect the first combination to perform better when $h_s^2 > h_c^2$ as explained earlier in section 5.2.2.1.

5.3.2.1 Neutralization and Feedback

In this case, Tx $j \in \{1,2\}$ sends

$$x_j^n(b) = \alpha_1 t_j^n(b) + \alpha_2 t_j^n(b+1) + \alpha_3 s_j^n(b) + \alpha_4 s_k^n(b-2), \tag{5.134}$$

where $\sum_{\ell=1}^4 \alpha_\ell^2 \leq 1$, and $j \neq k$. The relay removes the contribution of $t_j^n(b)$ and $s_k^n(b-2)$ first, then it decodes the sum of the F signals (one from each transmitter) followed by the sum of the CN signals leading to the rate constraints

$$R_s \leq C^+\left(\frac{h_s^2\alpha_3^2P}{1+2h_s^2\alpha_2^2P}-\frac{1}{2}\right) \tag{5.135}$$

$$R_t \leq C^+\left(h_s^2\alpha_2^2P-\frac{1}{2}\right). \tag{5.136}$$

The relay feeds back the sum of the F signals to the transmitters. This can be decoded successfully as long as

$$R_s \leq C\left(h_f^2P\right). \tag{5.137}$$

The relay also forwards the sum of the CN signals, scaled with $-{}^{h_c\alpha_1}/_{h_r}$, which requires ${}^{h_c^2\alpha_1^2}/_{h_r^2} \leq 1$. The decoding at the receivers starts with the desired F signal, then the desired CN signal, while treating the remaining signals as noise, leading to the constraints

$$R_s \leq C\left(\frac{h_c^2\alpha_4^2 P}{1+h_c^2(2\alpha_1^2+\alpha_2^2+\alpha_3^2)P}\right) \tag{5.138}$$

$$R_t \leq C\left(\frac{h_c^2\alpha_1^2 P}{1+h_c^2(\alpha_2^2+\alpha_3^2)P}\right). \tag{5.139}$$

For a given choice of α_ℓ, $\ell \in \{1,\dots,4\}$, each user achieves a rate of $R_t + R_s$ (satisfying (5.135)-(5.139)) using this combination of the CN strategy and the F strategy.

Proposition 5.1 (CN+F). *The combination of the CN strategy and the F strategy (CN+F) achieves a sum-rate given by*

$$R_\Sigma = 2 \max_{\substack{\sum_{\ell=1}^4 \alpha_\ell^2 \leq 1,\\ h_c^2\alpha_1^2 \leq h_r^2}} R_t + R_s. \tag{5.140}$$

5.3.2.2 Compute-forward and Feedback

In this case, the users employ the CF and the F strategies. Tx $j \in \{1,2\}$ sends

$$x_j^n(b) = \alpha_1 u_j^n(b) + \alpha_2 s_j^n(b) + \alpha_3 s_k^n(b-2), \tag{5.141}$$

where $\sum_{\ell=1}^3 \alpha_\ell^2 \leq 1$, and $j \neq k$. The relay removes the contribution of $s_k^n(b-2)$ first, and then it decodes the sum of the CF signals followed by the sum of the F signals leading to the rate constraints

$$R_u \leq C^+\left(\frac{h_s^2\alpha_1^2 P}{1+2h_s^2\alpha_2^2 P} - \frac{1}{2}\right) \tag{5.142}$$

$$R_s \leq C^+\left(h_s^2\alpha_2^2 P - \frac{1}{2}\right). \tag{5.143}$$

The relay then feeds back the sum of the F signals to the transmitters, which can decode it if

$$R_s \leq C\left(h_f^2 P\right). \tag{5.144}$$

The relay also sends the sum of the CF signals encoded in a signal $x_r^n(b+1)$ with power $\alpha_4^2 P \leq P$. Rx 1 receives

$$y_1^n(b) = h_c(u_2^n(b) + s_2^n(b) + s_1^n(b-2)) + h_r x_r^n(b) + z_1^n(b). \tag{5.145}$$

