Model-Based Fault Detection of Linear Discrete-Time Systems

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Contents

Notation

$\forall$	for all
$\in$	belong to
$\subset$	subset
$\equiv$	identically equal
$\approx$	approximately equal
$\Longrightarrow$	implies
$\Longleftrightarrow$	equivalent to
$>>$ ($<<$)	much greater (less) than
max (min)	maximum (minimum)
sup (inf)	supremum (infimum)
$\square$	end of proof
l_2	set of square summable sequences
l_∞	set of bounded sequences
$\mathbf{R}$	field of real numbers
$\mathbf{R}^n$	space of n-dimensional real vectors
$\mathbf{R}^{n\times m}$	space of n by m real matrices
$\mathbf{RH}_\infty, \mathbf{RH}_\infty^{n\times m}$	space of n by m proper and real rational stable transfer matrices
$\mathbf{RH}_2, \mathbf{RH}_2^{n\times m}$	space of n by m strictly proper and real rational stable transfer matrices
$\mathbf{RL}_\infty, \mathbf{RL}_\infty^{n\times m}$	space of n by m proper and real rational transfer matrices with no poles on the unit circle
X^T	transpose of X
X^{-1}	inverse of X
$X^\perp$	orthogonal complement of X
$rank(X)$	rank of X
$tr\,(X)$	trace of X
$\det(X)$	determinant of X
$\lambda\,(X)$	eigenvalue of X
$\bar{\lambda}\,(X)$ ($\lambda_{\max}$)	the largest eigenvalue of X
$\underline{\lambda}\,(X)$ ($\lambda_{\min}$)	the smallest eigenvalue of X
$\bar{\sigma}\,(X)$	the largest singular value of X
$\underline{\sigma}\,(X)$	the smallest least singular value of X
$\sigma_i\,(X)$	the i-th singular value of X
$\mathrm{Im}\,(X)$	image space of X
Γ^*	adjoint of operator Γ
$\langle\ ,\ \rangle$	inner product
prob($a < b$)	probability that $a < b$

$N(a, \sigma)$	Gaussian distribution with mean a and variance σ
χ^2	chi-square distribution
$\mathbf{E}\,(x)$	mean of x
$var\,(x)$	variance of x
$G(z)$	transfer matrix of a discrete time system
$G^*(e^{j\omega}) = G^T(e^{-j\omega})$	conjugate of $G(e^{j\omega})$
(A, B, C, D)	shorthand for $D + C\,(zI - A)^{-1}\,B$
$rank\,(G(z))$	normal rank of $G(z)$, see [91] for definition

List of Acronyms

CAN	controller area network
CCMS	central control and monitoring system
CIOF	co-inner-outer factorization
CSMA	carrier sense multiple access
DPRS	discrete-time periodic Riccati system
DTARE	discrete-time algebraic Riccati equation
DTARS	discrete-time algebraic Riccati system
FAR	false alarm rate
FD	fault detection
FDD	fault detection and diagnosis
FDF	fault detection filter
FDR	fault detection rate
GLR	generalized likelihood ratio
IIR	infinite impulse response
LDP	linear discrete-time periodic
LMI	linear matrix inequality
LPV	linear parameter varying
LTI	linear time-invariant
LTV	linear time-varying
MAC	medium access control
MDR	miss detection rate
MSD	multirate sampled-data
NCS	networked control systems
NSD	non-uniformly sampled-data
PCS	periodic communication sequence
PTF	parametric transfer function
QoS	quality of service
SD	sampled-data
SSD	single-rate sampled-data
SVD	singular value decomposition
TCP	transmission control protocol
TDMA	time division multiple access
UDP	user datagram protocol
WLAN	wireless local area network

Part I

FD of Discrete Linear Time-Invariant Systems

1

Introduction

With the increasing complexity of modern control systems, fault detection and diagnosis (FDD) has become an important research topic since the seventies [7, 14, 19, 29, 60, 61, 67, 82, 83, 121]. A *fault* can be understood as any undesired system behavior, such as a malfunction of sensors or actuators or some changes in the process itself, as shown in Fig. 1.1.

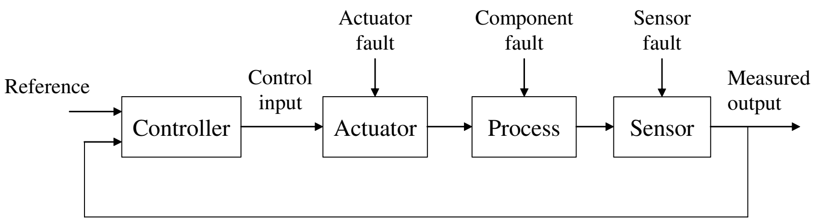

Fig. 1.1 Actuator, sensor and component fault in a control system

The FDD includes fault detection, fault isolation and fault identification. Timely detection and diagnosis of the malfunction in system components is essential for the prevention of fault propagation and the improvement of process safety, reliability and availability. In major industrial sectors, the FDD has become an important supporting technology and is replacing the traditional hardware redundancy technique in part or totally. As a standard functional module, FDD systems are increasingly integrated in modern technical systems and provide valuable information for condition-based predictive maintenance and asset management, higher level fault tolerant control and plant-wide process optimization.

This chapter will give a brief overview of the FDD technique and then describe the main content of this book.

1.1 Overview of fault detection and diagnosis

Due to the development of micro-electronics and driven by the pressure of reducing hardware costs, *software redundancy* based FDD are increasingly replacing *hardware redundancy* based FDD [61]. The basic idea of the software redundancy based FDD is to check the consistency of the online data with the help of the experience. The experience can exist in the form of models or historical data, typical signal characteristics, typical parameter range, etc. The current FDD approaches are often classified into [61]:

- signal processing based approaches,
- model based approaches, and
- data driven methods.

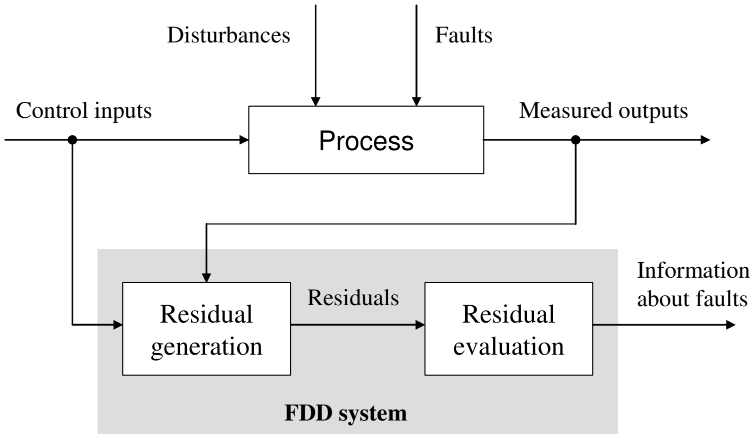

Fig. 1.2 General procedure of model-based FDD

The signal processing based approaches detect the changes in the system by analyzing the characteristics of some important signals and comparing them with the nominal values [69]. This can be done in the time domain by checking the mean value, the standard deviation, the higher order statistics, the upper and lower bound of the signals. In the frequency domain it can be done by analyzing the spectrum of the signals. The projection of the signals onto orthogonal basis functions and the wavelet technique also attract considerable attention in the research and applications.

The model based methods check the consistency between the input and output signals based on a mathematical model. In general, the model based FDD consists of two steps [19, 29, 67, 121], as shown in Fig. 1.2. In the first step, the so-called residual signals will be generated from the input and output signals with the help of the model. In the second step, the residual signals will be evaluated and the information about the faults will be extracted. The development of the model-based FDD is closely related to the development of the control theory and the filtering theory. On the other side, there are several distinct features of the model-based FDD problems that justify the efforts made in this field in the last years. The main objective of the filtering is to reduce the estimation error. In comparison, to achieve a good FDD performance, not only the robustness to the unknown disturbances and modelling errors but also the sensitivity to the faults is important. Another major difference between the controller/filter design and the FDD system design is that a decision procedure is an essential part of the FDD system. The main difference between the signal processing based FDD and the model-based FDD consists in that the former considers each signal separately, while the latter takes into account the interconnections among the signals. On the other side, the signal processing based change detection can be used to detect the changes in the residual signal of the model based FDD.

The data-driven approaches design the FDD system based on the historical data [127, 133], which include the PCA (principle component analysis), the PLS (partial least square), the FDA (Fischer discrimination analysis), the ICA (independent component analysis) and the artificial intelligence based approaches. The PCA and the PLS have been developed to cope with the highly correlated measurements and gained wide acceptance in the process industry. The basic idea of the PCA is to figure out the predominant independent linear relationships in the process variables through the singular value decomposition (SVD) and then check online whether the monitoring statistics, such as the SPE index and the T^2 index, are below the allowed level. In comparison, the PLS methods first divide the measured variables into descriptor variables and response variables and then find out a low-dimensional mapping between them. The standard PCA and PLS algorithms assume that the process is linear and in steady state. In order to take into account the process dynamics and the auto-correlation of process variables, dynamic PCA and dynamic PLS methods have been proposed. The FDA approach classifies the data into different classes by projecting the data onto a lower dimensional space based on mean value and covariance [72]. The basic idea of the ICA is to find

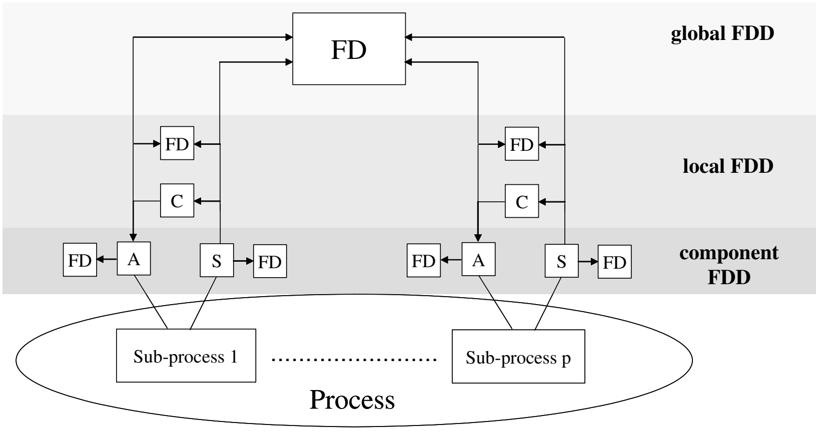

Fig. 1.3 Hierarchical FDD in complex control systems

out a set of latent variables, which are statistically independent and could explain the data. The process change is reflected in the change of these latent variables [95]. The artificial intelligence based approaches make use of neural networks, fuzzy logic or qualitative models to represent the system normal behavior and faulty behavior.

The development of the FDD theory is influenced by the development of many other scientific areas, such as the embedded electronics, the wired and wireless communication, etc. For complex systems, the FDD system are increasingly realized in a hierarchical structure, as shown in Fig. 1.3. At the component level, more and more self-diagnosis functions are integrated in the system components. For instance, the modern valves can realize some local diagnosis function by integrating a pressure difference sensor and a flow rate sensor. At the higher level, the FDD unit can make use of the information from different system components.

In practice, the main procedure of applying the model-based FDD is as follows:

- Classification of the signals in the system under consideration into measured outputs, known inputs (for instance, reference inputs, control inputs, measured disturbances) and unknown inputs (for instance, unmeasured disturbances)
- Specifications on the faults to be detected, such as the type of faults and the priority of detection
- Specifications on the implementation environment, such as the computational capacity, the data sampling scheme and the data transmission scheme
- Derivation of the system model, the associated model uncertainty and the fault effect
- Design of the residual generator (generation of indicator signals) and the residual evaluation scheme
- Off-line test of the FDD system with simulation data and/or real data
- On-line test of the FDD system
- Analysis of the FDD performance, such as the false alarm rate and the miss detection rate, and modification of the FD system.

1.2 Main content of this book

In this book, we shall concentrate on the model based fault detection (FD) technique for dynamic linear discrete-time systems. The FD methods discussed in this book can be used for the aim of fault isolation by designing a bank of residual generators, each of which is decoupled from or robust to a part of faults and sensitive to the other part of faults.

This book consists of five parts.

Part I considers the FD problems of discrete linear time-invariant systems. The main methods of residual generation and residual evaluation will be introduced, respectively, in Chapter 2 and

Chapter 3. Chapter 4 discusses the main criteria for the evaluation of the FD performance with focus on the evaluation of fault sensitivity. Chapter 5 shows different approaches to optimize the FD systems and studies the interconnections between different optimization problems. Chapter 6 introduces an approach to the design of the FD systems under multiple design objectives. Chapter 7 presents an approach to handle multiplicative model uncertainties.

Part II is about the FD of discrete linear periodic systems. The general properties of the discrete linear periodic systems will be introduced in Chapter 8. Due to the correspondence between the periodic systems and the time-invariant systems, the FD systems can be designed for the periodic system based on lifting, as introduced in Chapter 9. The direct methods of designing FD systems for the periodic systems are shown in Chapter 10. In Chapter 11, the model uncertainty problem is considered. In Chapter 12, it is shown how to directly identify periodic residual generators from the input and output data of the periodic systems.

Part III includes Chapter 13 and extends the FD approaches to discrete linear time-varying systems.

Part IV treats the FD problems of sampled-data systems. The single-rate sampled-data systems are considered in Chapter 14. The multirate sampled-data systems and the non-uniformly sampled-data systems are discussed in Chapter 15. Chapter 16 investigates the influence of the sampling period on the FD performance.

Part V discusses the FD of networked control systems. In Chapter 17, the network induced phenomena, such as time delays, packet loss, quantization error, are modelled from the FD viewpoint. Chapter 18 introduces the FD approaches for the networked control systems with a given network. In Chapter 19, several schemes for the integrated design of the FD systems and the communication scheme are introduced.

2

Residual generation

In this chapter, we consider discrete linear time-invariant (LTI) systems described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + E_d d(k) + E_f f(k) \\ y(k) &= Cx(k) + Du(k) + F_d d(k) + F_f f(k) \end{aligned} \tag{2.1}$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^{n_u}$ the control input vector, $y \in \mathbf{R}^m$ the measured output vector, $d \in \mathbf{R}^{n_d}$ the unknown disturbance vector, $f \in \mathbf{R}^{n_f}$ the fault vector to be detected, $A, B, C, D, E_d, E_f, F_d$ and F_f are known matrices of appropriate dimensions. Without loss of generality, we assume that (C, A) is detectable. The system (2.1) can be equivalently described by

$$y(z) = G_u(z)u(z) + G_d(z)d(z) + G_f(z)f(z) \tag{2.2}$$

where $G_u(z), G_d(z)$ and $G_f(z)$ denote the transfer function matrices from u, d and f to y, respectively.

The first step of fault detection (FD) is to generate a fault-indicating signal, called often as residual signal. One of the important tasks of residual generation is to eliminate the influence of the known control inputs on the residual signal. In the following, we shall introduce some often used approaches of residual generation [19, 29, 67, 123, 121].

2.1 Parity relation based residual generator

The parity space approach is initially proposed by [27, 28] and has been extensively studied since then [21, 47, 31, 33, 66, 111, 122, 159]. The essence of the parity space approach is to derive the so-called *parity relation*. Let s be an integer denoting the length of a moving time window. The parity relation of system (2.1) is the input-output relationship over the moving window $[k-s, k]$ expressed by

$$y_s(k) = H_{o,s}x(k-s) + H_{u,s}u_s(k) + H_{d,s}d_s(k) + H_{f,s}f_s(k) \tag{2.3}$$

where $x(k-s)$ is the initial state vector during the moving window, $u_s(k), d_s(k), f_s(k)$ and $y_s(k)$ are vectors obtained by stacking the corresponding input or output signals over the moving window as

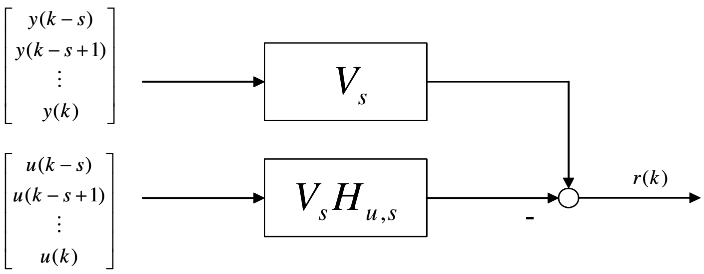

Fig. 2.1 Structure diagram of the parity relation based residual generator

$$y_s(k) = \begin{bmatrix} y(k-s) \\ y(k-s+1) \\ \vdots \\ y(k) \end{bmatrix}, u_s(k) = \begin{bmatrix} u(k-s) \\ u(k-s+1) \\ \vdots \\ u(k) \end{bmatrix} \tag{2.4}$$

$$d_s(k) = \begin{bmatrix} d(k-s) \\ d(k-s+1) \\ \vdots \\ d(k) \end{bmatrix}, f_s(k) = \begin{bmatrix} f(k-s) \\ f(k-s+1) \\ \vdots \\ f(k) \end{bmatrix}$$

$$H_{o,s} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^s \end{bmatrix}, \ H_{u,s} = \begin{bmatrix} D & O & \cdots & O \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}B & \cdots & CB & D \end{bmatrix}$$

$$H_{d,s} = \begin{bmatrix} F_d & O & \cdots & O \\ CE_d & F_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}E_d & \cdots & CE_d & F_d \end{bmatrix}, H_{f,s} = \begin{bmatrix} F_f & O & \cdots & O \\ CE_f & F_f & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}E_f & \cdots & CE_f & F_f \end{bmatrix}$$

2.1.1 Basic forms

Based on parity relation (2.3), a residual signal can be obtained as

$$r(k) = v_s(y_s(k) - H_{u,s}u_s(k)) \tag{2.5}$$

where $r \in \mathbf{R}$ is the residual signal, $v_s \in \mathbf{R}^{1\times m(s+1)}$ is a row vector. By the construction of the residual generator, the control inputs u have no influence on the residual r. To further eliminate the influence of the initial state $x(k-s)$ on the residual, the parameter v_s is selected to satisfy

$$v_s H_{o,s} = 0 \tag{2.6}$$

In the FD literature, the vector v_s satisfying (2.6) is often called *parity vector* and the left null space of the matrix $H_{o,s}$ is often called *parity space*, denoted by $\mathbf{P}_s$, i.e.

$$\mathbf{P}_s = \{v_s \mid v_s H_{o,s} = 0\} \tag{2.7}$$

The dynamics of the residual generator (2.5) is governed by

$$r(k) = v_s(H_{d,s}d_s(k) + H_{f,s}f_s(k)) \tag{2.8}$$

Note that the residual r obtained by the residual generator (2.5) is a scalar signal. To get more design freedom and keep the representation space of the residual, a matrix V_s can be used (see Fig. 2.1), i.e.

$$r(k) = V_s(y_s(k) - H_{u,s}u_s(k)) \tag{2.9}$$
$$= V_s(H_{d,s}d_s(k) + H_{f,s}f_s(k)) \tag{2.10}$$

where $V_s \in \mathbf{R}^{n_r \times m(s+1)}$ is a matrix called in the following as *parity matrix* which satisfies

$$V_s H_{o,s} = 0 \tag{2.11}$$

The number of independent rows in V_s is limited to $n_r \leq m(s+1) - rankH_{o,s}$. As the parity vector v_s can be regarded as a special case of the parity matrix V_s, a part of the subsequent discussion is based on the expressions (2.9)-(2.10).

2.1.2 Extended forms

In this subsection, we shall look at two extensions of the above parity relation based residual generators.

Extension I: From open loop to closed loop

The parity relation based residual generator (2.5) is in an open-loop form. By introducing the feedback terms we can extend it into a closed-loop form as

$$r(k) = v_s(y_s(k) - H_{u,s}u_s(k)) + g_s r(k-1) + \cdots + g_1 r(k-s) \tag{2.12}$$

where $g_1, g_2, \cdots, g_s$ are freely selectable constants. Compared with (2.5), (2.12) has more design freedom without increasing the order of the residual generator.

The dynamics of the residual generator (2.12) is governed by

$$r(k) = v_s(H_{d,s}d_s(k) + H_{f,s}f_s(k)) + g_s r(k-1) + \cdots + g_1 r(k-s) \tag{2.13}$$

As can be seen, the residual dynamics (2.13) is stable, if and only if all the roots of the characteristic equation $z^s - g_s z^{s-1} - \cdots - g_1 = 0$ are located inside the unit circle.

Extension II: Combination with observer

In principle, the parity space approach is applicable to both stable and unstable systems. However, it is worth noticing that numerical problem may be met for some systems, especially when A is unstable. To solve this problem, the parity relation based residual generator can be extended as follows

$$\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\
\hat{y}(k) &= C\hat{x}(k) + Du(k) \\
r_o(k) &= y(k) - \hat{y}(k) \\
r(k) &= V_{se} \begin{bmatrix} r_o(k-s) \\ r_o(k-s+1) \\ \vdots \\ r_o(k) \end{bmatrix}
\end{aligned} \tag{2.14}$$

where $\hat{x} \in \mathbf{R}^n$ and $\hat{y} \in \mathbf{R}^m$ denote, respectively, the state estimation and the output estimation, L is the observer gain matrix that stabilizes $A - LC$, V_{se} is the parity matrix. Let $e(k) = x(k) - \hat{x}(k)$ be the state estimation error. From (2.1) and (2.14), we get

$$\begin{aligned}
e(k+1) &= (A - LC)e(k) + (E_d - LF_d)d(k) + (E_f - LF_f)f(k) \\
r_o(k) &= Ce(k) + F_d d(k) + F_f f(k)
\end{aligned}$$

As a result, the dynamics of the residual generator (2.14) is governed by [45, 158]

$$r(k) = V_{se}(H_{L,o,s}e(k-s) + H_{L,d,s}d_s(k) + H_{L,f,s}f_s(k)) \tag{2.15}$$

where

$$H_{L,o,s} = \begin{bmatrix} C \\ C(A-LC) \\ \vdots \\ C(A-LC)^s \end{bmatrix}$$

$$H_{L,d,s} = \begin{bmatrix} F_d & O & \cdots & O \\ C(E_d - LF_d) & F_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C(A-LC)^{s-1}(E_d - LF_d) & \cdots & C(E_d - LF_d) & F_d \end{bmatrix}$$

$$H_{L,f,s} = \begin{bmatrix} F_f & O & \cdots & O \\ C(E_f - LF_f) & F_f & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C(A-LC)^{s-1}(E_f - LF_f) & \cdots & C(E_f - LF_f) & F_f \end{bmatrix}$$

Notice that the matrices $H_{L,o,s}, H_{L,d,s}, H_{L,f,s}$ in (2.15) are related to the matrices $H_{o,s}, H_{d,s}, H_{f,s}$ in (2.4) as

$$\begin{aligned} H_{L,o,s} &= Q_L H_{o,s} \\ H_{L,d,s} &= Q_L H_{d,s} \\ H_{L,f,s} &= Q_L H_{f,s} \end{aligned} \tag{2.16}$$

where Q_L is an invertible matrix

$$Q_L = \begin{bmatrix} I & O & \cdots & O \\ -CL & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ -C(A-LC)^{s-1}L & \cdots & -CL & I \end{bmatrix} \tag{2.17}$$

Therefore, the residual dynamics (2.15) can be equivalently re-written as

$$r(k) = V_{se}Q_L(H_{o,s}e(k-s) + H_{d,s}d_s(k) + H_{f,s}f_s(k)) \tag{2.18}$$

where $e(k-s)$ is influenced by $e(0), \{d(0), d(1), \cdots, d(k-s-1)\}$ and $\{f(0), f(1), \cdots, f(k-s-1)\}$. To get rid of the influence of $e(k-s)$ on $r(k)$, the parity matrix V_{se} in residual generator (2.14) should satisfy

$$V_{se}H_{L,o,s} = 0 \Leftrightarrow V_{se}Q_L H_{o,s} = 0 \Leftrightarrow V_{se}Q_L \in \mathbf{P}_s$$

Due to the invertibility of the matrix Q_L, the optimal performance and decouplability from the unknown disturbances will not be influenced by the observer structure, as will be shown later. Note that, if A is unstable, the matrices $H_{L,o,s}, H_{L,d,s}$ and $H_{L,f,s}$ have much better numerical property than the matrices $H_{o,s}, H_{d,s}$ and $H_{f,s}$. This kind of residual generators can be implemented either in the form of (2.14) or as

$$r(k) = V_{se}\left(Q_L y_s(k) - H_{L,u,s}u_s(k)\right)$$

where $H_{L,u,s}$ is given by

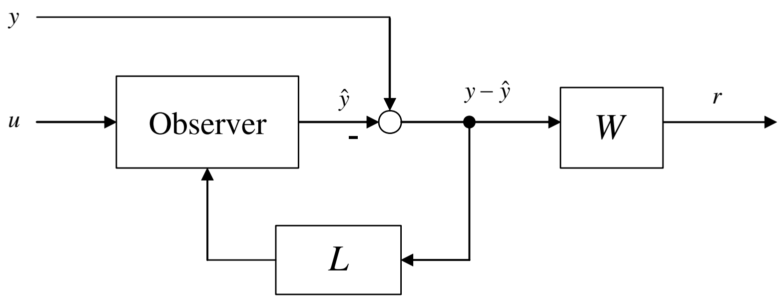

Fig. 2.2 Structure diagram of the full-order observer based residual generator

$$H_{L,u,s} = \begin{bmatrix} D & O & \cdots & O \\ C(B-LD) & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C(A-LC)^{s-1}(B-LD) & \cdots & C(B-LD) & D \end{bmatrix}$$

and also satisfies $H_{L,u,s} = Q_L H_{u,s}$.

2.2 Observer based residual generator

The basic idea of observer-based residual generator is to estimate the measured outputs with an observer and then compare the estimations with the measurements. For this purpose, in principle all kinds of observers can be used.

2.2.1 Full-order observer

A full-order observer based residual generator, called often as *fault detection filter (FDF)*, can be constructed as [19, 29, 67, 123, 121]

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) + Du(k) \\ r(k) &= W(y(k) - \hat{y}(k)) \end{aligned} \quad (2.19)$$

where $\hat{x} \in \mathbf{R}^n$ and $\hat{y} \in \mathbf{R}^m$ denote the state estimation and the output estimation, respectively, $r \in \mathbf{R}^{n_r}$ $(n_r \leq m)$ is the residual signal, the observer gain matrix L and the weighting matrix W are design parameters to be determined (see Fig. 2.2). Such an observer is based on a one-to-one reconstruction of the system model. While the control society mainly focuses on the selection of the feedback gain matrix L, the weighting matrix W plays an important role for the FD performance as well, as shown later in Chapter 5.

Let $e(k) = x(k) - \hat{x}(k)$ be the state estimation error. The dynamics of the residual generator (2.19) is governed by

$$\begin{aligned} e(k+1) &= (A - LC)e(k) + (E_d - LF_d)d(k) + (E_f - LF_f)f(k) \\ r(k) &= WCe(k) + WF_d d(k) + WF_f f(k) \end{aligned} \quad (2.20)$$

The influence of the initial state estimation error $e(0) = x(0) - \hat{x}(0)$ is asymptotically zero as long as $A - LC$ is stable.

2.2.2 Functional observer

To reduce the online computation and the implementation efforts, the following functional observer can also be used for residual generation [31, 60]

$$\begin{aligned} z(k+1) &= Gz(k) + Hu(k) + L_r y(k) \\ r(k) &= Wz(k) + Qu(k) + Py(k) \end{aligned} \tag{2.21}$$

where $r \in \mathbf{R}^{n_r}$ is the residual signal, $z \in \mathbf{R}^{n_z}$ is the state vector of the functional observer, G, H, L_r, W, Q and P are constant matrices. If G is stable and there exists a state transformation matrix $T \in \mathbf{R}^{n_z \times n}$ so that the following Luenberger equations

$$TA - GT = L_r C, \quad WT + PC = 0 \tag{2.22}$$

$$H = TB - L_r D, \quad Q = -PD \tag{2.23}$$

hold, then the residual dynamics is governed by

$$\begin{aligned} e(k+1) &= Ge(k) + (L_r F_d - TE_d)d(k) + (L_r F_f - TE_f)f(k) \\ r(k) &= We(k) + PF_d d(k) + PF_f f(k) \end{aligned} \tag{2.24}$$

where $e(k) = z(k) - Tx(k)$. As G is stable, the influence of $e(0)$ on $r(k)$ is asymptotically zero.

2.2.3 PI observer

In the recent years there are some discussions on using PI (proportional and integral) observers for the purpose of fault detection. The residual generator is built as

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + K_P(y(k) - \hat{y}(k)) + K_I \beta(k) \\ \beta(k+1) &= \beta(k) + (y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) + Du(k) \\ r(k) &= W(y(k) - \hat{y}(k)) \end{aligned} \tag{2.25}$$

where the design parameters are K_P and K_I, the coefficient matrices of the proportional term and the integral term. (2.25) can be re-written as

$$\begin{aligned} \begin{bmatrix} \hat{x}(k+1) \\ \beta(k+1) \end{bmatrix} &= \begin{bmatrix} A - K_P C & K_I \\ -C & I \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \beta(k) \end{bmatrix} + \begin{bmatrix} B - K_P D \\ -D \end{bmatrix} u(k) + \begin{bmatrix} K_P \\ I \end{bmatrix} y(k) \\ r(k) &= \begin{bmatrix} -WC & O \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \beta(k) \end{bmatrix} - WDu(k) + Wy(k) \end{aligned} \tag{2.26}$$

Let $e(k) = x(k) - \hat{x}(k)$. The residual dynamics is governed by

$$\begin{aligned} \begin{bmatrix} e(k+1) \\ \beta(k+1) \end{bmatrix} &= \begin{bmatrix} A - K_P C & -K_I \\ C & I \end{bmatrix} \begin{bmatrix} e(k) \\ \beta(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} E_d - K_P F_d \\ F_d \end{bmatrix} d(k) + \begin{bmatrix} E_f - K_P F_f \\ F_f \end{bmatrix} f(k) \\ r(k) &= \begin{bmatrix} WC & O \end{bmatrix} \begin{bmatrix} e(k) \\ \beta(k) \end{bmatrix} + WF_d d(k) + WF_f f(k) \end{aligned} \tag{2.27}$$

The parameters K_P and K_I should be selected to guarantee the stability of (2.27).

2.2.4 Simultaneous output and fault estimation

If there is some a priori knowledge of the faults, then such information can be incorporated in the observer design. Consider a simple example where $f(k)$ is constant, which can be modelled as $f(k+1) = f(k)$. Therefore, the system model (2.1) can be extended to

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ f(k+1) \end{bmatrix} &= \begin{bmatrix} A & E_f \\ O & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(k) \end{bmatrix} + \begin{bmatrix} B \\ O \end{bmatrix} u(k) + \begin{bmatrix} E_d \\ O \end{bmatrix} d(k) \\ y(k) &= \begin{bmatrix} C & F_f \end{bmatrix} \begin{bmatrix} x(k) \\ f(k) \end{bmatrix} + Du(k) + F_d d(k) \end{aligned} \tag{2.28}$$

If

$$rank \begin{bmatrix} zI - A & -E_f \\ O & zI - I \\ C & F_f \end{bmatrix} = n + n_f$$

for any z outside the unit circle, then an observer can be constructed as

$$\begin{aligned} \begin{bmatrix} \hat{x}(k+1) \\ \hat{f}(k+1) \end{bmatrix} &= \begin{bmatrix} A & E_f \\ O & I \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{f}(k) \end{bmatrix} + \begin{bmatrix} B \\ O \end{bmatrix} u(k) + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (y(k) - \hat{y}(k)) \\ \hat{y}(k) &= \begin{bmatrix} C & F_f \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{f}(k) \end{bmatrix} + Du(k) \\ r(k) &= W(y(k) - \hat{y}(k)) \end{aligned} \tag{2.29}$$

The residual dynamics is governed by

$$\begin{aligned} \begin{bmatrix} e(k+1) \\ e_f(k+1) \end{bmatrix} &= \left(\begin{bmatrix} A & E_f \\ O & I \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} C & F_f \end{bmatrix} \right) \begin{bmatrix} e(k+1) \\ e_f(k+1) \end{bmatrix} \\ &\quad + \left(\begin{bmatrix} E_d \\ O \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} F_d \right) d(k) \\ r(k) &= W \begin{bmatrix} C & F_f \end{bmatrix} \begin{bmatrix} e(k+1) \\ e_f(k+1) \end{bmatrix} + WF_d d(k) \end{aligned} \tag{2.30}$$

where

$$\begin{bmatrix} e(k) \\ e_f(k) \end{bmatrix} = \begin{bmatrix} x(k) - \hat{x}(k) \\ f(k) - \hat{f}(k) \end{bmatrix}$$

It is worth noticing that the observer (2.29) is rather similar to the PI observer discussed in the last subsection.

2.3 Parametrization of linear residual generators

2.3.1 General form

It is well-known that, in the robust control theory, the coprime factorization plays an important role for the parametrization of stabilizing controllers [59, 199]. It is pointed out by Ding et al. that all linear fault detection residual generators can also be parametrized with the help of the left coprime factorization [61].

Theorem 2.1 [61] Given a discrete LTI system described by (2.2). Let $(M_u(z), N_u(z))$ be a left coprime factorization of $G_u(z)$, $G_u(z) = M_u^{-1}(z)N_u(z)$, $M_u(z) \in \mathbf{RH}_\infty$, $N_u(z) \in \mathbf{RH}_\infty$. Then all discrete LTI residual generators can be written into the form of

$$r(z) = R(z)M_u(z)\left(y(z) - G_u(z)u(z)\right) \tag{2.31}$$

where $R(z) \in \mathbf{RH}_\infty$ is a post-filter that can be selected arbitrarily.

Substituting (2.2) into (2.31), the dynamics of the residual generator (2.31) is described by

$$r(z) = R(z)M_u(z)\left(G_d(z)d(z) + G_f(z)f(z)\right) \tag{2.32}$$

The unified expression of the residual generator and the residual dynamics enables a unified analysis and design of the optimal residual generators, as shown later in Chapter 5.

Assume that (A, B, C, D) is a state space realization of $G_u(z)$, i.e. $G_u(z) = C(zI - A)^{-1}B + D$. Then the left coprime factorization $(M_u(z), N_u(z))$ needed in the above parametrization can be obtained by [200]

$$\begin{aligned} M_u(z) &= I - C(zI - A + LC)^{-1}L \\ N_u(z) &= D + C(zI - A + LC)^{-1}(B - LD) \end{aligned} \tag{2.33}$$

where L is any matrix of compatible dimensions that stabilizes $A - LC$.

2.3.2 Specific selection of post-filters

In the following, we shall briefly show how the residual generators introduced in Section 2.1 and 2.2 correspond to (2.31).

Lemma 2.1 Let the parity vector v_s be partitioned as

$$v_s = \left[\, v_{s,0}\ v_{s,1}\ \cdots\ v_{s,s} \,\right],\ v_{s,i} \in \mathbf{R}^{1\times m},\ i = 0, 1, \cdots, s \tag{2.34}$$

Then the residual generator (2.5) can be expressed as (2.31) with

$$\begin{aligned} R(z) = {} & v_{s,s}I + (v_{s,s-1}I + v_{s,s}CL)z^{-1} + \cdots \\ & + \left(v_{s,0}I + v_{s,1}CL + \cdots + v_{s,s}CA^{s-1}L\right) z^{-s} \end{aligned} \tag{2.35}$$

Proof: Let $\rho_s = v_s H_{u,s}$. The residual generator (2.5) can be re-written as

$$r(k) = v_s y_s(k) - \rho_s u_s(k) \tag{2.36}$$

Partition the vector ρ_s also into $s+1$ blocks as follows

$$\rho_s = \left[\, \rho_{s,0}\ \rho_{s,1}\ \cdots\ \rho_{s,s} \,\right],\ \rho_{s,i} \in \mathbf{R}^{1\times m},\ i = 0, 1, \cdots, s \tag{2.37}$$

Then $\rho_s = v_s H_{u,s}$ can be expanded as

$$\begin{aligned} \rho_{s,s} &= v_{s,s}D \\ \rho_{s,s-1} &= v_{s,s-1}D + v_{s,s}CB \\ &\ \ \vdots \\ \rho_{s,0} &= v_{s,0}D + v_{s,1}CB + v_{s,2}CAB + \cdots + v_{s,s}CA^{s-1}B \end{aligned} \tag{2.38}$$

In the frequency domain the residual generator (2.36) can be expressed as

$$r(z) = v_s(z)y(z) - \rho_s(z)u(z) \tag{2.39}$$

where

$$v_s(z) = v_{s,s} + \cdots + v_{s,1}z^{-s+1} + v_{s,0}z^{-s} \tag{2.40}$$

$$\rho_s(z) = \rho_{s,s} + \cdots + \rho_{s,1}z^{-s+1} + \rho_{s,0}z^{-s} \tag{2.41}$$

In the next, we shall show that $R(z)M_u(z) = v_s(z)$ and $R(z)M_u(z)G_u(z) = \rho_s(z)$. As

$$M_u^{-1}(z) = I + C(zI - A)^{-1}L = I + CLz^{-1} + CALz^{-2} + \cdots$$

we get

$$\begin{aligned} v_s(z)M_u^{-1}(z) = v_{s,s}I &+ (v_{s,s-1}I + v_{s,s}CL)z^{-1} + \cdots \\ &+ \left(v_{s,0}I + v_{s,1}CL + \cdots + v_{s,s}CA^{s-1}L\right) z^{-s} \\ &+ \sum_{j=1}^{\infty} (v_{s,0}C + v_{s,1}CA + \cdots + v_{s,s}CA^s)A^{j-1}Lz^{-s-j} \end{aligned}$$

Recall that the parity vector v_s belongs to the parity space $\mathbf{P}_s$, i.e.

$$v_sH_{o,s} = v_{s,0}C + v_{s,1}CA + \cdots + v_{s,s}CA^s = 0 \tag{2.42}$$

Therefore, $R(z)$ given by (2.35) satisfies $R(z) = v_s(z)M_u^{-1}(z)$, i.e.

$$R(z)M_u(z) = v_s(z) \tag{2.43}$$

The term $R(z)M_u(z)G_u(z)$ can be expanded as

$$\begin{aligned} &R(z)M_u(z)G_u(z) \\ &= v_s(z)G_u(z) \\ &= v_{s,s}D + (v_{s,s-1}D + v_{s,s}CB)\, z^{-1} \cdots \\ &\quad + \left(v_{s,0}D + v_{s,1}CB + v_{s,2}CAB + \cdots + v_{s,s}CA^{s-1}B\right) z^{-s} \\ &\quad + \sum_{j=1}^{\infty} (v_{s,0}C + v_{s,1}CA + \cdots + v_{s,s}CA^s)A^{j-1}Bz^{-s-j} \end{aligned}$$

Due to (2.38) and (2.42), $R(z)M_u(z)G_u(z)$ reduces to

$$R(z)M_u(z)G_u(z) = \rho_{s,s} + \cdots + \rho_{s,1}z^{-s+1} + \rho_{s,0}z^{-s} = \rho_s(z) \tag{2.44}$$

Substituting (2.43) and (2.44) into (2.39) yields (2.31). □

Lemma 2.1 shows that in the parity space approach the post-filter $R(z)$ is a filter with finite impulse response. It is interesting to notice the role played by the condition $v_s \in \mathbf{P}_s$ in the derivation.

Lemma 2.2 The residual generator (2.12) can be expressed as (2.31) with

$$R(z) = \beta(z) \begin{pmatrix} v_{s,s}I + (v_{s,s-1}I + v_{s,s}CL)z^{-1} + \cdots \\ + \left(v_{s,0}I + v_{s,1}CL + \cdots + v_{s,s}CA^{s-1}L\right) z^{-s} \end{pmatrix} \tag{2.45}$$

where

$$\beta(z) = \frac{1}{1 - g_sz^{-1} - g_{s-1}z^{-2} - \cdots - g_1z^{-s}} \tag{2.46}$$

From Lemma 2.2 it can be seen that due to the feedback terms, the post-filter $R(z)$ becomes a filter with infinite impulse response. The parameters $g_1, g_2, \cdots, g_s$ can be designed by considering the desired frequency behaviour of the residual generator.

Lemma 2.3 Let the parity matrix V_{se} be partitioned as

$$V_{se} = \left[V_{se,0}\ V_{se,1}\ \cdots\ V_{se,s} \right],\ V_{se,i} \in \mathbf{R}^{n_r \times m},\ i = 0, 1, \cdots, s \tag{2.47}$$

Then the residual generator (2.14) can be expressed as (2.31) with

$$R(z) = V_{se,s} + \cdots + V_{se,1}z^{-s+1} + V_{se,0}z^{-s} \tag{2.48}$$

Proof: The full-order observer in (2.14) can be re-written as

$$\begin{aligned} \hat{x}(k+1) &= (A - LC)\hat{x}(k) + (B - LD)u(k) + Ly(k) \\ r_o(k) &= y(k) - \hat{y}(k) = -C\hat{x}(k) - Du(k) + y(k) \end{aligned} \tag{2.49}$$

Recalling (2.33), there is

$$r_o(z) = M_u(z)y(z) - N_u(z)u(z) = M_u(z)\left(y(z) - G_u(z)u(z)\right)$$

Hence,

$$\begin{aligned} r(z) &= \sum_{i=0}^{s} V_{se,i} r_o(z) z^{-(s-i)} = \left(\sum_{i=0}^{s} V_{se,i} z^{-(s-i)}\right) r_o(z) \\ &= \left(\sum_{i=0}^{s} V_{se,i} z^{-(s-i)}\right) M_u(z)\left(y(z) - G_u(z)u(z)\right) \end{aligned}$$

The lemma is thus proven. □

Lemma 2.4 The fault detection filter (2.19) can be expressed as (2.31) with

$$R(z) = W \tag{2.50}$$

Proof: As shown in the proof of Lemma 2.3, there is

$$y(k) - \hat{y}(k) = M_u(z)y(z) - N_u(z)u(z) = M_u(z)\left(y(z) - G_u(z)u(z)\right)$$

Therefore,

$$r(z) = W(y(k) - \hat{y}(k)) = WM_u(z)\left(y(z) - G_u(z)u(z)\right)$$

It is shown by Lemma 2.4 that the full-order observer based residual generator is a special case of (2.31), where the dynamic post-filter $R(z)$ reduces to a constant weighting matrix W.

