Binary and Semi-Fibonacci Partitions

by
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Dedicated to my friend, Ashok Agarwal, on the occasion of his 70th birthday

[1]

Abstract

It is proved that the partitions of n into powers of two with all parts appearing an odd number of times equals the number of Semi-Fibonacci partition of n. The parity of the number of such partitions is also exhibited.

1 Introduction

The set $\mathfrak{SF}(n)$ of semi-Fibonacci partitions is defined as follows: $\mathfrak{SF}(1) = \{1\}$, $\mathfrak{SF}(2) = \{2\}$. If $n > 2$ and even the $\mathfrak{SF}(n)$ consists of the partitions of $\mathfrak{SF}(\frac{n}{2})$ wherein each part has been multiplied by 2. If n is odd, $\mathfrak{SF}(n)$ arises from two sources: first a 1 is inserted in each partition of $n-1$ and second a 2 is added to the single odd part of $\mathfrak{S}(n-2)$ (note: it is easily seen by induction that semi-Fibonacci partitions have at most one odd part).

Thus here are the first seven $\mathfrak{SF}(n)$:

$$\mathfrak{SF}(1) = \{1\}$$
$$\mathfrak{SF}(2) = \{2\}$$
$$\mathfrak{SF}(3) = \{2+1, 3\}$$
$$\mathfrak{SF}(4) = \{4\}$$
$$\mathfrak{SF}(5) = \{4+1, 3+2, 5\}$$
$$\mathfrak{SF}(6) = \{4+2, 6\}$$
$$\mathfrak{SF}(7) = \{4+2+1, 6+1, 4+3, 5+2, 7\}$$

We now define

$$\mathrm{sf}(n) = |\mathfrak{SF}(n)|.$$

Thus $\mathrm{sf}(1) = \mathrm{sf}(2) = 1$, $\mathrm{sf}(3) = 2$, $\mathrm{sf}(4) = 1$, $\mathrm{sf}(5) = 3$, $\mathrm{sf}(6) = 2$, $\mathrm{sf}(7) = 5$.

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From the definition of $\mathfrak{S}\mathfrak{F}(n)$, we see that $\mathrm{sf}(-1) = 0$, $\mathrm{sf}(0) = 1$, and for $n > 0$

$$\mathrm{sf}(n) = \begin{cases} \mathrm{sf}(\frac{n}{2}) & \text{if } n \text{ is even} \\ \mathrm{sf}(n-1) + \mathrm{sf}(n-2) & \text{if } n \text{ is odd} \end{cases} \tag{1.1}$$

It appears that Jonathan Vos Post is the first one to consider the semi-Fibonacci sequence $\mathrm{sf}(n)$ [2]. Numerous properties are listed in [2] including a function equation for $g(x) = F(x) - 1$; however, both our theorems are not there.

George Beck [1] appears to be the first to consider semi-Fibonacci partitions, and in [1], he proves a nice theorem for a set of related polynomials. I thank George Beck for drawing my attention to semi-Fibonacci partitions.

Binary partitions are partitions into powers of 2. We let $\mathrm{ob}(n)$ denote the number of binary partitions of n in which each part appears an odd number of times. Thus $\mathrm{ob}(7) = 5$ because the relevant partitions are $4+2+1$, $4+1+1+1$, $2+2+2+1$, $2+1+1+1+1+1$, and $1+1+1+1+1+1+1$.

Theorem 1. *For each $n \geq 0$*

$$\mathrm{sf}(n) = \mathrm{ob}(n) \tag{1.2}$$

We shall also treat the parity of these sequences.

Theorem 2. *For each $n > 0$, $\mathrm{sf}(n)$ is even if $3|n$ and odd otherwise.*

Section 2 is devoted to the proof of Theorem 1. Section 3 treats Theorem 2, and Section 4 considers open questions.