Rx 1 decodes the interfering CF signal $u_2^n(b)$ first, followed by the interfering F signal $s_2^n(b)$, then the desired F signal $s_1^n(b-2)$, and finally, the relay signal $x_r^n(b)$, successively. This leads to the following rate constraints

$$R_u \leq C\left(\frac{h_c^2\alpha_1^2 P}{1+h_c^2(\alpha_2^2+\alpha_3^2)P + h_r^2\alpha_4^2 P}\right) \tag{5.146}$$

$$R_s \leq C\left(\frac{h_c^2\alpha_2^2 P}{1+h_c^2\alpha_3^2 P + h_r^2\alpha_4^2 P}\right) \tag{5.147}$$

$$R_s \leq C\left(\frac{h_c^2\alpha_3^2 P}{1+h_r^2\alpha_4^2 P}\right) \tag{5.148}$$

$$R_u \leq C\left(h_r^2\alpha_4^2 P\right). \tag{5.149}$$

Each transmitter achieves a rate of $R_u + R_s$ where R_u and R_s satisfy (5.142)-(5.149), for a given choice of α_ℓ, $\ell \in \{1, \dots, 4\}$.

Proposition 5.2 (CF+F). *The combination of the CF strategy and the F strategy (CF+F) achieves a sum-rate given by*

$$R_\Sigma = 2 \max_{\substack{\sum_{\ell=1}^{3} \alpha_\ell^2 \leq 1, \\ \alpha_4^2 \leq 1}} R_u + R_s. \tag{5.150}$$

Having derived the achievable sum-rate of the two different combinations (CN+F and CF+F), we can now compare their performance versus each other, and versus the sum-capacity upper bounds from Section 5.1.2 which can be easily evaluated for the Gaussian BFN using the 'Gaussian maximizes entropy' principle.

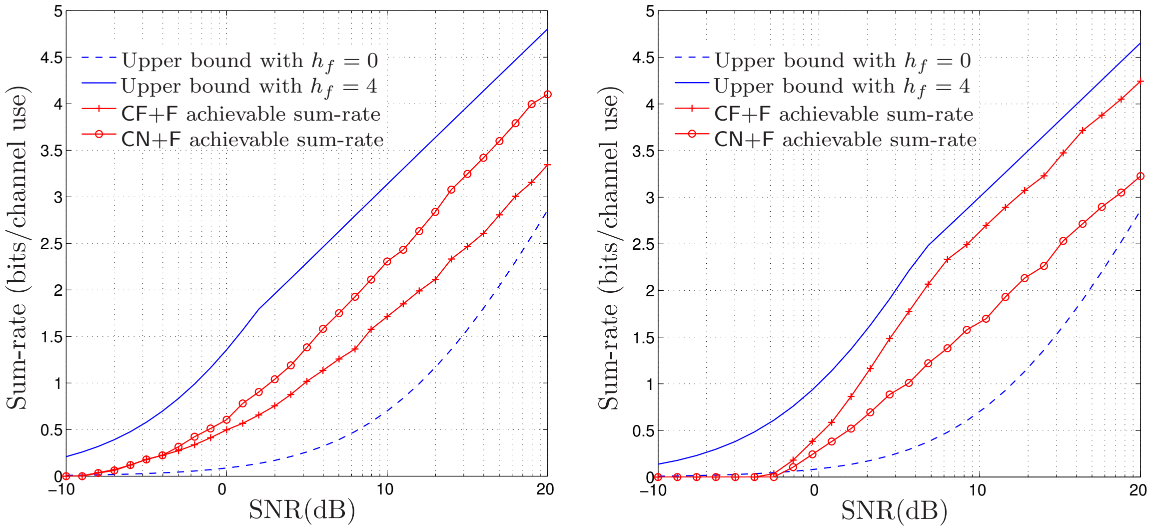
5.3.3 Numerical Analysis and Discussion

Figure 5.14(a) shows the achievable sum-rate for a G-BFN with $(h_s, h_c, h_r) = (2, 1, 1/4)$ with $h_f = 4$ as a function of SNR ($= P$). Additionally, the figure shows sum-capacity upper bounds as a benchmark for comparison. The upper bound with $h_f = 0$ is included to show the feedback gain. Without feedback, the achievable rate is at most equal to the dashed upper bound. The gain from feedback is obvious. This gain is a result of the alternative paths for information flow created by feedback [SGG12]. For instance, the path Tx 1→Relay→Tx 2→Rx 1 for information flow from Tx 1 to Rx 1 is enabled by feedback. Although both schemes achieve higher rates than the no-feedback upper bound, CN+F performs better than CF+F since $h_s^2 \geq h_c^2$ (see our discussion in Section 5.2.2.1). Note that in CF+F, the receivers need to decode four signals as in (5.146)-(5.149). This requires a strong cross channel h_c which is not the case in this example. On the other hand, if the channel to the relay h_s is stronger than the cross channel h_c, then the transmitters can send signals weaker than the noise at the receivers, while still decodable at the relay. This feature can be exploited by the CN strategy for sending the future CN signals ($t_j^n(b+1)$ in block b). Thus, in this example it is more convenient to use the CN strategy than the CF strategy.