Lemma 2.5 The functional observer based residual generator (2.21) can be expressed into the general form (2.31) with

$$R(z) = P + W(zI - G)^{-1}(L_r - TL) \tag{2.51}$$

Proof: As G is stable, $R(z) \in \mathbf{RH}_\infty$. Recalling (2.33), a state space realization of $R(z)M_u(z)$ is $(A_{RM}, B_{RM}, C_{RM}, D_{RM})$ with

$$\begin{aligned} A_{RM} &= \begin{bmatrix} G & -(L_r - TL)C \\ O & A - LC \end{bmatrix}, \; B_{RM} = \begin{bmatrix} L_r - TL \\ L \end{bmatrix} \\ C_{RM} &= \begin{bmatrix} W & -PC \end{bmatrix}, \; D_{RM} = P \end{aligned}$$

Do a similarity transformation with a nonsingular matrix

$$\bar{T} = \begin{bmatrix} I & T \\ O & I \end{bmatrix}$$

where T satisfies (2.22)-(2.23). Due to (2.22), there is

$$\begin{aligned} \bar{T} A_{RM} \bar{T}^{-1} &= \begin{bmatrix} G & TA - L_r C - GT \\ O & A - LC \end{bmatrix} = \begin{bmatrix} G & O \\ O & A - LC \end{bmatrix} \\ \bar{T} B_{RM} &= \begin{bmatrix} L_r \\ L \end{bmatrix} \\ C_{RM} \bar{T}^{-1} &= \begin{bmatrix} W & -WT - PC \end{bmatrix} = \begin{bmatrix} W \; O \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} R(z)M_u(z) &= (\bar{T} A_{RM} \bar{T}^{-1}, \bar{T} B_{RM}, C_{RM} \bar{T}^{-1}, D_{RM}) \\ &= W(zI - G)^{-1} L_r + P \end{aligned}$$

Similarly, by considering (2.23), it can be obtained that

$$\begin{aligned} R(z)N_u(z) &= (A_{RN}, B_{RN}, C_{RN}, D_{RN}) \\ &= \ (\bar{T}^{-1}A_{RN}\bar{T}, \bar{T}^{-1}B_{RN}, C_{RN}\bar{T}, D_{RN}) \\ &= -W(zI-G)^{-1}H - Q \end{aligned}$$

where

$$\begin{aligned} A_{RN} &= \begin{bmatrix} G & (L_r - TL)C \\ O & A - LC \end{bmatrix}, \ B_{RN} = \begin{bmatrix} (L_r - TL)D \\ B - LD \end{bmatrix} \\ C_{RN} &= \begin{bmatrix} W & PC \end{bmatrix}, \ D_{RN} = PD \end{aligned}$$

Hence, (2.21) can be written into (2.31) with $R(z)$ given by (2.51). □

Lemma 2.6 The PI observer based residual generator (2.25) can be expressed as (2.31) with

$$\begin{aligned} R(z) &= W - WC(zI - A + K_P C)^{-1} K_I \Delta^{-1} \\ \Delta &= zI - I + C(zI - A + K_P C)^{-1} K_I \end{aligned} \tag{2.52}$$

Proof: Note that

$$\begin{bmatrix} zI - A + K_P C & -K_I \\ C & (z-1)I \end{bmatrix}^{-1} = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ * & * \end{bmatrix}$$

where

$$\begin{aligned} \Pi_{11} &= (zI - A + K_P C)^{-1} \left(I - K_I \Delta^{-1} C (zI - A + K_P C)^{-1}\right) \\ \Pi_{12} &= (zI - A + K_P C)^{-1} K_I \Delta^{-1} \end{aligned}$$

and $*$ denotes the terms whose value is not of interest for the derivation. According to (2.26), there is

$$\begin{aligned} G_{ry} &= W + \begin{bmatrix} -WC & O \end{bmatrix} \begin{bmatrix} zI - A + K_P C & -K_I \\ C & (z-1)I \end{bmatrix}^{-1} \begin{bmatrix} K_P \\ I \end{bmatrix} \\ &= W - WC\Pi_{11}K_P - WC\Pi_{12} \\ &= \left(W - WC(zI - A + K_P C)^{-1} K_I \Delta^{-1}\right)\left(I - C(zI - A + K_P C)^{-1} K_P\right) \\ &= R(z)M_u(z) \\ G_{ru} &= -WD + \begin{bmatrix} -WC & O \end{bmatrix} \begin{bmatrix} zI - A + K_P C & -K_I \\ C & (z-1)I \end{bmatrix}^{-1} \begin{bmatrix} B - K_P D \\ -D \end{bmatrix} \\ &= -WD - WC\Pi_{11}(B - K_P D) + WC\Pi_{12} D \\ &= -\left(W - WC(zI - A + K_P C)^{-1} K_I \Delta^{-1}\right) \\ &\quad \times \left(D + C(zI - A + K_P C)^{-1}(B - K_P D)\right) \\ &= -R(z)N_u(z) \end{aligned}$$

Therefore, we get

$$r(z) = G_{ry}(z)y(z) + G_{ru}(z)u(z) = R(z)M_u(z)y(z) - R(z)N_u(z)u(z)$$

Lemma 2.6 shows that the PI observer corresponds to a higher order post-filter.

In summary, all discrete LTI residual generators (2.31) can be regarded as the cascade connection of a full order observer and a post-filter, as shown in Fig. 2.3.

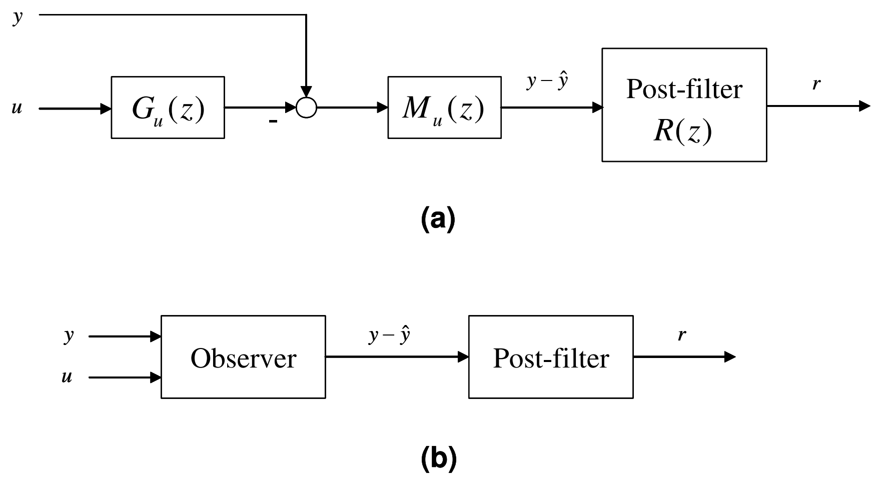

Fig. 2.3 (a) General description of discrete LTI residual generators (b) Equivalent structure

2.4 Interconnections between different residual generators

In the above sections, it has been shown that the residual generators can be implemented in different forms. As shown below, it is also possible to transform from one form directly to another form.

Assume that v_s is a parity vector, $v_s \in \mathbf{P}_s$ and $\rho_s = v_s H_s$. Partition v_s and ρ_s, respectively, as (2.34) and (2.37). Let a matrix T be given by [31]

$$T = \begin{bmatrix} v_{s,1} & \cdots & v_{s,s-1} & v_{s,s} \\ v_{s,2} & \cdots & v_{s,s} & 0 \\ \vdots & & & \vdots \\ v_{s,s} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0 & 0 & \cdots & 0 & g_1 \\ 1 & 0 & \ddots & \vdots & g_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & 0 & g_{s-1} \\ 0 & \cdots & 0 & 1 & g_s \end{bmatrix} \in \mathbf{R}^{s \times s}$$

$$H = \begin{bmatrix} \rho_{s,0} \\ \rho_{s,1} \\ \vdots \\ \rho_{s,s-1} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix} \rho_{s,s}$$

$$L_r = -\begin{bmatrix} v_{s,0} \\ v_{s,1} \\ \vdots \\ v_{s,s-1} \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix} v_{s,s}$$

$$W = \begin{bmatrix} 0 \; 0 \cdots 0 \; -1 \end{bmatrix}, \; Q = -\rho_{s,s}, \quad P = v_{s,s} \tag{2.53}$$

where $g_1, g_2, \cdots, g_s$ are freely selectable constants. Then, it can be verified that the matrices G, H, L_r, W, Q, P and T satisfy the equations (2.22)-(2.23). That means, with the help of (2.53), a functional observer based residual generator in the form of (2.21) can be readily obtained from a parity vector v_s.

With the parameters selected as (2.53), the dynamics of the functional observer based residual generator (2.21) would be

$$
\begin{aligned}
z_1(k-s+1) &= g_1 z_s(k-s) + (\rho_{s,0} + g_1\rho_{s,s})u(k-s) - (v_{s,0} + g_1 v_{s,s})y(k-s) \\
z_2(k-s+2) &= z_1(k-s+1) + g_2 z_s(k-s+1) \\
&\quad + (\rho_{s,1} + g_2\rho_{s,s})u(k-s+1) - (v_{s,1} + g_2 v_{s,s})y(k-s+1) \\
&\vdots \\
z_s(k) &= z_{s-1}(k-1) + g_s z_s(k-1) \\
&\quad + (\rho_{s,s-1} + g_s\rho_{s,s})u(k-1) - (v_{s,s-1} + g_s v_{s,s})y(k-1) \\
r(j) &= -z_s(j) - \rho_{s,s}u(j) \ + v_{s,s}y(j),\ j = k-s, \cdots, k
\end{aligned}
$$

Substituting $z_1(k-s+1), \cdots, z_s(k)$ into $r(k)$, we get

$$
r(k) = v_s(y_s(k) - H_{u,s}u_s(k)) + g_s r(k-1) + \cdots + g_1 r(k-s)
$$

It shows that the functional observer based residual generator (2.21) with the parameters (2.53) leads to exactly the *same* residual dynamics as the extended parity relation based residual generator (2.12). If $g_1 = g_2 = \cdots = g_s = 0$, then the residual generator (2.21) with the parameters (2.53) will yield the same dynamics as the traditional parity relation based residual generator (2.5).

2.5 Conclusion

In this chapter, the main techniques of model-based residual generation are introduced. It is shown that, based on the left coprime factorization, all these discrete LTI residual generators can be parametrized as an observer followed by a post-filter. At last, the interconnections between different residual generators are briefly commented.

The residual generation approach introduced here estimate the signals based on a (dynamic) model, and check the consistency between the estimations with the measurements. Another interesting way of residual generation, which is not explored here, is to estimate (identify) the model according to the measured signals and check the consistency between the estimated (identified) model with the nominal model. The model invalidation approach and the parameter estimation approach can be classified into this category [82, 113].

3

Residual evaluation

In the last chapter, we have shown how to generate the residual signal for the purpose of fault detection. The next step after residual generation is to analyze the changes in the residual signal and derive the information about the faults, which is often called residual evaluation [60, 29, 67]. In terms of fault detection, two kinds of conclusions may be made. The first kind is: "There is (not) a fault in the system". The second kind gives a conclusion with a certain probability of risk, such as "There is (not) a fault in the system with a probability of $x\%$". In this chapter, we shall introduce different residual evaluation schemes.

3.1 Basic principle

For the linear systems described by (2.2), the dynamics of the residual signal obtained by the residual generator (2.31) is

$$r(z) = G_{rd}(z)d(z) + G_{rf}(z)f(z) \tag{3.1}$$

where $G_{rd}(z)$ and $G_{rf}(z)$ denote, respectively, the transfer function matrices from the disturbances d and the faults f to the residual r, $G_{rd}(z) = R(z)M_u(z)G_d(z)$, $G_{rf}(z) = R(z)M_u(z)G_f(z)$.

If the residual r is decoupled from the disturbances d ($G_{rd}(z) = 0$, $G_{rf}(z) \neq 0$), then the existence of a fault is simply indicated by the non-zeroness of the residual signal, i.e.,

$$\begin{cases} r = 0 & \Rightarrow \text{ fault-free} \\ r \neq 0 & \Rightarrow \text{ faulty} \end{cases} \tag{3.2}$$

However, if a full decoupling from d is impossible ($G_{rd}(z) \neq 0$), then it is necessary to differentiate whether the variation in the residual signal is caused by the faults f or by the unknown disturbances d. For this purpose, an often adopted residual evaluation scheme is to compare some characteristic value (for instance, amplitude, energy, etc.) of the residual signal with a threshold. In this case, a residual evaluation scheme is composed of residual evaluation function, threshold and decision logic, as shown in Fig. 3.1. The *residual evaluation function* is the function used to calculate the characteristic value of the residual. The *threshold* characterizes the scope of variations in the residual signal caused by the unknown disturbances in the fault-free case. If a fault is large enough, then the characteristic value of the residual will surpass the threshold and an alarm signal will be triggered. Let $\|\cdot\|_{ev}$ denote the residual evaluation function and J_{th} the detection threshold. Then the *decision logic* is

$$\begin{cases} \|r\|_{ev} \leq J_{th} & \Rightarrow \text{ fault-free} \\ \|r\|_{ev} > J_{th} & \Rightarrow \text{ faulty} \end{cases} \tag{3.3}$$

Some often used residual evaluation functions in the model-based FD are

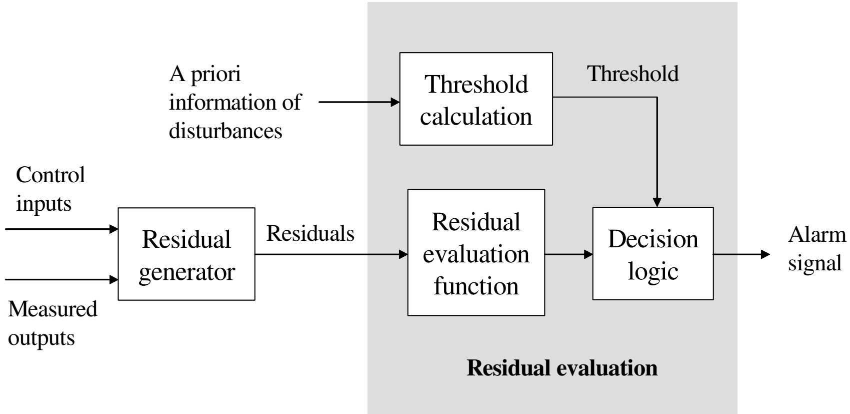

Fig. 3.1 Basic principle of residual evaluation

$$\|r\|_{ev,I} = \sqrt{\sum_{j=k-N}^{k} r^T(k)r(k)} \tag{3.4}$$

$$\|r\|_{ev,II} = \|r(k)\|_E = \sqrt{r^T(k)r(k)} \tag{3.5}$$

$$\|r\|_{ev,III} = \frac{1}{N+1}\sum_{j=k-N}^{k} r(k) \tag{3.6}$$

$$\|r\|_{ev,IV} = \sqrt{\frac{1}{N+1}\sum_{j=k-N}^{k} r^T(k)r(k)} \tag{3.7}$$

$\|r\|_{ev,I}$ is the l_2-norm of the residual r over a moving horizon of length N, $\|r\|_{ev,II}$ is the amplitude (measured by the Euclidean norm) of the residual signal, $\|r\|_{ev,III}$ is the mean value of the residual signal and often applied to systems with stochastic noise, and $\|r\|_{ev,IV}$ is the root mean square of the residual signal and is closely related to $\|r\|_{ev,I}$ by $\|r\|_{ev,I} = \sqrt{N+1}\,\|r\|_{ev,IV}$. In this chapter, the focus will be put on the residual evaluation functions (3.4) and (3.5).

To avoid false alarms, the threshold can be set as the maximal variation of $\|r\|_{ev}$ in the fault-free case, i.e.,

$$J_{th} = \max_{f=0} \|r\|_{ev} \tag{3.8}$$

Although it is one of the most popular schemes of threshold selection, it often leads to conservative threshold, which may result in a high miss detection rate.

To facilitate the detection of the incipient faults, another philosophy is to suitably reduce the value of the threshold to achieve a suitable compromise between the false alarm rate and the miss detection rate. In this case, the threshold $J_{th,\alpha}$ ($J_{th,\alpha} \leq J_{th}$) can be selected by integrating the available information about the disturbances and the faults.

3.2 Computation of threshold

In this section, we shall illustrate how to calculate the threshold J_{th} according to (3.8) with the help of three examples. For this purpose, the induced norms are a useful tool. Therefore, we call this kind of residual evaluation schemes *norm-based residual evaluation*.

Example 3.1 (3.4) is the residual evaluation function and the l_2-norm of the unknown disturbances over the moving window of length N is bounded by $\delta_{d,2}$, i.e. $\sup_k \sup_d \|d\|_{2,[k-N,k]} = \delta_{d,2}$.

It is known from the robust control theory that [199]

$$\sup_{x(k-N)=0, d\neq 0,\ d\in l_{2,[k-N,k]}} \frac{\|r\|_{2,[k-N,k]}}{\|d\|_{2,[k-N,k]}} \leq \sup_{x(0)=0, d\neq 0, d\in l_2} \frac{\|r\|_2}{\|d\|_2} = \|G_{rd}(z)\|_\infty$$

where $\|G_{rd}(z)\|_\infty$ is the $\mathbf{H}_\infty$-norm of $G_{rd}(z)$, which characterizes the maximal change in the residual energy caused by the unknown disturbances of unit energy. Therefore, the threshold J_{th} can be calculated by

$$J_{th} = \|G_{rd}(z)\|_\infty \sup_k \sup_d \|d\|_{2,[k-N,k]} = \|G_{rd}(z)\|_\infty \, \delta_{d,2} \tag{3.9}$$

That means, if

$$\|r\|_{2,[k-N,k]} = \sqrt{\sum_{j=k-N}^{k} r^T(k)r(k)} > J_{th} = \|G_{rd}(z)\|_\infty \, \delta_{d,2}$$

then it can be concluded that there is a fault in the system.

The $\mathbf{H}_\infty$-norm of discrete LTI systems can be calculated using standard algorithms in the robust control theory, for instance, singular value plot, Hamiltonian matrix or iteratively solving the linear matrix inequality (LMI) in the following lemma.

Lemma 3.1 [64] Given $G(z) = (A, B, C, D) \in \mathbf{RH}_\infty$ with zero initial conditions and a scalar $\gamma > 0$. Then $G(z)$ is stable and

$$\|G(z)\|_\infty < \gamma$$

if and only if there exists a symmetric matrix $X = X^T > 0$ such that

$$\begin{bmatrix} A^TXA - X + C^TC & A^TXB + C^TD \\ B^TXA + D^TC & B^TXB + D^TD - \gamma^2 I \end{bmatrix} < 0$$

Example 3.2 (3.5) is the residual evaluation function and the unknown disturbances are bounded by $\sup_k \sup_d \|d\|_{2,[k-N,k]} = \delta_{d,2}$.

In this case, the generalized $\mathbf{H}_2$ norm defined by [139]

$$\|G_{rd}(z)\|_g = \sup \left\{ \|r(T)\|_E : x(0) = 0, T \geq 0, \sum_{j=0}^{T} d^T(j)d(j) \leq 1 \right\}$$

is suitable for the calculation of the threshold. The generalized $\mathbf{H}_2$ norm measures the maximal change in the peak amplitude of the output signal caused by the input of unit energy. Hence,

$$J_{th} = \|G_{rd}(z)\|_g \sup_k \sup_d \|d\|_{2,[k-N,k]} = \|G_{rd}(z)\|_g \, \delta_{d,2} \tag{3.10}$$

If

$$\|r(k)\|_E = \sqrt{r^T(k)r(k)} > J_{th} = \|G_{rd}(z)\|_g \, \delta_{d,2}$$

then the changes in the peak amplitude of the residual signal can not be explained by the unknown disturbances and thus a fault is detected. The generalized $\mathbf{H}_2$ norm can be obtained from the state space realization according to Lemma 3.2.

Lemma 3.2 [37] Given $G(z) = (A, B, C, D) \in \mathbf{RH}_\infty$ with zero initial conditions and a scalar $\gamma > 0$. Then $G(z)$ is stable and $\|G(s)\|_g < \gamma$ if and only if there exists a symmetric matrix $X = X^T > 0$ such that

$$\begin{bmatrix} A^TXA - X & A^TXB \\ B^TXA & B^TXB - I \end{bmatrix} < 0 \tag{3.11}$$

$$\begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD \end{bmatrix} < \begin{bmatrix} P & O \\ O & \gamma^2 I \end{bmatrix} \tag{3.12}$$

Example 3.3 (3.5) is the residual evaluation function and $\|d(k)\|_E = \sqrt{d^T(k)d(k)} \le \delta_{d,\infty}$.

In this case, the relation between the peak amplitude of the input and the output is essential for the determination of the threshold. Recall that the peak to peak norm is defined by [139]

$$\|G_{rd}(z)\|_p = \sup_{d\neq 0, d\in l_\infty} \frac{\|r\|_{peak}}{\|d\|_{peak}}$$

Hence, the threshold can be set as

$$J_{th} = \|G_{rd}(z)\|_p \sup_d \|d\|_\infty = \|G_{rd}(z)\|_p \, \delta_{d,\infty} \tag{3.13}$$

A fault will be detected, if

$$\|r(k)\|_E = \sqrt{r^T(k)r(k)} > J_{th} = \|G_{rd}(z)\|_p \, \delta_{d,\infty}$$

Lemma 3.3 Given $G(z) = (A, B, C, D) \in \mathbf{RH}_\infty$ with zero initial conditions and a scalar $\beta > 0$. Then $G(z)$ is stable and $\|G(z)\|_p < \beta$, if there exists a symmetric matrix $X = X^T > 0$ and constants $\lambda > 0$, μ such that

$$\begin{bmatrix} A^TXA - X + \lambda X & A^TXB \\ B^TXA & B^TXB - \mu I \end{bmatrix} < 0 \tag{3.14}$$

$$\begin{bmatrix} \lambda P & O & C^T \\ O & (\beta-\mu)I & D^T \\ C & D & \beta I \end{bmatrix} > 0 \tag{3.15}$$

From the above discussion, we can see that the induced norm adopted for the calculation of the threshold is decided by the selected residual evaluation function and the available information of the disturbances.

3.3 Adaptive threshold

In (3.9), (3.10) and (3.13), the upper bound of the l_2 and l_∞-norm of the unknown disturbances is used in the threshold calculation. If the disturbances are strongly time-varying, then the threshold can be adapted to the level of the disturbances, if such information is available.

In case that the system model is not precisely known, the residual signal may be influenced by the control inputs. Let $G_{ru}(z)$ denote the transfer function matrix from the control inputs u to the residual r. In the fault-free case

$$\|r\|_{2,[k-N,k]} \le \|G_{ru}(z)\|_\infty \|u\|_{2,[k-N,k]} + \|G_{rd}(z)\|_\infty \|d\|_{2,[k-N,k]}$$

As the information of $\|u\|_{2,[k-N,k]}$ is usually online available, the threshold J_{th} can be set as

$$J_{th}(k) = \|G_{ru}(z)\|_\infty \|u\|_{2,[k-N,k]} + \|G_{rd}(z)\|_\infty \delta_{d,2} \tag{3.16}$$

which is adaptive to the changes in the control input signal. We would like to point out that $J_{th}(k)$ given by (3.16) is less conservative than

$$\left\| \begin{bmatrix} G_{ru}(z) & G_{rd}(z) \end{bmatrix} \right\|_\infty \left(\|u\|_{2,[k-N,k]} + \delta_{d,2} \right)$$

because

$$\left\| \begin{bmatrix} G_{ru}(z) & G_{rd}(z) \end{bmatrix} \right\|_\infty \ge \max\left\{ \|G_{ru}(z)\|_\infty, \|G_{rd}(z)\|_\infty \right\}$$

3.4 Risk-dependent threshold

As mentioned in Section 3.1, to reduce the conservatism, the threshold can be reduced by allowing a small percent of false alarms. Assume that the probability distribution of the l_2 or l_∞-norm of the unknown disturbances is known. Denote the cumulative distribution function as $F(\rho)$, i.e.,

$$F(\rho) = \text{Prob}\left\{\|d\| \leq \rho\right\}, \quad \rho \in [0, \delta_d]$$

where $\delta_d = \sup_d \|d\|$, $0 \leq F(\rho) \leq 1$ and $F(\rho)$ is a non-decreasing function. For a given allowable false alarm rate (FAR) level α $(0 \leq \alpha \leq 1)$, the threshold can be set as

$$J_{th,\alpha} = \|G_{rd}(z)\| F^{-1}(1-\alpha) \tag{3.17}$$

$J_{th,\alpha}$ guarantees that the false alarm rate lies under the allowed level, since in this case the false alarm rate is

$$\begin{aligned}
&\text{Prob}\left\{\|G_{rd}(z)d(z)\|_{ev} > J_{th,\alpha}\right\} \\
&\leq \text{Prob}\left\{\|G_{rd}(z)\| \|d\| > J_{th,\alpha}\right\} \\
&= \text{Prob}\left\{\|G_{rd}(z)\| \|d\| > \|G_{rd}(z)\| F^{-1}(1-\alpha)\right\} \\
&= \text{Prob}\left\{\|d\| > F^{-1}(1-\alpha)\right\} \\
&= 1 - \text{Prob}\left\{\|d\| \leq F^{-1}(1-\alpha)\right\} = \alpha
\end{aligned}$$

We call $J_{th,\alpha}$ a risk-dependent threshold. If

$$\|r\|_{ev} > J_{th,\alpha}$$

then there is a fault in the system with a probability not lower than $1-\alpha$. Specifically, if $\alpha = 0$, then the threshold is $J_{th,\alpha} = \|G_{rd}(z)\| F^{-1}(1) = \|G_{rd}(z)\| \delta_d$ and reduces to the one discussed in Section 3.2.

A randomized algorithm based approach to select a risk-dependent threshold has been introduced by [39].

3.5 Conclusion

Residual evaluation is an important integrated part of the FD systems [40, 39]. In this chapter, the basic principles of norm-based residual evaluation and threshold calculation, including adaptive threshold and risk-dependent threshold, are introduced.

The residual signal can also be evaluated in some other ways. If the probability distribution of the unknown disturbances d is completely known, then it is possible to figure out the probability distribution of the residual signal in the fault-free case. Based on that, *hypothesis testing* methods can be used to determine the threshold and evaluate the residual signal [7]. The advanced signal processing methods, such as the wavelet technique, can also be applied to detect the changes in the residual signal [194].

4

FD performance

Generally speaking, an FD system should be able to detect incipient faults timely and reliably. In this chapter, we shall consider how to evaluate the performance of the FD systems and, based on it, formulate the FD design problems.

4.1 FAR, FDR, robustness and sensitivity

The acceptance of the FD systems in practice is mainly decided by fault detection rate (FDR), false alarm rate (FAR) and detection delay [35, 184]. The FDR is the probability of the FD system correctly detecting an occuring fault, i.e. the probability that the evaluated residual signal surpasses the threshold if the system is faulty. The concept of the miss detection rate (MDR) is complementary to that of the FDR. The MDR is the probability that a fault goes undetected. The FAR, as briefly mentioned in Chapter 3, is the probability that an alarm is triggered in the fault-free case. The detection delay is the time elapsed before a fault is detected. A good FD system should achieve a high FDR (or equivalently, a low MDR), a low FAR and a short detection delay.

Assume that the residual dynamics is governed by $r(z) = G_{rd}(z)d(z) + G_{rf}(z)f(z)$, J_{th} is the threshold, the decision logic is (3.3) and a fault happens at time k_0, i.e.

$$f(k) \begin{cases} = 0, & k < k_0 \\ \neq 0, & k \geq k_0 \end{cases}$$

Then the FDR is given by

$$\begin{aligned} P_{FD} &= \text{Prob}\{\|r\|_{ev} > J_{th} \ \mid f \neq 0\} \\ &= \text{Prob}\{\|G_{rd}(z)d(z) + G_{rf}(z)f(z)\|_{ev} > J_{th} \ \mid f \neq 0\} \end{aligned} \tag{4.1}$$

and the MDR by

$$\begin{aligned} P_{MD} &= \text{Prob}\{\|r\|_{ev} \leq J_{th} \ \mid f \neq 0\} \\ &= \text{Prob}\{\|G_{rd}(z)d(z) + G_{rf}(z)f(z)\|_{ev} \leq J_{th} \ \mid f \neq 0\} \\ &= 1 - P_{FD} \end{aligned} \tag{4.2}$$

Different from the FDR and the MDR, the FAR is independent of the faults f. It is defined as

$$\begin{aligned} P_{FA} &= \text{Prob}\{\|r\|_{ev} > J_{th} \ \mid f = 0\} \\ &= \text{Prob}\{\|G_{rd}(z)d(z)\|_{ev} > J_{th} \ \} \end{aligned} \tag{4.3}$$

The FAR is mainly decided by the threshold J_{th}, the distribution of the disturbances d as well as the residual generator. The detection delay is

$$\tau = k_d - k_0$$

where k_d represents the time instant at which the fault is detected,

$$k_d = \min_k \; \{ \, k \mid \|r\|_{ev} > J_{th}, \; f \neq 0 \}$$

Comparing (4.1) and (4.3), it is easy to see that, for a given residual generator, a higher threshold J_{th} will reduce the FAR but possibly also decrease the FDR. To achieve simultaneously a high FDR and a low FAR, in the design of the FD systems we shall try to suppress the part $G_{rd}(z)d(z)$ (the influence of the unknown disturbances) and increase the part $G_{rf}(z)f(z)$ (the influence of the faults) in the residual signal.

To calculate P_{FD}, P_{MD} and P_{FA}, the statistics of f and d are needed, which are however often not available. Therefore, for the analysis we introduce several signal sets: $\Omega_{FA,d}$, the set of unknown disturbances that cause false alarms, $\Omega_{FD,f}$, the set of detectable faults, $\Omega_{FD,f,\max}$, the maximal set of detectable faults, and $\Omega_{FD,f,\min}$, the minimal set of detectable faults, i.e.

$$\begin{aligned}
\Omega_{FA,d} &= \{ \; d \mid \|G_{rd}(z)d(z)\|_{ev} > J_{th} \} \\
\Omega_{FD,f} &= \left\{ \; f \mid \|G_{rd}(z)d(z) + G_{rf}(z)f(z)\|_{ev} > J_{th} \right\} \\
\Omega_{FD,f,\min} &= \left\{ \; f \mid \|G_{rd}(z)d(z) + G_{rf}(z)f(z)\|_{ev} > J_{th}, \; \forall d \right\} \\
\Omega_{FD,f,\max} &= \left\{ \; f \mid \|G_{rd}(z)d(z) + G_{rf}(z)f(z)\|_{ev} > J_{th}, \; \exists d \right\}
\end{aligned} \tag{4.4}$$

The size of $\Omega_{FA,d}$ can be used to represent the FAR, while the size of the sets $\Omega_{FD,f}, \Omega_{FD,f,\min}, \Omega_{FD,f,\max}$ provides a measure of the FDR.

4.2 Definition of fault sensitivity index

In the framework of the model-based FD, the performance of the FD system is often evaluated by its robustness to the unknown disturbances and its sensitivity to the faults. The design objective is interpreted as to have a high sensitivity to the faults and simultaneously a strong robustness to the unknown disturbances. The robustness of the FD system to the unknown disturbances can be described by the norms of $G_{rd}(z)$ [19, 60, 48], such as the H_∞-norm, the peak-to-peak norm, etc, which are standard in the robust control theory [64, 139, 199]. For the characterisation of the influence of faults, at the very beginning the H_∞ norm of transfer function matrices was used [61]. Later it has been recognised that the H_∞ norm as a measurement for the maximum (possible) size of a transfer function matrix may fail for a fair evaluation of the sensitivity of the residual generator for the faults. In the initial work of [46] a kind of so-called minimal sensitivity indices have been introduced. After that, [76] proposed a linear matrix inequality (LMI) based approach to the H_-/H_∞ design of fault detection systems. More recently, a sufficient condition was given for the minimal sensitivity index by [129] in the form of LMI, which is derived based on Lyapunov function. [109] derived a necessary and sufficient condition for the minimal sensitivity index. It is shown by [182] that the so-called H_- index can be obtained from the coprime factorization.

In this section, we shall study how to characterize the sensitivity of the FD system to the faults in the discrete LTI systems.

In the parity space approach, the residual dynamics is described by $r(k) = V_s(H_{d,s}d_s(k) + H_{f,s}f_s(k))$ and $\|r(k)\|_E = \sqrt{r^T(k)r(k)}$ is usually used for residual evaluation. As

$$\begin{aligned}
\bar{\sigma}\left(V_s H_{f,s}\right) &= \sup_{d_s(k)=0, f_s(k)\neq 0} \frac{\|r(k)\|_E}{\|f_s(k)\|_E} \\
\underline{\sigma}\left(V_s H_{f,s}\right) &= \inf_{d_s(k)=0, f_s(k)\neq 0} \frac{\|r(k)\|_E}{\|f_s(k)\|_E}
\end{aligned} \tag{4.5}$$

$\bar{\sigma}(V_s H_{f,s})$ indicates the maximal fault sensitivity and $\underline{\sigma}(V_s H_{f,s})$ the minimal fault sensitivity. The singular values of $V_s H_{f,s}$ between $\bar{\sigma}(V_s H_{f,s})$ and $\underline{\sigma}(V_s H_{f,s})$ represent the fault sensitivity at different levels.

In the following discussion, the fault sensitivity is investigated based on the general description of the residual dynamics $r(z) = G_{rd}(z)d(z) + G_{rf}(z)f(z)$ and the l_2-norm as evaluation function. For other residual evaluation schemes the fault sensitivity can be analyzed in a similar way.

Note that

$$\|G_{rf}(z)f(z)\|_2 \leq \|G_{rf}(z)\|_\infty \|f(z)\|_2$$

where $\|G_{rf}(z)\|_\infty = \sup_{\omega\in[0,2\pi]} \bar{\sigma}(G_{rf}(e^{j\omega}))$, which describes the maximal influence of the faults f on the residual r. Therefore, $\|G_{rf}(z)\|_\infty$ can be regarded as the maximal fault sensitivity index.

In the next, we shall check under which condition

$$\|G_{rf}(z)f(z)\|_2 > J_{th} \tag{4.6}$$

is satisfied for any $f(z) \neq 0, \|f(z)\|_2 \geq \alpha(> 0)$. Note that $\forall f(z) \neq 0$

$$\begin{aligned}\|G_{rf}(z)f(z)\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{j\omega})G_{rf}^*(e^{j\omega})G_{rf}(e^{j\omega})f(e^{j\omega})d\omega \\ &\geq \inf_{\omega\in[0,2\pi]} \frac{\underline{\sigma}^2(G_{rf}(e^{j\omega}))}{2\pi}\int_0^{2\pi} f^*(e^{j\omega})f(e^{j\omega})d\omega \\ &= \inf_{\omega\in[0,2\pi]} \underline{\sigma}^2(G_{rf}(e^{j\omega}))\,\|f(z)\|_2^2\end{aligned}$$

Let

$$\|G_{rf}(z)\|_- = \inf_{\omega\in[0,2\pi]} \underline{\sigma}(G_{rf}(e^{j\omega})) \tag{4.7}$$

In the worst case (4.6) is satisfied if and only if

$$\alpha \geq \frac{J_{th}}{\|G_{rf}(z)\|_-} := \alpha_2$$

It can be seen that, by increasing $\|G_{rf}(z)\|_-$, the size (evaluated by the l_2-norm) of the detectable fault in the worst case can be reduced, i.e. the minimal sensitivity of the FD system to the fault is improved. Therefore, $\|G_{rf}(z)\|_-$ defined by (4.7) is called the minimal fault sensitivity index.

We would like to point out that

- $\|G_{rf}(z)\|_- > 0$ only if $m \geq n_f$, i.e. the number of independent outputs is not less than the number of the faults.
- Different from the $\mathbf{H}_\infty$ norm, the minimal sensitivity index $\|G_{rf}(z)\|_-$ is not a norm. Apparently, $\|G_{rf}(z)\|_-$ can be zero even if $G_{rf}(z) \neq 0$, for instance, if $n_f > m$ or if $G_{rf}(z)$ has zeros on the unit circle.
- Different from continuous-time systems [109], for discrete LTI systems D_{rf} has full column rank is not a necessary condition of $\|G_{rf}(z)\|_- > 0$.

4.3 Computation of minimal fault sensitivity index

In this section, several approaches are given to calculate the minimal fault sensitivity index $\|G_{rf}(z)\|_-$.

4.3.1 Singular value plot

According to the definition (4.7), $\|G_{rf}(z)\|_-$ can be obtained by evaluating $\underline{\sigma}(G_{rf}(e^{j\omega}))$ over a number of fine gridded frequency points $0 \le \omega_i < 2\pi$, $i = 1, 2, \cdots, N$, and then searching for the minimum, i.e.

$$\|G_{rf}(z)\|_- = \min_i \underline{\sigma}(G_{rf}(e^{j\omega_i}))$$

This can be realized in Matlab using the singular value plot.

4.3.2 Inversion based approach

Assume that $G_{rf}(z) \in \mathbf{RL}_\infty$ is $\mathbf{RL}_\infty$-left invertible, i.e., there exists $G_{rf}^{-1}(z) \in \mathbf{RL}_\infty$ such that $G_{rf}^{-1}(z)G_{rf}(z) = I$. According to the property of singular values [199], we know that for any $\omega \in [0, 2\pi]$,

$$\underline{\sigma}(G_{rf}(e^{j\omega})) = \frac{1}{\overline{\sigma}(G_{rf}^{-1}(e^{j\omega}))}$$

It follows

$$\|G_{rf}(z)\|_- = \inf_{\omega \in [0,2\pi]} \underline{\sigma}(G_{rf}(e^{j\omega})) = \frac{1}{\sup_{\omega \in [0,2\pi]} \overline{\sigma}(G_{rf}^{-1}(e^{j\omega}))}$$

Therefore, $\|G_{rf}(z)\|_-$ is the inverse of the $\mathbf{L}_\infty$ norm of $G_{rf}^{-1}(z)$. Note that $G_{rf}^{-1}(z)$ may be unstable.

4.3.3 Coprime factorization based approach

With the help of the coprime factorization, $\|G_{rf}(z)\|_-$ can also be transformed into the inverse of the $\mathbf{H}_\infty$ norm of a stable transfer function matrix. To show it, we use the following factorization from Ionescu *et al.*

Lemma 4.1 [80] Assume that $G(z) \in \mathbf{RL}_\infty^{m\times p}$, the rank of $G(z)$ is constant on the unit circle, the realization of $G(z) = (A, B, C, D)$ is stabilizable, has no invariant zeros on the unit circle and no unreachable null modes. Then there exists a right coprime factorization

$$G(z) = N(z)M^{-1}(z) \tag{4.8}$$

so that $N(z) \in \mathbf{RH}_\infty$, $M(z) \in \mathbf{RH}_\infty$, and $N(z)$ is an inner satisfying $N^*(e^{j\omega})N(e^{j\omega}) = I$ for all $\omega \in [0, 2\pi]$,

$$\begin{aligned} M(z) &= (A + BF, BV, F, V) \in \mathbf{RH}_\infty \\ N(z) &= (A + BF, BV, C + DF, DV) \in \mathbf{RH}_\infty \end{aligned} \tag{4.9}$$

where $Y = Y^T \ge 0$, (Y, F) is the stabilizing solution of the *discrete-time algebraic Riccati system* (DTARS)

$$\begin{bmatrix} A^TYA - Y + C^TC & A^TYB + C^TD \\ B^TYA + D^TC & B^TYB + D^TD \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} = 0 \tag{4.10}$$

and V is the right inverse of a full row rank matrix H $(HV = I)$ which satisfies

$$H^TH = D^TD + B^TYB. \tag{4.11}$$

The concept of the DTARS is an extension of the notion of discrete-time algebraic Riccati equation (DTARE) [80]. A numerically sound algorithm to solve the DTARS is available (see [80] and the references therein). The existence of the solution is guaranteed if the assumptions of the lemma are satisfied. If $B^TYB + D^TD$ is nonsingular, then the DTARS (4.10) reduces to a standard DTARE as follows

$$A^TYA - Y + C^TC - (A^TYB + C^TD)(B^TYB + D^TD)^{-1}(B^TYA + D^TC) = 0 \tag{4.12}$$

Lemma 4.2 Assume that $G(z) \in \mathbf{RL}_\infty^{m\times p}$ satisfies the assumptions in Lemma 4.1, $\|G(z)\|_- > 0$, $M(z)$ is obtained by (4.9)-(4.11). Then:

(1) $D^TD + B^TYB$ in (4.11) is invertible,
(2) H and V are square and invertible.

Proof: We shall first prove (1) by contradiction. Suppose that the rank of D^TD+B^TYB is not of full rank, $rank(D^TD+B^TYB) = \rho < p$. According to (4.11), $H \in \mathbf{R}^{\rho\times p}$ is short and of full row rank and $V \in \mathbf{R}^{p\times\rho}$ is tall and of full column rank. It follows that $M(z) = \big(I + F(zI - A - BF)^{-1}B\big)V \in \mathbf{RH}_\infty^{p\times\rho}$ is a tall matrix and its left inverse $M^{-1}(z) \in \mathbf{RL}_\infty^{\rho\times p}$ is a short matrix (not of full column rank). Then, $G(z)$ can not be of full column rank and $\|G(z)\|_-$ will be zero, which is contradictory to the known condition that $\|G(z)\|_- > 0$. Therefore, (1) holds. (2) follows directly from (1). □

Assume that $G_{rf}(z)$ satisfies the assumptions in Lemma 4.1 and $\|G_{rf}(z)\|_- > 0$. Then there exists a particular right coprime factorization of $G_{rf}(z)$ as follows

$$G_{rf}(z) = N_f(z)M_f^{-1}(z) \tag{4.13}$$

where $M_f(z) \in \mathbf{RH}_\infty, N_f(z) \in \mathbf{RH}_\infty$ and $N_f(z)$ is an inner. Because $\|G_{rf}(z)\|_- > 0$, $G_{rf}(e^{j\omega})$ is of full column rank for any $\omega \in [0, 2\pi]$. Then, $M_f^{-1}(e^{j\omega})$ must also be of full column rank and $\underline{\sigma}(M_f^{-1}(e^{j\omega})) \neq 0$, $\forall\omega \in [0, 2\pi]$. Therefore,

$$\underline{\sigma}(G_{rf}(e^{j\omega})) = \underline{\sigma}(N_f(e^{j\omega})M_f^{-1}(e^{j\omega})) = \underline{\sigma}(M_f^{-1}(e^{j\omega})) = \frac{1}{\overline{\sigma}(M_f(e^{j\omega}))}.$$

Taking the minimum over the frequency ω on both sides, it yields

$$\|G_{rf}(z)\|_- = \inf_{\omega\in[0,2\pi]} \underline{\sigma}(G_{rf}(e^{j\omega})) = \frac{1}{\sup_{\omega\in[0,2\pi]} \overline{\sigma}(M_f(e^{j\omega}))}$$

We have $\|G_{rf}(z)\|_- = \frac{1}{\|M_f(z)\|_\infty}$, i.e., $\|G_{rf}(z)\|_-$ is the inverse of the $\mathbf{H}_\infty$ norm of $M_f(z) \in \mathbf{RH}_\infty$. As a result, the minimal sensitivity index can be calculated according to the following theorem.

Theorem 4.1 Given a scalar $\beta > 0$ and a system $G_{rf}(z) \in \mathbf{RL}_\infty^{m\times p}$ with $m \geq p$ and no zeros on the unit circle. Assume that $G_{rf}(z) = N_f(z)M_f^{-1}(z)$ with $M_f(z), N_f(z) \in \mathbf{RH}_\infty$ and $N_f(z)$ inner. Then

(i) $\|G_{rf}(z)\|_- = \frac{1}{\|M_f(z)\|_\infty}$.