2 Proof of Theorem 1

We define

$$F(x) = \sum_{n\geq 0} \mathrm{sf}(n)x^n. \tag{2.1}$$

Then

$$\begin{aligned} F(x) &= \sum_{n\geq 0} \mathrm{sf}(2n)x^{2n} + \sum_{n\geq 0} \mathrm{sf}(2n+1)x^{2n+1} \\ &= \sum_{n\geq 0} \mathrm{sf}(n)x^{2n} + \sum_{n\geq 0} (\mathrm{sf}(2n) + \mathrm{sf}(2n-1))x^{2n+1} \\ &= F(x^2)(1+x) + x^2 \sum_{n\geq 0} \mathrm{sf}(2n+1)x^{2n+1} \\ &= F(x^2)(1+x) + \frac{x^2}{2}(F(x) - F(-x)). \end{aligned}$$

Also

$$\frac{1}{2}(F(x)+F(-x)) = \sum_{n\geq 0} \mathrm{sf}(2n)x^{2n} = \sum_{n\geq 0} \mathrm{sf}(n)x^{2n} = F(x^2) \tag{2.2}$$

From (2.2) we deduce that

$$F(-x) = 2F(x^2) - F(x). \tag{2.3}$$

Substituting (2.3) into (2.1), we find

$$F(x) = (1+x)F(x^2) + \frac{x^2}{2}F(x) - x^2\left(F(x^2) - \frac{1}{2}F(x)\right) \tag{2.4}$$

Simplifying we obtain

$$(1-x^2)F(x) = (1+x-x^2)F(x^2), \tag{2.5}$$

or

$$F(x) = \frac{1+x-x^2}{1-x^2}F(x^2), \tag{2.6}$$

and iterating (2.6), we obtain

$$F(x) = \prod_{n=0}^{\infty} \frac{1+x^{2^n}-x^{2^{n+1}}}{1-x^{2^{n+1}}} = \prod_{n=0}^{\infty}\left(1+\frac{x^{2^n}}{1-x^{2^{n+1}}}\right) = \prod_{n=0}^{\infty}\left(1+\sum_{m=0}^{\infty} x^{2^n(2m+1)}\right) = \sum_{m\geq 0} \mathrm{ob}(n)x^n. \tag{2.7}$$

Finally comparing coefficients in (2.1) and (2.7), we obtain Theorem 1. □

3 Proof of Theorem 2

Here we must utilize the fact that every positive integer is uniquely the sum of distinct powers of 2. In terms of generating functions, this is the identity

$$\frac{1}{1-x} = \prod_{n=0}^{\infty}(1+x^{2^n}). \tag{3.1}$$

We now proceed modulo 2 (where $-x \equiv x \pmod 2$). Hence

$$\sum_{n\geq 0} \mathrm{sf}(n)x^n = \prod_{n=0}^{\infty} \frac{(1+x^{2^n}-x^{2^{n-1}})}{1-x^{2^{n+1}}} \tag{3.2}$$

$$\begin{aligned}
&\equiv \prod_{n=0}^{\infty} \frac{1+x^{2^n}+x^{2^{n+1}}}{(1-x^{2^n})^2} \pmod 2 \\
&= \prod_{n=0}^{\infty} \frac{(1-x^{3\cdot 2^n})}{(1-x^{2^n})^3} \\
&\equiv \prod_{n=0}^{\infty} \frac{(1+x^{3\cdot 2^n})}{(1+x^{2^n})^3} \pmod 2 \\
&= \frac{(1-x)^3}{1-x^3} \\
&= 1 + \frac{-3x+3x^2}{1-x^3} \\
&\equiv 1 + \frac{x+x^2}{1-x^3} \pmod 2 \\
&= 1 + \sum_{n=0}^{\infty} (x^{3n+1}+x^{3n+2}),
\end{aligned}$$

and Theorem 2 follows by comparing coefficients in the extremes of (3.2).

4 Conclusion

It would be nice to have combinatorial proofs of both theorems. In light of the recursive nature of the definition of $\mathrm{sf}(n)$, this should be quite tractable. One only need note that the recurrence (1.1) for $\mathrm{ob}(n)$ works as follows. The top line follows by multiplying each part in the partition of $\frac{n}{2}$ by 2. The bottom line arises by inserting a 1 in the partitions of $n-1$ and inserting two ones in the partitions of $n-2$.

References

[1] G. Beck, http://demonstrations.wolfram.com/SemiFibonacciPartitions/

[2] On Line Encyclopedia of Integer Sequences, sequence A030067.

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