Figure 5.14(b) shows an opposite case, where h_c is the stronger channel. In this case, sending CN signals to the relay for the purpose of interference neutralization will cause strong interference at the receivers, so the CN strategy should be avoided. However, since the cross channel is strong, by using the CF strategy, the receiver can decode the F signals and the CF signals coming from the relay and the transmitters at a higher rate. This makes the CF strategy preferable in this case.

5.3.3.1 Net Feedback Gain

The form of feedback presented in this chapter provides net-gain. That is, the gain in sum-rate per feedback bit is larger than 1. To view this net gain, we define G as $G = C_{\mathsf{C_f}} - C_0$ where $C_{\mathsf{C_f}}$ is the sum-capacity of the G-BFN with a feedback channel of capacity $\mathsf{C_f} = C(h_f^2 P)$, and C_0 is the sum-capacity of the same network with no feedback, i.e., with $\mathsf{C_f} = 0$ or equivalently $h_f = 0$. Let the sum-capacity upper bound for the no-feedback case (obtained from Section 5.1.2) be denoted $\overline{C}_0$, i.e., $C_0 \leq \overline{C}_0$.



(a) $(h_s, h_c, h_r) = (2, 1, 1/4)$. (b) $(h_s, h_c, h_r) = (1, 2, 1/4)$.

Figure 5.14: Achievable sum-rates and sum-capacity upper bounds for the G-BFN in two examples. An upper bound for the G-BFN without feedback ($h_f = 0$) is also shown for comparison.

Then, the gain G can be bounded as follows

$$G = C_{\mathsf{C_f}} - C_0 \tag{5.151}$$
$$\geq C_{\mathsf{C_f}} - \overline{C}_0 \tag{5.152}$$
$$\geq R_\Sigma - \overline{C}_0 \tag{5.153}$$

where R_Σ is the sum-rate achieved by our schemes in Propositions 5.1 and 5.2 (the maximum of the two). This gain G is plotted in Figure 5.15 as a function of $\mathsf{C_f}$.

Starting from the point (0,0), the gain G increases linearly with a slope of 2. This means that we gain 2 bits in sum-capacity per feedback bit. Since the gain is greater than the cost, we say that we have *net-gain*. At some value of $\mathsf{C_f}$ (here at 0.6), the gain saturates to a value determined by the channel parameters h_s, h_c, and h_r. In conclusion, the sum-capacity gain is double the capacity of the feedback channel, as long as the network performance is not dominated by its forward coefficients.

5.4 Summary

In this chapter, we have characterized the capacity region of the linear-deterministic BFN with relay-source feedback. For this purpose, we developed a novel relaying strategy named cooperative interference neutralization. This strategy turns out to be essential for achieving the capacity of this network. We developed this new strategy for the BFN, however, it can be applied in the IRC in a straightforward manner.

By allowing relaying information backward to the sources (relay-source feedback), the capacity of the BFN can be increased. Our proposed feedback scheme which is based on bi-directional relaying is an efficient form of feedback since it provides a net gain. Namely, the increase in the sum-capacity of the network is twice the number of feedback bits as long as the forward channels are not the bottleneck of the channel capacity.