(ii) $\|G_{rf}(z)\|_- > \beta$ if and only if $\|M_f(z)\|_\infty < \frac{1}{\beta}$.

Theorem 4.1 shows that the minimal sensitivity index of $G_{rf}(z)$ can be computed in two steps: (i) do the coprime factorization (4.13) of $G_{rf}(z)$ to get $M_f(z) \in \mathbf{RH}_\infty$, (ii) calculate the $\mathbf{H}_\infty$ norm of $M_f(z)$ using the well-established methods and finally get $\|G_{rf}(z)\|_- = \frac{1}{\|M_f(z)\|_\infty}$. The factorization needed in the first step can be carried out based on a minimal state space realization (A_{rf}, B_{rf}, C_{rf}, D_{rf}) of $G_{rf}(z)$, as shown in Lemma 4.1. This factorization is unique up to a constant unitary multiple, which will have no influence on the $\mathbf{H}_\infty$-norm of $M_f(z)$.

Based on this observation, in the following an algorithm is given for the computation of the minimal sensitivity index.

Algorithm 4.1 Computation of the minimal sensitivity index $\|G_{rf}(z)\|_-$ for a given system $G_{rf}(z)$ with a minimal state space realization (A_{rf}, B_{rf}, C_{rf}, D_{rf}), no invariant zeros on the unit circle and no unreachable null modes:

- Solve the discrete-time algebraic Riccati equation

$$\begin{bmatrix} A_{rf}^TYA_{rf} - Y + C_{rf}^TC_{rf} & A_{rf}^TYB_{rf} + C_{rf}^TD_{rf} \\ B_{rf}^TYA_{rf} + D_{rf}^TC_{rf} & B_{rf}^TYB_{rf} + D_{rf}^TD_{rf} \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} = 0 \tag{4.14}$$

  for $Y \geq 0$ and F.
- If $D_{rf}^TD_{rf} + B_{rf}^TYB_{rf}$ is not invertible, then $\|G_{rf}(z)\|_- = 0$; otherwise, compute V by

$$HV = I,\ H^TH = D_{rf}^TD_{rf} + B_{rf}^TYB_{rf} \tag{4.15}$$

- Compute $\|M_f(z)\|_\infty$ with $M_f(z) = (A_{rf} + B_{rf}F, B_{rf}V, F, V)$
- Compute $\|G_{rf}(z)\|_- = \frac{1}{\|M_f(z)\|_\infty}$.

Dually, $\|G_{rf}(z)\|_-$ can also be calculated by means of a particular left coprime factorization of $G_{rf}(z)$, which is simultaneously a co-inner-outer factorization of $G_{rf}(z)$.

4.3.4 LMI based approach

In this subsection, we shall show that, after some transformations, the necessary and sufficient condition for $\|G_{rf}(z)\|_- > \beta$ $(\beta > 0)$ can be formulated as an LMI, as stated in Theorem 4.2.

Theorem 4.2 Given a scalar $\beta > 0$ and a system $G_{rf}(z) = (A_{rf}, B_{rf}, C_{rf}, D_{rf}) \in \mathbf{RL}_\infty^{m\times n_f}$ $(m \geq n_f)$ satisfying

$$\forall\omega \in [0, 2\pi], G_{rf}^*(e^{j\omega})G_{rf}(e^{j\omega}) > 0 \tag{4.16}$$

with no invariant zeros on the unit circle and no unreachable null modes. Then $\|G_{rf}(z)\|_- > \beta$ if and only if there exists a symmetric matrix $P = P^T$ such that

$$\begin{bmatrix} A_{rf}^T P A_{rf} - P + C_{rf}^T C_{rf} & A_{rf}^T P B_{rf} + C_{rf}^T D_{rf} \\ B_{rf}^T P A_{rf} + D_{rf}^T C_{rf} & B_{rf}^T P B_{rf} + D_{rf}^T D_{rf} - \beta^2 I \end{bmatrix} > 0 \tag{4.17}$$

Proof: According to Theorem 4.1, given a transfer function matrix $G_{rf}(z) \in \mathbf{RL}_\infty$, $\|G_{rf}(z)\|_- > \beta$ $(\beta > 0)$ is equivalent to $\|M_f(z)\|_\infty < \frac{1}{\beta}$, where $M_f(z) = (A_{rf} + B_{rf}F, B_{rf}V, F, V) \in \mathbf{RH}_\infty$. By Lemma 3.1, $\|M_f(z)\|_\infty < \frac{1}{\beta}$ if and only if there exists a matrix $X = X^T > 0$ such that

$$\Phi(X) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0 \tag{4.18}$$

where

$$\begin{aligned} \Phi_{11} &= (A_{rf} + B_{rf}F)^T X (A_{rf} + B_{rf}F) - X + F^T F \\ \Phi_{12} &= (A_{rf} + B_{rf}F)^T X B_{rf} V + F^T V \\ \Phi_{22} &= V^T B_{rf}^T X B_{rf} V + V^T V - \frac{1}{\beta^2} I \end{aligned}$$

Let

$$W = \begin{bmatrix} I & O \\ -HF & H \end{bmatrix} \tag{4.19}$$

and define

$$\bar{\Phi} := -\beta^2 W^T \Phi(X) W = \begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} \\ \bar{\Phi}_{12}^T & \bar{\Phi}_{22} \end{bmatrix} \tag{4.20}$$

According to Lemma 4.2, H is nonsingular if $\|G_{rf}(z)\|_- > 0$. As a result, W is non-singular and $\Phi < 0$ is equivalent to $\bar{\Phi} > 0$. Taking into account (4.14) and (4.15), there is

$$\begin{aligned} \bar{\Phi}_{11} &= -\beta^2 \left(\Phi_{11} - F^T H^T \Phi_{12}^T - (\Phi_{12} - F^T H^T \Phi_{22}) HF \right) \\ &= A_{rf}^T (Y - \beta^2 X) A_{rf} - (Y - \beta^2 X) + C_{rf}^T C_{rf} \\ \bar{\Phi}_{12} &= -\beta^2 (\Phi_{12} - F^T H^T \Phi_{22}) H = A_{rf}^T (Y - \beta^2 X) B_{rf} + C_{rf}^T D_{rf} \\ \bar{\Phi}_{22} &= -\beta^2 H^T \Phi_{22} H = B_{rf}^T (Y - \beta^2 X) B_{rf} + D_{rf}^T D_{rf} - \beta^2 I \end{aligned}$$

Let

$$P = Y - \beta^2 X \tag{4.21}$$

We get

$$\bar{\Phi} = \begin{bmatrix} A_{rf}^T P A_{rf} - P + C_{rf}^T C_{rf} & A_{rf}^T P B_{rf} + C_{rf}^T D_{rf} \\ B_{rf}^T P A_{rf} + D_{rf}^T C_{rf} & B_{rf}^T P B_{rf} + D_{rf}^T D_{rf} - \beta^2 I \end{bmatrix} \tag{4.22}$$

Thus, $\|G_{rf}(z)\|_- > \beta$ ($\beta > 0$) if and only if $\bar{\Phi} > 0$ for some symmetric matrix P. □

From the above derivation, it can be seen that, though $Y \geq 0$ and $X > 0$, the definiteness of $P = Y - \beta^2 X$ is uncertain. Therefore, in the necessary and sufficient condition of $\|G_{rf}(z)\|_- > \beta$ ($\beta > 0$) the matrix P is only required to be symmetric. Moreover, it follows from the above discussion that

- The condition that
$$\forall \omega \in [0, 2\pi], G_{rf}^*(e^{j\omega}) G_{rf}(e^{j\omega}) > 0 \tag{4.23}$$
or equivalently in its state space form
$$\forall \omega \in [0, 2\pi], \begin{bmatrix} A_{rf} - e^{j\omega} I & B_{rf} \\ C_{rf} & D_{rf} \end{bmatrix} \text{ has full column rank} \tag{4.24}$$
is necessary to ensure that $\|G_{rf}(z)\|_- > 0$.
- The assumption $G_{rf}(z) \in \mathbf{RH}_\infty$ is not necessary for the achieved results.

According to Theorem 4.2, $\|G_{rf}(z)\|_-$ can be obtained as the largest value of β that satisfies (4.17). This can be realized as follows:

- Set the initial value of β as $\beta = \beta_0$.
- Check the feasibility of (4.17). If (4.17) has a symmetric matrix P as solution, then the value of β can be increased. Otherwise, reduce the value of β.

The iteration can be implemented using the well-known bisection algorithm.

4.3.5 Example

In this subsection, we shall give an example to illustrate the above results.

Example 4.1 Consider a system $G_{rf}(z) = (A_{rf}, B_{rf}, C_{rf}, D_{rf})$ with

$$A_{rf} = \begin{bmatrix} 0.5 & -0.3 & 0.2 \\ 0 & 0.4 & 0.2 \\ 0.7 & 0 & 0.6 \end{bmatrix}, B_{rf} = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix}$$
$$C_{rf} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, D_{rf} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

It is a stable system. We shall first determine the minimal sensitivity index of $G_{rf}(z)$ according to the definition (4.7). The singular value of $G_{rf}(e^{j\omega})$ is plotted in Fig.4.1. The step size of the frequency points is chosen as $\Delta\omega = 0.005$. As can be seen from the figure, the smallest singular value is 0.37609. Therefore, $\|G_{rf}(z)\|_- = \inf_{\omega \in [0,\infty]} \underline{\sigma}(G_{rf}(e^{j\omega})) = 0.37609$.

Using Algorithm 4.1 to calculate $\|G_{rf}(z)\|_-$, we first solve the Riccati equation (4.14) using the Matlab function *dare*. The stabilizing solution to (4.14) is

$$X = \begin{bmatrix} 1.1103 & 1.0218 & 0.2276 \\ 1.0218 & 2.1127 & 1.1198 \\ 0.2276 & 1.1198 & 1.3568 \end{bmatrix} > 0$$

By (4.15), we get

$$F = \begin{bmatrix} -1.1351 & -0.3225 & -0.9301 \end{bmatrix}$$
$$H = 0.8772, V = 1.1400$$

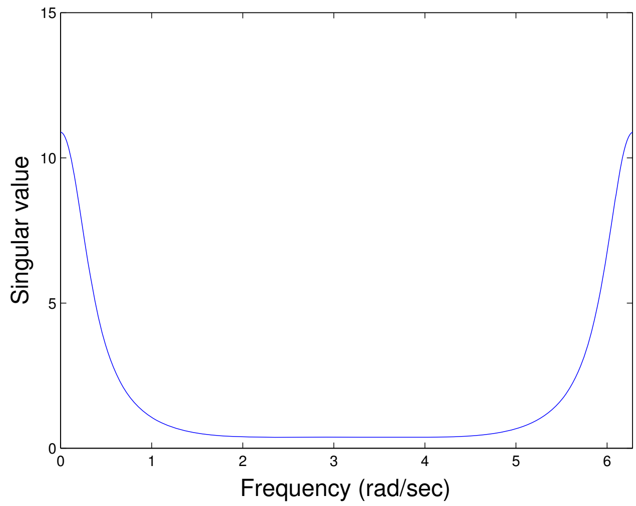

Fig. 4.1 Singular values of $G_{rf}(e^{j\omega})$

Then, using the Matlab function *norm* we obtain that the H_∞ norm of the system $M_f(z) = (A_{rf} + B_{rf}F, B_{rf}V, F, V)$ is $\|M_f(z)\|_\infty = 2.6589$. Therefore, $\|G_{rf}(z)\|_- = 1/\|M_f(z)\|_\infty = 0.37609$.

To apply Theorem 4.2 to the computation of $\|G_{rf}(z)\|_-$, we solve the LMI (4.17) (without requirement on the definiteness of the symmetric matrix P) iteratively to find the maximal β satisfying (4.17). It gives also $\|G_{rf}(z)\|_- = 0.37608$. The corresponding matrix is

$$P = \begin{bmatrix} -0.6118 & 1.8350 & 1.1945 \\ 1.8350 & 0.0710 & -0.9671 \\ 1.1945 & -0.9671 & -0.9295 \end{bmatrix}$$

The eigenvalues of the matrix P are at -3.1843, 0.1178, 1.5963. It is neither positive definite nor negative definite. In comparison, the LMI (4.17) has a positive definite symmetric matrix P as feasible solution for $\beta \leq 0.3751 = \beta_1$ and (4.17) has a negative definite symmetric matrix P as feasible solution only for $\beta \leq 0.0634 = \beta_2$. Both β_1 and β_2 are smaller than the real value of $\|G_{rf}(z)\|_-$.

4.4 FD problem formulation

After introducing the concept of the fault sensitivity, we can formulate the FD system design problem. The discussion is carried out based on the general description of the residual dynamics $r(z) = G_{rd}(z)d(z) + G_{rf}(z)f(z)$.

4.4.1 Full decoupling

The residual signal r is said to be fully decoupled from the unknown disturbances d if the parameters of the residual generator can be selected in such a way that [31]

$$G_{rd}(z) = 0,\ G_{rf}(z) \neq 0 \tag{4.25}$$

i.e. $r \neq 0$ if and only if $f \neq 0$.

Recalling (2.31) and (2.32), that means $R(z)$ should satisfy

$$R(z)M_u(z)G_d(z) = 0,\ \ R(z)M_u(z)G_f(z) \neq 0 \tag{4.26}$$

According to (2.33), $M_u(z)$ is a square matrix of full rank. Therefore, there exists a post-filter $R(z)$ satisfying (4.26) if and only if [31]

$$rank\left[G_d(z)\ G_f(z) \right] > rank G_d(z) \tag{4.27}$$

Suppose that (A, E_d, C, F_d) and (A, E_f, C, F_f) are, respectively, the state space realizations of $G_d(z)$ and $G_f(z)$. Then the condition of full decoupling (4.27) is equivalent to

$$rank \begin{bmatrix} zI - A & E_d & E_f \\ C & F_d & F_f \end{bmatrix} > rank \begin{bmatrix} zI - A & E_d \\ C & F_d \end{bmatrix} \tag{4.28}$$

Specifically, in case of the parity relation based residual generator (2.5) or (2.9), the condition of full decoupling is

$$rank\left[H_{o,s}\ H_{d,s}\ H_{f,s} \right] > rank\left[H_{o,s}\ H_{d,s} \right] \tag{4.29}$$

In case of a full decoupling, the residual dynamics is only influenced by the faults, i.e.

$$r(z) = G_{rf}(z)f(z) = R(z)M_u(z)G_f(z)f(z) \tag{4.30}$$

Assume that

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n_f} \end{bmatrix}, \ G_f(z) = \left[G_{f1}(z)\ G_{f2}(z) \cdots G_{fn_f}(z) \right]$$

Then

$$r(z) = \sum_{i=1}^{n_f} R(z)M_u(z)G_{fi}(z)f_i(z) \tag{4.31}$$

Depending on the fault sensitivity, the faults $f_i, i = 1, \cdots, n_f$, can be divided into two sets. The set of faults

$$\mathcal{F}_1 = \{ f_i \mid G_{fi}(z) \not\subseteq \operatorname{Im}(G_d(z)) \} \tag{4.32}$$

can be easily detected, as any nonzero $f_i \in \mathcal{F}_1$ will cause a deviation of the residual signal from zero. However, the sensitivity of the FD system is zero to the set of faults

$$\mathcal{F}_2 = \{ f_i \mid G_{fi}(z) \subset \operatorname{Im}(G_d(z)) \} \tag{4.33}$$

because the residual is decoupled from the faults in $\mathcal{F}_2$ as well. Therefore, in the design of a full decoupling FD system, the analysis of the fault sensitivity is important. To ensure that each fault f_i is detectable, the full decoupling condition (4.25) is strengthened into

$$G_{rd}(z) = 0,\ G_{rf_i}(z) \neq 0,\ \forall i \tag{4.34}$$

In the parity space approach, the full decoupling problem can be solved very easily. In the observer based approach, it can be solved by the unknown input observer approach, the geometric approach or the eigenstructure assignment approach [19, 75, 115].

4.4.2 Optimization problems

If a full decoupling is not achievable, in order to enhance the robustness of the FD system to the unknown disturbances and simultaneously improve its sensitivity to the faults, it is straightforward to formulate the optimal design problem as a minmax problem

$$\|G_{rd}(z)\| \to \min, \ \ \|G_{rf}(z)\| \to \max \tag{4.35}$$

where $\|\bullet\|$ denotes some suitably selected norm or index for the evaluation of the robustness and the sensitivity. The optimization problem (4.35) is a multiobjective optimization problem.

To facilitate the derivation of a solution, (4.35) is often simplified as [19, 76, 111], for a given β (> 0),

$$\|G_{rf}(z)\| > \beta, \quad \|G_{rd}(z)\| \to \min \tag{4.36}$$

or, for a given γ (> 0)

$$\|G_{rd}(z)\| < \gamma, \quad \|G_{rf}(z)\| \to \max \tag{4.37}$$

or,

$$\|G_{rf}(z)\| - \|G_{rd}(z)\| \to \max \tag{4.38}$$

In the FD literature, it is also proposed to transform the fault detection problem into a fault estimation problem [19, 42], i.e. design $R(z)$, such that

$$r - W_f f = R(z)M_u(z)G_d(z)d(z) + (R(z)M_u(z)G_f(z) - W_f(z))\, f(z) \to 0 \tag{4.39}$$

where W_f is a dynamic weighting matrix that characterizes the frequency band of the faults of interest. The design problem is thus

$$\left\| \begin{bmatrix} R(z)M_u(z)G_d(z) & R(z)M_u(z)G_f(z) - W_f(z) \end{bmatrix} \right\| \to \min \tag{4.40}$$

Aiming at unifying the design objectives $\min \|G_{rd}(z)\|$ and $\max \|G_{rf}(z)\|$, the following *ratio type performance index* has been introduced and received much attention in the FD study [19, 44, 61, 62, 34, 60, 137, 159, 172]

$$J = \frac{\text{influence of the faults on the residual}}{\text{influence of the disturbances on the residual}} = \frac{\|G_{rf}(z)\|}{\|G_{rd}(z)\|} \tag{4.41}$$

The numerator of J represents the fault sensitivity and the denominator the robustness to the unknown disturbances. The higher the fault sensitivity is and the stronger the robustness to the unknown disturbances is, the bigger will be the value of J. Therefore, the optimal design problem can be formulated as

$$\max J = \max \frac{\|G_{rf}(z)\|}{\|G_{rd}(z)\|} \tag{4.42}$$

To handle the model uncertainty, it is proposed by Ding and co-workers to design the FD system, so that the residual signal approximates the best residual dynamics in the ideal case of no model uncertainty [190, 197], i.e.

$$\begin{aligned} r - r_{opt} &= r - (W_d d + W_f f) \\ &= (R(z)M_u(z)G_d(z) - W_d(z))d(z) + (R(z)M_u(z)G_f(z) - W_f(z))\, f(z) \to 0 \end{aligned} \tag{4.43}$$

where W_d and W_f represent the best residual dynamics achievable if there is no model uncertainty. The design problem is formulated as

$$\left\| \begin{bmatrix} R(z)M_u(z)G_d(z) - W_d(z) & R(z)M_u(z)G_f(z) - W_f(z) \end{bmatrix} \right\| \to \min \tag{4.44}$$

The optimization problems (4.35)-(4.38), (4.40) and (4.44) can be solved *iteratively* with the LMI technique, which is well-established in the robust control theory. In the next chapter, our focus will be mainly put on the ratio type optimization problems (4.41).

Recall that the optimality is only "optimal" in some sense. In the selection of the norms and the indexes for the optimization, it is necessary to take into account how the residual signal will be evaluated later. For instance, if the $\mathbf{H}_\infty$-norm of $G_{rd}(z)$ is minimized but the residual signal is evaluated by its amplitude, then the optimality doesn't contribute much to the improvement of the overall FD performance. Therefore, for the successful application of the FD systems, it is essential to design the residual generation and the residual evaluation in an integrated way.

It is well-known that, for a given system, the input signals can be re-constructed from the output signals, only if certain conditions are satisfied. Therefore, a reasonable solution can be obtained for the fault estimation problem (4.39) only in some cases. A post-analysis of the FD performance of the optimal solution to (4.40) is recommendable.

4.4.3 Optimal compromise between the FDR and the FAR

From the application viewpoint, the optimal design of the FD systems aims to improve the FDR and simultaneously reduce the FAR [35, 184], i.e.

$$P_{FD} \to \max, \; P_{FA} \to \min \tag{4.45}$$

(4.45) is often simplified as:

- Given an allowable FAR, design the FD system so that the FDR is maximized, or the dual problem
- Given an expected FDR, design the FD system so that the FAR is minimized.

4.5 Conclusion

This chapter gives an introduction to the main criteria for the evaluation of the FD performance and different formulations of the FD design problem. The key of designing model-based FD systems is to achieve a high fault detection rate (FDR) and simultaneously a low false alarm rate (FAR). It is equivalent to enhance the robustness of the FD system to the unknown disturbances and simultaneously guarantee its sensitivity to the faults. Because the robustness has been extensively studied in the robust control theory, we have focused our discussion on the fault sensitivity. Several methods for the calculation of the minimal fault sensitivity index are presented. In the next chapter, we shall concentrate on the optimal design of the FD systems.

5

Optimization of FD systems

In the last chapter, it has been shown that different kinds of optimization problems can be formulated to improve the FD performance. The central task is to improve the sensitivity of the FD system to the faults and simultaneously enhance its robustness to the disturbances. In this chapter, we shall discuss how to solve these optimization problems.

5.1 Optimization of parity relation based residual generators

5.1.1 Parity vector

Recall that the dynamics of the parity relation based residual generator (2.5) is described by

$$r(k) = v_s H_{d,s} d_s(k) + v_s H_{f,s} f_s(k)$$

where $v_s \in \mathbf{P}_s$ is the so-called parity vector, r is a scalar residual signal. In the framework of the parity space approach, the amplitude of the residual signal

$$\|r(k)\|_E = \sqrt{r^T(k)r(k)} \tag{5.1}$$

is often used for residual evaluation. Hence, a ratio type performance index can be defined as [159]

$$J_{PS} = \frac{\sup_{d_s(k)=0, f_s(k)\neq 0} \frac{r^T(k)r(k)}{f_s^T(k)f_s(k)}}{\sup_{f_s(k)=0, d_s(k)\neq 0} \frac{r^T(k)r(k)}{d_s^T(k)d_s(k)}} = \frac{v_s H_{f,s} H_{f,s}^T v_s^T}{v_s H_{d,s} H_{d,s}^T v_s^T} \tag{5.2}$$

The optimization problem is thus formulated as

$$\max_{v_s \in \mathbf{P}_s} J_{PS} = \max_{v_s, v_s H_{o,s}=0} \frac{v_s H_{f,s} H_{f,s}^T v_s^T}{v_s H_{d,s} H_{d,s}^T v_s^T} \tag{5.3}$$

Let N_{basis} denote the basis matrix of the parity space $\mathbf{P}_s$. Then, any parity vector $v_s \in \mathbf{P}_s$ can be expressed by

$$v_s = p_s N_{basis} \tag{5.4}$$

where $p_s \in \mathbf{R}^{1\times(m(s+1)-rankH_{o,s})}$ is a freely selectable vector. By substituting (5.4) into (5.3), the constrained optimization problem (5.3) is transformed into an unconstrained one

$$\max_{p_s} J_{PS} = \max_{p_s} \frac{p_s N_{basis} H_{f,s} H_{f,s}^T N_{basis}^T p_s^T}{p_s N_{basis} H_{d,s} H_{d,s}^T N_{basis}^T p_s^T} \tag{5.5}$$

If $N_{basis}H_{d,s}$ is not of full row rank, then p_s can be selected in the left null space of $N_{basis}H_{d,s}$ to achieve a full decoupling and in this case $J_{PS,opt} = \infty$. If $N_{basis}H_{d,s}$ is of full row rank, then according to the Rayleigh-ratio theorem [103], the performance index J_{PS} varies in the range

$$\lambda_{\min} \leq J_{PS} \leq \lambda_{\max} \tag{5.6}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are, respectively, the maximal and the minimal generalized eigenvalue of the following generalized eigenvalue-eigenvector problem

$$p_s(N_{basis}H_{f,s}H_{f,s}^T N_{basis}^T - \lambda N_{basis}H_{d,s}H_{d,s}^T N_{basis}^T) = 0 \tag{5.7}$$

Denote with $p_{s,\max}$ the eigenvector corresponding to the generalized eigenvalue $\lambda_{\max}$. The maximum $J_{PS} = \lambda_{\max}$ is achieved when $p_s = p_{s,\max}$. Therefore, the optimization problem (5.3) is solved by

$$v_{s,opt} = p_{s,\max} N_{basis} \tag{5.8}$$

and the optimal performance index is

$$J_{PS,opt} = \lambda_{\max} \tag{5.9}$$

Theorem 5.1 Given the LTI system (2.1) and the residual generator (2.5). Assume that N_{basis} is the basis matrix of the parity space $\mathbf{P}_s$ defined by (2.7) and $N_{basis}H_{d,s}$ is of full row rank. The parity vector $v_{s,opt}$ given by (5.7)-(5.8) solves the optimization problem (5.3). The corresponding optimal performance index is (5.9).

Let

$$\delta_d = \max_k \|d_s(k)\|_E = \max_k \|d\|_{2,[k-s,k]} \tag{5.10}$$

Following (3.8), the threshold for the residual generator (2.5) can be set as

$$J_{th} = \sup_{f=0,d} \|r(k)\|_E = \left(v_s H_{d,s} H_{d,s}^T v_s^T\right)^{\frac{1}{2}} \delta_d \tag{5.11}$$

If

$$\|r(k)\|_E = \sqrt{r^T(k) r(k)} > J_{th} = \left(v_s H_{d,s} H_{d,s}^T v_s^T\right)^{\frac{1}{2}} \delta_d$$

then it is deduced that a fault has happened.

5.1.2 Parity matrix

If a parity matrix V_s is used in the residual generation, then the residual signal obtained by residual generator (2.9) is a vector signal with dynamics governed by [33]

$$r(k) = V_s H_{d,s} d_s(k) + V_s H_{f,s} f_s(k)$$

Denote the singular values of the matrix $V_s H_{f,s}$ in descending order as $\sigma_1(V_s H_{f,s}) \geq \sigma_2(V_s H_{f,s}) \geq \cdots \geq \sigma_{n_r}(V_s H_{f,s})$. As

$$\bar{\sigma}(V_s H_{f,s}) = \sigma_1(V_s H_{f,s}) = \sup_{d_s(k)=0, f_s(k)\neq 0} \frac{\|r(k)\|_E}{\|f_s(k)\|_E}$$
$$\underline{\sigma}(V_s H_{f,s}) = \sigma_{n_r}(V_s H_{f,s}) = \inf_{d_s(k)=0, f_s(k)\neq 0} \frac{\|r(k)\|_E}{\|f_s(k)\|_E}$$

$\bar{\sigma}(V_s H_{f,s})$ and $\underline{\sigma}(V_s H_{f,s})$ represent, respectively, the maximal and the minimal influence of the vector $f_s(k)$ on $r(k)$. Those singular values between $\bar{\sigma}(V_s H_{f,s})$ and $\underline{\sigma}(V_s H_{f,s})$ describe the influence of $f_s(k)$ on $r(k)$ at intermediate levels. Therefore, the ratio type optimization problem (4.42) can be defined in this case as

$$\max_{V_s,V_sH_{o,s}=0} J_{PS,\infty/\infty} = \max_{V_s,V_sH_{o,s}=0} \frac{\sup_{d_s(k)=0,f_s(k)\neq 0} \frac{r^T(k)r(k)}{f_s^T(k)f_s(k)}}{\sup_{f_s(k)=0,d_s(k)\neq 0} \frac{r^T(k)r(k)}{d_s^T(k)d_s(k)}}$$
$$= \max_{V_s,V_sH_{o,s}=0} \frac{\bar{\sigma}^2(V_sH_{f,s})}{\bar{\sigma}^2(V_sH_{d,s})} \tag{5.12}$$

$$\max_{V_s,V_sH_{o,s}=0} J_{PS,-/\infty} = \max_{V_s,V_sH_{o,s}=0} \frac{\inf_{d_s(k)=0,f_s(k)\neq 0} \frac{r^T(k)r(k)}{f_s^T(k)f_s(k)}}{\sup_{f_s(k)=0,d_s(k)\neq 0} \frac{r^T(k)r(k)}{d_s^T(k)d_s(k)}}$$
$$= \max_{V_s,V_sH_{o,s}=0} \frac{\underline{\sigma}^2(V_sH_{f,s})}{\bar{\sigma}^2(V_sH_{d,s})} \tag{5.13}$$

$$\max_{V_s,V_sH_{o,s}=0} J_{PS,i/\infty} = \max_{V_s,V_sH_{o,s}=0} \frac{\sigma_i^2(V_sH_{f,s})}{\bar{\sigma}^2(V_sH_{d,s})} \tag{5.14}$$

The maximal fault sensitivity is considered in (5.12), while the fault sensitivity in the worst case is considered in (5.13). (5.14) is a generalization of (5.12) and (5.13), which takes into account the fault sensitivity in different directions.

The constrained optimization problems (5.12)-(5.14) can be first transformed into unconstrained optimization problems by expressing V_s as $V_s = P_sN_{basis}$, where $P_s \in \mathbf{R}^{n_r\times(m(s+1)-rankH_{o,s})}$ can be arbitrarily selected. Correspondingly, (5.12)-(5.14) are equivalent to

$$\max_{V_s,V_sH_{o,s}=0} J_{PS,\infty/\infty} = \max_{P_s} \frac{\bar{\sigma}^2(P_sN_{basis}H_{f,s})}{\bar{\sigma}^2(P_sN_{basis}H_{d,s})} \tag{5.15}$$

$$\max_{V_s,V_sH_{o,s}=0} J_{PS,-/\infty} = \max_{P_s} \frac{\underline{\sigma}^2(P_sN_{basis}H_{f,s})}{\bar{\sigma}^2(P_sN_{basis}H_{d,s})} \tag{5.16}$$

$$\max_{V_s,V_sH_{o,s}=0} J_{PS,i/\infty} = \max_{P_s} \frac{\sigma_i^2(P_sN_{basis}H_{f,s})}{\bar{\sigma}^2(P_sN_{basis}H_{d,s})} \tag{5.17}$$

Similar to the parity vector case, if $N_{basis}H_{d,s}$ is not of full row rank, then a full decoupling may be achieved. In case that $N_{basis}H_{d,s}$ is of full row rank, its singular value decomposition (SVD) is

$$N_{basis}H_{d,s} = U\begin{bmatrix} S & O \end{bmatrix} V^T,\ S = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_\gamma \end{bmatrix} \tag{5.18}$$

where $\sigma_1, \cdots, \sigma_\gamma$ are nonzero singular values, U and V are unitary matrices. Because S and U are both invertible, P_s can be re-written as $P_s = \bar{P}_sS^{-1}U^T$. Then we have

$$\bar{\sigma}(V_sH_{d,s}) = \bar{\sigma}(P_sN_{basis}H_{d,s}) = \bar{\sigma}\left(\bar{P}_sS^{-1}U^TU\begin{bmatrix} S\ O \end{bmatrix}V^T\right) = \bar{\sigma}(\bar{P}_s)$$

Notice that, for any two matrices X_1 and X_2 of compatible dimensions, the following relations always hold

$$\bar{\sigma}(X_1X_2) \leq \bar{\sigma}(X_1)\bar{\sigma}(X_2) \tag{5.19}$$
$$\underline{\sigma}(X_1X_2) \leq \bar{\sigma}(X_1)\underline{\sigma}(X_2) \tag{5.20}$$
$$\sigma_i(X_1X_2) \leq \bar{\sigma}(X_1)\sigma_i(X_2) \tag{5.21}$$

Moreover, the equality in (5.19)-(5.21) holds if X_1 is a unitary matrix, i.e. $X_1^TX_1 = X_1X_1^T = I$. Taking into account (5.19)-(5.21), there is

$$\begin{aligned}
J_{PS,\infty/\infty} &= \frac{\bar{\sigma}^2(\bar{P}_s S^{-1}U^T N_{basis}H_{f,s})}{\bar{\sigma}^2(\bar{P}_s)} \le \frac{\bar{\sigma}^2(\bar{P}_s)\bar{\sigma}^2\left(S^{-1}U^T N_{basis}H_{f,s}\right)}{\bar{\sigma}^2(\bar{P}_s)} \\
&= \bar{\sigma}^2\left(S^{-1}U^T N_{basis}H_{f,s}\right) \\
J_{PS,-/\infty} &= \frac{\underline{\sigma}^2(\bar{P}_s S^{-1}U^T N_{basis}H_{f,s})}{\bar{\sigma}^2(\bar{P}_s)} \le \frac{\bar{\sigma}^2(\bar{P}_s)\underline{\sigma}^2\left(S^{-1}U^T N_{basis}H_{f,s}\right)}{\bar{\sigma}^2(\bar{P}_s)} \\
&= \underline{\sigma}^2\left(S^{-1}U^T N_{basis}H_{f,s}\right) \\
J_{PS,i/\infty} &= \frac{\sigma_i^2(\bar{P}_s S^{-1}U^T N_{basis}H_{f,s})}{\bar{\sigma}^2(\bar{P}_s)} \le \frac{\bar{\sigma}^2(\bar{P}_s)\sigma_i^2\left(S^{-1}U^T N_{basis}H_{f,s}\right)}{\bar{\sigma}^2(\bar{P}_s)} \\
&= \sigma_i^2\left(S^{-1}U^T N_{basis}H_{f,s}\right)
\end{aligned}$$

The upper bounds of $J_{PS,\infty/\infty}, J_{PS,-/\infty}$ and $J_{PS,i/\infty}$ are achieved if $\bar{P}_s$ is a unitary matrix. Therefore, optimization problems (5.15)-(5.17) are simultaneously solved by

$$V_{s,opt} = \bar{P}_s S^{-1}U^T N_{basis} \tag{5.22}$$

where $\bar{P}_s$ is any unitary matrix of compatible dimensions.

Theorem 5.2 [33] Given the LTI system (2.1) and the residual generator (2.9). Assume that N_{basis} is the basis matrix of the parity space $\mathbf{P}_s$ defined by (2.7) and $N_{basis}H_{d,s}$ is of full row rank. The parity matrix given by (5.22) solves the optimization problems (5.12)-(5.14) simultaneously. The corresponding optimal performance indices are

$$\begin{aligned}
J_{PS,\infty/\infty,opt} &= \bar{\sigma}^2\left(S^{-1}U^T N_{basis}H_{f,s}\right) \\
J_{PS,-/\infty,opt} &= \underline{\sigma}^2\left(S^{-1}U^T N_{basis}H_{f,s}\right) \\
J_{PS,i/\infty,opt} &= \sigma_i^2\left(S^{-1}U^T N_{basis}H_{f,s}\right)
\end{aligned} \tag{5.23}$$

It is interesting to notice that the optimal solution to (5.13) (or equivalently to (5.16)) is not unique. Assume that $N_{basis}H_{f,s}$ is left invertible and its left inverse is denoted by $(N_{basis}H_{f,s})^{-1}$, i.e.

$$(N_{basis}H_{f,s})^{-1} N_{basis}H_{f,s} = I$$

Let

$$V_{s,-/\infty} = \bar{Q}_s (N_{basis}H_{f,s})^{-1} N_{basis} \tag{5.24}$$

where $\bar{Q}_s$ is a matrix of compatible dimensions. Because

$$\begin{aligned}
\bar{\sigma}(V_{s,-/\infty}H_{d,s}) &= \bar{\sigma}(\bar{Q}_s (N_{basis}H_{f,s})^{-1} N_{basis}H_{d,s}) \\
&\ge \underline{\sigma}(\bar{Q}_s)\bar{\sigma}((N_{basis}H_{f,s})^{-1} N_{basis}H_{d,s})
\end{aligned} \tag{5.25}$$

there is

$$\begin{aligned}
J_{PS,-/\infty} &= \frac{\underline{\sigma}^2(V_{s,-/\infty}H_{f,s})}{\bar{\sigma}^2(V_{s,-/\infty}H_{d,s})} \le \frac{\underline{\sigma}^2(\bar{Q}_s (N_{basis}H_{f,s})^{-1} N_{basis}H_{f,s})}{\underline{\sigma}^2(\bar{Q}_s)\bar{\sigma}^2((N_{basis}H_{f,s})^{-1} N_{basis}H_{d,s})} \\
&= \frac{1}{\bar{\sigma}^2((N_{basis}H_{f,s})^{-1} N_{basis}H_{d,s})}
\end{aligned} \tag{5.26}$$

If $\bar{Q}_s$ is a unitary matrix, then the equality in (5.25) holds and $J_{PS,-/\infty}$ achieves the upper bound in (5.26). As

$$\frac{1}{\bar{\sigma}^2\left((N_{basis}H_{f,s})^{-1}N_{basis}H_{d,s}\right)}$$
$$= \frac{1}{\bar{\sigma}^2\left((N_{basis}H_{f,s})^{-1}U\left[\,S\;O\,\right]V^T\right)}$$
$$= \frac{1}{\bar{\sigma}^2\left((N_{basis}H_{f,s})^{-1}US\right)}$$
$$= \underline{\sigma}^2\left(S^{-1}U^TN_{basis}H_{f,s}\right)$$

the upper bound in (5.26) is exactly the same with $J_{PS,-/\infty,opt}$ given in (5.23). Therefore, the optimization problem (5.16) is also solved by

$$V_{s,-/\infty,opt} = \bar{Q}_s\left(N_{basis}H_{f,s}\right)^{-1}N_{basis} \tag{5.27}$$

where $\bar{Q}_s$ is an arbitrary unitary matrix.

Theorem 5.3 [172] Given the LTI system (2.1) and the residual generator (2.9). Assume that N_{basis} is the basis matrix of the parity space $\mathbf{P}_s$ defined by (2.7), $N_{basis}H_{f,s}$ is of full column rank and $(N_{basis}H_{f,s})^{-1}$ is the left inverse of $N_{basis}H_{f,s}$. Then the parity matrix given by (5.27) leads to

$$J_{PS,-/\infty}\left(V_{s,-/\infty,opt}\right) = J_{PS,-/\infty,opt}$$

and solves the optimization problem (5.13).

In comparison, $V_{s,-/\infty,opt}$ given in (5.27) solves only the optimization problem (5.16), while the optimization problems (5.15), (5.16) and (5.17) are simultaneously solved by $V_{s,opt}$ given in (5.22). Therefore, in the following discussion $V_{s,opt}$ is called the *unified parity space solution*.

Following (3.8), the threshold for the residual generator (2.9) is set as

$$\begin{aligned} J_{th} &= \sup_{f=0,d}\|r(k)\|_E = \bar{\sigma}\left(V_sH_{d,s}\right)\delta_d \\ &= \begin{cases} \delta_d, & \text{if } V_s = V_{s,opt} \\ \bar{\sigma}\left((N_{basis}H_{f,s})^{-1}N_{basis}H_{d,s}\right)\delta_d, & \text{if } V_s = V_{s,-/\infty,opt} \end{cases} \end{aligned} \tag{5.28}$$

where δ_d is given by (5.10). The decision logic is

$$\begin{cases} \|r(k)\|_E = \sqrt{r^T(k)r(k)} > J_{th} & \Rightarrow \text{ faulty} \\ \|r(k)\|_E = \sqrt{r^T(k)r(k)} \le J_{th} & \Rightarrow \text{ fault-free} \end{cases} \tag{5.29}$$

At the end of this subsection, we shall briefly discuss the relation between the optimum of the optimization problems (5.3) and (5.12).

Lemma 5.1 [33] Given the LTI system (2.1) and the residual generators (2.5) and (2.9). Denote with $J_{PS,opt}$ and $J_{PS,\infty/\infty,opt}$ the optimal performance indices of the optimization problems (5.3) and (5.12), respectively. Then, $J_{PS,opt} = J_{PS,\infty/\infty,opt}$.

Proof: According to (5.5),

$$J_{PS} = \frac{p_sN_{basis}H_{f,s}H_{f,s}^TN_{basis}^Tp_s^T}{p_sN_{basis}H_{d,s}H_{d,s}^TN_{basis}^Tp_s^T}$$

where p_s is an arbitrary vector of compatible dimensions. Because $S^{-1}U^T$ is an invertible matrix, p_s can always be re-written as $p_s = \bar{p}_sS^{-1}U^T$. Taking into account (5.18), we get

$$J_{PS} = \frac{\bar{p}_sS^{-1}U^TN_{basis}H_{f,s}H_{f,s}^TN_{basis}^TUS^{-1}\bar{p}_s^T}{\bar{p}_s\bar{p}_s^T}$$

Therefore,

$$\begin{aligned} J_{PS,opt} &= \max_{\bar{p}_s} J_{PS} = \bar{\lambda}\left(S^{-1}U^T N_{basis} H_{f,s} H_{f,s}^T N_{basis}^T U S^{-1}\right) \\ &= \bar{\sigma}^2\left(S^{-1}U^T N_{basis} H_{f,s}\right) = J_{PS,\infty/\infty,opt} \end{aligned}$$

i.e. the optimization problems (5.3) and (5.12) achieve the same optimal performance index. □

5.1.3 Extended form

As shown in (2.14), an observer structure can be introduced in the parity space approach to improve the numerical properties. Based on the residual dynamics

$$r(k) = V_{se}(H_{L,d,s} d_s(k) + H_{L,f,s} f_s(k)) \tag{5.30}$$

the optimization problems can be formulated as

$$\begin{aligned} \max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} J_{PSE,\infty/\infty} &= \max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} \frac{\sup_{d_s(k)=0, f_s(k)\neq 0} \frac{r^T(k)r(k)}{f_s^T(k)f_s(k)}}{\sup_{f_s(k)=0, d_s(k)\neq 0} \frac{r^T(k)r(k)}{d_s^T(k)d_s(k)}} \\ &= \max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} \frac{\bar{\sigma}^2(V_{se}H_{L,f,s})}{\bar{\sigma}^2(V_{se}H_{L,d,s})} \end{aligned} \tag{5.31}$$

$$\begin{aligned} \max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} J_{PSE,-/\infty} &= \max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} \frac{\inf_{d_s(k)=0, f_s(k)\neq 0} \frac{r^T(k)r(k)}{f_s^T(k)f_s(k)}}{\sup_{f_s(k)=0, d_s(k)\neq 0} \frac{r^T(k)r(k)}{d_s^T(k)d_s(k)}} \\ &= \max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} \frac{\underline{\sigma}^2(V_{se}H_{L,f,s})}{\bar{\sigma}^2(V_{se}H_{L,d,s})} \end{aligned} \tag{5.32}$$

$$\max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} J_{PSE,i/\infty} = \max_{\substack{V_{se},L \\ V_{se}H_{L,o,s}=0}} \frac{\sigma_i^2(V_{se}H_{L,f,s})}{\bar{\sigma}^2(V_{se}H_{L,d,s})} \tag{5.33}$$

At first we shall show that the observer gain L has no influence on the optimal FD performance. Suppose that, for a given observer gain matrix $\bar{L}$, the parity matrix $\bar{V}_{se}$ ($\bar{V}_{se}H_{\bar{L},o,s} = 0$) solves (5.31)-(5.33) and generates a residual $\bar{r}$ with optimal dynamics

$$\bar{r}(k) = \bar{V}_{se}(H_{\bar{L},d,s} d_s(k) + H_{\bar{L},f,s} f_s(k))$$

Assume now the observer gain is selected as $L \neq \bar{L}$. Note that there exists always an invertible matrix Q given by

$$Q = Q_{\bar{L}} Q_L^{-1} = \begin{bmatrix} I & O & \cdots & O \\ C(L-\bar{L}) & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C(A-\bar{L}C)^{s-1}(L-\bar{L}) & \cdots & C(L-\bar{L}) & I \end{bmatrix}$$

such that

$$\begin{aligned} QH_{L,o,s} &= H_{\bar{L},o,s} \\ QH_{L,f,s} &= H_{\bar{L},f,s} \\ QH_{L,d,s} &= H_{\bar{L},d,s} \end{aligned}$$

Let $V_{se} = \bar{V}_{se}Q$. Then $V_{se}H_{L,o,s} = \bar{V}_{se}QH_{L,o,s} = \bar{V}_{se}H_{\bar{L},o,s} = 0$ and

$$\begin{aligned} r(k) &= V_{se}\left(H_{L,d,s}d_s(k) + H_{L,f,s}f_s(k)\right) \\ &= \bar{V}_{se}Q\left(H_{L,d,s}d_s(k) + H_{L,f,s}f_s(k)\right) = \bar{r}(k) \end{aligned}$$

i.e. the residual dynamics will keep to be the same if $V_{se} = \bar{V}_{se}Q$. Therefore, as long as the observer gain matrix L stabilizes $A-LC$, the optimal residual $\bar{r}$ can always be obtained by a suitable selection of V_{se}. As a result, the optimization problems (5.31)-(5.33) can be solved in two steps:

- set the observer gain matrix L,
- calculate the optimal parity matrix V_{se} using the approaches introduced in the last subsection.