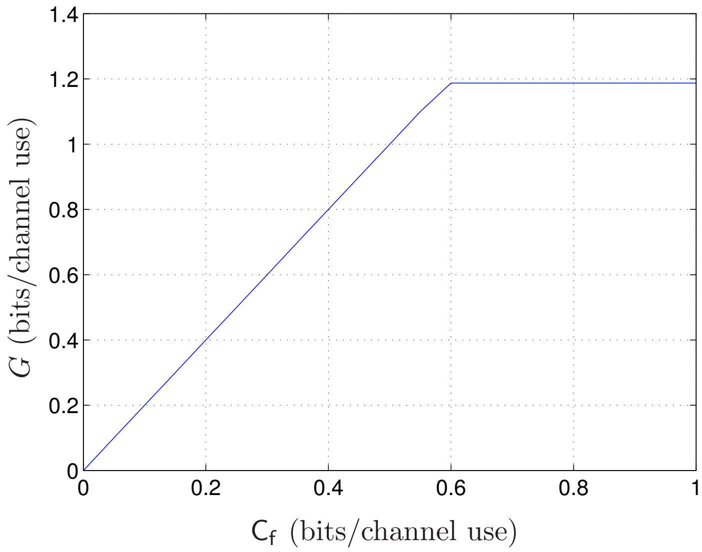


Figure 5.15: The feedback gain versus the capacity of the feedback channel C_f for a G-BFN with $(h_s, h_c, h_r) = (1, 2, 1/4)$

We also extended the proposed strategies to the Gaussian BFN using nested-lattice codes. We compared the performance of these strategies with sum-capacity upper bounds numerically, where their good performance can be clearly noticed.

Chapter 6

Conclusion

In this thesis, we have studied the capacity of the interference relay channel (IRC). We used the linear-deterministic (LD) approximation of the IRC as a stepping stone towards the more involved Gaussian IRC. For the LD-IRC, we have developed a compute-forward (CF) strategy which characterizes the sum-capacity for all cases where the source-relay channel is weaker than the cross channel. Then we extended this result to the Gaussian IRC using nested-lattice codes, for which we characterized the GDoF under the same condition above. By characterizing the GDoF, we have shown that CF achieves the high SNR asymptotic approximation of the sum-capacity of the G-IRC, and therefore, CF is optimal from this point-of-view. This characterization covers half the space of all possible channel parameters. It turns out that while the relay does not increase the degrees-of-freedom of the IRC [CJ09], it indeed increases its GDoF.

We have also studied the BFN, which is an IRC with no direct channels, and thus, with more emphasis on the relaying component of the network. We characterized the capacity region of the LD-BFN where a new strategy named cooperative interference neutralization (CN) was required. We also extended the developed strategies for the Gaussian BFN.

These results solve the main problem of the thesis given in Chapter 3, i.e., identifying optimal relaying strategies. The optimality of the proposed strategies is shown for a wide range of channel parameters. The obtained results are important from both a theoretical and a practical point of view. From a theoretical point of view, identifying optimal relaying strategies for the IRC might help us to approach the capacity of this network, and possibly larger interference networks with relays. From a practical point of view, finding optimal relaying strategies and characterizing the capacity of interference networks with relays can be very beneficial. For instance, given a wireless interference network which suffers from coverage problems, bad channel quality, and high interference, where some specified target rates must be achieved, knowing the capacity of the network allows us to judge whether the given network limitations can be overcome by relays (otherwise one would look for another cooperation strategy). If this turns out to be the case, then the next question would be how to adjust relay parameters in order to achieve the given target rates. These parameters include the relay power and the relay location for instance. If these parameters are properly set, then installing relays does not only overcome the network limitations, but also allows achieving higher target while still ensuring reliable communication.

By analyzing our strategies, it can be noted that the task of the relay is focused on interference cancellation, indirectly in the CF strategy, and directly in the CN strategy. This is in contrast to classical strategies where the relay tries to either increase the power of the desired signals or the power of interference in an attempt to increase the achievable rates. Moreover, the relay in our strategies takes care of specific signals that are constructed especially for the task of relaying. It does not repeat source transmissions, nor does it decode the source messages. This is also in contrast to classical strategies where the relay operates on all the signals transmitted by the transmitters which do not have any special construction designed for relaying purposes.

This study also shows the capabilities of structured codes versus random codes for the purpose of relaying. In our characterization, we were successful in finding the GDoF if the IRC by using structured (lattice) codes, earlier work focused on random codes and did not reach such a characterization.

On the other hand, the following similarity can be noted between our strategies and classical ones. In CF, signals from successive transmission blocks are independent, and decoding is done by considering two blocks at a time. From this perspective, this is similar to compress-forward (only from this aspect). On the other hand, in CN, signals from each two successive transmission blocks are dependent (block Markov encoding), and decoding is done by considering only one block at a time. Thus, CN shares this property with decode-forward. Due to the delay at the relay, this structure has to be used in general in relay networks.