Moreover, we have the following lemma.

Lemma 5.2 Given the LTI system (2.1), the residual generators (2.9) and (2.14). Then

$$\begin{aligned} J_{PSE,\infty/\infty,opt} &= J_{PS,\infty/\infty,opt} \\ J_{PSE,-/\infty,opt} &= J_{PS,-/\infty,opt} \\ J_{PSE,i/\infty,opt} &= J_{PS,i/\infty,opt} \end{aligned}$$

Lemma 5.2 shows that the introduction of the observer structure in the parity space approach will not change the optimal achievable performance.

5.1.4 Optimizations in terms of FAR and FDR

In this subsection, we shall consider how to solve the FD design problems from the viewpoint of the FAR and the FDR in the framework of the parity space approach [172].

At first, we shall look at the optimization problem of *maximization of the FDR under a given FAR*.

To guarantee that the FAR lies under the allowed level α ($P_{FA} \leq \alpha$, $0 < \alpha \leq 1$), the threshold can be set as

$$J_{th,\alpha} = \bar{\sigma}\left(V_s H_{d,s}\right) F^{-1}(1-\alpha) \tag{5.34}$$

where $F^{-1}(\rho)$ is the inverse function of $F(\rho)$, and $F(\rho)$ is the cumulative distribution function of $\|d_s(k)\|_E$. According to the decision logic (5.29), a fault f is detected by a residual generator with parity matrix V_s, if

$$\|V_s\left(H_{d,s}d_s(k) + H_{f,s}f_s(k)\right)\|_E > J_{th,\alpha} = \bar{\sigma}\left(V_s H_{d,s}\right) F^{-1}(1-\alpha) \tag{5.35}$$

As mentioned before, any parity matrix V_s can always be re-written as $V_s = \bar{P}_s S^{-1} U^T N_{basis}$, where $\bar{P}_s$ is some matrix of compatible dimensions, S and U are defined by (5.18) and thus both invertible, N_{basis} is the basis matrix of the parity space $\mathbf{P}_s$. Then (5.35) is equivalent to

$$\begin{aligned} &\left\|\bar{P}_s\left[\, I \; O \,\right] V^T d_s(k) + \bar{P}_s S^{-1} U^T N_{basis} H_{f,s} f_s(k)\right\|_E \\ &> \bar{\sigma}\left(\bar{P}_s\left[\, I \; O \,\right] V^T\right) F^{-1}(1-\alpha) = \bar{\sigma}\left(\bar{P}_s\right) F^{-1}(1-\alpha) \end{aligned} \tag{5.36}$$

According to the property of the maximal singular value,

$$\begin{aligned} &\|\bar{P}_s\left[\, I \; O \,\right] V^T d_s(k) + \bar{P}_s S^{-1} U^T N_{basis} H_{f,s} f_s(k)\|_E \\ &\leq \bar{\sigma}\left(\bar{P}_s\right) \left\|\left[\, I \; O \,\right] V^T d_s(k) + S^{-1} U^T N_{basis} H_{f,s} f_s(k)\right\|_E \end{aligned} \tag{5.37}$$

Hence, (5.36) holds, only if

$$\left\|\left[\, I \; O \,\right] V^T d_s(k) + S^{-1} U^T N_{basis} H_{f,s} f_s(k)\right\|_E > F^{-1}(1-\alpha) \tag{5.38}$$

Note that, the equality in (5.37) holds and the condition (5.38) becomes both necessary and sufficient, if $\bar{P}_s$ is a unitary matrix. In this case, V_s is exactly $V_{s,opt}$ given by (5.22). Let the set of detectable faults be denoted by

$$\begin{aligned} \Omega_{FD,f}(V_s) &= \{ \ f \mid f \text{ satisfies } (5.35) \ \} \\ \Omega_{FD,f}(V_{s,opt}) &= \{ \ f \mid f \text{ satisfies } (5.38) \ \} \end{aligned}$$

Because any fault f satisfying (5.35) will also satisfy (5.38), $\Omega_{FD,f}(V_s) \subseteq \Omega_{FD,f}(V_{s,opt})$. Therefore, the set of detectable faults delivered by (5.22) is the biggest. In other words, the FDR is maximized by (5.22).

Theorem 5.4 Given the LTI system (2.1) and the residual generator (2.9). The parity matrix $V_{s,opt}$ given by (5.22) ensures that

$$\Omega_{FD,f}(V_{s,opt}) \supseteq \Omega_{FD,f}(V_s), \ \forall V_s \tag{5.39}$$

and achieves the maximal FDR under a given FAR.

In the next, we shall show that the dual optimization problem of *minimization of the FAR under a given FDR* is solved by (5.27).

Suppose that $\underline{\sigma}\left(V_s H_{f,s}\right) \neq 0$ and $d = 0$. Because

$$\|r\|_E = \|V_s H_{f,s} f_s(k)\|_E \geq \underline{\sigma}\left(V_s H_{f,s}\right) \|f_s(k)\|_E \tag{5.40}$$

Set the threshold as

$$J_{th,\alpha} = \alpha \underline{\sigma}\left(V_s H_{f,s}\right) \tag{5.41}$$

Then the faults that satisfy

$$\|f_s(k)\|_E > \alpha$$

ensure $\|r\|_E \geq J_{th,\alpha}$ and can always be detected. Hence, the size of α indicates the percentage of the faults that can always be detected. Correspondingly, the FDR can be fixed by setting α. A false alarm will happen if in the fault-free case

$$\|r\|_E = \|V_s H_{d,s} d_s(k)\|_E > J_{th,\alpha} \tag{5.42}$$

i.e.

$$\|V_s H_{d,s} d_s(k)\|_E - \alpha \underline{\sigma}\left(V_s H_{f,s}\right) > 0 \tag{5.43}$$

Denote the set of all disturbances d which satisfy (5.43) by $\Omega_{FA,d}(V_s)$, i.e.

$$\Omega_{FA,d}(V_s) = \{d \mid d \text{ satisfies } (5.43)\}$$

Since $\Omega_{FA,d}(V_s)$ is the set of all disturbances that would cause false alarms, its size is a reasonable measurement of the FAR. In this context, the size of $\Omega_{FA,d}(R)$ is interpreted as the FAR. Let $V_s = \bar{Q}_s \left(N_{basis} H_{f,s}\right)^{-1} N_{basis}$. (5.43) can be re-written into

$$\left\| \bar{Q}_s \left(N_{basis} H_{f,s}\right)^{-1} N_{basis} H_{d,s} d_s(k) \right\|_E - \alpha \underline{\sigma}\left(\bar{Q}_s\right) > 0 \tag{5.44}$$

Note that

$$\begin{aligned} &\left\| \bar{Q}_s \left(N_{basis} H_{f,s}\right)^{-1} N_{basis} H_{d,s} d_s(k) \right\|_E \\ &\geq \underline{\sigma}\left(\bar{Q}_s\right) \left\| \left(N_{basis} H_{f,s}\right)^{-1} N_{basis} H_{d,s} d_s(k) \right\|_E \end{aligned} \tag{5.45}$$

There is

$$\begin{aligned} &\left\| \bar{Q}_s \left(N_{basis} H_{f,s}\right)^{-1} N_{basis} H_{d,s} d_s(k) \right\|_E - \alpha \underline{\sigma}\left(\bar{Q}_s\right) \\ &\geq \underline{\sigma}\left(\bar{Q}_s\right) \left(\left\| \left(N_{basis} H_{f,s}\right)^{-1} N_{basis} H_{d,s} d_s(k) \right\|_E - \alpha \right) \end{aligned} \tag{5.46}$$

As a result,

$$\left\| (N_{basis}H_{f,s})^{-1} N_{basis}H_{d,s}d_s(k) \right\|_E - \alpha > 0 \tag{5.47}$$

is sufficient for (5.43). That means, any d satisfying (5.47) will result in

$$\begin{aligned} & \|V_s H_{d,s} d_s(k)\|_E - \alpha \underline{\sigma} (V_s H_{f,s}) \\ = & \left\| \bar{Q}_s (N_{basis}H_{f,s})^{-1} N_{basis}H_{d,s}d_s(k) \right\|_E - \alpha \underline{\sigma} \left(\bar{Q}_s\right) > 0 \end{aligned}$$

Considering that the equality in (5.45) and (5.46) can be achieved by selecting $\bar{Q}_s$ as a unitary matrix, i.e. if V_s is given by (5.27). Let

$$\Omega_{FA,d}(V_{s,-/\infty,opt}) = \{d \mid d \text{ satisfies } (5.47)\}$$

Any $d \in \Omega_{FA,d}(V_{s,-/\infty,opt})$ will also belong to the set $\Omega_{FA,d}(V_s)$. Therefore,

$$\Omega_{FA,d}\left(V_{s,-/\infty,opt}\right) \subseteq \Omega_{FA,d}\left(V_s\right)$$

i.e. $V_{s,-/\infty,opt}$ delivers the smallest set of disturbances that would cause false alarms and therefore ensures the lowest FAR.

Theorem 5.5 Given the LTI system (2.1) and the residual generator (2.9). The parity matrix $V_{s,-/\infty,opt}$ given by (5.27) ensures that

$$\Omega_{FA,d}\left(V_{s,-/\infty,opt}\right) \subseteq \Omega_{FA,d}\left(V_s\right), \ \forall V_s \tag{5.48}$$

and achieves the minimal FAR under a given FDR.

5.1.5 Example

Example 5.1 Consider the FD problem of a system in the form of (2.1) with matrices

$$\begin{aligned} A &= \begin{bmatrix} 1 & -0.34 & -1 & 2 \\ -1 & 0.6 & 4 & -2.3 \\ -0.2 & 0.2 & 0.8 & 0 \\ -1.5 & 0.1 & 3.6 & 1 \end{bmatrix}, B = \begin{bmatrix} -0.1 \\ 0.8 \\ 0.2 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ E_d &= \begin{bmatrix} 0.3 & 1.2 \\ 0 & 0.2 \\ 1 & 0.4 \\ 0.8 & 0 \end{bmatrix}, E_f = \begin{bmatrix} 0.5 & 0 \\ -1.5 & 0.3 \\ -0.6 & 1 \\ 0 & 0.4 \end{bmatrix} \\ F_d &= \begin{bmatrix} 0.3 & 0 \\ 0.1 & 0.5 \end{bmatrix}, F_f = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix} \end{aligned} \tag{5.49}$$

Suppose that $s = 2$. The basis matrix of the parity space is

$$N_{basis} = \begin{bmatrix} 0.6744 & -0.1799 & -0.5903 & -0.3099 & 0.2580 & 0.0420 \\ 0.0474 & -0.3496 & 0.5450 & -0.5552 & 0.1303 & 0.5033 \end{bmatrix}$$

The optimization problem (5.3) is solved by the parity vector

$$v_{s,opt} = \begin{bmatrix} 0.0224 & -0.1597 & 0.2479 & -0.2537 & 0.0597 & 0.2297 \end{bmatrix}$$

which leads to the optimal performance index $J_{PS,opt} = 1.2689$.

If a parity matrix is used, then

$$V_{s,opt} = \begin{bmatrix} 0.0850 & -0.1702 & 0.1823 & -0.2726 & 0.0816 & 0.2243 \\ 0.6670 & -0.1043 & -0.7120 & -0.1898 & 0.2306 & -0.0679 \end{bmatrix}$$

solves the optimization problems (5.12)-(5.14) simultaneously, $J_{PS,\infty/\infty,opt} = 1.2689$ and $J_{PS,-/\infty,opt} = 0.0868$.

In this example, the eigenvalues of the matrix A are $1.5720 + 1.7932i$, $1.5720 - 1.7932i$, -0.4532, 0.7091. For large s, for instance, $s = 50$, there is possibly a numerical problem. As shown in Section 5.1.3, an observer structure can be introduced to improve the numerical property. Let

$$L = \begin{bmatrix} 1.4275 & 0.3019 \\ -1.7337 & -0.5996 \\ -0.0688 & 0.2774 \\ -0.1553 & 0.7671 \end{bmatrix}$$

the eigenvalues of $A - LC$ are located at 0.5, 0.6, 0.65, 0.7. The optimal solution to the optimization problems (5.31)-(5.33) is

$$V_{s,opt} = \begin{bmatrix} -0.0180 & 0.0728 & 0.0674 & -0.2576 & -0.0611 & 0.2240 \\ 0.1020 & 0.0220 & -0.3118 & -0.0787 & 0.2369 & 0.0690 \end{bmatrix}$$

The optimal performance index are, respectively,

$$J_{PSE,\infty/\infty,opt} = 1.2689$$
$$J_{PSE,-/\infty,opt} = 0.0868$$

It shows that the observer gain matrix L has indeed no influence on the optimal performance index.

5.2 Optimization of post-filters

It is shown in Chapter 2 that, for the systems described by (2.1) or (2.2), all LTI residual generators can be unifiedly expressed in the form of (2.31) and the residual dynamics is governed by

$$r(z) = R(z)M_u(z)\left(G_d(z)d(z) + G_f(z)f(z)\right)$$

where $M_u(z) = I - C(zI - A + LC)^{-1}L$ is decided by a matrix L that stabilizes $A - LC$, which can be interpreted as an observer gain matrix.

At first, we would like to point out that the parameter L is not important for the optimization of the residual dynamics [32, 190]. Suppose that $\bar{L}$, together with $\bar{R}(z)$, generates a residual $\bar{r}$ with optimal dynamics

$$\bar{r}(z) = \bar{R}(z)\bar{M}_u(z)\left(G_d(z)d(z) + G_f(z)f(z)\right)$$
$$\bar{M}_u(z) = I - C(zI - A + \bar{L}C)^{-1}\bar{L}$$

If now a different matrix L is used, then the optimal residual dynamics $\bar{r}$ can still be achieved if we just let $R(z) = \bar{R}(z)Q(z)$, where

$$Q(z) = I + C(sI - A + \bar{L}C)^{-1}(L - \bar{L}) \in \mathbf{RH}_\infty$$

because $Q(z)M_u(z) = \bar{M}_u(z)$. That means, as long as L stabilizes $A - LC$, the optimal residual $\bar{r}(z)$ can always be obtained by a suitable selection of the post-filter. Therefore, in this section, we shall concentrate on the optimal selection of the post-filter $R(z)$ in the sense of (4.42).

5.2.1 H_∞/H_∞, H_-/H_∞ and H_i/H_∞ design

The H_∞/H_∞, H_-/H_∞ and H_i/H_∞ optimal design problems are formulated, respectively, as

$$\begin{aligned}\max_{L,R(z)\in\mathbf{RH}_\infty} J_{FRE,\infty/\infty} &= \max_{L,R(z)\in\mathbf{RH}_\infty} \frac{\sup_{d=0,f\neq 0}\frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,d\neq 0}\frac{\|r\|_2}{\|d\|_2}} \\ &= \max_{L,R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)M_u(z)G_f(z)\|_\infty}{\|R(z)M_u(z)G_d(z)\|_\infty} \end{aligned} \quad (5.50)$$

$$\begin{aligned}\max_{L,R(z)\in\mathbf{RH}_\infty} J_{FRE,-/\infty} &= \max_{L,R(z)\in\mathbf{RH}_\infty} \frac{\inf_{d=0,f\neq 0}\frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,d\neq 0}\frac{\|r\|_2}{\|d\|_2}} \\ &= \max_{L,R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)M_u(z)G_f(z)\|_-}{\|R(z)M_u(z)G_d(z)\|_\infty} \end{aligned} \quad (5.51)$$

$$\max_{L,R(z)\in\mathbf{RH}_\infty} J_{FRE,i/\infty} = \max_{L,R(z)\in\mathbf{RH}_\infty} \frac{\sigma_i(R(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega}))}{\|R(z)M_u(z)G_d(z)\|_\infty} \quad (5.52)$$

In the above optimization problems, the robustness is evaluated by the $\mathbf{H}_\infty$ norm, while the fault sensitivity is evaluated differently. In (5.50) the fault sensitivity is evaluated by the $\mathbf{H}_\infty$ norm, which can here be interpreted as the best case sensitivity. In (5.51), the minimal fault sensitivity index introduced in Chapter 4 is adopted. In (5.52), a number of optimization problems are considered. The numerator of $J_{FRE,i/\infty}$ reflects the sensitivity of $G_{rf}(z)$ at different frequencies and in different directions. As for a given post filter $R(z)$

$$\begin{aligned}&\|R(z)M_u(z)G_f(z)\|_\infty \\ &= \sup_{\omega\in[0,2\pi]} \bar{\sigma}(R(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega})) \\ &\geq \sigma_i(R(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega})) \\ &\geq \inf_{\omega\in[0,2\pi]} \underline{\sigma}(R(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega})) \\ &= \|R(z)M_u(z)G_f(z)\|_-\end{aligned}$$

there is the following relation between the performance indexes

$$J_{FRE,\infty/\infty} \geq J_{FRE,i/\infty} \geq J_{FRE,-/\infty},\ \forall R(z)\in\mathbf{RH}_\infty \quad (5.53)$$

Assume that

$$M_u(z)G_d(z) = G_{do}(z)G_{di}(z) \quad (5.54)$$

is a co-inner-outer factorization (CIOF) of $M_u(z)G_d(z)$, where $G_{do}(z)$ is the co-outer and $\mathbf{RH}_\infty$-left-invertible, and $G_{di}(z)$ is the co-inner satisfying

$$G_{di}(e^{j\omega})G_{di}^*(e^{j\omega}) = I,\ \forall\omega\in[0,2\pi]$$

Let

$$R(z) = Q(z)G_{do}^{-1}(z) \quad (5.55)$$

Notice that

$$\begin{aligned}\|Q(z)G_{do}^{-1}(z)M_u(z)G_f(z)\|_\infty &\leq \|Q(z)\|_\infty\|G_{do}^{-1}(z)M_u(z)G_f(z)\|_\infty \\ \|Q(z)G_{do}^{-1}(z)M_u(z)G_f(z)\|_- &\leq \|Q(z)\|_\infty\|G_{do}^{-1}(z)M_u(z)G_f(z)\|_- \\ \sigma_i(Q(e^{j\omega})G_{do}^{-1}(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega})) &\leq \|Q(z)\|_\infty\sigma_i(G_{do}^{-1}(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega}))\end{aligned} \quad (5.56)$$

We have

$$\begin{aligned}
J_{FRE,\infty/\infty} &= \frac{\|R(z)M_u(z)G_f(z)\|_\infty}{\|R(z)G_{do}(z)G_{di}(z)\|_\infty} \\
&= \frac{\left\|Q(z)G_{do}^{-1}(z)M_u(z)G_f(z)\right\|_\infty}{\|Q(z)\|_\infty} \\
&\le \left\|G_{do}^{-1}(z)M_u(z)G_f(z)\right\|_\infty \\
J_{FRE,-/\infty} &= \frac{\|R(z)M_u(z)G_f(z)\|_-}{\|R(z)M_u(z)G_d(z)\|_\infty} \\
&= \frac{\left\|Q(z)G_{do}^{-1}(z)M_u(z)G_f(z)\right\|_-}{\|Q(z)\|_\infty} \\
&\le \left\|G_{do}^{-1}(z)M_u(z)G_f(z)\right\|_- \\
J_{FRE,i/\infty} &= \frac{\sigma_i(R(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega}))}{\|R(z)M_u(z)G_d(z)\|_\infty} \\
&= \frac{\sigma_i(Q(z)G_{do}^{-1}(z)M_u(z)G_f(z))}{\|Q(z)\|_\infty} \\
&\le \sigma_i(G_{do}^{-1}(z)M_u(z)G_f(z))
\end{aligned}$$

The upper bounds in the above inequalities will be achieved if $Q(z)$ is a unitary matrix. Therefore, optimization problems (5.50)-(5.52) are solved simultaneously by

$$R_{opt}(z) = Q_d G_{do}^{-1}(z) \tag{5.57}$$

where Q_d is a unitary matrix of compatible dimensions.

Theorem 5.6 [34, 180, 176] Given the LTI system (2.1) and the residual generator (2.31). Assume that

$$G_d(e^{j\omega})G_d^*(e^{j\omega}) > 0,\ \forall\omega \in [0, 2\pi]$$

Then the post-filter $R_{opt}(z)$ given by (5.57) solves the optimization problems (5.50)-(5.52) simultaneously. The corresponding optimal performance indices are

$$\begin{aligned}
J_{FRE,\infty/\infty,opt} &= \left\|G_{do}^{-1}(z)M_u(z)G_f(z)\right\|_\infty \\
J_{FRE,-/\infty,opt} &= \left\|G_{do}^{-1}(z)M_u(z)G_f(z)\right\|_- \\
J_{FRE,i/\infty,opt}(\omega) &= \sigma_i(G_{do}^{-1}(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega}))
\end{aligned} \tag{5.58}$$

Moreover, for any i and ω, it always holds

$$J_{FRE,\infty/\infty,opt} \ge J_{FRE,i/\infty,opt}(\omega) \ge J_{FRE,-/\infty,opt} \tag{5.59}$$

The optimal solution $R_{opt}(z)$ lead to

$$\begin{aligned}
R_{opt}(z)M_u(z)G_d(z) &= Q_d G_{di}(z) \\
\|R_{opt}(z)M_u(z)G_d(z)\|_\infty &= \|Q_d G_{di}(z)\|_\infty = 1
\end{aligned}$$

We would like to emphasize that the optimal post-filter $R_{opt}(z)$ is not unique in the sense that, depending on different co-inner-outer factorization approaches, different $R_{opt}(z)$ may be obtained.

The optimal solution to the $\mathbf{H}_-/\mathbf{H}_\infty$ problem (5.51) is not unique. Assume that $M_u(z)G_f(z)$ has no zeros on the unit circle and is of full column rank, i.e. $\|M_u(z)G_f(z)\|_- > 0$. Do the CIOF of $M_u(z)G_f(z)$ as follows

$$M_u(z)G_f(z) = G_{fo}(z)G_{fi}(z)$$

where $G_{fo}(z) \in \mathbf{RH}_\infty$ is co-outer and its left inverse $G_{fo}^{-1}(z) \in \mathbf{RH}_\infty, G_{fi}(z) \in \mathbf{RH}_\infty$ is co-inner. Let

$$R(z) = Q(z)G_{fo}^{-1}(z), Q(z) \in \mathbf{RH}_\infty \tag{5.60}$$

There is

$$J_{FRE,-/\infty} = \frac{\|R(z)M_u(z)G_f(z)\|_-}{\|R(z)M_u(z)G_d(z)\|_\infty} = \frac{\|Q(z)G_{fi}(z)\|_-}{\|Q(z)G_{fo}^{-1}(z)M_u(z)G_d(z)\|_\infty}$$

Due to

$$\|Q(z)G_{fo}^{-1}(z)M_u(z)G_d(z)\|_\infty \geq \|Q(z)\|_- \|G_{fo}^{-1}(z)M_u(z)G_d(z)\|_\infty \tag{5.61}$$

it holds

$$J_{FRE,-/\infty} \leq \frac{1}{\|G_{fo}^{-1}(z)M_u(z)G_d(z)\|_\infty}$$

Moreover, $J_{FRE,-/\infty}$ achieves the maximum if the equality in (5.61) holds. This is the case if $Q(z)$ is a unitary matrix. Notice that

$$\begin{aligned}
&\frac{1}{\|G_{fo}^{-1}(z)M_u(z)G_d(z)\|_\infty} \\
&= \frac{1}{\|G_{fo}^{-1}(z)G_{do}(z)G_{di}(z)\|_\infty} = \frac{1}{\|G_{fo}^{-1}(z)G_{do}(z)\|_\infty} \\
&= \|G_{do}^{-1}(z)G_{fo}(z)\|_- = \left\|G_{do}^{-1}(z)G_{fo}(z)G_{fi}(z)\right\|_- \\
&= \left\|G_{do}^{-1}(z)M_u(z)G_f(z)\right\|_- = J_{FRE,-/\infty,opt}
\end{aligned}$$

Hence,

$$R_{opt,-/\infty}(z) = Q_f G_{fo}^{-1}(z) \tag{5.62}$$

with unitary matrix Q_f of compatible dimensions is another solution to the optimization problem (5.51).

Theorem 5.7 [172] Given the LTI system (2.1) and the residual generator (2.31). Assume that

$$G_f^*(e^{j\omega})G_f(e^{j\omega}) > 0, \ \forall\omega \in [0, 2\pi] \tag{5.63}$$

Then the post-filter $R_{opt,-/\infty}(z)$ given by (5.62) solves the optimization problem (5.51) and $J_{FRE,-/\infty}(R_{opt,-/\infty}(z)) = J_{FRE,-/\infty,opt}$.

Let $\delta_{d,2} = \max \|d\|_2$. Using the l_2-norm as evaluation function and applying (3.8), the threshold is set as

$$\begin{aligned}
J_{th} &= \sup_{f=0,d} \|r\|_2 = \|R(z)M_u(z)G_d(z)\|_\infty \, \delta_{d,2} \\
&= \begin{cases} \delta_{d,2}, & \text{if } R(z) = R_{opt}(z) \\ \|G_{fo}^{-1}(z)M_u(z)G_d(z)\|_\infty \delta_{d,2}, & \text{if } R(z) = R_{opt,-/\infty}(z) \end{cases}
\end{aligned} \tag{5.64}$$

The decision logic is

$$\begin{cases} \|r\|_2 > J_{th} & \Rightarrow \text{ faulty} \\ \|r\|_2 \leq J_{th} & \Rightarrow \text{ fault-free} \end{cases} \tag{5.65}$$

Finally, we are going to remove the assumption (5.63) and extend the solution (5.62) so that it can be applied for any system described by (2.1). This extension is of practical interest and will enhance the applicability of the proposed approach considerably.

To achieve the optimal solution $R_{opt,-/\infty}(z)$, it is required that

$$\|M_u(z)G_f(z)\|_- \neq 0$$

Note that in the case of $\|M_u(z)G_f(z)\|_- = 0$, there exists a class of faults which are, independent of their size, structurally not detectable. They can be, for $n_f > m$, vectors in the right null subspace of $M_u(z)G_f(z)$, or for $\text{rank}\big(M_u(e^{j\omega})G_f(e^{j\omega})\big) < m$, those vectors corresponding to the zeros $M_u(e^{j\omega})G_f(e^{j\omega})$ on the unit circle. *The basic idea behind the extension study is to exclude these* *faults and consider only the structurally detectable faults.* For this purpose, an extended CIOF of $M_u(z)G_f(z)$ can be used, which is described by

$$M_u(z)G_f(z) = G_{fo}(z)G_C(z)G_{fi}(z) \tag{5.66}$$

where $G_{fi}(z)$ is co-inner, $G_{fo}(z)$ has a left inverse in $\mathbf{RH}_\infty$, $G_C(z)$ has the same zeros on the unit circle as $M_u(z)G_f(z)$. Considering that $\|G_C(z)G_{fi}(z)\|_- = 0$, it is reasonable to define

$$f^*(z) = \frac{G_C(z)}{\|G_C(z)\|_\infty} G_{fi}(z)f(z) \tag{5.67}$$

$$\Longrightarrow \|f^*\|_2 \leq \|G_{fi}(z)f(z)\|_2 \leq \|f\|_2 \tag{5.68}$$

and re-formulate the fault detection problem as finding $R(z)$ such that the residual generator

$$r(z) = R(z)\left(M_u(z)G_d(z)d(z) + \bar{G}_{fo}(z)f^*(z)\right)$$

with $\bar{G}_{fo}(z) = G_{fo}(z)\|G_C(z)\|_\infty$ is optimal in the sense of minimizing the FAR under a given FDR. This problem can then be solved using Theorem 5.7 and the optimal solution is given by $R_{opt}(z) = Q_f \bar{G}_{fo}^{-1}(z)$.

5.2.2 H_2/H_2 design

The H_2/H_2 optimal design problem is formulated as

$$\sup_{R(z)\in\mathbf{RH}_\infty^{1\times m}} J_{FRE,2/2} \tag{5.69}$$

$$= \sup_{R(z)\in\mathbf{RH}_\infty^{1\times m}} \frac{\int_0^{2\pi} R(e^{j\omega})M_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})M_u^*(e^{j\omega})R^*(e^{j\omega})d\omega}{\int_0^{2\pi} R(e^{j\omega})M_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})M_u^*(e^{j\omega})R^*(e^{j\omega})d\omega}$$

The optimal solution is presented in the following theorem [43, 178].

Theorem 5.8 Given the LTI system (2.1) and the residual generator (2.31). Assume that $\lambda_{\max}(\omega)$ and $v_{\max}(e^{j\omega})$ are, respectively, the maximal eigenvalue and the corresponding eigenvector of the generalized eigenvalue-eigenvector problem

$$v_{\max}(e^{j\omega})(M_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})M_u^*(e^{j\omega}) - \lambda_{\max}(\omega)M_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})M_u^*(e^{j\omega})) = 0 \tag{5.70}$$

ω_0 is the frequency at which $\lambda_{\max}(\omega)$ achieves the maximum, i.e.

$$\lambda_{\max}(\omega_0) = \sup_\omega \lambda_{\max}(\omega) \tag{5.71}$$

and $f_{\omega_0}(z)$ is an ideal frequency selector satisfying

$$\forall q(z) \in \mathbf{RH}_\infty^{1\times m}, \quad f_{\omega_0}(e^{j\omega})q(e^{j\omega}) = 0, \quad \omega \neq \omega_0$$

$$\int_0^{2\pi} f_{\omega_0}(e^{j\omega})q(e^{j\omega})q^*(e^{j\omega})f_{\omega_0}^*(e^{j\omega})d\omega = q(e^{j\omega_0})q^*(e^{j\omega_0})$$

Then, the optimization problem (5.69) is solved by

$$R_{opt,2/2}(z) = f_{\omega_0}(z)v_{\max}(z) \tag{5.72}$$

and the optimal H_2/H_2 performance index is

$$J_{FRE,2/2,opt} = \lambda_{\max}(\omega_0)$$

Proof: For any given ω, the matrices $M_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})M_u^*(e^{j\omega})$ and $M_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})M_u^*(e^{j\omega})$ are positive semi-definite Hermitian matrices. Therefore, the generalized eigenvalue $\lambda(\omega)$ in (5.70) is always real. It is easy to verify that $J_{FRE,2/2}\left(R_{opt,2/2}(z)\right) = \lambda_{\max}(\omega_0)$. On the other side, for any $\lambda > \lambda_{\max}(\omega_0)$ and $v(z)$ we have

$$\begin{aligned} v(e^{j\omega})(M_u(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})M_u^*(e^{j\omega}) - \\ \lambda M_u(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})M_u^*(e^{j\omega}))v^*(e^{j\omega}) < 0 \end{aligned} \quad (5.73)$$

which shows $J_{FRE,2/2} < \lambda$. This demonstrates that (5.72) is the optimal solution to (5.69). □

In practice, a bandpass filter $\tilde{f}_{\omega_0}(z)$ with a very narrow frequency bandwidth is usually used to approximate the ideal frequency selector $f_{\omega_0}(z)$. Using such a frequency selector may, however, lead to a high miss detection rate if the frequency spectrum of the fault strongly differs from the frequency range defined by the frequency selector, as pointed out by [35]. For this reason, a bank of frequency selectors should be used in practice to ensure a high sensitivity of the FD system to the possible faults over a wide frequency range.

5.2.3 Optimizations in terms of FAR and FDR

In this subsection, we shall first consider the optimization problem of maximizing FDR under a given FAR. Let the l_2 norm be the residual evaluation function and set the fault detection logic as

$$\begin{cases} \|r\|_2 \le J_{th} \Longrightarrow \text{fault-free} \\ \|r\|_2 > J_{th} \Longrightarrow \text{faulty} \end{cases}$$

As discussed in Chapter 3, to guarantee that the FAR is lower than the allowed level α, the threshold can be set as

$$J_{th,\alpha} = \|R(z)M_u(z)G_d(z)\|_\infty F^{-1}(1-\alpha)$$

where $F(\rho)$ is the cumulative distribution function of $\|d\|_2$, and $F^{-1}(\rho)$ is the inverse function of $F(\rho)$. A fault f is detected by the residual generator (2.31), if

$$\begin{aligned} &\|R(z)M_u(z)\left(G_d(z)d(z) + G_f(z)f(z)\right)\|_2 \\ &> J_{th,\alpha} = \|R(z)M_u(z)G_d(z)\|_\infty F^{-1}(1-\alpha) \end{aligned} \quad (5.74)$$

Let the post-filter $R(z)$ be re-written as $R(z) = Q(z)G_{do}^{-1}(z)$, where $Q(z) \in \mathbf{RH}_\infty$ is an arbitrary filter of compatible dimensions, $G_{do}^{-1}(z)$ is the inverse of the co-outer of $M_u(z)G_d(z)$. Then a fault f is detected if and only if

$$\left\|Q(z)G_{di}(z)d(z) + Q(z)G_{do}^{-1}(z)M_u(z)G_f(z)f(z)\right\|_2 > \|Q(z)\|_\infty F^{-1}(1-\alpha) \quad (5.75)$$

Because

$$\begin{aligned} &\left\|Q(z)G_{di}(z)d(z) + Q(z)G_{do}^{-1}(z)M_u(z)G_f(z)f(z)\right\|_2 \\ &\le \|Q(z)\|_\infty \left\|G_{di}(z)d(z) + G_{do}^{-1}(z)M_u(z)G_f(z)f(z)\right\|_2 \end{aligned} \quad (5.76)$$

(5.75) holds, only if

$$\left\|G_{di}(z)d(z) + G_{do}^{-1}(z)M_u(z)G_f(z)f(z)\right\|_2 > F^{-1}(1-\alpha) \quad (5.77)$$

The equality in (5.76) holds and the condition (5.77) becomes both necessary and sufficient, if $Q(z)$ is a unitary matrix. In this case, $R(z)$ is exactly $R_{opt}(z)$ given by (5.57). Let

$$\begin{aligned} \Omega_{FD,f}(R(z)) &= \{\ f \mid f \text{ satisfies (5.75)}\ \} \\ \Omega_{FD,f}(R_{opt}(z)) &= \{\ f \mid f \text{ satisfies (5.77)}\ \} \end{aligned}$$

Because any fault f satisfying (5.75) will also satisfy (5.77), $\Omega_{FD,f}(R(z)) \subseteq \Omega_{FD,f}(R_{opt}(z))$. Therefore, (5.57) achieves the biggest set of detectable faults. In other words, the FDR is maximized by (5.57).

Theorem 5.9 Given $M_u(z)G_d(z) \in \mathbf{RH}_\infty$ and $M_u(z)G_f(z) \in \mathbf{RH}_\infty$. Assume that $G_d(e^{j\omega})$ has no zero on the unit circle, i.e.

$$\forall \omega \in [0, 2\pi], G_d(e^{j\omega})G_d^*(e^{j\omega}) > 0$$

Then $R_{opt}(z) = Q_d G_{do}^{-1}(z)$ given by (5.57) ensures that

$$\Omega_{FD,f}(R_{opt}(z)) \supseteq \Omega_{FD,f}(R(z)), \ \forall R(z) \in \mathbf{RH}_\infty \tag{5.78}$$

where $G_{do}(z)$ is the co-outer of $M_u(z)G_d(z)$, Q_d is an arbitrary unitary matrix of compatible dimensions.

Now we shall consider the dual optimization problem of minimizing FAR under a given FDR.

Suppose that $\|R(z)M_u(z)G_f(z)\|_- \neq 0$ and $d = 0$. It follows from (2.32) that any fault f can be definitively detected only if

$$\|r\|_2 = \|R(z)M_u(z)G_f(z)f(z)\|_2 \geq \|R(z)M_u(z)G_f(z)\|_- \|f\|_2 > J_{th} \tag{5.79}$$

In practice, it is often required that any fault whose size is larger than a tolerant range should be detected. The percentage of the faults whose size is beyond the tolerant range represents the FDR. Therefore, setting the tolerance range is understood as fixing the FDR. Bearing this in mind, to be sure that all faults whose size is equal to or larger than α are detected, the threshold J_{th} should be set as

$$J_{th} = \alpha \|R(z)M_u(z)G_f(z)\|_- \tag{5.80}$$

For a given J_{th} the size of detachable faults is fixed, i.e. the FDR is fixed.

Recall that a false alarm will be created if in the fault-free case

$$\|r\|_2 = \|R(z)M_u(z)G_d(z)d(z)\|_2 > J_{th} = \alpha \|R(z)M_u(z)G_f(z)\|_- \tag{5.81}$$

i.e.

$$\|R(z)M_u(z)G_d(z)d(z)\|_2 - \alpha \|R(z)M_u(z)G_f(z)\|_- > 0 \tag{5.82}$$

Let $\Omega_{FA,d}(R)$ denote the set of all disturbances that would cause false alarms, i.e.

$$\Omega_{FA}(R(z)) = \{ \ d \mid d \text{ satisfies (5.82)} \ \} \tag{5.83}$$

The size of $\Omega_{FA,d}(R)$ is a reasonable measurement of the FAR. The optimization problem is reformulated as: Given α, find $R(z)$ so that the size of $\Omega_{FA}(R(z))$ is minimized. This problem will be solved using the factorization technique, as presented in the following theorem.

Theorem 5.10 Given $M_u(z)G_f(z) \in \mathbf{RH}_\infty$ and $M_u(z)G_d(z) \in \mathbf{RH}_\infty$. Assume that

$$\forall \omega \in [0, 2\pi], G_f^*(e^{j\omega})G_f(e^{j\omega}) > 0$$

Then $R_{opt,-/\infty}(z) = Q_f G_{fo}^{-1}(z)$ given by (5.62) ensures that $\forall R(z) \in \mathbf{RH}_\infty$ with $\|R(z)M_u(z)G_f(z)\|_- > 0$,

$$\Omega_{FA,d}(R_{opt,-/\infty}(z)) \subseteq \Omega_{FA,d}(R(z)) \tag{5.84}$$

where $G_{fo}(z)$ is the co-outer of $M_u(z)G_f(z)$, Q_f is an arbitrary unitary matrix of compatible dimensions.

Theorem 5.10 suggests that, if $R_{opt,-/\infty}(z)$ is selected as the post-filter, then the set of disturbances that would cause false alarms will be the smallest. Therefore, $R_{opt,-/\infty}(z)$ ensures the lowest FAR.

Proof: It follows from Lemma 4.1 that there exists a particular left coprime factorization of $M_u(z)G_f(z) = G_{fo}(z)G_{fi}(z)$, where $G_{fo}^{-1}(z) \in \mathbf{RH}_\infty, G_{fi}(z) \in \mathbf{RH}_\infty$ and $G_{fi}(z)$ is a co-inner. Assume that

$$R(z) = Q(z)G_{fo}^{-1}(z), Q(z) \in \mathbf{RH}_{\infty}$$

Then, (5.82) can be re-written into

$$\left\| Q(z)G_{fo}^{-1}(z)M_u(z)G_d(z)d(z) \right\|_2 - \alpha \left\| Q(z)G_{fi}(z) \right\|_- > 0 \tag{5.85}$$

Note that

$$\left\| Q(z)G_{fi}(z) \right\|_- \leq \left\| Q(z) \right\|_- \left\| G_{fi}(z) \right\|_\infty = \left\| Q(z) \right\|_-$$
$$\left\| Q(z)G_{fo}^{-1}(z)M_u(z)G_d(z)d(z) \right\|_2 \geq \left\| Q \right\|_- \left\| G_{fo}^{-1}(z)M_u(z)G_d(z)d(z) \right\|_2$$

It turns out

$$\left\| Q(z)G_{fo}^{-1}(z)M_u(z)G_d(z)d(z) \right\|_2 - \alpha \left\| Q(z)G_{fi}(z) \right\|_-$$
$$\geq \left\| Q \right\|_- \left(\left\| G_{fo}^{-1}(z)M_u(z)G_d(z)d(z) \right\|_2 - \alpha \right)$$

As a result,

$$\left\| Q(z) \right\|_- > 0, \left\| G_{fo}^{-1}(z)\bar{G}_d(z)d(z) \right\|_2 - \alpha > 0 \tag{5.86}$$

lead to (5.85) that is equivalent to (5.82). In other words, (5.86) is a sufficient condition for (5.82). Hence any d satisfying (5.86) will result in

$$\left\| R(z)M_u(z)G_d(z)d(z) \right\|_2 - \alpha \left\| R(z)M_u(z)G_f(z) \right\|_- =$$
$$\left\| Q(z)G_{fo}^{-1}(z)M_u(z)G_d(z)d(z) \right\|_2 - \alpha \left\| Q(z)G_{fi}(z) \right\|_- > 0$$

Considering that (5.86) can be achieved by setting $R(z) = Q_f G_{fo}^{-1}(z)$, where Q_f is an arbitrary unitary matrix, we finally have $\forall Q(z) \in \mathbf{RH}_\infty$ with $\left\| Q(z) \right\|_- > 0$,

$$\Omega_{FA,d}\left(Q_f G_{fo}^{-1}\right) \subseteq \Omega_{FA,d}\left(Q(z)G_{fo}^{-1}\right)$$

which is equivalent to (5.84). The theorem is proven. □

Theorem 5.10 provides us with an approach, by which we can achieve an optimal trade-off in the sense of minimizing the FAR under a given FDR in the context of norm based residual evaluation. It is interesting to notice that the role of the post-filter $R_{opt,-/\infty}(z)$ is in fact to inverse the magnitude profile of $\bar{G}_f(z)$. As a result, we have

$$\left\| R_{opt,-/\infty}(z)M_u(z)G_f(z) \right\|_- = \left\| R_{opt,-/\infty}(z)M_u(z)G_f(z) \right\|_\infty = 1 \tag{5.87}$$

Moreover, the residual dynamics is governed by

$$r(z) = Q_f G_{fo}^{-1}(z)\bar{G}_d(z)d(z) + Q_f G_{fi}(z)f(z)$$

and the threshold J_{th} should be set, according to (5.80), as

$$J_{th} = \alpha \left\| R_{opt,-/\infty}(z)M_u(z)G_f(z) \right\|_- = \alpha \tag{5.88}$$

Note that in case of weak disturbances, $R_{opt,-/\infty}(z)$ also delivers an estimation of the size of the fault (i.e. the energy of the fault), as

$$\left\| r \right\|_2 \approx \left\| Q_f G_{fi}(z)f(z) \right\|_2 = \left\| f \right\|_2 \tag{5.89}$$

We would like to mention that the application of the well-established factorization technique to the problem solution is very helpful for getting a deep insight into the optimization problem. Different from the LMI solutions, the interpretation of (5.62) as the inverse of the magnitude profile of $\bar{G}_f(z)$ is evident. From the computational viewpoint, solution (5.62) is an analytical one.