6.1 Summary of Contributions

The main problem whose solution was sought in this thesis is optimal relaying strategies for the IRC. For achieving this goal, we have started by studying the LD-IRC where:

1. We derived new sum-capacity upper bounds. While we used genie-aided approaches and cooperative approaches to develop the new upper bounds, some of the new upper bounds used new ideas that have not been used earlier.

2. We developed a new transmission scheme which recovers the sum-capacity of the IC, and uses computation at the relay to improve upon this capacity.

3. We have shown that our strategy is sum-capacity achieving for a wide range of channel parameters.

This establishes the optimality of our scheme for the given parameter regime, thus giving a solution for the main problem.

We then extended the results to the G-IRC leading to an approximation of its sum-capacity in the form of a generalized degrees-of-freedom characterization.

1. We first extended the upper bounds to the Gaussian IRC.

2. Then, we translated our transmission scheme to the Gaussian IRC where nested-lattice codes. We also expressed the achievable sum-rate of the strategy.

3. Finally, we showed that our proposed novel strategy achieves the GDoF of the Gaussian IRC for a wide range of channel parameters.

As a result, we have shown that our proposed strategy is GDoF optimal.

As for the BFN, we have studied the capacity region of its linear-deterministic approximation.

1. We used the upper bounds for the IRC in the BFN, but we also developed some new bounds that are necessary for the capacity characterization.

2. Then, we developed transmission strategies for this network. To approach the capacity region, we developed the new CN strategy, which focuses on neutralizing interference at the receivers. The achievable rate region of a combination of different strategies is given.

3. Finally, we showed that our proposed strategies achieves the capacity region of the LD-BFN.

4. We extended the proposed strategies to the Gaussian BFN, and we have discussed their performance using numerical examples.

6.2 Outlook

This thesis covered half the space of the channel parameters of the IRC, namely, all cases where the channel to the relay is weaker than the cross channel. For this regime, CF has been shown to be optimal. The other half-space is left for future work. Notice that the performance of the CN strategy developed for the BFN has not been studied in the IRC. Thus, it is interesting to know whether this strategy (in conjunction with other strategies such as CF) achieves the capacity of the LD-IRC or the GDoF of the G-IRC in the remaining regime, and thus completes the GDoF characterization of the IRC. Otherwise, one might try other relaying strategies such as noisy network coding [EGK11] for instance. It is worth to note that the study of the capacity of the IRC is still an ongoing process [ZD13, ZD12, Dab12].

In this study, we have focused on symmetric channels for simplicity of exposition. Results are usually easier to express in a compact form for symmetric channels than for asymmetric ones. Nevertheless, the developed transmission strategies can be extended to asymmetric channels as well with some modification. In both the CN and CF strategies, the relay decodes the sum of the transmit signals in the symmetric case. Decoding the sum is suitable due to the symmetry of the channel. In the asymmetric case however, the relay has to decode a weighted sum of the transmitted signals, where the weights are dictated by the channel. This is possible by using computation [NG11] which is not limited to decoding sums of codewords, but also weighted sums. The relay can choose the most suitable linear combination of transmitted signals and compute it. The optimality of the developed strategies in asymmetric channels is open.

The next step which follows naturally after GDoF characterization is tightening the results to obtain a capacity characterization. This progressive approach from GDoF characterization to capacity characterization has has been used for instance in the IC where the results of [ETW08] (GDoF results) have been used as stepping stones to obtain a capacity characterization for the noisy IC in [SKC09, MK09, AV09]. Our results on the GDoF of the IRC can be used similarly as stepping stones towards characterizing the capacity of the network, which is the ultimate goal in information theory.

Recall that the elemental network considered in this thesis (the IRC) has 2 users, whereas practical interference networks are likely to have more users. Although it is natural to start with a small model that has all the necessary ingredients (interference, relaying), it is beneficial to extend the results of this model to larger networks. For instance, in practice one might encounter interference relay networks with more users, relays, antennas, or a combination thereof. Studying the optimality of our strategies (in addition to other new or old strategies) in such networks is important from both theoretical and practical view points.

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