5.2.4 State space realization of post-filters

For the practical implementation, it is often useful to get a state space representation of $R_{opt}(z)$ and $R_{opt,-/\infty}(z)$. For this purpose, the dual version of the inner-outer factorization approach of Ionescu et al. (1996) is applied [80]. Considering

$$M_u(z)G_d(z) = F_d + C(zI - A + LC)^{-1}(E_d - LF_d)$$

it yields

$$G_{do}(z) = H_d - C(zI - A + LC)^{-1}F^T H_d$$

where H_d is of full column rank,

$$H_d H_d^T = CX_dC^T + F_dF_d^T \tag{5.90}$$

and (X_d, F) is the stabilizing solution to a discrete-time algebraic Riccati system (DTARS)

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} = 0 \tag{5.91}$$

where

$$\begin{aligned} \Theta_{11} &= (A - LC)X_d(A - LC)^T - X_d + (E_d - LF_d)(E_d - LF_d)^T \\ \Theta_{12} &= (A - LC)X_dC^T + (E_d - LF_d)F_d^T \\ \Theta_{22} &= CX_dC^T + F_dF_d^T \end{aligned}$$

Let

$$L_d = F - L^T \tag{5.92}$$

and denote the left inverse of H_d by W_d, i.e.

$$W_d H_d = I \tag{5.93}$$

Then, the optimal post filter $R_{opt}(z) = Q_d G_{do}^{-1}(z)$ is represented by

$$R_{opt}(z) = Q_d W_d + Q_d W_d C(zI - A - L_d^T C)^{-1}(L_d^T + L) \tag{5.94}$$

and the DTARS (5.91) reduces to the following one

$$\begin{bmatrix} AX_dA^T - X_d + E_dE_d^T & AX_dC^T + E_dF_d^T \\ CX_dA^T + F_dE_d^T & CX_dC^T + F_dF_d^T \end{bmatrix} \begin{bmatrix} I \\ L_d \end{bmatrix} = 0 \tag{5.95}$$

Lemma 5.3 Under the same conditions as in Theorem 5.6, assume that $M_u(z)$ is given by (2.33). Then the state space realization of the optimal post-filter (5.57) is given by (5.94), where Q_d is an arbitrary unitary matrix, L_d and W_d are determined by (5.95), (5.90) and (5.93).

Therefore, a state space realization of $R_{opt}(z)$ in the form of (5.94) can be obtained by:

- solving the DTARS (5.95) for the stabilizing solution (X_d, L_d),
- finding the full column rank matrix H_d satisfying $H_d H_d^T = CX_dC^T + F_dF_d^T$,
- solving $W_d H_d = I$ for W_d.

Similarly, a state space realization of $R_{opt,-/\infty}(z)$ can be found, as stated in the following lemma.

Lemma 5.4 Under the same conditions as in Theorem 5.7, assume that $m = n_f$ and $M_u(z)$ is given by (2.33). Then the state space realization of the post-filter (5.62) is given by

$$R_{opt,-/\infty}(z) = Q_f W_f + Q_f W_f C(zI - A - L_f^T C)^{-1}(L_f^T + L) \tag{5.96}$$

where Q_f is an arbitrary unitary matrix, W_f is the left inverse of a full column rank matrix H_f satisfying $H_f H_f^T = CX_fC^T + F_fF_f^T$, and (X_f, L_f) is the stabilizing solution to the DTARS

$$\begin{bmatrix} AX_fA^T - X_f + E_fE_f^T & AX_fC^T + E_fF_f^T \\ CX_fA^T + F_fE_f^T & CX_fC^T + F_fF_f^T \end{bmatrix} \begin{bmatrix} I \\ L_f \end{bmatrix} = 0 \tag{5.97}$$

In case that $m > n_f$, the state space realization of $R_{opt,-/\infty}(z)$ should be obtained by other algorithms of co-inner-outer factorization.

5.2.5 Optimal residual dynamics

As mentioned before, if $R(z) = R_{opt}(z)$, then $R_{opt}(z)M_u(z)G_d(z) = Q_d G_{di}(z)$ is a co-inner. Assume that $(A_{rd}, B_{rd}, C_{rd}, D_{rd})$ is a state space realization of $R_{opt}(z)M_u(z)G_d(z)$. As a dual result to [200], there exist a matrix $X_{rd} \geq 0$ that satisfies

$$A_{rd}X_{rd}A_{rd}^T - X_{rd} + B_{rd}B_{rd}^T = 0 \tag{5.98}$$

$$A_{rd}X_{rd}C_{rd}^T + B_{rd}D_{rd}^T = 0 \tag{5.99}$$

$$C_{rd}X_{rd}C_{rd}^T + D_{rd}D_{rd}^T - I = 0 \tag{5.100}$$

The optimal residual dynamics is

$$\begin{aligned} \hat{x}(k+1) &= (A + L_d^T C)\hat{x}(k) + (E_d + L_d^T F_d)d(k) + (E_f + L_d^T F_f)f(k) \\ r(k) &= Q_d W_d (C\hat{x}(k) + F_d d(k) + F_f f(k)) \end{aligned} \tag{5.101}$$

5.3 Optimization of observer based residual generators

As shown by Lemma 2.4, the observer-based residual generator (2.19) is a special case of (2.31), where the post-filter $R(z)$ reduces to a constant weighting matrix W. Therefore, the optimization of the observer-based residual generator (2.19) can be derived from the results in the last section.

The residual dynamics of (2.19) is governed by

$$r(z) = W(G_{d,L}(z)d(z) + G_{f,L}(z)f(z)) \tag{5.102}$$

where

$$\begin{aligned} G_{d,L}(z) &= F_d + C(zI - A + LC)^{-1}(E_d - LF_d) \\ G_{f,L}(z) &= F_f + C(zI - A + LC)^{-1}(E_f - LF_f) \end{aligned}$$

5.3.1 H_∞/H_∞, H_-/H_∞ and H_i/H_∞ design

The H_∞/H_∞, H_-/H_∞ and H_i/H_∞ optimization problems are in this case, respectively, formulated as [34, 32, 61, 180, 176]

$$\max_{L,W} J_{OBS,\infty/\infty} = \max_{L,W} \frac{\|WG_{f,L}(z)\|_\infty}{\|WG_{d,L}(z)\|_\infty} \tag{5.103}$$

$$\max_{L,W} J_{OBS,-/\infty} = \max_{L,W} \frac{\|WG_{f,L}(z)\|_-}{\|WG_{d,L}(z)\|_\infty} \tag{5.104}$$

$$\max_{L,W} J_{OBS,i/\infty} = \max_{L,W} \frac{\sigma_i(WG_{f,L}(e^{j\omega}))}{\|WG_{d,L}(z)\|_\infty} \tag{5.105}$$

A unified solution to the optimization problem (5.103)-(5.105) is given in the following theorem.

Theorem 5.11 [179] Given the LTI system (2.1), assume that (A, E_d, C, F_d) is detectable and has no invariant zeros on the unit circle and no unobservable modes at the origin. Then

$$L_{opt} = -L_d^T, \ W_{opt} = Q_d W_d \tag{5.106}$$

solves the optimization problem (5.103)-(5.105) simultaneously, where Q_d is a unitary matrix of compatible dimensions, W_d is the left inverse of a full column rank matrix H_d satisfying $H_d H_d^T = CX_dC^T + F_dF_d^T$, and (X_d, L_d) is the stabilizing solution to the DTARS (5.95). Moreover,

$$\Omega_{FD,f}(L_{opt}, W_{opt}) \supseteq \Omega_{FD,f}(L, W), \ \forall L, W \tag{5.107}$$

Proof: At first assume that L is a fixed stabilizing observer gain matrix. Recall that

$$G_{d,L}(z) = M_u(z)G_d(z), G_{f,L}(z) = M_u(z)G_f(z)$$

and $R_{opt}(z)$ given by (5.94) is the unified optimal solution to optimization problems (5.50)-(5.52), where Q_d is an arbitrary unitary matrix of compatible dimensions. Note that, if L is fixed to $-L_d^T$, then $R_{opt}(z)$ reduces to the constant matrix Q_dW_d. That means

$$\frac{\|Q_dW_dG_{f,-L_d^T}(z)\|_\infty}{\|Q_dW_dG_{d,-L_d^T}(z)\|_\infty} = \max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)G_{f,-L_d^T}(z)\|_\infty}{\|R(z)G_{d,-L_d^T}(z)\|_\infty} \tag{5.108}$$

$$\frac{\|Q_dW_dG_{f,-L_d^T}(z)\|_-}{\|Q_dW_dG_{d,-L_d^T}(z)\|_\infty} = \max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)G_{f,-L_d^T}(z)\|_-}{\|R(z)G_{d,-L_d^T}(z)\|_\infty} \tag{5.109}$$

$$\frac{\sigma_i\left(Q_dW_dG_{f,-L_d^T}(e^{j\omega})\right)}{\|Q_dW_dG_{d,-L_d^T}(z)\|_\infty} = \max_{R(z)\in\mathbf{RH}_\infty} \frac{\sigma_i\left(R(e^{j\omega})G_{f,-L_d^T}(e^{j\omega})\right)}{\|R(z)G_{d,-L_d^T}(z)\|_\infty} \tag{5.110}$$

Moreover, for any given observer gain matrix L, there exists always an $\mathbf{RH}_\infty$-invertible matrix

$$Q(z) = I + C(zI - A - L_d^TC)^{-1}(L + L_d^T) \tag{5.111}$$

such that

$$\begin{aligned} G_{d,-L_d^T}(z) &= Q(z)G_{d,L}(z) \\ G_{f,-L_d^T}(z) &= Q(z)G_{f,L}(z) \end{aligned} \tag{5.112}$$

On the other side, note that $R(z)$ is a dynamic system and includes the constant matrix as its special case. Hence, once the stabilizing observer gain matrix L is fixed (i.e. $G_{d,L}(z)$ and $G_{f,L}(z)$ fixed), there is

$$\max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)G_{f,L}(z)\|_\infty}{\|R(z)G_{d,L}(z)\|_\infty} \geq \max_W \frac{\|WG_{f,L}(z)\|_\infty}{\|WG_{d,L}(z)\|_\infty} \tag{5.113}$$

$$\max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)G_{f,L}(z)\|_-}{\|R(z)G_{d,L}(z)\|_\infty} \geq \max_W \frac{\|WG_{f,L}(z)\|_-}{\|WG_{d,L}(z)\|_\infty} \tag{5.114}$$

$$\max_{R(z)\in\mathbf{RH}_\infty} \frac{\sigma_i(R(e^{j\omega})G_{f,L}(e^{j\omega}))}{\|R(z)G_{d,L}(z)\|_\infty} \geq \max_W \frac{\sigma_i(WG_{f,L}(e^{j\omega}))}{\|WG_{d,L}(z)\|_\infty} \tag{5.115}$$

From (5.108), (5.112) and (5.113) it can be obtained that, $\forall L$,

$$\begin{aligned} &\frac{\|Q_dW_dG_{f,-L_d^T}(z)\|_\infty}{\|Q_dW_dG_{d,-L_d^T}(z)\|_\infty} \\ &= \max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)G_{f,-L_d^T}(z)\|_\infty}{\|R(z)G_{d,-L_d^T}(z)\|_\infty} \\ &= \max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)Q(z)G_{f,L}(z)\|_\infty}{\|R(z)Q(z)G_{d,L}(z)\|_\infty} \\ &= \max_{\hat{R}(z)\in\mathbf{RH}_\infty} \frac{\|\hat{R}(z)G_{f,L}(z)\|_\infty}{\|\hat{R}(z)G_{d,L}(z)\|_\infty} \\ &\geq \max_W \frac{\|WG_{f,L}(z)\|_\infty}{\|WG_{d,L}(z)\|_\infty} \end{aligned}$$

As a result,

$$\frac{\|WG_{f,L}(z)\|_\infty}{\|WG_{d,L}(z)\|_\infty}\Big|_{L=-L_d^T, W=Q_dW_d} \geq \max_{L,W} \frac{\|WG_{f,L}(z)\|_\infty}{\|WG_{d,L}(z)\|_\infty}$$

It shows that $L = -L_d^T, W = Q_d W_d$ is the optimal solution to the optimization problem (5.103). Similarly, it can be proved that $L = -L_d^T$ and $W = Q_d W_d$ solve (5.104)-(5.105). □

From the above derivation, we obtain the following lemma as a byproduct.

Lemma 5.5 Denote the optimal values of the optimization problems (5.103)-(5.105), respectively, by $J_{OBS,\infty/\infty,opt}$, $J_{OBS,-/\infty,opt}$, $J_{OBS,i/\infty,opt}(\omega)$ and those of the the optimization problems (5.50)-(5.52), respectively, by $J_{FRE,\infty/\infty,opt}$, $J_{FRE,-/\infty,opt}$, $J_{FRE,i/\infty,opt}(\omega)$. Then

$$\begin{aligned} J_{OBS,\infty/\infty,opt} &= J_{FRE,\infty/\infty,opt} \\ J_{OBS,-/\infty,opt} &= J_{FRE,-/\infty,opt} \\ J_{OBS,i/\infty,opt}(\omega) &= J_{FRE,i/\infty,opt}(\omega) \\ J_{OBS,\infty/\infty,opt} &\geq J_{OBS,i/\infty,opt}(\omega) \geq J_{OBS,-/\infty,opt} \end{aligned} \tag{5.116}$$

Moreover, the optimal solution L_{opt} and W_{opt} given by (5.106) leads to

$$\begin{aligned} G_{rd}(z) &= W_{opt}F_d + W_{opt}C(zI - A + L_{opt}C)^{-1}(E_d - L_{opt}F_d) \\ \|G_{rd}(z)\|_\infty &= 1 \end{aligned} \tag{5.117}$$

and $G_{rd}(z)$ is a co-inner.

In Theorem 5.11, the assumption that (A, E_d, C, F_d) has no invariant zeros on the unit circle is a standard assumption in the robust control theory. The other assumptions are not restrictive and do not lead to a loss of generality [81]. If $CX_dC^T + F_dF_d^T$ is nonsingular, then the DTARS (5.95) reduces to a standard DTARE as follows

$$\begin{aligned} AX_dA^T - X_d + E_dE_d^T - (AX_dC^T + E_dF_d^T) \\ \times (CX_dC^T + F_dF_d^T)^{-1}(CX_dA^T + F_dE_d^T) = 0 \end{aligned} \tag{5.118}$$

which can be solved in Matlab with the function *dare*.

In the same way, we can prove the following theorem.

Theorem 5.12 Given the LTI system (2.1), assume that $m = n_f$ and (A, E_f, C, F_f) is detectable and has no invariant zeros on the unit circle, no unreachable modes on the unit circle, and no unobservable modes at the origin. Then

$$L_{opt,-/\infty} = -L_f^T, \; W_{opt,-/\infty} = Q_f W_f \tag{5.119}$$

solves the optimization problem (5.104), where Q_f is a unitary matrix, W_f is the left inverse of a full column rank matrix H_f satisfying $H_f H_f^T = CX_fC^T + F_fF_f^T$, and (X_f, L_f) is the stabilizing solution to the DTARS (5.97). Moreover,

$$\Omega_{FA,d}(L_{opt,-/\infty}, W_{opt,-/\infty}) \subseteq \Omega_{FA,d}(L, W), \; \forall L, W \tag{5.120}$$

Considering that (5.106) solves the optimization problem (5.103)-(5.105) simultaneously, it is called the *unified optimal observer-based residual generator*.

The threshold can be set as

$$\begin{aligned} J_{th} &= \sup_{f=0,d} \|r\|_2 \\ &= \begin{cases} \delta_{d,2}, & \text{if } L = L_{opt}, W = W_{opt} \\ \|Q_f W_f G_{d,-L_f^T}(z)\|_\infty \delta_{d,2}, & \text{if } L = L_{opt,-/\infty}, W = W_{opt,-/\infty} \end{cases} \end{aligned} \tag{5.121}$$

5.3.2 Example

Example 5.2 For the same system as in Example 5.1, design observer based H_∞/H_∞, H_-/H_∞ and H_i/H_∞ optimal FD systems.

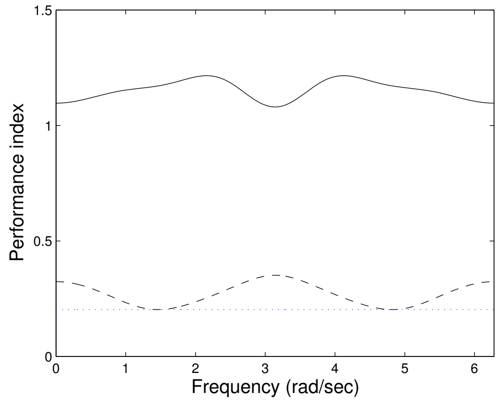

Fig. 5.1 Performance index $J_{1,\omega}(L_{opt}, W_{opt})$ (solid line), $J_{2,\omega}(L_{opt}, W_{opt})$ (dashed line), $J_{OBS,-/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) = J_{1,\omega}(L_{opt,-/\infty}, W_{opt,-/\infty}) = J_{2,\omega}(L_{opt,-/\infty}, W_{opt,-/\infty}) = J_{OBS,\infty/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) \equiv 0.2031$ (dotted line)

From Theorem 5.11, we get the optimal gain matrix

$$L_{opt} = \begin{bmatrix} 1.6883 & -0.1281 \\ -1.6101 & 0.7840 \\ 0.0369 & 0.3677 \\ 0.7917 & 1.3489 \end{bmatrix}, W_{opt} = \begin{bmatrix} 0.2499 & 0 \\ 0.0194 & 0.2507 \end{bmatrix} \tag{5.122}$$

which solves (5.103)-(5.105) simultaneously. According to Theorem 5.12, (5.104) is also solved by

$$L_{opt,-/\infty} = \begin{bmatrix} 1.6261 & -0.1786 \\ -1.8671 & 0.8734 \\ -0.1963 & 0.4029 \\ 0.6697 & 1.3317 \end{bmatrix}, W_{opt,-/\infty} = \begin{bmatrix} 0.6914 & 0 \\ -0.5288 & 0.2955 \end{bmatrix} \tag{5.123}$$

The optimal performance indexes, as obtained by solving (5.103)-(5.105), are shown in Fig.5.1. It can be seen that, $\forall\omega$,

$$\begin{aligned} 1.2156 &= J_{OBS,\infty/\infty}(L_{opt}, W_{opt}) = J_{1,\omega=2.160}(L_{opt}, W_{opt}) \geq J_{1,\omega}(L_{opt}, W_{opt}) \\ &\geq J_{2,\omega}(L_{opt}, W_{opt}) \geq J_{2,\omega=1.460}(L_{opt}, W_{opt}) = J_{OBS,-/\infty}(L_{opt}, W_{opt}) \\ &= J_{OBS,-/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) = J_{1,\omega}(L_{opt,-/\infty}, W_{opt,-/\infty}) \\ &= J_{2,\omega}(L_{opt,-/\infty}, W_{opt,-/\infty}) = J_{OBS,\infty/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) = 0.2031 \end{aligned}$$

These results verify Lemma 5.5.

5.4 Interconnections between optimization problems

In this section, we shall discuss the relationship between different optimization approaches.

5.4.1 J_{PS} and $J_{FRE,2/2}$

The relationship between the optimal solutions of the parity space approach and the H_2/H_2 approach can be analyzed as follows [195].

Suppose that $\{g_d(0), g_d(1), \cdots\}$ is the impulse response of system (2.1) to the unknown disturbances. Apparently,

$$g_d(0) = F_d,\ g_d(1) = CE_d,\ \cdots,\ g_d(s) = CA^{s-1}E_d, \cdots \tag{5.124}$$

The matrix $H_{d,s}$ can then be expressed in terms of the impulse response as follows

$$H_{d,s} = \begin{bmatrix} g_d(0) & O & \cdots & O \\ g_d(1) & g_d(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ g_d(s) & \cdots & g_d(1) & g_d(0) \end{bmatrix}$$

Partition the parity vector v_s as

$$v_s = \begin{bmatrix} v_{s,0} \ v_{s,1} \ \cdots \ v_{s,s} \end{bmatrix}$$

where the row vector $v_{s,i} \in \mathbf{R}^{1\times m}$, $i = 0, 1, \cdots s$.

Then, we have

$$v_s H_{d,s} = \begin{bmatrix} \varphi(s) \ \varphi(s-1) \ \cdots \ \varphi(0) \end{bmatrix}$$

where

$$\varphi(i) = \sum_{l=0}^{i} \rho_{i-l} g_d(l), \quad \rho_i = v_{s,s-i}, \quad i = 0, 1, \cdots s$$

Let s go to infinity. It leads to

$$\lim_{s\to\infty} v_s H_{d,s} = \begin{bmatrix} \varphi(\infty) \ \cdots \ \varphi(0) \end{bmatrix} \tag{5.125}$$

and in this case

$$\varphi(i) = \sum_{l=0}^{i} \rho_{i-l} g_d(l) = \rho(i) \otimes g_d(i) = \mathcal{Z}^{-1}(P(z)G_d(z)) \tag{5.126}$$

$$P(z) = \mathcal{Z}[\rho(i)], \quad \rho(i) = \{\rho_0, \rho_1, \cdots\} \tag{5.127}$$

where $\otimes$ denotes the convolution. Equation (5.127) means that $P(z)$ is the z-transform of the sequence $\{\rho_0, \rho_1, \cdots\}$.

According to the Parseval Theorem, we have

$$\begin{aligned} \lim_{s\to\infty} v_s H_{d,s} H_{d,s}^T v_s^T &= \sum_{i=0}^{\infty} \varphi(i)\varphi^T(i) \\ &= \frac{1}{2\pi} \int_0^{2\pi} P(e^{j\omega})G_d(e^{j\omega})G_d^*(e^{j\omega})P^*(e^{j\omega})d\omega \end{aligned} \tag{5.128}$$

Similarly, it can be proven that

$$\lim_{s\to\infty} v_s H_{f,s} H_{f,s}^T v_s^T = \frac{1}{2\pi} \int_0^{2\pi} P(e^{j\omega})G_f(e^{j\omega})G_f^*(e^{j\omega})P^*(e^{j\omega})d\omega \tag{5.129}$$

On the other side, if given a residual generator (2.31), we can always construct a parity vector, as stated in Lemma 5.6.

Lemma 5.6 Given system (2.1) and a residual generator (2.31) with $R(z) \in \mathbf{RH}_\infty^{1\times m}$. Then the row vector defined by

$$v = \begin{bmatrix} \cdots & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & \bar{D} \end{bmatrix} \tag{5.130}$$

where $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is the state space realization of $R(z)M_u(z)$, belongs to the parity space P_s $(s \to \infty)$.

Proof: Assume that (A_r, B_r, C_r, D_r) is a state space realization of $R(z)$. Recalling (2.33), we know that

$$\bar{A} = \begin{bmatrix} A - LC & O \\ -B_r C & A_r \end{bmatrix}, \ \bar{B} = \begin{bmatrix} L \\ B_r \end{bmatrix}, \ \bar{C} = \begin{bmatrix} -D_r C & C_r \end{bmatrix}, \ \bar{D} = D_r$$

It can be easily obtained that

$$\begin{aligned} \lim_{s\to\infty} vH_{o,s} &= \lim_{s\to\infty} \begin{bmatrix} \cdots & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & \bar{D} \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \\ &= \lim_{s\to\infty} \begin{bmatrix} \cdots & C_r A_r B_r & C_r B_r & D_r \end{bmatrix} \begin{bmatrix} C \\ C(A-LC) \\ C(A-LC)^2 \\ \vdots \end{bmatrix} \end{aligned} \tag{5.131}$$

For a linear discrete-time system

$$\begin{aligned} \lambda(k+1) &= (A - LC)\lambda(k) \\ \delta(k) &= C\lambda(k) \end{aligned} \tag{5.132}$$

with any initial state vector $\lambda(0) = \lambda_0 \in \mathbf{R}^n$, apparently

$$\begin{aligned} \delta(0) &= C\lambda_0 \\ \delta(1) &= C(A-LC)\lambda_0 \\ \delta(2) &= C(A-LC)^2\lambda_0, \cdots \end{aligned}$$

Since $R(z) \in \mathbf{RH}_\infty^{1\times m}$ and L is selected to ensure the stability of $A - LC$, the cascade connection of system (5.132) and $R(z)$ is stable. So

$$\lim_{k\to\infty} \mathcal{Z}^{-1}\{R(z)\delta(z)\} = 0$$

Note that

$$\lim_{k\to\infty} \mathcal{Z}^{-1}\{R(z)\delta(z)\} = \lim_{s\to\infty} \begin{bmatrix} \cdots & C_r A_r B_r & C_r B_r & D_r \end{bmatrix} \begin{bmatrix} C\lambda_0 \\ C(A-LC)\lambda_0 \\ C(A-LC)^2\lambda_0 \\ \vdots \end{bmatrix}$$

we get

$$\lim_{s\to\infty} \begin{bmatrix} \cdots & C_r A_r B_r & C_r B_r & D_r \end{bmatrix} \begin{bmatrix} C \\ C(A-LC) \\ C(A-LC)^2 \\ \vdots \end{bmatrix} \lambda_0 = 0$$

for any initial state vector $\lambda_0 \in \mathbf{R}^n$. Thus it can be concluded that

$$\lim_{s\to\infty} \begin{bmatrix} \cdots & C_r A_r B_r & C_r B_r & D_r \end{bmatrix} \begin{bmatrix} C \\ C(A-LC) \\ C(A-LC)^2 \\ \vdots \end{bmatrix} = 0$$

At last, from (5.131) we obtain

$$\lim_{s\to\infty} vH_{o,s} = 0$$

i.e. the vector v defined by (5.130) belongs to the parity space P_s ($s \to \infty$). The lemma is thus proven. □

It is of interest to note that the vector v is indeed composed of the impulse response of the residual generator $R(z)M_u(z) = \bar{D} + \bar{C}(zI - \bar{A})^{-1}\bar{B}$, which is given by $\left\{\bar{D}, \bar{C}\bar{B}, \bar{C}\bar{A}\bar{B}, \bar{C}\bar{A}^2\bar{B}, \cdots\right\}$.

Based on the above analysis, the following theorem can be obtained.

Theorem 5.13 [195] Given system (2.1) and assume that $v_{s,opt}$, $J_{PS,opt}$ and $R_{opt}(z)$, $J_{FRE,2/2,opt}$ are, respectively, the optimal solutions of optimization problems (5.3) and (5.69). Then

$$\lim_{s\to\infty} J_{PS,opt} = J_{FRE,2/2,opt} \tag{5.133}$$

$$P(z) = R_{opt}(z)M_u(z) \tag{5.134}$$

where

$$P(z) = \mathcal{Z}[\rho(i)], \quad \rho(i) = \{v_{s\to\infty,opt,s}, v_{s\to\infty,opt,s-1}, \cdots, v_{s\to\infty,opt,0}\} \tag{5.135}$$

Proof: Let $v_{s\to\infty,opt}$ denote the optimal solution of optimization problem (5.3) as $s \to \infty$, then it follows from (5.127)-(5.129) that for any left coprime factorization of $G_u(z) = M_u^{-1}(z)N_u(z)$, the post-filter $R_o(z)$ given by

$$R_o(z) = P(z)M_u^{-1}(z)$$

where $P(z)$ is defined by (5.135), leads to

$$J_{FRE,2/2} \mid_{R(z)=R_o(z)} = \lim_{s\to\infty} J_{PS,opt} = \lim_{s\to\infty} \max_{v_s\in\mathbf{P}_s} J_{PS} \tag{5.136}$$

$$= \max_s \max_{v_s\in\mathbf{P}_s} J_{PS} \leq \max_{R(z)\in\mathbf{RH}_\infty^{1\times m}} J_{FRE,2/2} \tag{5.137}$$

We now demonstrate that

$$J_{FRE,2/2} \mid_{R(z)=R_o(z)} = J_{FRE,2/2,opt} = \max_{R(z)\in\mathbf{RH}_\infty^{1\times m}} J_{FRE,2/2} \tag{5.138}$$

Suppose that (5.138) does not hold. Then, the optimal solution of optimization problem (5.69), denoted by $R_c(z) \in \mathbf{RH}_\infty^{1\times m}$ and different from $R_o(z)$, should lead to

$$J_{FRE,2/2} \mid_{R(z)=R_c(z)} = \max_{R(z)\in\mathbf{RH}_\infty^{1\times m}} J_{FRE,2/2} > J_{FRE,2/2} \mid_{R(z)=R_o(z)} \tag{5.139}$$

According to Lemma 5.6, we can find a parity vector $v \in P_s$ whose components are just a rearrangement of the impulse response of $R_c(z)M_u(z)$. Moreover, because of (5.127)-(5.129), we have

$$J_{PS} \mid_{v_s=v} = J_{FRE,2/2} \mid_{R(z)=R_c(z)} \tag{5.140}$$

As a result, it follows from (5.136), (5.139) and (5.140) that

$$J_{PS} \mid_{v_s=v} > \max_s \max_{v_s\in P_s} J_{PS,2/2}$$

which is an obvious contradiction. Thus we can conclude that

$$J_{FRE,2/2,opt} = \max_{R(z)\in\mathbf{RH}_\infty^{1\times m}} J_{FRE,2/2} = J_{FRE,2/2} \mid_{R(z)=R_o(z)} = \lim_{s\to\infty} J_{PS,opt}$$

and

$$R_o(z) = P(z)M_u^{-1}(z) := R_{opt}(z)$$

solve optimization problem (5.69). The theorem is thus proven. □

Theorem 5.13 gives a deeper insight into the relationship between the parity space approach and the H_2/H_2 approach and reveals some very interesting facts when the order of the parity relation s increases:

- The optimal performance index $J_{PS,opt}$ of the parity space approach converges to a limit which is just the optimal performance index $J_{FRE,2/2,opt}$ of the H_2/H_2 approach.
- There is a one-to-one relationship between the optimal solutions of optimization problems (5.3) and (5.69) when the order of the parity relation $s \to \infty$. Since $R_{opt}(z)$ is a band-limited filter, the frequency response of $v_{s\to\infty,opt}$ is also band-limited.

The above result can be applied in several ways, for instance:

- For multi-dimensional systems, the optimal solution of the H_2/H_2 approach can be approximately computed by at first calculating the optimal solution of the parity space approach with a high order s and then doing the z-transform of the optimal parity vector.
- In the parity space approach, a high order s will improve the performance index $J_{PS,opt}$ but, on the other side, increase the online computational effort. To determine a suitable trade-off between performance and implementation effort, the optimal performance index $J_{FRE,2/2,opt}$ of the H_2/H_2 approach can be used as a reference value.
- Based on the property that the frequency response of $v_{s\to\infty,opt}$ is band-limited, advanced parity space approaches can be developed to achieve both a good performance and a low order parity vector. For instance, the parameters $g_1, \cdots, g_s$ in the extended parity relation based residual generator (2.12) can be selected in such a way that

$$\beta(z) = \frac{1}{1 - g_s z^{-1} - g_{s-1} z^{-2} - \cdots - g_1 z^{-s}}$$

is a band-pass filter centering around ω_0 given by (5.71). In the references [161]-[163] wavelet transform have been introduced to design optimized parity vectors of low order and good performance.

Example 5.2 Given a discrete-time system modelled by (2.1), where

$$A = \begin{bmatrix} 1 & -1.30 \\ 0.25 & -0.25 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$E_d = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, E_f = \begin{bmatrix} 0.6 \\ 0.1 \end{bmatrix}, D = F_d = F_f = 0. \tag{5.141}$$

As system (5.141) is stable, matrix L in (2.33) can be selected to be zero matrix and thus $M_u(z)$ is an identity matrix. To solve the generalized eigenvalue-eigenvector problem (5.70) to get ω_0 that achieves $\sigma_{\max}(\omega_0) = \sup_\omega \sigma_{\max}(\omega)$, note that

$$\sigma_{\max}(\omega) = \frac{0.0125 + 0.01 \cos\omega}{0.41 - 0.4 \cos\omega}$$

Therefore, the optimal performance index of the H_2/H_2 approach is $J_{FRE,2/2,opt} = 2.2502$ and the selective frequency is $\omega_0 = 0$.

Fig.5.2 demonstrates the change of the optimal performance index $J_{PS,opt}$ with respect to the order of the parity relation s. From the figure it can be seen that $J_{PS,opt}$ decreases with the increase of s and, moreover, $J_{PS,opt}$ converges to $J_{FRE,2/2,opt}$ when $s \to \infty$. Fig. 5.3 shows the frequency responses of the optimal parity vector $v_{s,opt}$ when s is chosen as $20, 50, 100$ and 200 respectively. We see that the bandwidth of the frequency response of $v_{s,opt}$ becomes narrower and narrower with the increase of s.

5.4.2 $J_{PS,\infty/\infty}$, $J_{PS,-/\infty}$ and $J_{FRE,\infty/\infty}$, $J_{FRE,-/\infty}$

Let $J_{PS,\infty/\infty,opt}$, $J_{PS,-/\infty,opt}$, $J_{FRE,\infty/\infty,opt}$, $J_{FRE,-/\infty,opt}$ denote, respectively, the optimal performance indices of (5.15), (5.16), (5.50) and (5.51). To study the relationship between the performance indices $J_{FRE,\infty/\infty}$, $J_{FRE,-/\infty}$ and $J_{PS,\infty/\infty}$, $J_{PS,-/\infty}$, recall the time domain interpretation of $J_{FRE,\infty/\infty}$ and $J_{FRE,-/\infty}$.

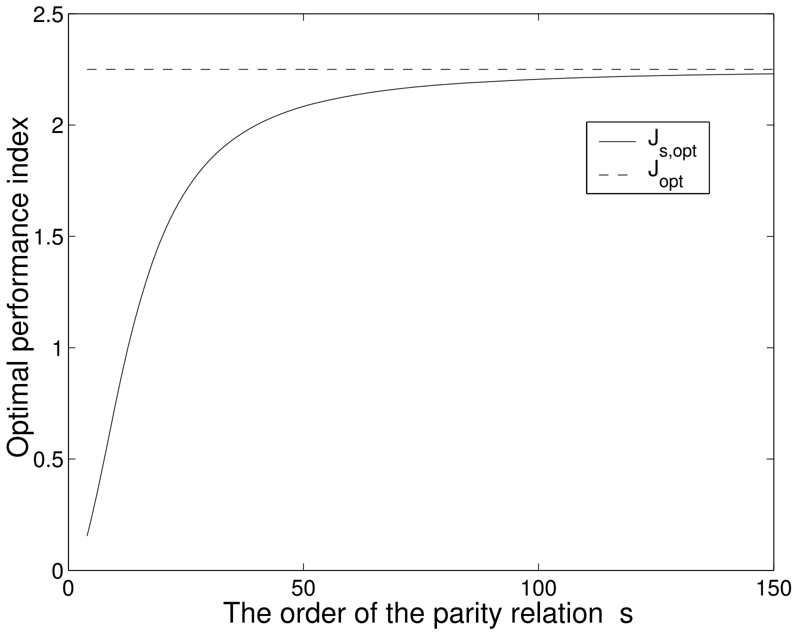

Fig. 5.2 The change of the optimal performance index $J_{PS,opt}$ with respect to s

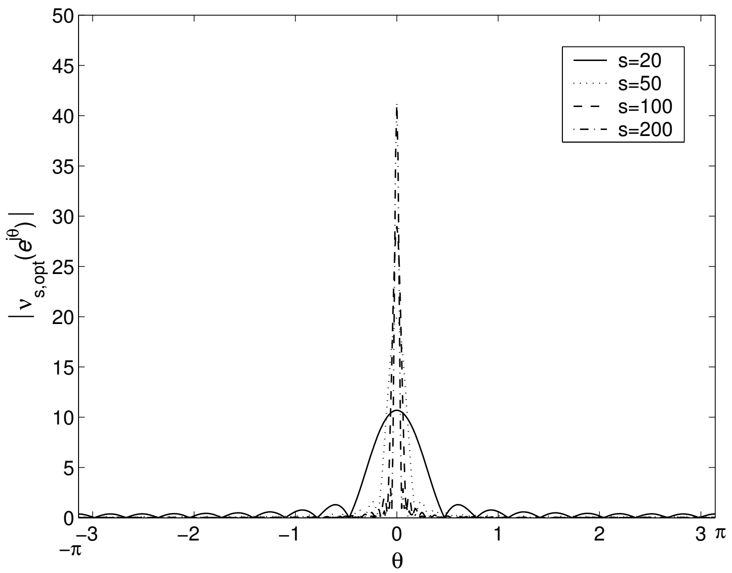

Fig. 5.3 The frequency response of the optimal parity vector $v_{s,opt}$ with respect to s

Assume that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is the state space realization of $\bar{R}(z) = R(z)M_u(z)$. The impulse response of $R(z)M_u(z)$ and $G_d(z)$ are, respectively, $\{g_{\bar{R}}(0),\ g_{\bar{R}}(1),\ g_{\bar{R}}(2),\ \cdots\}$ and $\{g_d(0), g_d(1), g_d(2), \cdots\}$. Then,

$$\begin{aligned} g_d(0) &= F_d,\ g_d(1) = CE_d,\ \cdots,\ g_d(s) = CA^{s-1}E_d, \cdots \\ g_{\bar{R}}(0) &= \bar{D},\ g_{\bar{R}}(1) = \bar{C}\bar{B},\ \cdots, g_{\bar{R}}(s) = \bar{C}\bar{A}^{s-1}\bar{B}, \cdots \end{aligned} \tag{5.142}$$

The impulse response of $R(z)M_u(z)G_d(z)$ is

$$\{g_{\bar{R}}(0)g_d(0), g_{\bar{R}}(0)g_d(1) + g_{\bar{R}}(1)g_d(0), \cdots, \sum_{j=0}^{s} g_{\bar{R}}(j)g_d(s-j), \cdots\}$$

Under the assumption of $x(0) = 0$, the residual dynamics (2.32) can be re-written as

$$r_\infty = H_{\bar{R}}\left(H_d d_\infty + H_f f_\infty\right) \tag{5.143}$$

where

$$r_\infty = \begin{bmatrix} r(0) \\ r(1) \\ r(2) \\ \vdots \end{bmatrix},\ d_\infty = \begin{bmatrix} d(0) \\ d(1) \\ d(2) \\ \vdots \end{bmatrix},\ f_\infty = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \end{bmatrix}$$

$$H_d = \begin{bmatrix} g_d(0) & O & O & \cdots \\ g_d(1) & g_d(0) & O & \vdots \\ g_d(2) & g_d(1) & g_d(0) & \ddots \\ \vdots & \cdots & \ddots & \ddots \end{bmatrix},\ H_f = \begin{bmatrix} g_f(0) & O & O & \cdots \\ g_f(1) & g_f(0) & O & \vdots \\ g_f(2) & g_f(1) & g_f(0) & \ddots \\ \vdots & \cdots & \ddots & \ddots \end{bmatrix}$$

$$H_{\bar{R}} = \begin{bmatrix} g_{\bar{R}}(0) & O & O & \cdots \\ g_{\bar{R}}(1) & g_{\bar{R}}(0) & O & \vdots \\ g_{\bar{R}}(2) & g_{\bar{R}}(1) & g_{\bar{R}}(0) & \ddots \\ \vdots & \cdots & \ddots & \ddots \end{bmatrix} \tag{5.144}$$

Therefore, there is

$$\begin{aligned} \sup_{d_\infty=0, f_\infty\neq 0} \frac{r_\infty^T r_\infty}{f_\infty^T f_\infty} &= \sup_{d_\infty=0, f_\infty\neq 0} \frac{f_\infty^T H_f^T H_{\bar{R}}^T H_{\bar{R}} H_f f_\infty}{f_\infty^T f_\infty} = \bar{\sigma}^2(H_{\bar{R}}H_f) \\ \inf_{d_\infty=0, f_\infty\neq 0} \frac{r_\infty^T r_\infty}{f_\infty^T f_\infty} &= \inf_{d_\infty=0, f_\infty\neq 0} \frac{f_\infty^T H_f^T H_{\bar{R}}^T H_{\bar{R}} H_f f_\infty}{f_\infty^T f_\infty} = \underline{\sigma}^2(H_{\bar{R}}H_f) \end{aligned}$$

As $\delta_\infty^T \delta_\infty = \sum_{k=0}^{\infty} \delta^T(k)\delta(k) = \|\delta\|_2^2$ $(\delta = r, d, f)$, it follows

$$\begin{aligned} \bar{\sigma}^2(H_{\bar{R}}H_f) &= \sup_{x(0)=0, d=0, f\neq 0} \frac{\|r\|_2}{\|f\|_2} = \|R(z)M_u(z)G_f(z)\|_\infty^2 \\ \underline{\sigma}^2(H_{\bar{R}}H_f) &= \inf_{x(0)=0, d=0, f\neq 0} \frac{\|r\|_2}{\|f\|_2} = \|R(z)M_u(z)G_f(z)\|_-^2 \end{aligned}$$

Comparing (5.144) with (2.4), we see that

$$H_d = \lim_{s\to\infty} H_{d,s}, \quad H_f = \lim_{s\to\infty} H_{f,s}$$

Hence,

$$\begin{aligned}
J^2_{FRE,\infty/\infty,opt} &= \max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)M_u(z)G_f(z)\|_\infty^2}{\|R(z)M_u(z)G_d(z)\|_\infty^2} = \max_{R(z)\in\mathbf{RH}_\infty} \frac{\bar{\sigma}^2(H_{\bar{R}}H_f)}{\bar{\sigma}^2(H_{\bar{R}}H_d)} \\
&= \lim_{s\to\infty} \max_{R(z)\in\mathbf{RH}_\infty} \frac{\bar{\sigma}^2(H_{\bar{R}}H_{f,s})}{\bar{\sigma}^2(H_{\bar{R}}H_{d,s})} \geq \lim_{s\to\infty} \max_{V_s} \frac{\bar{\sigma}^2(V_sH_{f,s})}{\bar{\sigma}^2(V_sH_{f,s})} \\
J^2_{FRE,-/\infty,opt} &= \max_{R(z)\in\mathbf{RH}_\infty} \frac{\|R(z)M_u(z)G_f(z)\|_-^2}{\|R(z)M_u(z)G_d(z)\|_\infty^2} = \max_{R(z)\in\mathbf{RH}_\infty} \frac{\underline{\sigma}^2(H_{\bar{R}}H_f)}{\bar{\sigma}^2(H_{\bar{R}}H_d)} \\
&= \lim_{s\to\infty} \max_{R(z)\in\mathbf{RH}_\infty} \frac{\underline{\sigma}^2(H_{\bar{R}}H_{f,s})}{\bar{\sigma}^2(H_{\bar{R}}H_{d,s})} \geq \lim_{s\to\infty} \max_{V_s} \frac{\underline{\sigma}^2(V_sH_{f,s})}{\bar{\sigma}^2(V_sH_{d,s})} \qquad (5.145)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
J^2_{FRE,\infty/\infty,opt} &\geq \lim_{s\to\infty} J_{PS,\infty/\infty,opt} \\
J^2_{FRE,-/\infty,opt} &\geq \lim_{s\to\infty} J_{PS,-/\infty,opt} \qquad (5.146)
\end{aligned}$$

Recall Theorem 5.13, we get further

$$J^2_{FRE,\infty/\infty,opt} \geq J_{FRE,2/2,opt} \qquad (5.147)$$

As the influence of the initial condition can often be neglected if $s \to \infty$, especially for stable systems, there is often

$$\begin{aligned}
J^2_{FRE,\infty/\infty,opt} &= \lim_{s\to\infty} J_{PS,\infty/\infty,opt} \\
J^2_{FRE,-/\infty,opt} &= \lim_{s\to\infty} J_{PS,-/\infty,opt} \\
J^2_{FRE,\infty/\infty,opt} &= J_{FRE,2/2,opt} \qquad (5.148)
\end{aligned}$$

5.4.3 Kalman filter based FD

Assume that an LTI system with stochastic noises is described by

$$\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + w(k) + E_f f(k), \\
y(k) &= Cx(k) + Du(k) + v(k) + F_f f(k), \qquad (5.149)
\end{aligned}$$

where $w \in \mathbf{R}^n, v \in \mathbf{R}^m$ are white Gaussian distributed noises with covariances

$$\begin{aligned}
&\mathbf{E}\left\{ \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w^T(j) \ v^T(j) \end{bmatrix} \right\} \\
&= \begin{bmatrix} E_wE_w^T & E_wF_v^T \\ F_vE_w^T & F_vF_v^T \end{bmatrix} \delta_{kj}
\end{aligned}$$

and independent of $u(k)$. It is well-known that Kalman filter in the form of [88, 92]

$$\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\
\hat{y}(k) &= C\hat{x}(k) + Du(k), \qquad (5.150)
\end{aligned}$$

with $L = (AXC^T + E_wF_v^T)(CXC^T + F_vF_v^T)^{-1}$ and X solving the Riccati equation

$$\begin{aligned}
&AXA^T - X + E_wE_w^T - (AXC^T + E_wF_v^T) \\
&\qquad \times (CXC^T + F_vF_v^T)^{-1}(CXA^T + F_vE_w^T) = 0 \qquad (5.151)
\end{aligned}$$

will give a state estimate with the minimum error covariance matrix $P_e = \mathbf{E}\{e(k)e^T(k)\}$, where $e(k) = x(k) - \hat{x}(k)$, and $e(k)$ is white and Gaussian distributed.

Comparing the above Kalman filter with the optimal parameters of the observer-based residual generator given in Theorem 5.11, we find out that they are strikingly similar. In the following, we shall study the physical meaning behind it.

Let $r_o(k) = y(k) - \hat{y}(k)$. In the fault free case, we have

$$\begin{aligned} e(k+1) &= Ae(k) + w(k) - Lr_o(k) \\ r_o(k) &= Ce(k) + v(k) \end{aligned}$$

As a result,

$$\mathbf{E}\{r_o(k)r_o^T(j)\} = (CPC^T + F_vF_v^T)\delta_{kj} = P_{r_o}\delta_{kj}$$

i.e, $r_o(k)$ is also white, Gaussian distributed and of minimum covariance among all output estimation errors. Let $\bar{r}_o(k)$ be the stacked residual vector as follows

$$\bar{r}_o(k) = \begin{bmatrix} r_o(k) \\ \vdots \\ r_o(k-N) \end{bmatrix}$$

Note that

$$\mathbf{E}\left\{\bar{r}_o(k)\bar{r}_o^T(k)\right\} = diag\{P_{r_o}, \cdots, P_{r_o}\} = P_{\bar{r}_o}$$

The problem of fault detection is transformed into a hypothesis testing problem of detecting the mean value change in the output estimation error $r_o(k)$. For this purpose, consider the generalized likelihood ratio (GLR) method,

$$\begin{aligned} S_N &= \frac{\ln p_{\bar{r}_o, f\neq 0}(\bar{r}_{o,f\neq 0})}{\ln p_{\bar{r}_o, f=0}(\bar{r}_{o,f=0})} \\ &= \frac{\ln \frac{1}{\sqrt{(2\pi)^{m(N+1)}\det(P_{\bar{r}_o})}} e^{-\frac{1}{2}(\bar{r}_o - \bar{r}_{fo})^T P_{\bar{r}_o}^{-1}(\bar{r}_o - \bar{r}_{fo})}}{\ln \frac{1}{\sqrt{(2\pi)^{m(N+1)}\det(P_{\bar{r}_o})}} e^{-\frac{1}{2}\bar{r}_o^T P_{\bar{r}_o}^{-1}\bar{r}_o}} \\ &= \frac{1}{2}\bar{r}_o^T P_{\bar{r}_o}^{-1}\bar{r}_o - \frac{1}{2}(\bar{r}_o - \bar{r}_{fo})^T P_{\bar{r}_o}^{-1}(\bar{r}_o - \bar{r}_{fo}) \end{aligned}$$

If $\bar{r}_{fo} = \bar{r}_o$, S_N achieves the maximum $S_N = \frac{1}{2}\bar{r}_o^T P_{\bar{r}_o}^{-1}\bar{r}_o = \frac{1}{2}(P_{\bar{r}_o}^{-\frac{1}{2}}\bar{r}_o)^T(P_{\bar{r}_o}^{-\frac{1}{2}}\bar{r}_o)$. The fault decision follows from

$$S_N = \begin{cases} < J_{th}, & H_0 \text{ (fault-free) is accepted} \\ > J_{th}, & H_1 \text{ (faulty) is accepted} \end{cases}$$

Suppose that the allowed FAR is α. Because

$$\begin{aligned} r_N = P_{\bar{r}_o}^{-\frac{1}{2}}\bar{r}_o &= diag\{P_{r_o}^{-\frac{1}{2}}, \cdots, P_{r_o}^{-\frac{1}{2}}\} \begin{bmatrix} r_o(k) \\ \vdots \\ r_o(k-N) \end{bmatrix} \\ &= \begin{bmatrix} P_{r_o}^{-\frac{1}{2}} r_o(k) \\ \vdots \\ P_{r_o}^{-\frac{1}{2}} r_o(k-N) \end{bmatrix} \sim N(0, I), \end{aligned}$$

Let

$$r(k) = P_{r_o}^{-\frac{1}{2}} r_o(k) = (CPC^T + F_vF_v^T)^{-\frac{1}{2}}(y(k) - \hat{y}(k))$$

Then

$$S_N = \frac{1}{2} r_N^T r_N = \frac{1}{2} \sum_{j=k-N}^{k} r^T(j) r(j) \sim \frac{1}{2} \chi^2(m(N+1))$$

and the threshold J_{th} can be set according to the requirement on the FAR. It is well-known the GLR method achieves the minimum MDR under given FAR.

The above discussion shows that the unified solution and the Kalman filter use not only the same way to calculate the optimal parameters, but also are equivalent in the sense of minimizing the MDR under a given FAR.

5.4.4 Connection with other optimization problems

It has been pointed out in Chapter 4 (Section 4.4) that the FD problem can be formulated as a number of optimization problems from different viewpoints. In this chapter, our main attention is put on the ratio type optimization problems in the form of (4.41). In this subsection, we shall show that the optimal solution to the ratio-type optimization problems (4.41) can also be used to solve the optimization problems (4.35)-(4.38).

We take the minmax problem (4.35) as an example. Suppose that the observer based residual generator (2.19) is used and the maximal fault sensitivity is under consideration. The design objective is concretized to

$$\|WG_{d,L}(z)\|_\infty < \gamma \to \min, \quad \|WG_{f,L}(z)\|_\infty > \beta \to \max \tag{5.152}$$

Assume that the optimal solution to (5.152) is L_c and W_c, which make

$$\begin{aligned} \|W_c G_{d,L_c}(z)\|_\infty &= \gamma_c (< \gamma) \\ \|W_c G_{f,L_c}(z)\|_\infty &= \beta_c (> \beta) \end{aligned}$$

Let

$$L_g = L_{opt}, \quad W_g = \gamma_c W_{opt} \tag{5.153}$$

with L_{opt}, W_{opt} given by (5.106). Because L_{opt}, W_{opt} maximize the H_∞/H_∞ index $J_{OBS,\infty/\infty}$, there is

$$\frac{\|W_{opt} G_{f,L_{opt}}(z)\|_\infty}{\|W_{opt} G_{d,L_{opt}}(z)\|_\infty} \geq \frac{\|W_c G_{f,L_c}(z)\|_\infty}{\|W_c G_{d,L_c}(z)\|_\infty}$$

Then

$$\begin{aligned} \|W_g G_{d,L_g}(z)\|_\infty &= \gamma_c \|W_{opt} G_{d,L_{opt}}(z)\|_\infty = \gamma_c \\ \|W_g G_{f,L_g}(z)\|_\infty &= \|W_g G_{d,L_g}(z)\|_\infty \frac{\|W_g G_{f,L_g}(z)\|_\infty}{\|W_g G_{d,L_g}(z)\|_\infty} \\ &= \gamma_c \frac{\|W_{opt} G_{f,L_{opt}}(z)\|_\infty}{\|W_{opt} G_{d,L_{opt}}(z)\|_\infty} \\ &\geq \gamma_c \frac{\|W_c G_{f,L_c}(z)\|_\infty}{\|W_c G_{d,L_c}(z)\|_\infty} \\ &= \gamma_c \frac{\beta_c}{\gamma_c} = \beta_c \end{aligned}$$

i.e.

$$\begin{aligned} \|W_g G_{d,L_g}(z)\|_\infty &= \|W_c G_{d,L_c}(z)\|_\infty \\ \|W_g G_{f,L_g}(z)\|_\infty &\geq \|W_c G_{f,L_c}(z)\|_\infty \end{aligned}$$

It demonstrates that L_g, W_g given by (5.153) will also be optimal in the sense of (5.152).

The optimal solution to (4.36)-(4.38) can be obtained in a similar way based on the optimal solution to (4.41).

5.4.5 Comparison with LMI based design

The LMI technique has been much studied in the framework of robust control theory due to its numerical advantage. As the constraints

$$\|G_{rd}(z)\| < \gamma$$
$$\|G_{rf}(z)\| < \beta \text{ or } \|G_{rf}(z)\| > \beta$$
$$\left\| \begin{bmatrix} R(z)M_u(z)G_d(z) & R(z)M_u(z)G_f(z) - W_f(z) \end{bmatrix} \right\| < \alpha$$
$$\left\| \begin{bmatrix} R(z)M_u(z)G_d(z) - W_d(z) & R(z)M_u(z)G_f(z) - W_f(z) \end{bmatrix} \right\| < \alpha$$

can be re-formulated as LMI, the optimization problems (4.35)-(4.38), (4.40) and (4.44) can be solved with the help of LMI technique. In this section, we shall take the LMI based solution of the H_∞/H_∞ and H_-/H_∞ problem as examples to illustrate the basic idea of the LMI based design. We shall also compare the LMI based solution with the direct solution to the ratio type optimization problems.

Suppose that the H_∞ norm is used to evaluate both the robustness of the FD system to the unknown disturbances and the sensitivity to the faults. The constraints in the optimization problems (4.35)-(4.38) come down to

$$\|G_{rd}(z)\|_\infty < \gamma \tag{5.154}$$
$$\|G_{rf}(z)\|_\infty < \beta \tag{5.155}$$

which can be re-formulated as follows.

Theorem 5.14 Given the system (2.1), the observer based residual generator (2.19) with dynamics (5.102). Then the residual dynamics is stable and satisfies (5.154)-(5.155), if and only if there exists symmetric matrices $P_d = P_d^T > 0$, $P_f = P_f^T > 0$ and matrices L and W such that

$$\begin{bmatrix} (A-LC)^T P_d (A-LC) - P_d + C^T W^T W C & (A-LC)^T P_d (E_d - LF_d) + C^T W^T W F_d \\ (E_d - LF_d)^T P_d (A-LC) + F_d^T W^T W C & (E_d - LF_d)^T P_d (E_d - LF_d) + F_d^T W^T W F_d - \gamma^2 I \end{bmatrix} < 0 \tag{5.156}$$

$$\begin{bmatrix} (A-LC)^T P_f (A-LC) - P_f + C^T W^T W C & (A-LC)^T P_f (E_f - LF_f) + C^T W^T W F_f \\ (E_f - LF_f)^T P_f (A-LC) + F_f^T W^T W C & (E_f - LF_f)^T P_f (E_f - LF_f) + F_f^T W^T W F_f - \beta^2 I \end{bmatrix} < 0 \tag{5.157}$$

The proof follows directly from Lemma 3.1 and thus omitted here.

Using Schur Lemma, (5.156)-(5.157) are, respectively equivalent to

$$\begin{bmatrix} -P_d & P_d(A-LC) & P_d(E_d - LF_d) \\ (A-LC)^T P_d & -P_d + C^T W^T W C & C^T W^T W F_d \\ (E_d - LF_d)^T P_d & F_d^T W^T W C & F_d^T W^T W F_d - \gamma^2 I \end{bmatrix} < 0$$
$$\begin{bmatrix} -P_f & P_f(A-LC) & P_f(E_f - LF_f) \\ (A-LC)^T P_f & -P_f + C^T W^T W C & C^T W^T W F_f \\ (E_f - LF_f)^T P_f & F_f^T W^T W C & F_f^T W^T W F_f - \beta^2 I \end{bmatrix} < 0 \tag{5.158}$$

Suppose that the H_∞ norm is used to evaluate the robustness of the FD system to the unknown disturbances and the H_- index is used to characterize the minimal sensitivity to the faults. Then the constraints in the optimization problems (4.35)-(4.38) are

$$\|G_{rd}(z)\|_\infty < \gamma \tag{5.159}$$
$$\|G_{rf}(z)\|_- > \beta \tag{5.160}$$

which can also be re-formulated as matrix inequalities.

Theorem 5.15 Given the system (2.1), the observer based residual generator (2.19) with dynamics (5.102). Then the residual dynamics is stable and satisfies (5.159)-(5.160), if and only if there exists symmetric matrices $P_d = P_d^T > 0$, $P_f = P_f^T$ and matrices L and W such that

$$\begin{bmatrix} (A-LC)^T P_d (A-LC) - P_d + C^T W^T W C & (A-LC)^T P_d (E_d - LF_d) + C^T W^T W F_d \\ (E_d - LF_d)^T P_d (A-LC) + F_d^T W^T W C & (E_d - LF_d)^T P_d (E_d - LF_d) + F_d^T W^T W F_d - \gamma^2 I \end{bmatrix} < 0 \tag{5.161}$$

$$\begin{bmatrix} (A-LC)^T P_f (A-LC) - P_f + C^T W^T W C & (A-LC)^T P_f (E_f - LF_f) + C^T W^T W F_f \\ (E_f - LF_f)^T P_f (A-LC) + F_f^T W^T W C & (E_f - LF_f)^T P_f (E_f - LF_f) + F_f^T W^T W F_f - \beta^2 I \end{bmatrix} > 0 \tag{5.162}$$

Proof: According to Lemma 3.1, $A - LC$ is stable and $\|G_{rd}(z)\|_\infty < \gamma$ holds if and only if (5.161) holds for $P_d = P_d^T > 0$. As stated by Theorem 4.2, (5.162) is the necessary and sufficient condition for $\|G_{rf}(z)\|_- > \beta$. □

Notice that, as P_f is not definite, Schur Complement can not be applied to (5.162).

In comparison, the optimal solutions given in Section 5.3 are obtained by solving a DTARS, while the LMI solution needs to be solved in an iterative procedure. On the other side, the LMI based solution could provide advantage in integrating other design objectives and in handling model uncertainties.

5.5 Conclusion

This chapter focuses on the optimal design of model-based FD systems under different performance indices. It is very interesting to notice the close relationship among the different optimization problems. The similarity between the Kalman filter and the optimal solution (5.106) can be interpreted from the viewpoint of the MDR and the FAR.

6

Multiobjective design

A complete decoupling of the residual from the unknown disturbances can be achieved by the unknown input observer based approach or the unknown input decoupling parity space approach. However, it is only possible if the number of independent measurements is larger than the number of the unknown disturbances. It is also worth noticing that, after a full decoupling, the residual is also decoupled from the faults lying in the space spanned by the unknown disturbances and can not detect these faults. To keep the sensitivity of the FD system to the faults, it is advisable that only the unknown disturbances of strong influence are decoupled, while the effect of other unknown disturbances is minimized in consideration of the sensitivity of the residual generator to the faults.

In this chapter, the multi-objective design problem of FD systems will be considered. The process under consideration is described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + E_{d1}d_1(k) + E_{d2}d_2(k) + E_f f(k) \\ y(k) &= Cx(k) + Du(k) + F_{d1}d_1(k) + F_{d2}d_2(k) + F_f f(k) \end{aligned} \tag{6.1}$$

where $d_1 \in \mathbf{R}^{n_{d1}}$ denotes the unknown disturbances to be decoupled, $d_2 \in \mathbf{R}^{n_{d2}}$ the unknown disturbances whose influence need to be suppressed but not necessarily decoupled. The main objective of this chapter is to design an FD system, whose residual is decoupled from the unknown disturbances d_1 and simultaneously achieves a suitable trade-off between the robustness against the unknown disturbances d_2 and the sensitivity to the faults f.

6.1 Basic idea

Recall the discussion in Section 2.1 and 2.4, the approach that we shall discuss here consists of two steps. In the first step, a full decoupling from d_1 is realized in the framework of the parity space approach. In the second step, the rest freedom is used to meet other specifications on the robustness and the sensitivity. Depending on the residual evaluation schemes, different optimization problems can be formulated. Due to the close relationship between the parity space approach and the observer based approach, the order of the residual generator can be kept at a low and flexible level.

6.2 Design procedure

The parity relation of the system (6.1) is

$$y_s(k) = H_{o,s}x(k-s) + H_{u,s}u_s(k) + H_{d1,s}d_{1s}(k) + H_{d2,s}d_{2s}(k) + H_{f,s}f_s(k) \tag{6.2}$$

where $x(k-s), u_s(k), f_s(k), y_s(k), H_{o,s}, H_{u,s}, H_{f,s}$ are defined by (2.4),

$$d_{1s}(k) = \begin{bmatrix} d_1(k-s) \\ d_1(k-s+1) \\ \vdots \\ d_1(k) \end{bmatrix}, \; d_{2s}(k) = \begin{bmatrix} d_2(k-s) \\ d_2(k-s+1) \\ \vdots \\ d_2(k) \end{bmatrix}$$

$$H_{d1,s} = \begin{bmatrix} F_{d1} & O & \cdots & O \\ CE_{d1} & F_{d1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}E_{d1} & \cdots & CE_{d1} & F_{d1} \end{bmatrix}$$

$$H_{d2,s} = \begin{bmatrix} F_{d2} & O & \cdots & O \\ CE_{d2} & F_{d2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}E_{d2} & \cdots & CE_{d2} & F_{d2} \end{bmatrix}$$

Based on (6.2), a temporary residual signal $\bar{r}(k)$ is generated as

$$\bar{r}(k) = v_s(y_s(k) - H_{u,s}u_s(k)) \tag{6.3}$$

If the rank condition

$$rank \begin{bmatrix} H_{o,s} & H_{d1,s} & H_{f,s} \end{bmatrix} > rank \begin{bmatrix} H_{o,s} & H_{d1,s} \end{bmatrix}$$

is satisfied, then $\bar{r}(k)$ can be decoupled from the unknown disturbances d_1. The parity vector v_s is chosen as

$$v_s \in \mathbf{P}_{sd1} = \{v_s \mid v_s \begin{bmatrix} H_{o,s} & H_{d1,s} \end{bmatrix} = 0, \; v_s H_{f,s} \neq 0\} \tag{6.4}$$

With such a selection of v_s, there is

$$\bar{r}(k) = v_s(H_{d2,s}d_{2s}(k)) + H_{f,s}f_s(k)) \tag{6.5}$$

i.e. $\bar{r}(k)$ is decoupled from the initial state $x(k-s)$ and the unknown disturbances d_1.

The design freedom after the decoupling is represented by the free selectability of $v_s \in \mathbf{P}_{sd1}$. The question now is how to achieve the compromise between the sensitivity to the faults f and the robustness to the unknown disturbances d_2. In Section 4.4, it has been pointed out that the formulation of the optimization problem should take into account the residual evaluation scheme. Here we shall discuss the subsequent design under two different evaluation schemes.

In the first case, assume that $r(k) = \bar{r}(k)$ and the amplitude of the residual signal is used as residual evaluation function, i.e.

$$r_{e1}(k) = \|r(k)\|_E \tag{6.6}$$

Aiming at the optimal compromise between the robustness against d_2 and the sensitivity to f, the optimization problem can be formulated as

$$\max_{v_s \in \mathbf{P}_{sd1}} J_1 = \max_{v_s \in \mathbf{P}_{sd1}} \frac{\sup_{d_{2s}(k)=0, f_s(k)\neq 0} \frac{r^T(k)r(k)}{f_s^T(k)f_s(k)}}{\sup_{f_s(k)=0, d_{2s}(k)\neq 0} \frac{r^T(k)r(k)}{d_{2s}^T(k)d_{2s}(k)}} = \max_{v_s \in \mathbf{P}_{sd1}} \frac{v_s H_{f,s} H_{f,s}^T v_s^T}{v_s H_{d2,s} H_{d2,s}^T v_s^T} \tag{6.7}$$

Denote the basis matrix of the decoupling space $\mathbf{P}_{sd1}$ by N_{sd1}. The optimization problem (6.7) is equivalent to

$$\max_{v_s \in \mathbf{P}_{sd1}} J_1 = \max_{p_s} \frac{p_s N_{sd1} H_{f,s} H_{f,s}^T N_{sd1}^T p_s^T}{p_s N_{sd1} H_{d2,s} H_{d2,s}^T N_{sd1}^T p_s^T} \tag{6.8}$$

where p_s can be arbitrarily selected to maximize J_1. Thus, the optimal solution to the optimization problem (6.7) is given by

$$v_{s,opt} = p_{s,\max} N_{sd1} \tag{6.9}$$

where $p_{s,\max}$ is the eigenvector corresponding to the maximal eigenvalue of the generalized eigenvalue-eigenvector problem

$$p_{s,\max}(N_{sd1}H_{f,s}H_{f,s}^TN_{sd1}^T - \lambda_{\max}N_{sd1}H_{d2,s}H_{d2,s}^TN_{sd1}^T) = 0 \tag{6.10}$$

Correspondingly, the threshold can be determined as

$$\begin{aligned} J_{th,1} &= \sup_{f=0,d_2} r_{e1}(k) = \sup_{f=0,d_2} \|v_sH_{d2,s}d_{2s}(k)\|_E \\ &= (v_sH_{d2,s}H_{d2,s}^Tv_s^T)^{1/2}\max_k \max_{d_2}\|d_{2s}(k)\|_E \end{aligned} \tag{6.11}$$

Algorithm 6.1 Given the system (6.1) and the residual evaluation function (6.6), design an FD system so that the residual signal is decoupled from d_1 and simultaneously achieves a trade-off between the robustness to d_2 and the sensitivity to f in the sense of (6.7).

- Determine the basis matrix N_{sd1} of the decoupling space $\mathbf{P}_{sd1}$ defined by (6.4).
- Solve the generalized eigenvalue-eigenvector problem (6.10) for $p_{s,\max}$.
- Compute the vector $v_{s,opt}$ by (6.9) and generate the residual signal by

$$r(k) = v_{s,opt}(y_s(k) - H_{u,s}u_s(k)) \tag{6.12}$$

- Determine the threshold $J_{th,1}$ by (6.11).

In the second case, assume that d_2 is energy bounded, $\|d_2\|_2 \le \delta_{d2}$, and the l_2-norm of the residual signal is used for residual evaluation, i.e.

$$r_{e2}(k) = \|r(k)\|_2 \tag{6.13}$$

In this case, it is difficult to continue the design in the framework of the parity space approach. Therefore, we first transform the parity relation based residual generator (6.3) into an observer based residual generator. Recalling the analysis in Section 2.4, we partition the vector v_s obtained in (6.4) and $\rho_s = v_sH_{u,s}$ into

$$\begin{aligned} v_s &= \begin{bmatrix} v_{s,0} \; v_{s,1} \; \cdots \; v_{s,s} \end{bmatrix}, \; v_{s,i} \in \mathbf{R}^{1\times m} \\ \rho_s &= \begin{bmatrix} \rho_{s,0} \; \rho_{s,1} \; \cdots \; \rho_{s,s} \end{bmatrix}, \; \rho_{s,i} \in \mathbf{R}^{1\times n_u} \end{aligned} \tag{6.14}$$

An observer based residual generator can be readily constructed as follows

$$\begin{aligned} z(k+1) &= Gz(k) + Ju(k) + Ly(k) \\ r(k) &= wz(k) + pu(k) + vy(k) \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} G &= \begin{bmatrix} 0 & \cdots & 0 & g_1 \\ 1 & \cdots & 0 & g_2 \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & g_s \end{bmatrix}, \; J = \begin{bmatrix} \rho_{s,0} \\ \rho_{s,1} \\ \vdots \\ \rho_{s,s-1} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix} \rho_{s,s} \\ L &= -\begin{bmatrix} v_{s,0} \\ v_{s,1} \\ \vdots \\ v_{s,s-1} \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix} v_{s,s} \\ w &= \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix}, \; v = v_{s,s}, \; p = -\rho_{s,s} \end{aligned} \tag{6.16}$$

and $g_1, \cdots, g_s$ are free parameters that represent the design freedom and should guarantee the stability of matrix G. The dynamics of the residual r got by (6.15) is governed by

$$\begin{aligned} e(k+1) &= Ge(k) + (LF_f - TE_f)f(k) + (LF_{d2} - TE_{d2})d_2(k) \\ r(k) &= we(k) + vF_f f(k) + vF_{d2}d_2(k) \end{aligned} \tag{6.17}$$

where

$$T = \begin{bmatrix} v_{s,1} & \cdots & v_{s,s-1} & v_{s,s} \\ v_{s,2} & \cdots & v_{s,s} & 0 \\ \vdots & & & \vdots \\ v_{s,s} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{bmatrix}$$

It shows that, no matter what $g_1, \cdots, g_s$ are, $r(k)$ is decoupled from the unknown disturbances d_1. By substituting (6.16), (6.17) can be re-written as

$$\begin{aligned} r(k) = {} & v_s(H_{d2,s}d_{2s}(k) + H_{f,s}f_s(k)) \\ & + g_s r(k-1) + g_{s-1}r(k-2) + \cdots + g_1 r(k-s) \end{aligned} \tag{6.18}$$

If $g_1, \cdots, g_s$ are all zero, then the residual generator (6.15) has exactly the same dynamics as (6.3), $r = \bar{r}$. However, if $g_1, \cdots, g_s$ are set to be nonzero values, then (6.15) is indeed a generalization of the IIR (infinite impulse response) filter based residual generator proposed by [164]. The remaining problem is how to make advantage of the freedom provided by $g_1, \cdots, g_s$, so that a suitable trade-off between the robustness to the unknown disturbances d_2 and the sensitivity to the faults f can be achieved. To this aim, the residual dynamics (6.17) is re-written into

$$\begin{aligned} e(k+1) &= G_o e(k) + (L_o F_{d2} - TE_{d2})d_2(k) + (L_o F_f - TE_f)f(k) - gr(k) \\ r(k) &= we(k) + vF_{d2}d_2(k) + vF_f f(k) \end{aligned} \tag{6.19}$$

where

$$G_o = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad L_o = -\begin{bmatrix} v_{s,0} \\ v_{s,1} \\ \vdots \\ v_{s,s-1} \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix} \tag{6.20}$$

Note that G_o, L_o are independent of the vector g, and (G_o, w) is observable, as

$$rank \begin{bmatrix} w \\ wG_o \\ \vdots \\ wG_o^{s-1} \end{bmatrix} = s.$$

Thus, (6.19) shows a typical structure of observer dynamics with the vector g as feedback gain. Realizing this, we formulate the robustness and sensitivity problem as

$$\max_{g \in \mathbf{R}^{s \times 1}} J_2 = \max_{g \in \mathbf{R}^{s \times 1}} \frac{\sigma_i(G_{rf}(e^{j\omega}))}{\|G_{rd2}(z)\|_\infty} \tag{6.21}$$

where $\sigma_i(G_{rf}(e^{j\omega}))$ denotes the singular value of $G_{rf}(z)$ at $z = e^{j\omega}$, $G_{rd2}(z)$ and $G_{rf}(z)$ are, respectively, the transfer function matrices from the unknown disturbances d_2 and the faults f to the residual r,

$$\begin{aligned} G_{rd2}(z) &= vF_{d2} + w(zI - G_o + gw)^{-1}(L_o F_{d2} - TE_{d2} - gvF_{d2}) \\ G_{rf}(z) &= vF_f + w(zI - G_o + gw)^{-1}(L_o F_f - TE_f - gvF_f) \end{aligned} \tag{6.22}$$

Applying the results in Chapter 5, the optimal solution to optimization problem (6.21) is

$$g = -q^T \tag{6.23}$$

where (X, q) is the stabilizing solution of the discrete-time algebraic Riccati system (DTARS)

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ q \end{bmatrix} = 0. \tag{6.24}$$

with

$$\begin{aligned}
\Phi_{11} &= G_o X G_o^T - X + (L_o F_{d2} - T E_{d2})(L_o F_{d2} - T E_{d2})^T \\
\Phi_{12} &= G_o X w^T + (L_o F_{d2} - T E_{d2}) F_{d2}^T v^T \\
\Phi_{22} &= w X w^T + v F_{d2} F_{d2}^T v^T
\end{aligned}$$

If g is selected as (6.23), the threshold can be set as

$$J_{th,2} = \sup_{f=0,d_2} r_{e2}(k) = \sup_{f=0,d_2} \|r\|_2 = \|G_{rd2}(z)\|_\infty \delta_{d2} = \delta_{d2} \tag{6.25}$$

Algorithm 6.2 Given the system (6.1) and the residual evaluation function (6.13), design an FD system so that the residual signal is decoupled from d_1 and simultaneously achieves a trade-off between the robustness to d_2 and the sensitivity to f in the sense of (6.21).

- Solve (6.4) for a decoupling parity vector v_s.
- Compute $\rho_s = v_s H_{u,s}$ and partition v_s, ρ_s as (6.14).
- Build the matrices G_o, L_o by (6.20) and w, v, p, T by (6.16).
- Solve the DTARS (6.24) for the stabilizing solution (X, q).
- Let $g = -q^T$. Compute the matrices G, L, J according to (6.16) and generate the residual signal by (6.15).
- Determine the threshold $J_{th,2}$ by (6.25).

6.3 Example

Example 6.1 Consider an LTI system described by (6.1) with

$$A = \begin{bmatrix} 0 & 0.5 & 1 & 0 \\ -1 & -1 & 0.25 & 1 \\ 1 & 0.2 & -2 & 1 \\ 0.25 & 1 & -0.3 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$
$$E_{d1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0.2 \end{bmatrix}, \quad E_{d2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0.5 \\ 0.3 & 0.4 \end{bmatrix}, \quad E_f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0.5 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$F_{d1} = \begin{bmatrix} 0.3 \\ 0.4 \\ 0 \end{bmatrix}, \quad F_{d2} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0 \\ 0.5 & 0.1 \end{bmatrix}, \quad F_f = O \tag{6.26}$$

The basis matrix of the decoupling space $\mathbf{P}_{sd1}$ is

$$N_{sd1} = \begin{bmatrix} 0.5197 & -0.6324 & 0.1930 & -0.0114 & -0.3247 & -0.3664 & 0.0443 & -0.0332 & 0.2235 \\ 0.1548 & 0.2494 & -0.1709 & -0.5678 & 0.2617 & -0.0986 & -0.0366 & 0.0274 & 0.6942 \end{bmatrix}$$

If the amplitude of the residual signal is used as the residual evaluation function, then the optimal parity vector is obtained by solving the generalized eigenvalue-eigenvector problem (6.10) as

$$v_{s,opt} = \begin{bmatrix} 1.7245 & -3.1413 & 1.1845 & 1.3068 & -1.9313 & -1.2406 & 0.2656 & -0.1992 & -0.7535 \end{bmatrix}$$

$v_{s,opt} \in \mathbf{P}_{sd1}$ and maximizes J_1, i.e. $J_{1,opt} = \max_{v_s \in \mathbf{P}_{sd1}} \frac{v_s H_{f,s} H_{f,s}^T v_s^T}{v_s H_{d2,s} H_{d2,s}^T v_s^T} = 0.9018$. The residual signal is generated by (6.12).

If the l_2-norm of the residual signal is used as the residual evaluation function, choose a parity vector v_s in the decoupling space $\mathbf{P}_{sd1}$ as

$$v_s = \begin{bmatrix} 0.5197 & -0.6324 & 0.1930 & -0.0114 & -0.3247 & -0.3664 & 0.0443 & -0.0332 & 0.2235 \end{bmatrix}$$

Correspondingly,

$$\rho_s = v_s H_{u,s} = \begin{bmatrix} 0.4331 & -0.0004 & 0.0443 \end{bmatrix}$$

Partition v_s, ρ_s as (6.14). Then we get

$$G_o = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ J_o = \begin{bmatrix} 0.4331 \\ -0.0004 \end{bmatrix}, \ L_o = \begin{bmatrix} -0.5197 & 0.6324 & -0.1930 \\ 0.0114 & 0.3247 & 0.3664 \end{bmatrix}$$
$$w = \begin{bmatrix} 0 & -1 \end{bmatrix}, \ p = -0.0443, \ v = \begin{bmatrix} 0.0443 & -0.0332 & 0.2235 \end{bmatrix} \tag{6.27}$$

Substitute G_o, L_o, w, v into the DTARS (6.24). The stabilizing solution to (6.24) is

$$X = \begin{bmatrix} 0.0824 & 0.0625 \\ 0.0625 & 0.0825 \end{bmatrix}, \ q = \begin{bmatrix} -0.3279 & 0.5125 \end{bmatrix}$$

It follows that

$$g = \begin{bmatrix} 0.3279 \\ -0.5125 \end{bmatrix}$$

As a result, we get

$$G = \begin{bmatrix} 0 & 0.3279 \\ 1 & -0.5125 \end{bmatrix}, \ J = \begin{bmatrix} 0.4476 \\ -0.0231 \end{bmatrix}$$
$$L = \begin{bmatrix} -0.5342 & 0.6432 & -0.2663 \\ 0.0342 & 0.3077 & 0.4809 \end{bmatrix}$$

The residual signal is generated by (6.15). The residual is decoupled from d_1 and maximizes $J_2 = \frac{\sigma_i(G_{rf}(e^{j\omega}))}{\|G_{rd2}(z)\|_\infty}$ for any i and ω, for instance, $J_{2,opt} = \frac{\|G_{rf}(z)\|_\infty}{\|G_{rd2}(z)\|_\infty} = 5.6838$.

6.4 Conclusion

In this chapter, we have considered the multiobjective design of robust FD systems. In the literature, several authors have discussed the design of residual generators that are decoupled from deterministic unknown disturbances on the one side and make the estimation error variance minimal in the presence of noise on the other side [18, 77]. The method suggested by [18] divides the observer feedback gain into two parts, one concerning the full decoupling and the other concerning the minimal variance. The approach in [77] is developed based on an equivalent disturbance free model of the original system. More recently, [97] has given an approach to design observer-based residual generators with mixed H_2/H_∞ robustness against different kinds of unknown disturbances. *We have here concentrated on deterministic unknown disturbances*, which are classified into two parts: the part of unknown disturbances that should be decoupled and the part of unknown disturbances that needs not to be decoupled. Unlike [18, 77, 97], in the design the sensitivity of the FD system to the faults is also taken into account. In the first step, a decoupling is achieved between the residual and a part of the unknown disturbances within the framework of the parity space approach. Then, different optimizations can be carried out to improve the robustness of residual generator to the rest part of the unknown disturbances and the sensitivity to the faults, depending on the residual evaluation schemes adopted. The algorithms introduced in this chapter are characterized by an easy implementation and a flexible determination of the order of residual generators.

7

Probabilistic design

In this chapter, we shall discuss the fault detection problem of uncertain discrete LTI systems described by

$$
\begin{aligned}
x(k+1) &= A(\Delta)x(k) + B(\Delta)u(k) + E_f f(k) \\
y(k) &= C(\Delta)x(k) + D(\Delta)u(k) + F_f f(k)
\end{aligned}
\tag{7.1}
$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^{n_u}$ the control input vector, $y \in \mathbf{R}^m$ the measured output vector, and $f \in \mathbf{R}^{n_f}$ the vector of faults to be detected, E_f, F_f are known constant matrices, $A(\Delta), B(\Delta), C(\Delta), D(\Delta)$ are system matrices dependent on *unknown but bounded* parameter vector

$$
\Delta = \begin{bmatrix} \delta_1 \ \delta_2 \ \cdots \ \delta_l \end{bmatrix}^T \in \mathbf{R}^l
$$

It is assumed that the probability distribution of Δ is known *a priori* and denoted by $f_\Delta(\Delta)$. But there is no restriction on how Δ enters into the matrices A, B, C, D.

7.1 Construction of residual generator

Let

$$
\begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix} = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix} + \begin{bmatrix} F_a(\Delta) & F_b(\Delta) \\ F_c(\Delta) & F_d(\Delta) \end{bmatrix}
$$

where F_a, F_b, F_c, F_d are Δ-dependent unknown matrices, A_o, B_o, C_o, D_o are constant matrices representing the nominal behavior of the system. For instance, A_o, B_o, C_o, D_o can be defined by

$$
\begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix} = \begin{bmatrix} A(\bar{\Delta}) & B(\bar{\Delta}) \\ C(\bar{\Delta}) & D(\bar{\Delta}) \end{bmatrix}
\tag{7.2}
$$

where $\bar{\Delta}$ denotes the mean value of Δ that can be computed according to the probability distribution $f_\Delta(\Delta)$.

An observer-based residual generator can be constructed as

$$
\begin{aligned}
\hat{x}(k+1) &= A_o\hat{x}(k) + B_o u(k) + L(y(k) - \hat{y}(k)) \\
\hat{y}(k) &= C_o\hat{x}(k) + D_o u(k) \\
r(k) &= W(y(k) - \hat{y}(k))
\end{aligned}
\tag{7.3}
$$

where $r \in \mathbf{R}^{n_r}$ is the residual signal, L and W are, respectively, the observer gain matrix and the weighting matrix to be designed. The residual will be evaluated by the following logic

$$
\begin{cases} \|r\|_2 \le J_{th} \ \Rightarrow \ \text{fault-free} \\ \|r\|_2 > J_{th} \ \Rightarrow \ \text{alarm} \end{cases}
\tag{7.4}
$$

where the threshold J_{th} is set as

$$J_{th} = \sup_{f=0,\ \Delta} \|r\|_2 \tag{7.5}$$

The dynamics of the residual generator (7.3) is governed by

$$\begin{aligned}
\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} &= \begin{bmatrix} A & O \\ F_a - LF_c & A_o - LC_o \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} \\
&\quad + \begin{bmatrix} B \\ F_b - LF_d \end{bmatrix} u(k) + \begin{bmatrix} E_f \\ E_f - LF_f \end{bmatrix} f(k) \\
r(k) &= W \begin{bmatrix} F_c & C_o \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} + WF_d u(k) + WF_f f(k)
\end{aligned} \tag{7.6}$$

where $e(k) = x(k) - \hat{x}(k)$. It can be seen from (7.6) that the residual dynamics is internally stable, only if $A(\Delta)$ is stable for any Δ. Therefore, in the following, we assume that (A_o, C_o) is observable and that $A(\Delta)$ is stable for any Δ. Due to the existence of the model uncertainty, the residual signal r is influenced not only by the faults f but also by the control inputs u. The design parameters L and W should be selected in such a way that the influence of u on r is suppressed and the influence of f on r is strengthened. Hence, the FD problem is formulated as:

Problem 7.1 Given $\alpha > 0$, find the parameter L and W of the residual generator (7.3), so that

$$\|G_{ru}(z)\|_\infty < \alpha \tag{7.7}$$

Problem 7.2 Given $\alpha > 0$ and $\beta > 0$, find the parameter L and W of the residual generator (7.3), so that

$$\begin{aligned}
\|G_{ru}(z)\|_\infty &< \alpha \\
\|G_{rf}(z)\|_- &> \beta
\end{aligned} \tag{7.8}$$

The physical meaning behind Problem 1 is to attenuate influence of the non-fault factors. Problem 2 is a multi-objective design, which considers not only the robustness of the FD system to non-fault factors but also the sensitivity of the FD system to faults. Note that the transfer function matrices $G_{ru}(z)$ and $G_{rf}(z)$ are dependent on the uncertainty Δ.

In the following, we shall consider Problem 7.1. The basic idea of its solution can be extended to solve Problem 7.2.

7.2 Optimal parameter selection

In this section, we shall present an approach to find a solution to Problem 7.1 for the system (7.1) with arbitrary uncertainty structure by exploring the sequential subgradient approach.

7.2.1 Formulation of the constraint as LMI

As the first step, the constraint (7.7) is formulated as an LMI.

Lemma 7.1 Given the system (7.1), the residual generator (7.3) and $\alpha > 0$. Let $W = I$. The residual dynamics (7.6) is stable and $\|G_{ru}(z)\|_\infty < \alpha$, if there exist matrices $X_1 = X_1^T > 0$, $X_2 = X_2^T > 0$, P and a scalar $\epsilon > 0$, such that

$$\begin{aligned}
&V(X_1, X_2, P, \Delta) \\
&= \begin{bmatrix}
-X_1 + \epsilon I & O & X_1 A(\Delta) & O & X_1 B(\Delta) & O \\
O & -X_2 + \epsilon I & V_{23} & V_{24} & V_{25} & O \\
A^T(\Delta) X_1 & V_{23}^T & -X_1 & O & O & F_c^T(\Delta) \\
O & V_{24}^T & O & -X_2 & O & C_o^T \\
B^T(\Delta) X_1 & V_{25}^T & O & O & -\alpha^2 I & F_d^T(\Delta) \\
O & O & F_c(\Delta) & C_o & F_d(\Delta) & -I
\end{bmatrix} \leq O
\end{aligned} \tag{7.9}$$

where

$$
\begin{aligned}
V_{23} &= X_2 F_a(\Delta) - PF_c(\Delta) \\
V_{24} &= X_2 A_o - PC_o \\
V_{25} &= X_2 F_b(\Delta) - PF_d(\Delta)
\end{aligned}
$$

Proof: According to Lemma 3.1, $\|G_{ru}(z)\|_\infty < \alpha$, if and only if there exists a symmetric positive-definite matrix $X = X^T > 0$ such that

$$
\begin{bmatrix} -X & XA_{ru} & XB_{ru} & O \\ A_{ru}^T X & -X & O & C_{ru}^T \\ B_{ru}^T X & O & -\alpha^2 I & D_{ru}^T \\ O & C_{ru} & D_{ru} & -I \end{bmatrix} < 0 \tag{7.10}
$$

where

$$
\begin{aligned}
A_{ru} &= \begin{bmatrix} A & O \\ F_a - LF_c & A_o - LC_o \end{bmatrix}, \; B_{ru} = \begin{bmatrix} B \\ F_b - LF_d \end{bmatrix} \\
C_{ru} &= \begin{bmatrix} F_c \; C_o \end{bmatrix}, \; D_{ru} = F_d
\end{aligned}
$$

Assume

$$
X = \begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix} \tag{7.11}
$$

and let $P = X_2 L$. Then, (7.9) is obtained. □

Remark 7.1 The assumption (7.11) will introduce some conservatism in the design. If a post filter $R(z)$ is used in the residual generator (7.3) [190], then a less conservative result can be achieved.

Note that the matrix inequality (7.9) is linear with respect to the unknowns X_1, X_2, P and ϵ. The problem now is to find a solution X so that the LMI (7.9) is satisfied for any Δ. For certain kinds of structured uncertainty this is a well-studied problem and can be easily solved. However, for general uncertainty structure (for instance, nonlinear dependency of the matrices on Δ), the solution can be obtained by (i) overbounding the uncertainty by a structured one and then solving it, or (ii) looking for solutions satisfying the LMIs at a large number of samples of uncertainty. The former may introduce conservatism by overbounding. The latter needs to solve a large amount of LMIs simultaneously. However, due to lack of knowledge of the model uncertainty structure, it is not easy to solve (7.9). In the next, we shall apply the probabilistic robustness technique to solve this problem.

7.2.2 Preliminary of probabilistic robustness theory

The probabilistic robustness theory provides a new philosophy of control system analysis and synthesis. The idea of probabilistic robustness originated at the beginning of the eighties [144]. The important concepts like "probability of instability" has been introduced by [130, 145]. In the past several years the probabilistic robustness theory has been intensively investigated and developed [17, 63, 94, 100, 119, 125, 147, 152]. In this framework, based on random samples of uncertainty generated according to its distribution, the probability of performance can be estimated. The accuracy of the estimate can be guaranteed with a specified confidence level by taking enough amount of samples. Approaches have also been developed to solve controller synthesis problem, i.e. to find a controller which meets specification on robustness performance in a probabilistic sense.

The approaches to controller synthesis can be divided into two major classes:

- learning theory based approach, and
- sequential stochastic approach.

The basic idea of the learning theory based approach is to take random samples both in the uncertainty space and in the controller parameter space. By estimating the performance achieved by each trial (sample) of controller parameter, the best one will be selected. The learning theory based approach is conceptually straightforward. Its performance strongly depends on efficient generation of samples in the controller parameter space.

In the sequential stochastic approach, the basic idea is to use the subgradient method to iteratively update the controller parameter based on random samples of the uncertainty. The sequential subgradient approach has been proved to be efficient in finding solutions to LMI, BMI (bilinear matrix inequality) with unknown or varying parameters and employed in robust $\mathbf{H}_2$, $\mathbf{H}_\infty$ controller design. It can overcome the difficulty of sampling design parameters and will be applied here. Therefore, we shall at first briefly describe the mechanism of this kind of solution algorithm (see [17, 63, 94, 100, 119, 125] and the references therein).

The sequential subgradient approach finds the solution to $V(X, \Delta) \leq O$, $\forall \Delta$, in the following way [17, 125]:

- setting an initial value X^0 of the unknown matrix X,
- generating a random sample Δ^k of the uncertainty Δ according to the probability distribution of Δ,
- updating X^k based on subgradient of the *convex* objective function with respect to the unknown X.

It is proven that the algorithm converges in finite steps with probability 1, i.e.

$$\text{Prob}\{\exists k_0 < \infty, \text{ s.t. } V(X^k, \Delta) \leq O, \forall \Delta \text{ and } \forall k \geq k_0\} = 1$$

if the following two conditions holds: (i) The solution set is nonempty, and (ii) the probability that the LMI is not satisfied for some Δ is nonzero, as long as X is not a feasible solution. In case that a feasible solution is not found, a good approximately feasible candidate can be obtained through the above algorithm [17, 63].

7.2.3 Computation of subgradient

In this subsection, we apply the above introduced sequential subgradient approach to find the observer gain matrix L that satisfies (7.9) for arbitrary uncertainty structure.

Let the objective function be defined as

$$v(X_1, X_2, P, \Delta) = \left\| V^+(X_1, X_2, P, \Delta) \right\|_F \tag{7.12}$$

where V^+ denotes the projection of the symmetric matrix V onto the space of positive semi-definite matrices, and $\|V^+\|_F$ denotes the Frobenius norm of the matrix V^+. If $V \leq 0$, then $V^+ = O$ and $v = 0$. Otherwise, $V^+ \geq O$ and $v > 0$. The function $v(X_1, X_2, P, \Delta)$ is a convex scalar function of the unknowns X_1, X_2, P. If a set of matrices X_1, X_2, P can be found such that $v = 0$, then a feasible solution of (7.9) is found. Given a symmetric matrix V, the projection V^+ can be computed via solving an eigenvalue-eigenvector problem. Partition the matrix V^+ as $[V_{ij}^+]$, $i, j = 1, \cdots, 6$, corresponding to the dimensions of the blocks in (7.9).

Theorem 7.1 The subgradients of $v(X_1, X_2, P, \Delta)$ defined by (7.12) and (7.9) with respect to X_1, X_2, P are as follows. If $v(X_1, X_2, P, \Delta) > 0$, then

$$\begin{aligned}
&\partial_{X_1} v(X_1, X_2, P, \Delta) \\
&= -V_{11}^+ - V_{33}^+ + A(\Delta)V_{31}^+ + V_{31}^{+^T} A^T(\Delta) + B(\Delta)V_{51}^+ + V_{51}^{+^T} B^T(\Delta) \\
&\partial_{X_2} v(X_1, X_2, P, \Delta) \\
&= -V_{22}^+ - V_{44}^+ + F_a(\Delta)V_{32}^+ + A_o V_{42}^+ + F_b(\Delta)V_{52}^+ + V_{32}^{+^T} F_a^T(\Delta) \\
&+ V_{42}^{+^T} A_o^T + V_{52}^{+^T} F_b^T(\Delta) \\
&\partial_P v(X_1, X_2, P, \Delta) = -2V_{32}^{+^T} F_c^T(\Delta) - 2V_{42}^{+^T} C_o^T - 2V_{52}^{+^T} F_d^T(\Delta)
\end{aligned} \tag{7.13}$$

If $v(X_1, X_2, P, \Delta) = 0$, then

$$\begin{aligned} &\partial_{X_1} v(X_1, X_2, P, \Delta) = 0,\ \partial_{X_2} v(X_1, X_2, P, \Delta) = 0 \\ &\partial_P v(X_1, X_2, P, \Delta) = 0 \end{aligned} \tag{7.14}$$

Proof: Let the parameters X_1, X_2, P subject to small changes $\delta X_1, \delta X_2$ and δP, respectively. Then

$$V(X_1 + \delta X_1, X_2 + \delta X_2, P + \delta P, \Delta) = V(X_1, X_2, P) + \delta V$$

where

$$\delta V = \Lambda_1 H_1 + H_1^T \Lambda_1 + \Lambda_2 H_2 + H_2^T \Lambda_2 + H_3 + H_3^T$$

$$H_1 = \begin{bmatrix} -\frac{I}{2} & O & A & O & B & O \\ O & O & O & O & O & O \\ O & O & -\frac{I}{2} & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{bmatrix}$$

$$H_2 = \begin{bmatrix} O & O & O & O & O & O \\ O & -\frac{I}{2} & F_a(\Delta) & A_o & F_b(\Delta) & O \\ O & O & O & O & O & O \\ O & O & O & -\frac{I}{2} & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{bmatrix}$$

$$H_3 = \begin{bmatrix} O & O & O & O & O & O \\ O & O & -\delta P F_c & -\delta P C_o & -\delta P F_d & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{bmatrix}$$

$$\Lambda_1 = diag\{\delta X_1, O, \delta X_1, O, O, O\}$$
$$\Lambda_2 = diag\{O, \delta X_2, O, \delta X_2, O, O\}$$

Due to the differentiability of v, there is [63, 94]

$$\begin{aligned} &v(X_1 + \delta X_1, X_2 + \delta X_2, P + \delta P, \Delta) \\ &= \left\| [V(X_1, X_2, P) + \delta V]^+ \right\|_F \\ &= v(X_1, X_2, P) + \left\langle V^+(X_1, X_2, P), \delta V \right\rangle + o(\|\delta V\|_F) \end{aligned}$$

where $\langle P_1, P_2 \rangle = tr(P_1 P_2)$ denotes the inner product of the matrices P_1 and P_2. It can be derived that

$$\begin{aligned} &\left\langle V^+(X_1, X_2, P), \delta V \right\rangle \\ &= \left\langle V^+(X_1, X_2, P), \Lambda_1 H_1 \right\rangle + \left\langle V^+(X_1, X_2, P), H_1^T \Lambda_1 \right\rangle \\ &\quad + \left\langle V^+(X_1, X_2, P), \Lambda_2 H_2 \right\rangle + \left\langle V^+(X_1, X_2, P), H_2^T \Lambda_2 \right\rangle \\ &\quad + \left\langle V^+(X_1, X_2, P), H_3 \right\rangle + \left\langle V^+(X_1, X_2, P), H_3^T \right\rangle \\ &= \left\langle H_1 V^+(X_1, X_2, P), \Lambda_1 \right\rangle + \left\langle V^+(X_1, X_2, P) H_1^T, \Lambda_1 \right\rangle \\ &\quad + \left\langle H_2 V^+(X_1, X_2, P), \Lambda_2 \right\rangle + \left\langle V^+(X_1, X_2, P) H_2^T, \Lambda_2 \right\rangle \\ &\quad + \left\langle V^+(X_1, X_2, P), H_3 \right\rangle + \left\langle V^+(X_1, X_2, P), H_3^T \right\rangle \end{aligned}$$

Therefore

$$\begin{aligned}\partial_{X_1}\left\|V^{+}(X_1,X_2,P)\right\|_F &= U_{11}+U_{11}^{T}+U_{33}+U_{33}^{T}\\ \partial_{X_2}\left\|V^{+}(X_1,X_2,P)\right\|_F &= Q_{22}+Q_{22}^{T}+Q_{44}+Q_{44}^{T}\\ \partial_{P}\left\|V^{+}(X_1,X_2,P)\right\|_F &= W_{22}\end{aligned}$$

where

$$\begin{aligned}U_{11} &= -\frac{1}{2}V_{11}^{+}+A(\Delta)V_{31}^{+}+B(\Delta)V_{51}^{+}\\ Q_{22} &= -\frac{1}{2}V_{22}^{+}+F_a(\Delta)V_{32}^{+}+A_oV_{42}^{+}+F_b(\Delta)V_{52}^{+}\\ U_{33} &= -\frac{1}{2}V_{33}^{+},\quad Q_{44}=-\frac{1}{2}V_{44}^{+}\\ W_{22} &= -2{V_{32}^{+}}^{T}F_c^{T}(\Delta)-2{V_{42}^{+}}^{T}C_o^{T}-2{V_{52}^{+}}^{T}F_d^{T}(\Delta)\end{aligned}$$

The theorem is thus proven. □

7.2.4 Design procedure

Based on Theorem 7.1, the sequential subgradient approach can be applied to find out the solution of Problem 1.

Algorithm 7.1 Given the system (7.1) with the uncertainty Δ described by the probability distribution $f_\Delta(\Delta)$ and $\alpha > 0$, an observer-based residual generator (7.3) can be designed as follows:

Step 1 Set the value of ϵ and select an initial value of X_1^0, X_2^0, P^0.

Step 2 Generate a sample of model uncertainty Δ^k according to the probability distribution $f_\Delta(\Delta)$.

Step 3 Compute the projection $V^{+}(X_1^k, X_2^k, P^k, \Delta^k)$ and the value of the objective function

$$v(X_1^k,X_2^k,P^k,\Delta^k)=\left\|V^{+}(X_1^k,X_2^k,P^k,\Delta^k)\right\|_F$$

Step 4 Compute the subgradients

$$\begin{aligned}&\partial_{X_1}v(X_1^k,X_2^k,P^k,\Delta^k)\\ &\partial_{X_2}v(X_1^k,X_2^k,P^k,\Delta^k)\\ &\partial_{P}v(X_1^k,X_2^k,P^k,\Delta^k)\end{aligned}$$

according to Theorem 7.1.

Step 5 Calculate

$$\begin{aligned}\beta^k &= \Big(\left\|\partial_{X_1}v(X_1^k,X_2^k,P^k,\Delta^k)\right\|_F^2+\left\|\partial_{X_2}v(X_1^k,X_2^k,P^k,\Delta^k)\right\|_F^2\\ &\quad+\left\|\partial_{P}v(X_1^k,X_2^k,P^k,\Delta^k)\right\|_F^2\Big)^{1/2}\\ \alpha^k &= \frac{v(X_1^k,X_2^k,P^k,\Delta^k)}{\beta^k}+r_p\end{aligned}$$

where $r_p > 0$ is the radius of a ball inside the feasible solution set.

Step 6 If $v(X_1^k, X_2^k, P^k, \Delta^k) = 0$, let

$$X_1^{k+1}=X_1^k,\ X_2^{k+1}=X_2^k,\ P^{k+1}=P^k$$

Otherwise, update the variables X_1^k, X_2^k, P^k by

$$X_1^{k+1} = X_1^k - \frac{\alpha^k}{\beta^k}\partial_{X_1} v(X_1^k, X_2^k, P^k, \Delta^k)$$
$$X_2^{k+1} = X_2^k - \frac{\alpha^k}{\beta^k}\partial_{X_2} v(X_1^k, X_2^k, P^k, \Delta^k)$$
$$P^{k+1} = P^k - \frac{\alpha^k}{\beta^k}\partial_{P} v(X_1^k, X_2^k, P^k, \Delta^k)$$

Step 7 If the above algorithm converges, then the observer gain matrix is obtained as

$$L = (X_2^k)^{-1} P^k$$

Otherwise, set $k = k + 1$ and return to step 2.

Remark 7.2 The necessary iterations may be reduced by using the approach proposed by [119].

Remark 7.3 If a feasible solution is not found for the given α after a sufficiently large number of iterations, the approximately feasible candidate obtained through the algorithm can be used as initial value for starting the next iteration with a larger α.

Remark 7.4 In case that the probability distribution $f_\Delta(\Delta)$ of the bounded uncertainty Δ is unavailable, a uniform distribution can be assumed [6].

To evaluate the residual based on (3.3), an adaptive threshold J_{th} can be determined. If a gain matrix L that satisfies (7.7) is found, then J_{th} can be set as

$$J_{th} = \alpha \|u\|_{2,[k-N,k]}$$

which guarantees the false alarm rate FAR defined by

$$P_{FA} = \text{Prob}\{\|r\|_{2,[k-N,k]} > J_{th} \mid f = 0\}$$

to be zero with probability 1, because $\|r\|_{2,[k-N,k]} \le \|G_{ru}(z)\|_\infty \|u\|_{2,[k-N,k]}$. Alternatively, the threshold J_{th} can be selected to guarantee the false alarm rate be under a user defined level using the approach developed by [39].

7.3 Example

In this section, two examples will be given to illustrate the proposed design procedure.

Example 7.1 Consider the FD problem of a system in the form of (7.1) with

$$A = \begin{bmatrix} 0.7+\theta_1 & 0 & \theta_9 & \theta_6 \\ 0 & 0.8+\theta_2\theta_3 & 0 & \theta_7 \\ 0 & 0 & 0.6+\theta_4 & \theta_8 \\ 0 & 0 & 0 & 0.5+\theta_5 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix}, \; E_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; D = O, \; F_f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The nominal value of the parameter vector is $\theta_1 = -0.5$, $\theta_2 = -0.55$, $\theta_3 = 0.28$, $\theta_4 = 0.086$, $\theta_5 = -0.11$, $\theta_6 = 0.1$, $\theta_7 = -0.042$, $\theta_8 = 0.601$, $\theta_9 = -0.29$. The parameter change is smaller than 10% of the nominal value and is of uniform distribution. Given $\alpha = 2.5$.

Select A_o according to (7.2) and set $M = 5000$, $\epsilon = 0.01$, $r_p = 0.001$. The proposed design procedure yields

$$X_1 = \begin{bmatrix} 1.0075 & -0.0049 & 0.1860 & -0.0242 \\ -0.0049 & 0.1928 & 0.0099 & 0.0179 \\ 0.1860 & 0.0099 & 0.2604 & 0.0018 \\ -0.0242 & 0.0179 & 0.0018 & 0.4236 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1.3000 & 0.1582 & 0.0512 & -0.1764 \\ 0.1582 & 0.8767 & -0.1318 & -0.0777 \\ 0.0512 & -0.1318 & 1.5097 & 0.3835 \\ -0.1764 & -0.0777 & 0.3835 & 1.9179 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.4219 & -0.2738 & 0.1000 \\ -0.1262 & -0.1045 & -0.2821 \\ -0.0717 & 0.7187 & 0.8349 \\ 0.0523 & 0.1242 & 0.5551 \end{bmatrix}$$

Finally, an observer-based residual generator that satisfies (7.7) is obtained as

$$\begin{aligned} \hat{x}(k+1) &= A_o\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) \\ r(k) &= y(k) - \hat{y}(k) \end{aligned}$$

with

$$A_o = \begin{bmatrix} 0.2 & 0 & -0.29 & 0.1 \\ 0 & 0.646 & 0 & -0.042 \\ 0 & 0 & 0.686 & 0.601 \\ 0 & 0 & 0 & 0.39 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.3646 & -0.2372 & 0.1153 \\ -0.2180 & -0.0066 & -0.2536 \\ -0.0970 & 0.4980 & 0.4776 \\ 0.0714 & -0.0569 & 0.1943 \end{bmatrix}$$

The next example shows that a residual generator which minimizes α can be found by iteratively using the proposed design procedure.

Example 7.2 The system under consideration is the vehicle lateral dynamics which is described by the so-called bicycle model [36]:

$$\begin{bmatrix} \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -\frac{C_{\alpha V}+C_{\alpha H}}{mv_{ref}} & \frac{l_H C_{\alpha H} - l_V C_{\alpha V}}{mv_{ref}^2} - 1 \\ \frac{l_H C_{\alpha H} - l_V C_{\alpha V}}{I_z} & -\frac{l_V^2 C_{\alpha V} + l_H^2 C_{\alpha H}}{I_z v_{ref}} \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} \frac{C_{\alpha V}}{mv_{ref}} \\ \frac{l_V C_{\alpha V}}{I_z} \end{bmatrix} \delta_L^* \qquad (7.15)$$

where β denotes the vehicle side slip angle, γ the yaw rate and δ_L^* the steering angle. The original vehicle parameters of a car have been adopted. It is assumed that only a yaw rate sensor is available.

It is well-known that among the parameters in the model (7.15) the front cornering stiffness $C_{\alpha V}$ and the rear cornering stiffness $C_{\alpha H}$ may vary over a large range, depending on the road condition and the driving maneuvers [36]. This causes a strong model uncertainty in the bicycle model (7.15). It is assumed that $C_{\alpha H} = kC_{\alpha V}, k = 1.7278$ and $C_{\alpha V} = C_{\alpha V}^o + \Delta C_{\alpha V}$, $C_{\alpha V}^o = 103600N/rad$, $\Delta C_{\alpha V} \in [-b_1, 0]$ is a random number with uniform distribution with b_1 representing the maximal size of parameter change.

Model (7.15) can be re-written into the form

$$\begin{aligned} \dot{x} &= (A + \Delta A)x + (B + \Delta B)u, \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned} \qquad (7.16)$$

with $x = \begin{bmatrix} \beta \ \gamma \end{bmatrix}^T$, $u = \delta_L^*$, $y = \gamma$ and

$$A = \begin{bmatrix} -\frac{(1+k)C_{\alpha V}^o}{mv_{ref}} & \frac{(kl_H - l_V)C_{\alpha V}^o}{mv_{ref}^2} - 1 \\ \frac{(kl_H - l_V)C_{\alpha V}^o}{I_z} & -\frac{(l_V^2 + kl_H^2)C_{\alpha V}^o}{I_z v_{ref}} \end{bmatrix}, \ \Delta A = \begin{bmatrix} -\frac{1+k}{mv_{ref}} & \frac{kl_H - l_V}{mv_{ref}^2} \\ \frac{kl_H - l_V}{I_z} & -\frac{l_V^2 + kl_H^2}{I_z v_{ref}} \end{bmatrix} \Delta C_{\alpha V}$$

$$B = \begin{bmatrix} \frac{C_{\alpha V}^o}{mv_{ref}} \\ \frac{l_V C_{\alpha V}^o}{I_z} \end{bmatrix}, \ \Delta B = \begin{bmatrix} \frac{1}{mv_{ref}} \\ \frac{l_V}{I_z} \end{bmatrix} \Delta C_{\alpha V}$$

Because the sampling period of the system is $T = 0.01$ second, the discretized model is

$$\begin{aligned} x(k+1) &= (A_d + F_a(\Delta C_{\alpha V}))x(k) + (B_d + F_b(\Delta C_{\alpha V}))u(k), \\ y(k) &= \begin{bmatrix} 0 \ 1 \end{bmatrix} x(k) \end{aligned} \tag{7.17}$$

where

$$A_d = e^{AT}, \ B_d = \int_0^T e^{At} B dt$$

$$F_a(\Delta C_{\alpha V}) = e^{(A+\Delta A)T} - e^{AT}$$

$$F_b(\Delta C_{\alpha V}) = \int_0^T e^{(A+\Delta A)t}(B + \Delta B)dt - \int_0^T e^{At} B dt$$

Although $\Delta A, \Delta B$ in the continuous-time model (7.16) depend linearly on the uncertain parameter $\Delta C_{\alpha V}$, the model uncertainties F_a, F_b in the discretized model (7.17) depend on $\Delta C_{\alpha V}$ nonlinearly.

For the purpose of residual generation, the following observer is used

$$\begin{bmatrix} \hat{\beta}(k+1) \\ \hat{\gamma}(k+1) \end{bmatrix} = A_d \begin{bmatrix} \hat{\beta}(k) \\ \hat{\gamma}(k) \end{bmatrix} + B_d \delta_L^*(k) + L(\gamma - \hat{\gamma})$$

$$r = \gamma - \hat{\gamma}$$

where L is the design parameter.

Assume that $\epsilon = 0.01$, $r_p = 0.001$, $M = 1000$. For $b_1 = 30000$, the minimal achievable α is 0.22 and the resulting observer gain matrix is

$$L = \begin{bmatrix} 0.3241 \\ 0.8874 \end{bmatrix}$$

To verify the result, we have generated 30000 random samples of $\Delta C_{\alpha V}$ uniformly distributed in $[-30000, 0]$. The design procedure is also carried out under other values of b_1. It should be emphasized that the selection of ε, r_p, M will influence the convergence rate [63]. Therefore, the achieved minimal α is only sub-optimal.

7.4 Conclusion

This chapter studies the fault detection problem of uncertain linear systems with arbitrary uncertainty structure. With the aid of probabilistic robustness technique, an algorithm is developed to determine the parameter of observer-based residual generators. The results can be extended to handle systems with both multiplicative uncertainty and unknown disturbances. Future study will be focused on the multi-objective design of observer-based fault detection systems directly guaranteeing specified false alarm rate and miss detection rate.

Part II

FD of Discrete-Time Linear Periodic Systems

8

Introduction to periodic systems

Linear periodic systems are the simplest class of linear systems next to LTI systems and exist widely in different areas, for instance, aeronautics and aerospace, telecommunication and signal processing [9]. Recently, the interest in the periodic systems has been renewed both in theory and application [89, 25, 68, 96, 132, 134, 154]. For instance, the parametric transfer function (PTF) theory [132], the $\mathbf{H}_2$, $\mathbf{H}_\infty$ theory [24, 143, 102] and the polynomial approach [11] have been developed for the periodic systems. Typical applications are helicopter vibration control [4], satellite attitude control [112], wind turbine [146] as well as in networked control systems [131, 166].

In this chapter, we shall introduce some basic properties of linear discrete-time periodic systems described by

$$\Sigma : \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), \ x(0) = x_0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases} \tag{8.1}$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^{n_u}$ the control input vector and $y \in \mathbf{R}^m$ the measured output vector, $A(k), B(k)$, $C(k), D(k)$ are known bounded and real periodic matrices of period T, i.e. $\forall k$,

$$\begin{aligned} A(k+T) &= A(k), \ B(k+T) = B(k) \\ C(k+T) &= C(k), \ D(k+T) = D(k) \end{aligned} \tag{8.2}$$

8.1 Time and frequency response

Assume that $\tau_1 \geq \tau_0$. It is easy to derive that the solution to the state equation (8.1) is

$$x(\tau_1) = \Psi(\tau_1, \tau_0)x(\tau_0) + \sum_{j=\tau_0+1}^{\tau_1} \Psi(\tau_1, j)B(j-1)u(j-1) \tag{8.3}$$

where $\Psi(\tau_1, \tau_0)$ is the state transition matrix defined by

$$\Psi(\tau_1, \tau_0) = \begin{cases} I, & \text{if } \tau_1 = \tau_0 \\ A(\tau_1 - 1)A(\tau_1 - 2)\cdots A(\tau_0), & \text{if } \tau_1 > \tau_0 \end{cases} \tag{8.4}$$

Hence, the system response in the time domain is

$$x(k) = \Psi(k,0)x(0) + \sum_{j=1}^{k} \Psi(k,j)B(j-1)u(j-1) \tag{8.5}$$

$$y(k) = C(k)\Psi(k,0)x(0) + \sum_{j=1}^{k} C(k)\Psi(k,j)B(j-1)u(j-1) + D(k)u(k)$$

Let the initial condition be $x(0) = 0$. Then

$$\begin{aligned} y(0) &= D(0)u(0) \\ y(1) &= C(1)B(0)u(0) + D(1)u(1) \\ y(2) &= C(2)A(1)B(0)u(0) + C(2)B(1)u(1) + D(2)u(2), \cdots \end{aligned}$$

Doing the z-transform of the output sequence gives

$$\begin{aligned} y(z) &= y(0) + y(1)z + y(2)z^2 + y(3)z^3 \cdots + y(k)z^k + \cdots \\ &= D(0)u(0) + (C(1)B(0)u(0) + D(1)u(1))\, z \\ &\quad + (C(2)A(1)B(0)u(0) + C(2)B(1)u(1) + D(2)u(2))\, z^2 \\ &\quad + \begin{pmatrix} C(3)A(2)A(1)B(0)u(0) + C(3)A(2)B(1)u(1) \\ +C(3)B(2)u(2) + D(3)u(3) \end{pmatrix} z^3 + \cdots \\ &= (D(0) + C(1)B(0)z + C(2)A(1)B(0)z^2 + \cdots) \\ &\quad \times (u(0) + u(T)z^T + u(2T)z^{2T} + \cdots) \\ &\quad + (D(1) + C(2)B(1)z + C(3)A(2)B(1)z^2 + \cdots) \\ &\quad \times \left(u(1)z + u(T+1)z^{T+1} + u(2T+1)z^{2T+1} + \cdots\right) + \cdots \\ &\quad + (D(T-1) + C(T)B(T-1)z + C(T+1)A(T)B(T-1)z^2 + \cdots) \\ &\quad \times \left(u(T-1)z^{T-1} + u(2T-1)z^{2T-1} + u(3T-1)z^{3T-1} + \cdots\right) \end{aligned}$$

Define

$$\begin{aligned} G_\tau(z) &= D(\tau) + C(\tau+1)B(\tau)z + C(\tau+2)A(\tau+1)B(\tau)z^2 + \cdots \\ u_\tau(z) &= u(\tau)z^\tau + u(\tau+T)z^{\tau+T} + u(\tau+2T)z^{\tau+2T} + \cdots \end{aligned}$$

where $\tau = 0, 1, \cdots, T-1$. Therefore, in the frequency domain the input and output are related by

$$y(z) = \sum_{\tau=0}^{T-1} G_\tau(z) u_\tau(z) \tag{8.6}$$

8.2 Stability, observability and reachability

For any integer τ, the state transition matrix over one period $\Psi(\tau + T, \tau)$ is called monodromy matrix. It is known from linear algebra that, for any given real matrices A and B of dimensions $n \times n$, matrices AB and BA have the same eigenvalues. Therefore, the eigenvalues of $\Psi(\tau + T, \tau)$ are independent of τ and referred to as *characteristic multipliers* in the literature.

The system (8.1) is asymptotically *stable*, if $\lim_{k\to\infty} x(k) = 0$, if $u \equiv 0$ and $\forall x_0$. Note that $\Psi(k, 0) = \Psi(k, nT)\Psi^n(T, 0)$, if $k = nT + j$, $0 \le j \le T-1$. Because $0 \le k - nT \le T-1$ and $A(k)$ is bounded for any k, $\Psi(k, nT)$ is always bounded. Hence, system (8.1) is asymptotically *stable*, if and only if all the eigenvalues of $\Psi(T, 0)$ are inside the unit circle, i.e., all characteristic multipliers are located inside the unit circle.

The stability of the system (8.1) can also be checked with the aid of the Lyapunov theorem. Let $u(k) \equiv 0$ and define a Lyapunov function

$$V(k) = x^T(k)P(k)x(k)$$

where $P(k)$ is a periodic positive-definite matrix, $P(k) = P(k+T) > 0$, $\forall k$. Note that

$$V(k+1) - V(k) = x^T(k)\left(A^T(k)P(k+1)A(k) - P(k)\right)x(k)$$

Therefore, system (8.1) is asymptotically stable, if and only if there exists a periodic positive-definite matrix $P(k) = P(k+T) > 0$ such that [9, 143]

$$A^T(k)P(k+1)A(k) - P(k) < 0, \;\; k = 0, 1, \cdots, T-1 \tag{8.7}$$

A characteristic multiplier λ is said to be $(A(k), C(k))$ *unobservable* at time τ, if there exists a column vector $\zeta \neq 0$ such that

$$\begin{bmatrix} \lambda I - \Psi(\tau+T,\tau) \\ C(\tau)\Psi(\tau,\tau) \\ C(\tau+1)\Psi(\tau+1,\tau) \\ \vdots \\ C(\tau+T-1)\Psi(\tau+T-1,\tau) \end{bmatrix} \zeta = 0 \tag{8.8}$$

The system (8.1) is *observable* at time τ, if all characteristic multipliers are observable at time τ.

Dually, λ is said to be $(A(k), B(k))$ *unreachable* at time τ, if there exists a column vector $\eta \neq 0$ such that

$$\begin{bmatrix} \lambda I - \Psi^T(\tau+T,\tau) \\ B^T(\tau)\Psi^T(\tau+T,\tau+1) \\ B^T(\tau+1)\Psi^T(\tau+T,\tau+2) \\ \vdots \\ B^T(\tau+T-1)\Psi^T(\tau+T,\tau+T) \end{bmatrix} \eta = 0$$

System (8.1) is *reachable* at time τ, if all characteristic multipliers are reachable at time τ.

The observability and reachability of nonzero characteristic multipliers are time independent. To illustrate it, we take the observability as an example. Assume that $\lambda \neq 0$ is a nonzero characteristic multiplier of system (8.1) and, together with vector $\zeta_\tau \neq 0$, it satisfies (8.8) at time τ. Let $1 \leq j \leq T-1$ and

$$\zeta_{\tau+j} = \Psi(\tau+j,\tau)\zeta_\tau$$

As

$$\Psi(\tau+T,\tau)\zeta_\tau = \Psi(\tau+T,\tau+j)\zeta_{\tau+j} = \lambda\zeta_\tau \neq 0$$

it can be seen that $\zeta_{\tau+j} \neq 0$. At time $\tau+j$, there is

$$\begin{aligned} &\Psi(\tau+j+T,\tau+j)\zeta_{\tau+j} \\ &= \Psi(\tau+j+T,\tau)\zeta_\tau = \Psi(\tau+j+T,\tau+T)\Psi(\tau+T,\tau)\zeta_\tau \\ &= \lambda\Psi(\tau+j+T,\tau+T)\zeta_\tau = \lambda\zeta_{\tau+j} \end{aligned}$$

For any $0 \leq l \leq T-1-j$, it is easy to get

$$C(\tau+j+l)\Psi(\tau+j+l,\tau+j)\zeta_{\tau+j} = C(\tau+j+l)\Psi(\tau+j+l,\tau)\zeta_\tau = 0$$

For $T-j \leq l \leq T-1$, there is

$$\begin{aligned} &C(\tau+j+l)\Psi(\tau+j+l,\tau+j)\zeta_{\tau+j} \\ &= C(\tau+j+l)\Psi(\tau+j+l,\tau)\zeta_\tau \\ &= C(\tau+j+l-T)\Psi(\tau+j+l,\tau+T)\Psi(\tau+T,\tau)\zeta_\tau \\ &= C(\tau+j+l-T)\Psi(\tau+j+l,\tau+T)\lambda\zeta_\tau \\ &= \lambda C(\tau+j+l-T)\Psi(\tau+j+l-T,\tau)\zeta_\tau = 0 \end{aligned}$$

Hence, $\lambda \neq 0$ and $\zeta_{\tau+j} \neq 0$ satisfies (8.8) at time $\tau+j$, i.e. the characteristic multiplier $\lambda \neq 0$ is also unobservable at time $\tau+j$, for any $1 \leq j \leq T-1$. However, such a property doesn't exist if $\lambda = 0$. The observability and reachability of zero characteristic multipliers are time dependent.

Detectability and stabilizability are defined by focusing on the unstable characteristic multipliers. The system (8.1) is *detectable (stabilizable)*, if all characteristic multipliers lying on or outside the unit circle are observable (reachable) at each time. The time independence of the observability and reachability of nonzero characteristic multipliers shows that the detectability and stabilizability of periodic systems are time independent.

8.3 LTI reformulation of periodic systems

The correspondence between the periodic system (8.1) and discrete LTI systems is well-recognized. In this section, we shall introduce several LTI reformulations of periodic systems in the time domain and in the frequency domain [8, 10].

8.3.1 Time domain lifting

Let τ denote the initial time and, for a given signal δ, an augmented signal $\bar{\delta}_\tau$ is defined as

$$\bar{\delta}_\tau(k) = \begin{bmatrix} \delta_{\tau 1}(k) \\ \vdots \\ \delta_{\tau T}(k) \end{bmatrix} \tag{8.9}$$

$$\delta_{\tau i}(k) = \begin{cases} \delta(k), & \text{if } \frac{k-\tau-i+1}{T} \text{ is integer} \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \cdots, T$$

The *cyclic reformulation* of the periodic system (8.1) is

$$\begin{aligned} \bar{x}_\tau(k+1) &= \bar{A}_\tau \bar{x}_\tau(k) + \bar{B}_\tau \bar{u}_\tau(k) \\ \bar{y}_\tau(k) &= \bar{C}_\tau \bar{x}_\tau(k) + \bar{D}_\tau \bar{u}_\tau(k) \end{aligned} \tag{8.10}$$

where

$$\begin{aligned} \bar{A}_\tau &= \begin{bmatrix} O & \cdots & O & A(\tau+T-1) \\ A(\tau) & \cdots & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & A(\tau+T-2) & O \end{bmatrix} \\ \bar{B}_\tau &= \begin{bmatrix} O & \cdots & O & B(\tau+T-1) \\ B(\tau) & \cdots & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & B(\tau+T-2) & O \end{bmatrix} \\ \bar{C}_\tau &= \begin{bmatrix} C(\tau) & O & \cdots & O \\ O & \ddots & \ddots & \vdots \\ \vdots & \ddots & C(\tau+T-2) & O \\ O & \cdots & O & C(\tau+T-1) \end{bmatrix} \\ \bar{D}_\tau &= \begin{bmatrix} D(\tau) & O & \cdots & O \\ O & \ddots & \ddots & \vdots \\ \vdots & \ddots & D(\tau+T-2) & O \\ O & \cdots & O & D(\tau+T-1) \end{bmatrix} \end{aligned} \tag{8.11}$$

The *lifted reformulation* of the periodic system (8.1) is

$$\begin{aligned} \tilde{x}_\tau(k+1) &= \tilde{A}_\tau \tilde{x}_\tau(k) + \tilde{B}_\tau \tilde{u}_\tau(k) \\ \tilde{y}_\tau(k) &= \tilde{C}_\tau \tilde{x}_\tau(k) + \tilde{D}_\tau \tilde{u}_\tau(k) \end{aligned} \tag{8.12}$$

where $\tilde{x}_\tau(k) = x(kT+\tau)$, the input and output signals are lifted to

$$\tilde{y}_\tau(k) = \begin{bmatrix} y(kT+\tau) \\ y(kT+\tau+1) \\ \vdots \\ y(kT+\tau+T-1) \end{bmatrix}, \quad \tilde{u}_\tau(k) = \begin{bmatrix} u(kT+\tau) \\ u(kT+\tau+1) \\ \vdots \\ u(kT+\tau+T-1) \end{bmatrix} \tag{8.13}$$

and the matrices are defined by

$$
\begin{aligned}
\tilde{A}_\tau &= \Psi(\tau+T,\tau) \\
\tilde{B}_\tau &= \left[\Psi(\tau+T,\tau+1)B(\tau)\ \Psi(\tau+T,\tau+2)B(\tau+1)\ \cdots\ B(\tau+T-1)\right] \\
\tilde{C}_\tau &= \begin{bmatrix} C(\tau) \\ C(\tau+1)\Psi(\tau+1,\tau) \\ \vdots \\ C(\tau+T-1)\Psi(\tau+T-1,\tau) \end{bmatrix} \\
\tilde{D}_\tau &= \begin{bmatrix} \tilde{D}_{\tau,1,1} & O & \cdots & O \\ \tilde{D}_{\tau,2,1} & \tilde{D}_{\tau,2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ \tilde{D}_{\tau,T,1} & \cdots & \tilde{D}_{\tau,T,T-1} & \tilde{D}_{\tau,T,T} \end{bmatrix} \\
\tilde{D}_{\tau,i,j} &= \begin{cases} D(\tau+i-1), & \text{if } i=j \\ C(\tau+i-1)\Psi(\tau+i-1,\tau+j)B(\tau+j-1), & \text{if } i>j \end{cases}
\end{aligned}
\tag{8.14}
$$

Note that the dimension of the state vector keeps to be n in the lifted reformulation, while it is lifted to nT in the cyclic reformulation.

It is easy to get the correspondence between the structural properties of the original periodic system and its LTI reformulations.

- The poles of the lifted reformulation (8.12) are exactly the same with the characteristic multipliers of the periodic system (8.1). As the characteristic polynomial of the cyclic reformulation (8.10) is

  $$\left|\lambda I - \bar{A}_\tau\right| = (\lambda^T)^n - \Psi(\tau+T,\tau)$$

  If λ_i is a characteristic multipliers of the periodic system (8.1), then $(\lambda_i)^{1/T}$ is a pole of cyclic reformulation (8.10).
- The periodic system (8.1) is stable if and only if the cyclic reformulation (8.10) (the lifted reformulation (8.12)) is stable.
- The periodic system (8.1) is observable (reachable) at time τ if and only if the lifted reformulation (8.12) is observable (reachable) at time τ, because

  $$\begin{bmatrix} \lambda I - \tilde{A}_\tau \\ \tilde{C}_\tau \end{bmatrix} = \begin{bmatrix} \lambda I - \Psi(\tau+T,\tau) \\ C(\tau)\Psi(\tau,\tau) \\ C(\tau+1)\Psi(\tau+1,\tau) \\ \vdots \\ C(\tau+T-1)\Psi(\tau+T-1,\tau) \end{bmatrix}$$

  Hence, if λ is an unobservable characteristic multiplier of periodic system (8.1) at time τ, it will be an unobservable pole of the lifted system (8.12) with initial time τ, and vice versa.
- The periodic system (8.1) is observable (reachable) at each time if and only if the cyclic reformulation (8.10) is observable (reachable). The observability matrix of the cyclically reformulated LTI system $(\bar{A}_\tau, \bar{B}_\tau, \bar{C}_\tau, \bar{D}_\tau)$ is

  $$\Gamma_{cyc} = \begin{bmatrix} \bar{C}_\tau \\ \bar{C}_\tau \bar{A}_\tau \\ \vdots \\ \bar{C}_\tau \bar{A}_\tau^{nT-1} \end{bmatrix}$$

  After interchanging the rows, it can be brought into the following form

$$Q\Gamma_{cyc} = \begin{bmatrix} \Gamma_{lift,\tau} & O & \cdots & O \\ O & \Gamma_{lift,\tau+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & \Gamma_{lift,\tau+T-1} \end{bmatrix}$$

$$\Gamma_{lift,j} = \begin{bmatrix} \tilde{A}_j \\ \tilde{C}_j \tilde{A}_j \\ \vdots \\ \tilde{C}_j \tilde{A}_j^{n-1} \end{bmatrix}, \quad j = \tau, \tau+1, \cdots, \tau+T-1$$

where Q represents the primary transformation, which doesn't change the rank of the observability matrix Γ_{cyc}, and $\Gamma_{lift,\tau}$ is indeed the observability matrix of the lifted reformulation (8.12) at time τ. Therefore, Γ_{cyc} is of full column rank means that there is no unobservable characteristic multiplier at any time τ.

- The periodic system (8.1) is detectable (stabilizable) if and only if the cyclic reformulation (8.10) (the lifted reformulation (8.12)) is detectable (stabilizable).

The notation of transmission zeros of the periodic system (8.1) can be defined based on either the cyclic reformulation (8.10) or the lifted reformulation (8.12), called respectively *cyclic zeros* or *lifted zeros*. Using similar arguments as those in the analysis of observability and reachability, we see that the cyclic zeros are time independent, while the lifted zeros are time dependent. However, regarding the nonzero transmission zeros, these two definitions are equivalent.

8.3.2 Frequency domain lifting

Assume that the z-transform of u and y is, respectively, $u(z)$ and $y(z)$. Let

$$\hat{u}(e^{j\omega}) = \begin{bmatrix} u(e^{j\omega}) \\ u(e^{j(\omega-\omega_T)}) \\ \vdots \\ u(e^{j(\omega-(T-1)\omega_T)}) \end{bmatrix}, \quad \hat{y}(e^{j\omega}) = \begin{bmatrix} y(e^{j\omega}) \\ y(e^{j(\omega-\omega_T)}) \\ \vdots \\ y(e^{j(\omega-(T-1)\omega_T)}) \end{bmatrix} \tag{8.15}$$

where $\omega_T = \frac{2\pi}{T}$.

Recalling (8.6), the following relation holds

$$\hat{y}(e^{j\omega}) = \hat{G}(e^{j\omega})\hat{u}(e^{j\omega}) \tag{8.16}$$

where

$$\hat{G}(e^{j\omega}) = \begin{bmatrix} G_0(e^{j\omega}) & G_1(e^{j\omega}) & \cdots & G_{T-1}(e^{j\omega}) \\ G_{T-1}(e^{j(\omega-\omega_T)}) & G_0(e^{j(\omega-\omega_T)}) & \cdots & G_{T-2}(e^{j(\omega-\omega_T)}) \\ \vdots & \vdots & & \vdots \\ G_1(e^{j(\omega-(T-1)\omega_T)}) & G_2(e^{j(\omega-(T-1)\omega_T)}) & \cdots & G_0(e^{j(\omega-(T-1)\omega_T)}) \end{bmatrix}$$

The matrix $\hat{G}(e^{j\omega})$ is called the frequency domain lifting of the system (8.1). The $(i+1, l+1)$ block of $\hat{G}(e^{j\omega})$ describes the output at frequency $\omega - i\omega_T$, $i = 0, 1, \cdots, T-1$, in response to an input at frequency $\omega - l\omega_T$, $l = 0, 1, \cdots, T-1$.

There is a one to one correspondence between the frequency domain lifted system, called also as modulated transfer function, and the frequency response of the time domain lifted system (lifted LTI reformulation and cyclic LTI reformulation), i.e.

$$\hat{G}(e^{j\omega}) = W_s^{-1}(e^{j\omega}) G_{cyc}(e^{j\omega T}) W_s(e^{j\omega}) \tag{8.17}$$

where

$$G_{cyc}(e^{j\omega T}) = \bar{D}_\tau + \bar{C}_\tau \left(e^{j\omega T} I - \bar{A}_\tau\right)^{-1} \bar{B}_\tau$$

$$W_s(e^{j\omega}) = \begin{bmatrix} I & I & \cdots & I \\ e^{j\omega} I & e^{j(\omega-\omega_T)} I & \cdots & e^{j(\omega-(T-1)\omega_T)} I \\ \vdots & \vdots & & \vdots \\ e^{j(T-1)\omega} I & e^{j(T-1)(\omega-\omega_T)} I & \cdots & e^{j(T-1)(\omega-(T-1)\omega_T)} I \end{bmatrix}$$

8.4 Norms and robustness

To analyze the influence of unknown disturbances on periodic system behaviour, the model (8.1) is extended to

$$\Sigma_d : \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) + E_d(k)d(k), \ x(0) = x_0 \\ y(k) = C(k)x(k) + D(k)u(k) + F_d(k)d(k) \end{cases} \tag{8.18}$$

where $d \in \mathbf{R}^{n_d}$ denotes the disturbance vector. In the literature different norms have been proposed to characterize the robustness.

One of the most often discussed norms is the $\mathbf{H}_\infty$-norm defined by

$$\|\Sigma_d\|_\infty = \sup_{d \in l_\infty, d \neq 0} \frac{\|y\|_2}{\|d\|_2} \tag{8.19}$$

where $\|\xi\|_2 = \sqrt{\sum_{k=0}^{\infty} \xi^T(k)\xi(k)}$ denotes the l_2-norm of ξ ($\xi = y, d$). The $\mathbf{H}_\infty$-norm can be defined equivalently in the frequency domain as [165]

$$\|\Sigma_d\|_\infty = \left\|\hat{G}(z)\right\|_\infty = \sup_{0 \le \omega \le 2\pi} \sigma_{\max}(\hat{G}(e^{j\omega})) \tag{8.20}$$

The peak to peak norm of the periodic system (8.1) is an induced norm with both the output signal and the input signal measured by the maximal amplitude, i.e.,

$$\|\Sigma_d\|_{peak} = \sup_{d \in l_\infty, d \neq 0} \frac{\|y\|_\infty}{\|d\|_\infty} \tag{8.21}$$

where $\|\xi\|_\infty = \sup_k \sqrt{\xi^T(k)\xi(k)}$ denotes the l_∞-norm of ξ ($\xi = y, d$).

The generalized $\mathbf{H}_2$ norm of the periodic system (8.1) is the induced norm with the input signal measured by the energy and the output signal measured by the maximal amplitude, i.e.,

$$\|\Sigma_d\|_g = \sup_{d \in l_2, d \neq 0} \frac{\|y\|_\infty}{\|d\|_2} \tag{8.22}$$

The above norms can be characterized as follows.

Theorem 8.1 [13] Given the periodic system (8.18) with zero initial conditions and a real number $\alpha > 0$, then the system (8.18) is stable and $\|\Sigma_d\|_\infty < \alpha$, if and only if there exist a T-periodic matrix $P(k) = P(k+T) > 0$, such that

$$\begin{bmatrix} -P(k) & O & A^T(k)P(k+1) & C^T(k) \\ O & -\alpha^2 I & E_d^T(k)P(k+1) & F_d{}^T(k) \\ P(k+1)A(k) & P(k+1)E_d(k) & -P(k+1) & O \\ C(k) & F_d(k) & O & -I \end{bmatrix} < 0 \tag{8.23}$$

The proof of Theorem 8.1 can be found in [13].

Theorem 8.2 Given the periodic system (8.18) with zero initial conditions and a real number $\beta > 0$, then the system (8.18) is stable and $\|\Sigma_d\|_{peak} < \beta$, if there exist a T-periodic matrix $P(k) = P(k+T) > 0$ and T-periodic real numbers $\lambda(k) = \lambda(k+T) > 0$ and $\mu(k) = \mu(k+T)$, such that

$$\begin{bmatrix} -P(k) + \lambda(k)P(k) & O & A^T(k)P(k+1) \\ O & -\mu(k)I & E_d^T(k)P(k+1) \\ P(k+1)A(k) & P(k+1)E_d(k) & -P(k+1) \end{bmatrix} < 0 \quad (8.24)$$

$$\begin{bmatrix} \lambda(k)P(k) & O & C^T(k) \\ O & (\beta - \mu(k))I & F_d^T(k) \\ C(k) & F_d(k) & \beta I \end{bmatrix} > 0 \quad (8.25)$$

Proof: Define a periodic Lyapunov function for the periodic system (8.1) as

$$V(x(k)) = x^T(k)P(k)x(k) \quad (8.26)$$

where $P(k+T) = P(k) > 0$. According to the Schur lemma, (8.24) is equivalent to

$$\Psi(k) = \begin{bmatrix} A^T(k)P(k+1)A(k) - P(k) + \lambda(k)P(k) & A^T(k)P(k+1)E_d(k) \\ E_d^T(k)P(k+1)A(k) & E_d^T(k)P(k+1)E_d(k) - \mu(k)I \end{bmatrix} < 0$$

From $P(k) > 0$ and $A^T(k)P(k+1)A(k) - P(k) < 0$, it is clear that Σ_d is stable. Moreover,

$$\begin{aligned} & V(x(k+1)) - V(x(k)) + \lambda(k)V(x(k)) - \mu(k)d^T(k)d(k) \\ & = \begin{bmatrix} x^T(k) & d^T(k) \end{bmatrix} \Psi(k) \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} < 0, \quad \forall x, d \end{aligned}$$

which means that

$$\lambda(k)V(x(k)) - \mu(k)d^T(k)d(k) < 0, \quad \forall x, d$$

As (8.25) is equivalent to

$$\begin{bmatrix} \lambda(k)P(k) & 0 \\ 0 & (\beta - \mu(k))I \end{bmatrix} > \beta^{-1} \begin{bmatrix} C^T(k) \\ F_d^T(k) \end{bmatrix} \begin{bmatrix} C(k) & F_d(k) \end{bmatrix}$$

there is

$$\begin{aligned} & y^T(k)y(k) \\ & < \beta \begin{bmatrix} x^T(k) & d^T(k) \end{bmatrix} \begin{bmatrix} \lambda(k)P(k) & 0 \\ 0 & (\beta - \mu(k))I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \\ & = \beta^2 d^T(k)d(k) + \beta\left(\lambda(k)V(x(k)) - \mu(k)d^T(k)d(k)\right) \\ & < \beta^2 d^T(k)d(k) \end{aligned}$$

Thus, $\|\Sigma_d\|_{peak} < \beta$. □

Theorem 8.3 Given the periodic system (8.18) with zero initial conditions and a real number $\gamma > 0$, then the system (8.18) is stable and $\|\Sigma_d\|_g < \gamma$, if and only if there exists a T-periodic matrix $P(k) = P(k+T) > 0$ such that

$$\begin{bmatrix} -P(k) & O & A^T(k)P(k+1) \\ O & -I & E_d^T(k)P(k+1) \\ P(k+1)A(k) & P(k+1)E_d(k) & -P(k+1) \end{bmatrix} < 0 \quad (8.27)$$

$$\begin{bmatrix} P(k) & O & C^T(k) \\ O & I & F_d^T(k) \\ C(k) & F_d(k) & \gamma^2 I \end{bmatrix} > 0 \quad (8.28)$$

Proof: Assume that (8.26) is a periodic Lyapunov function of the periodic system (8.18). Note that (8.27) is equivalent to

$$\Psi(k) = \begin{bmatrix} A^T(k)P(k+1)A(k) - P(k) & A^T(k)P(k+1)E_d(k) \\ E_d^T(k)P(k+1)A(k) & E_d^T(k)P(k+1)E_d(k) - I \end{bmatrix} < 0$$

$P(k) > 0$ and $A^T(k)P(k+1)A(k) - P(k) < 0$ imply the stability of Σ_d. Moreover, (8.27) holds if and only if

$$\begin{aligned} & V(x(k+1)) - V(x(k)) - d^T(k)d(k) \\ &= \begin{bmatrix} x^T(k) \; d^T(k) \end{bmatrix} \Psi(k) \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} < 0, \quad \forall x, d \end{aligned}$$

which means that

$$V(x(k)) < \sum_{j=0}^{k-1} d^T(j)d(j), \quad \forall x, d$$

Taking (8.28) into account, we have

$$\begin{aligned} y^T(k)y(k) &= \begin{bmatrix} x^T(k) \; d^T(k) \end{bmatrix} \begin{bmatrix} C^T(k) \\ F_d^T(k) \end{bmatrix} \begin{bmatrix} C(k) \; F_d(k) \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \\ &< \gamma^2 \left(V(x(k)) + d^T(k)d(k) \right) < \gamma^2 \sum_{j=0}^{k} d^T(j)d(j) \end{aligned}$$

i.e., $\|\Sigma_d\|_g < \gamma$. □

With the so-called relaxation variables [26], the Lyapunov variable can be decoupled from the system matrices, as shown below.

Theorem 8.4 Given periodic system (8.18) with zero initial conditions and a real number $\beta > 0$, then the system (8.18) is stable and $\|\Sigma_d\|_{peak} < \beta$, if there exist T-periodic matrices $P(k) = P(k+T) > 0, G(k) = G(k+T)$, and T-periodic real numbers $\lambda(k) = \lambda(k+T) > 0, \mu(k) = \mu(k+T)$, so that (8.25) and the following matrix inequality hold

$$\begin{bmatrix} -P(k) + \lambda(k)P(k) & O & A^T(k)G^T(k) \\ O & -\mu(k)I & E_d^T(k)G^T(k) \\ G(k)A(k) & G(k)E_d(k) & P(k+1) - G(k) - G^T(k) \end{bmatrix} < 0 \tag{8.29}$$

Proof: Assume that (8.25) and (8.29) hold for some $P(k) > 0, G(k), \lambda(k) > 0, \mu(k)$. Pre- and postmultiplying (8.29) by $\Gamma(k)$ and $\Gamma^T(k)$, respectively, where

$$\Gamma(k) = \begin{bmatrix} I & O & A^T(k) \\ O & I & E_d^T(k) \end{bmatrix}$$

As $\Gamma(k)$ is a matrix of full row rank, we get (8.24), i.e., the same matrices $P(k) > 0, \lambda(k) > 0, \mu(k)$ satisfy (8.24)-(8.25). Recalling Theorem 8.2, the periodic system (8.18) is stable and its peak to peak norm is smaller than β. On the other side, if (8.24)-(8.25) hold for some $P(k) > 0, \lambda(k) > 0$ and $\mu(k)$, then (8.29) and (8.25) are satisfied by the same $P(k) > 0, \lambda(k) > 0, \mu(k)$ and $G(k) = P(k+1)$. That means, the conditions (8.29) and (8.25) are equivalent with (8.24)-(8.25). □

Theorem 8.5 Given the periodic system (8.18) with zero initial conditions and a real number $\alpha > 0$, then the system (8.18) is stable and $\|\Sigma_d\|_\infty < \alpha$, if and only if there exist a T-periodic matrix $P(k) = P(k+T) > 0$ and $G(k) = G(k+T)$, such that

$$\begin{bmatrix} -P(k) & O & A^T(k)G^T(k) & C^T(k) \\ O & -\alpha^2 I & E_d^T(k)G^T(k) & F_d^T(k) \\ G(k)A(k) & G(k)E_d(k) & P(k+1) - G(k) - G^T(k) & O \\ C(k) & F_d(k) & O & -I \end{bmatrix} < 0$$

Theorem 8.6 Given periodic system (8.18) with zero initial conditions and a real number $\gamma > 0$, then the system (8.18) is stable and $\|\Sigma_d\|_g < \gamma$, if and only if there exist T-periodic matrices $P(k) = P(k+T) > 0$ and $G(k)$, so that (8.28) and the following LMI hold

$$\begin{bmatrix} -P(k) & O & A^T(k)G^T(k) \\ O & -I & E_d^T(k)G^T(k) \\ G(k)A(k) & G(k)E_d(k) & P(k+1) - G(k) - G^T(k) \end{bmatrix} < 0 \tag{8.30}$$

The proof of Theorem 8.5 and 8.6 is similar to that of Theorem 8.4 and thus omitted here.

A natural question that may arise is whether the norms of the periodic system (8.1) can be obtained from its LTI reformulations. Note that the $\mathbf{H}_\infty$-norm, the peak to peak norm and the generalized $\mathbf{H}_2$ norm are induced norms defined based on the l_2 or l_∞ norm of the input and output signals. For a signal δ, let $\bar{\delta}$ and $\tilde{\delta}$ be defined, respectively, by (8.9) and (8.13). Then

$$\begin{cases} \|\bar{\delta}_\tau\|_2 = \left\|\tilde{\delta}_\tau\right\|_2 = \|\delta\|_2 , \text{ if } \delta \in l_2 \\ \|\bar{\delta}_\tau\|_\infty = \|\delta\|_\infty , \text{ if } \delta \in l_\infty \end{cases} \tag{8.31}$$

However, $\left\|\tilde{\delta}_\tau\right\|_\infty$ may be different from $\|\delta\|_\infty$. Therefore, there is

$$\begin{cases} \|\Sigma_d\|_\infty = \|\bar{G}_{yd}(z)\|_\infty = \left\|\tilde{G}_{yd}(z)\right\|_\infty \\ \|\Sigma_d\|_{peak} = \|\bar{G}_{yd}(z)\|_{peak} \\ \|\Sigma_d\|_g = \|\bar{G}_{yd}(z)\|_g \end{cases} \tag{8.32}$$

where

$$\bar{G}_{yd}(z) = \bar{F}_{d,\tau} + \bar{C}_\tau(zI - \bar{A}_\tau)^{-1}\bar{E}_{d,\tau}$$
$$\tilde{G}_{yd}(z) = \tilde{F}_{d,\tau} + \tilde{C}_\tau(zI - \tilde{A}_\tau)^{-1}\tilde{E}_{d,\tau}$$

and $\bar{E}_{d,\tau}, \bar{F}_{d,\tau}, \tilde{E}_{d,\tau}, \tilde{F}_{d,\tau}$ are defined, respectively, similar to $\bar{B}_\tau, \bar{D}_\tau, \tilde{B}_\tau, \tilde{D}_\tau$.

8.5 Periodic observer

If the state vector $x(k)$ of the periodic system (8.1) is not measurable, then a periodic observer of the form

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) + L(k)(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C(k)\hat{x}(k) + D(k)u(k) \end{aligned} \tag{8.33}$$

can be used to estimate $x(k)$, where the observer gain matrix $L(k)$ is a periodic matrix. As always, we design the observer based on an analysis of the dynamics of the estimation error $e(k) = x(k) - \hat{x}(k)$, which is given by

$$e(k+1) = (A(k) - L(k)C(k))e(k) \tag{8.34}$$

The error dynamics is stable, i.e. $\lim_{k\to\infty} e(k) = 0$ for any initial estimation error $e(0)$, if and only if the characteristic multipliers of $A(k) - L(k)C(k)$ are located inside the unit circle. In the following, we shall introduce approaches for the design of the periodic gain matrix $L(k)$.

8.5.1 Pole placement approach

Assume that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the desired locations of the characteristic multipliers, $|\lambda_i| < 1$, $i = 1, \cdots, n$. Note that the monodromy matrix of the error dynamics is

$$\Psi_{err}(T,0) = (A(T-1) - L(T-1)C(T-1))(A(T-2) - L(T-2)C(T-2)) \\ \cdots (A(1) - L(1)C(1))(A(0) - L(0)C(0)) \quad (8.35)$$

which can be re-written as

$$\begin{aligned} &\Psi_{err}(T,0) = A(T-1)A(T-2)A(T-3)\cdots A(1)A(0) \\ &-L(T-1)C(T-1)A(T-2)A(T-3)\cdots A(1)A(0) \\ &-(A(T-1) - L(T-1)C(T-1))L(T-2)C(T-2)A(T-3)\cdots A(1)A(0) \\ &-(A(T-1) - L(T-1)C(T-1))\cdots(A(2) - L(2)C(2))L(1)C(1)A(0) \\ &-(A(T-1) - L(T-1)C(T-1))\cdots(A(1) - L(1)C(1))L(0)C(0) \end{aligned}$$

Recall that, if $\tau = 0$, then the coefficient matrices of the lifted reformulation (9.2) are

$$\tilde{A}_0 = A(T-1)A(T-2)\cdots A(1)A(0)$$

$$\tilde{C}_0 = \begin{bmatrix} C(0) \\ C(1)A(0) \\ \vdots \\ C(T-1)A(T-2)\cdots A(1)A(0) \end{bmatrix}$$

Therefore, $\Psi_{err}(T,0)$ can be further expressed as

$$\Psi_{err}(T,0) = \tilde{A}_0 - \tilde{L}_0\tilde{C}_0$$

where

$$\begin{aligned} \tilde{L}_0 &= \left[\, \tilde{L}_{\tau,0} \cdots \tilde{L}_{\tau,T-2}\ \tilde{L}_{\tau,T-1} \,\right] \\ \tilde{L}_{\tau,0} &= (A(T-1) - L(T-1)C(T-1))\cdots(A(1) - L(1)C(1))L(0), \cdots \\ \tilde{L}_{\tau,T-2} &= (A(T-1) - L(T-1)C(T-1))L(T-2) \\ \tilde{L}_{\tau,T-1} &= L(T-1) \end{aligned}$$

Motivated by the above observation, the periodic observer gain $L(k)$ can be designed by the following algorithm.

Algorithm 8.1 Given the periodic system (8.1), the periodic observer (8.33) and desired characteristic multipliers $\lambda_1, \lambda_2, \cdots, \lambda_n$. Design of the periodic observer gain matrix $L(k)$:

- For $\tau = 0$ calculate matrices $\tilde{A}_0$ and $\tilde{C}_0$.
- Calculate the matrix $\tilde{L}_0$ to assign the eigenvalues of $\tilde{A}_0 - \tilde{L}_0\tilde{C}_0$ at $\lambda_1, \lambda_2, \cdots, \lambda_n$ using the standard pole placement algorithms for LTI systems.
- Partition $\tilde{L}_0$ into
$$\tilde{L}_0 = \left[\, \tilde{L}_{\tau,0}\ \tilde{L}_{\tau,1} \cdots \tilde{L}_{\tau,T-1} \,\right],\ \tilde{L}_{\tau,i} \in \mathbf{R}^{n\times m}$$
- Get the periodic gain matrix $L(k)$ by
$$\begin{aligned} L(T-1) &= \tilde{L}_{\tau,T-1} \\ L(T-2) &= (A(T-1) - L(T-1)C(T-1))^{-1}\,\tilde{L}_{\tau,T-2} \\ &\vdots \\ L(0) &= ((A(T-1) - L(T-1)C(T-1))\cdots(A(1) - L(1)C(1)))^{-1}\,\tilde{L}_{\tau,0} \end{aligned}$$

Note that in the above algorithm to calculate $L(j-1), j = 1, \cdots, T-1$, it is required that $A(j) - L(j)C(j)$ is invertible.

8.5.2 LMI based approach

According to (8.7), $A(k) - L(k)C(k)$ is stable and the estimation error $e(k)$ governed by (8.34) converges to zero, if and only if a periodic matrix $P(k) = P(k+T) > 0$ can be found so that

$$(A(k) - L(k)C(k))^T P(k+1) (A(k) - L(k)C(k)) - P(k) < 0 \tag{8.36}$$

holds for $k = 0, 1, \cdots, T-1$. Using Schur Lemma, (8.36) is equivalent to

$$\begin{bmatrix} P(k+1) & P(k+1)(A(k) - L(k)C(k)) \\ (A(k) - L(k)C(k))^T P(k+1) & P(k) \end{bmatrix} > 0 \tag{8.37}$$
$$k = 0, 1, \cdots, T-1$$

In (8.37), the unknown variables $P(k+1)$ and $L(k)$ are coupled together. In order to transform (8.37) into LMI, let

$$Y(k) = P(k+1)L(k)$$

Therefore, the periodic observer gain matrix $L(k)$ stabilizes $A(k) - L(k)C(k)$, if and only if there exists periodic matrices $P(k) = P(k+T) > 0$ and $Y(k) = Y(k+T)$ so that the following LMIs are satisfied

$$\begin{bmatrix} P(k+1) & P(k+1)A(k) - Y(k)C(k) \\ A^T(k)P(k+1) - C^T(k)Y^T(k) & P(k) \end{bmatrix} > 0 \tag{8.38}$$
$$k = 0, 1, \cdots, T-1$$

Based on it, $L(k)$ can be designed using the algorithm below.

Algorithm 8.2 Given the periodic system (8.1), the periodic observer (8.33). Design of the periodic observer gain matrix $L(k)$ that stabilizes the error dynamics (8.34):

- Solve the set of LMIs (8.38) for periodic matrices $P(k) = P(k+T) > 0$ and $Y(k) = Y(k+T)$.
- Let
$$L(k) = P^{-1}(k+1)Y(k), \ k = 0, 1, \cdots, T-1$$

8.5.3 Robust design

Assume that the periodic system is described by (8.18). Due to the disturbances d, the error dynamics of observer (8.33) is governed by

$$e(k+1) = (A(k) - L(k)C(k))e(k) + (E_d(k) - L(k)F_d(k))d(k) \tag{8.39}$$

To suppress the influence of the disturbances on the error dynamics, the norms introduced in the last section can be applied. The robust design problem can be, for instance, formulated as looking for the gain matrix $L(k)$ such that

$$\|\Sigma_{ed}\|_\infty < \alpha \tag{8.40}$$

The smaller the constant α is, the weaker is the worst case influence of the unknown disturbances d on the state estimation error e.

According to Theorem 8.1, the error dynamics (8.39) is stable and (8.40) holds if and only if there exist a T-periodic matrix $P(k) = P(k+T) > 0$, such that

$$\begin{bmatrix} -P(k) & O & \Phi_{31}^T & I \\ O & -\alpha^2 I & \Phi_{32}^T & O \\ \Phi_{31} & \Phi_{32} & -P(k+1) & O \\ I & O & O & -I \end{bmatrix} < 0 \tag{8.41}$$
$$k = 0, 1, \cdots, T-1$$

where

$$\begin{aligned} \Phi_{31} &= P(k+1)(A(k) - L(k)C(k)) \\ \Phi_{32} &= P(k+1)(E_d(k) - L(k)F_d(k)) \end{aligned}$$

To decouple $P(k+1)$ and $L(k)$, let $Y(k) = P(k+1)L(k)$. Applying Schur Lemma, (8.41) is equivalent to

$$\begin{bmatrix} -P(k)+I & O & \bar{\Phi}_{31}^T \\ O & -\alpha^2 I & \bar{\Phi}_{32}^T \\ \bar{\Phi}_{31} & \bar{\Phi}_{32} & -P(k+1) \end{bmatrix} < 0 \tag{8.42}$$

$$\begin{aligned} \bar{\Phi}_{31} &= P(k+1)A(k) - Y(k)C(k) \\ \bar{\Phi}_{32} &= P(k+1)E_d(k) - Y(k)F_d(k) \\ k &= 0, 1, \cdots, T-1 \end{aligned}$$

Algorithm 8.3 Given the periodic system (8.18), the periodic observer (8.33) and a constant $\alpha > 0$. Design of the periodic observer gain matrix $L(k)$ so that the error dynamics (8.39) is stable and (8.40) holds:

- Solve the set of LMIs (8.42) for periodic matrices $P(k) = P(k+T) > 0$ and $Y(k) = Y(k+T)$.
- If there is no feasible solutions, then the value of α is too small. Otherwise, let $L(k) = P^{-1}(k+1)Y(k)$, $k = 0, 1, \cdots, T-1$.

Due to the property of linear matrix inequalities, the above design procedure can be easily extended to take into account multiple design objectives.

The design problem (8.40) can also be solved by first getting the cyclic reformulation of the residual dynamics (8.39) and then applying the robust theory for discrete LTI systems. However, it is difficult to solve (8.40) by lifting (8.39), because the system matrices after the lifting involve the multiplication of $L(i)$ and $L(j)$.

It is a misunderstanding that designing an observer for the periodic system (8.18) is equivalent to designing an observer for its lifted LTI reformulation. The lifted LTI reformulation of (8.18) is

$$\begin{aligned} \tilde{x}_\tau(k+1) &= \tilde{A}_\tau \tilde{x}_\tau(k) + \tilde{B}_\tau \tilde{u}_\tau(k) + \tilde{E}_{d,\tau} \tilde{d}_\tau(k) \\ \tilde{y}_\tau(k) &= \tilde{C}_\tau \tilde{x}_\tau(k) + \tilde{D}_\tau \tilde{u}_\tau(k) + \tilde{F}_{d,\tau} \tilde{d}_\tau(k) \end{aligned} \tag{8.43}$$

Recall that $\tilde{x}_\tau(k) = x(kT+\tau)$. To get the estimation of $x(k)$ at each time instant, considering that

$$\begin{bmatrix} x(kT+\tau) \\ x(kT+\tau+1) \\ \vdots \\ x(kT+\tau+T-1) \end{bmatrix} = \tilde{C}_{x,\tau} \tilde{x}_\tau(k) + \tilde{F}_{x,\tau} \tilde{d}_\tau(k) \tag{8.44}$$

$$\tilde{C}_{x,\tau} = \begin{bmatrix} I \\ \Psi(\tau+1,\tau) \\ \vdots \\ \Psi(\tau+T-1,\tau) \end{bmatrix}$$

$$\tilde{F}_{x,\tau} = \begin{bmatrix} O & O & \cdots & O \\ E_d(\tau) & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ \Psi(\tau+T-1,\tau+1)E_d(\tau) & \cdots & E_d(\tau) & O \end{bmatrix}$$

an observer can be built as

$$\begin{aligned}\hat{\tilde{x}}_\tau(k+1) &= \tilde{A}_\tau\hat{\tilde{x}}_\tau(k) + \tilde{B}_\tau\tilde{u}_\tau(k) + \tilde{L}_\tau\left(\tilde{y}_\tau(k) - \hat{\tilde{y}}_\tau(k)\right)\\ \hat{\tilde{y}}_\tau(k) &= \tilde{C}_\tau\hat{\tilde{x}}_\tau(k) + \tilde{D}_\tau\tilde{u}_\tau(k)\\ \begin{bmatrix}\hat{x}(kT+\tau)\\ \hat{x}(kT+\tau+1)\\ \vdots\\ \hat{x}(kT+\tau+T-1)\end{bmatrix} &= \tilde{C}_{x,\tau}\hat{\tilde{x}}_\tau(k)\end{aligned} \tag{8.45}$$

Let

$$\begin{aligned}\eta_\tau(k) &= \tilde{x}_\tau(k) - \hat{\tilde{x}}_\tau(k)\\ \eta_{x,\tau}(k) &= \begin{bmatrix}x(kT+\tau)\\ x(kT+\tau+1)\\ \vdots\\ x(kT+\tau+T-1)\end{bmatrix} - \begin{bmatrix}\hat{x}(kT+\tau)\\ \hat{x}(kT+\tau+1)\\ \vdots\\ \hat{x}(kT+\tau+T-1)\end{bmatrix}\end{aligned}$$

The dynamics of the observer (8.45) is governed by

$$\begin{aligned}\eta_\tau(k+1) &= \left(\tilde{A}_\tau - \tilde{L}_\tau\tilde{C}_\tau\right)\eta_\tau(k) + \left(\tilde{E}_{d,\tau} - \tilde{L}_\tau\tilde{F}_{d,\tau}\right)\tilde{d}_\tau(k)\\ \eta_{x,\tau}(k) &= \tilde{C}_{x,\tau}\eta_\tau(k) + \tilde{F}_{x,\tau}\tilde{d}_\tau(k)\end{aligned} \tag{8.46}$$

In comparison, the lifting of the dynamics of the periodic observer (8.39) is

$$\begin{aligned}\tilde{e}_\tau(k+1) &= \Psi_{err}(\tau+T,\tau)\tilde{e}_\tau(k) + \tilde{E}_{e,\tau}\tilde{d}_\tau(k)\\ \begin{bmatrix}e(kT+\tau)\\ e(kT+\tau+1)\\ \vdots\\ e(kT+\tau+T-1)\end{bmatrix} &= \tilde{C}_{e,\tau}\tilde{e}_\tau(k) + \tilde{F}_{e,\tau}\tilde{d}_\tau(k)\end{aligned} \tag{8.47}$$

where

$$\begin{aligned}\tilde{e}_\tau(k) &= e(kT+\tau)\\ \tilde{E}_{e,\tau} &= \begin{bmatrix}\Psi_{err}(\tau+T,\tau+1)E_{d,L}(\tau) & \Psi_{err}(\tau+T,\tau+2)E_{d,L}(\tau+1)\\ \cdots & E_{d,L}(\tau+T-1)\end{bmatrix}\\ \tilde{C}_{e,\tau} &= \begin{bmatrix}I\\ \Psi_{err}(\tau+1,\tau)\\ \vdots\\ \Psi_{err}(\tau+T-1,\tau)\end{bmatrix}\\ \tilde{F}_{e,\tau} &= \begin{bmatrix}O & O & \cdots & O\\ E_{d,L}(\tau) & O & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ \Psi(\tau+T-1,\tau+1)E_{d,L}(\tau) & \cdots & E_{d,L}(\tau+T-1) & O\end{bmatrix}\\ E_{d,L}(\tau) &= E_d(\tau) - L(\tau)F_d(\tau)\end{aligned}$$

Let

$$\begin{aligned}\tilde{L}_\tau &= \left[\tilde{L}_{\tau,0}\ \cdots\ \tilde{L}_{\tau,T-2}\ \tilde{L}_{\tau,T-1}\right]\\ \tilde{L}_{\tau,0} &= \Psi_{err}(\tau+T,\tau+1)L(\tau),\cdots\\ \tilde{L}_{\tau,T-2} &= \Psi_{err}(\tau+T,\tau+T-1)L(\tau+T-2)\\ \tilde{L}_{\tau,T-1} &= L(\tau+T-1)\end{aligned} \tag{8.48}$$

then

$$\begin{aligned}\Psi_{err}(\tau+T,\tau) &= \tilde{A}_\tau - \tilde{L}_\tau \tilde{C}_\tau \\ \tilde{E}_{e,\tau} &= \tilde{E}_{d,\tau} - \tilde{L}_\tau \tilde{F}_{d,\tau} \\ \tilde{C}_{e,\tau} &= \tilde{C}_{x,\tau} + L_\triangle \tilde{C}_\tau \\ \tilde{F}_{e,\tau} &= \tilde{F}_{x,\tau} + L_\triangle \tilde{F}_{d,\tau}\end{aligned}$$

where

$$L_\triangle = \begin{bmatrix} O & O & \cdots & O \\ -L(\tau) & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ -\Psi(\tau+T-1,\tau+1)L(\tau) & \cdots & -L(\tau+T-1) & O \end{bmatrix}$$

If

$$\tilde{e}_\tau(0) = \eta_\tau(0)$$

then $\tilde{e}_\tau(k) \equiv \eta_\tau(k)$ and

$$\begin{bmatrix} e(kT+\tau) \\ e(kT+\tau+1) \\ \vdots \\ e(kT+\tau+T-1) \end{bmatrix} = \eta_{x,\tau}(k) + L_\triangle \left(\tilde{y}_\tau(k) - \hat{\tilde{y}}_\tau(k)\right)$$

It shows clearly that the difference in the estimation error dynamics is caused by the feedback terms, i.e. the different way how the measurement information is used to modify the state estimation. In (8.33) the measurement information is taken into account at each time instant, while in (8.45) the measurement information is taken into account with an interval of T.

In the following, we shall give two examples to illustrate the observer design.

Example 8.1 Consider the periodic system (8.1) with period $T = 2$ and

$$\begin{aligned}&A(0) = \begin{bmatrix} 1 & 1 \\ 0 & 0.5 \end{bmatrix},\ A(1) = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix},\ B(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B(1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\ &C(0) = C(1) = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D(0) = D(1) = 0\end{aligned}$$

Design the observer (8.33) with stable estimation error dynamics.

We shall first apply the pole placement approach. Let $\tau = 0$. Lift $A(k)$ and $C(k)$ into

$$\tilde{A}_0 = \begin{bmatrix} 0.5 & 0.75 \\ 0 & 0.5 \end{bmatrix},\ \tilde{C}_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{8.49}$$

Assume that the desired characteristic multipliers of the error dynamics are 0.2 and 0.3. This can be achieved by

$$\tilde{L}_0 = \begin{bmatrix} -0.55 & 0.75 \\ -0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} \tilde{L}_{0,0} & \tilde{L}_{0,1} \end{bmatrix}$$

Then,

$$L(0) = \begin{bmatrix} 4.0 \\ 0.9 \end{bmatrix},\ L(1) = \begin{bmatrix} 0.75 \\ 0.3 \end{bmatrix} \tag{8.50}$$

It can be verified that the eigenvalues of the matrix

$$(A(1) - L(1)C(1))\,(A(0) - L(0)C(0))$$

are located at $0.2, 0.3$.

If the LMI based approach is applied to stabilize the estimation error dynamics (8.34), we solve at first (8.38) and get a feasible solution

$$P(0) = \begin{bmatrix} 1.4259 & -0.3709 \\ -0.3709 & 1.3302 \end{bmatrix}, \; Y(1) = \begin{bmatrix} 0.8027 \\ 0.0793 \end{bmatrix}$$
$$P(1) = \begin{bmatrix} 0.9286 & -0.4237 \\ -0.4237 & 1.8654 \end{bmatrix}, \; Y(0) = \begin{bmatrix} 1.1455 \\ -0.2767 \end{bmatrix}$$

The observer gain matrix is obtained as

$$L(0) = \begin{bmatrix} 1.3008 \\ 0.1472 \end{bmatrix}, \; L(1) = \begin{bmatrix} 0.6237 \\ 0.2335 \end{bmatrix} \tag{8.51}$$

The eigenvalues of the matrix $(A(1) - L(1)C(1))\,(A(0) - L(0)C(0))$ are 0.0001 and 0.2300 and (8.34) is stable.

Example 8.2 Consider the periodic system in Example 8.1. Assume that the disturbances are described by

$$E_d(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; E_d(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, F_d(0) = 1, \; F_d(1) = 0$$

Design the observer (8.33) so that the error dynamics is stable and the $\mathbf{H}_\infty$-norm from the disturbances to the state estimation error is minimized.

Solving (8.42) iteratively, we get the minimal feasible constant $\alpha = 1.5001$ and, correspondingly,

$$P(0) = \begin{bmatrix} 2368362.747 & 1.233 \\ 1.233 & 2.250 \end{bmatrix}, \; Y(1) = \begin{bmatrix} 1776271.438 \\ 0.200 \end{bmatrix}$$
$$P(1) = \begin{bmatrix} 405955.534 & -811909.069 \\ -811909.069 & 1623819.142 \end{bmatrix}, \; Y(0) = \begin{bmatrix} 405955.516 \\ -811909.034 \end{bmatrix}$$

Finally, the robust observer gain matrix is given by

$$L(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; L(1) = \begin{bmatrix} 0.75 \\ 0.50 \end{bmatrix} \tag{8.52}$$

The characteristic multipliers of $(A(1) - L(1)C(1))\,(A(0) - L(0)C(0))$ are $0, 0$ and the $\mathbf{H}_\infty$-norm from the disturbances to the state estimation error is 1.5001. In comparison, for the gain matrix given by (8.50) and (8.51), the $\mathbf{H}_\infty$-norm is, respectively, 4.7151 and 2.0229.

8.6 Conclusion

This chapter has introduced some basic properties of the linear discrete-time periodic (LDP) systems. The isomorphism between the LDP systems and the discrete LTI systems is very helpful for the analysis and design of the LDP systems. Good tutorials on periodic systems can be found in [8, 9, 10], which give an extensive overview of the development in this field before 2000. [13, 143] are the first papers that use LMI to solve the H_∞ control and filtering problems in the LDP systems. The pole placement problem of the LDP systems is considered in [23, 74, 149].

9

FD schemes based on lifted LTI reformulation

In this and the next chapter, we shall study the fault detection problem of linear discrete-time periodic systems described by

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + E_d(k)d(k) + E_f(k)f(k), \\ y(k) &= C(k)x(k) + D(k)u(k) + F_d(k)d(k) + F_f(k)f(k) \end{aligned} \tag{9.1}$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^{n_u}$ the control input vector, $y \in \mathbf{R}^m$ the measured output vector, $d \in \mathbf{R}^{n_d}$ the unknown disturbance vector, and $f \in \mathbf{R}^{n_f}$ the vector of faults to be detected, $A(k), B(k), C(k), D(k), E_d(k),\ E_f(k), F_d(k), F_f(k)$ are known real bounded periodic matrices of period T and with appropriate dimensions, i.e., for all integers $k \geq 0$, they satisfy

$$\begin{aligned} &\begin{bmatrix} A(k+T) & B(k+T) & E_d(k+T) & E_f(k+T) \\ C(k+T) & D(k+T) & F_d(k+T) & F_f(k+T) \end{bmatrix} \\ &= \begin{bmatrix} A(k) & B(k) & E_d(k) & E_f(k) \\ C(k) & D(k) & F_d(k) & F_f(k) \end{bmatrix} \end{aligned}$$

Recalling the strong correspondence between discrete-time periodic systems and discrete LTI systems, the FD system design for periodic system (9.1) can be carried out as follows:

- lift periodic system (9.1) into a discrete LTI system,
- design residual generator(s) based on the lifted LTI reformulation,
- use a bank of residual generators, or select the parameters of the residual generator to satisfy the causality condition (that means, to generate the residual signal based on the available inputs and outputs) to facilitate a periodic implementation.

Using the technique introduced in the last chapter, periodic system (9.1) can be lifted into a discrete LTI system described by

$$\begin{aligned} \tilde{x}_\tau(k+1) &= \tilde{A}_\tau \tilde{x}_\tau(k) + \tilde{B}_\tau \tilde{u}_\tau(k) + \tilde{E}_{d,\tau} \tilde{d}_\tau(k) + \tilde{E}_{f,\tau} \tilde{f}_\tau(k), \\ \tilde{y}_\tau(k) &= \tilde{C}_\tau \tilde{x}_\tau(k) + \tilde{D}_\tau \tilde{u}_\tau(k) + \tilde{F}_{d,\tau} \tilde{d}_\tau(k) + \tilde{F}_{f,\tau} \tilde{f}_\tau(k) \end{aligned} \tag{9.2}$$

where τ is an integer between 0 and $T-1$ denoting the initial time, the state vector of the lifted system is $\tilde{x}_\tau(k) = x(kT+\tau)$, $\tilde{\eta}_\tau$ with η standing for u, y, d, f is the augmented signal defined by

$$\tilde{\eta}_\tau(k) = \begin{bmatrix} \eta(kT+\tau) \\ \eta(kT+\tau+1) \\ \vdots \\ \eta(kT+\tau+T-1) \end{bmatrix}$$

The matrices $\tilde{A}_\tau, \tilde{B}_\tau, \tilde{C}_\tau, \tilde{D}_\tau$ are defined in (8.14), $\tilde{E}_{d,\tau}, \tilde{E}_{f,\tau}$ are defined in a way similar to $\tilde{B}_\tau$, and $\tilde{F}_{d,\tau}, \tilde{F}_{f,\tau}$ similar to $\tilde{D}_\tau$.

9.1 Observer-based FD system design and implementation

Assume that $(A(k), C(k))$ is detectable. Then $(\tilde{A}_\tau, \tilde{C}_\tau)$ is detectable and an observer based LTI residual generator can be designed based on lifted reformulation (9.2) as

$$\begin{aligned}\hat{\tilde{x}}_\tau(k+1) &= \tilde{A}_\tau \hat{\tilde{x}}_\tau(k) + \tilde{B}_\tau \tilde{u}_\tau(k) + L_\tau(\tilde{y}_\tau(k) - \hat{\tilde{y}}_\tau(k)),\\ \hat{\tilde{y}}_\tau(k) &= \tilde{C}_\tau \hat{\tilde{x}}_\tau(k) + \tilde{D}_\tau \tilde{u}_\tau(k)\\ r_\tau(k) &= W_\tau(\tilde{y}_\tau(k) - \hat{\tilde{y}}_\tau(k))\end{aligned} \tag{9.3}$$

where L_τ and W_τ are constant matrices and can be designed with the FD approaches for the discrete LTI systems introduced in Part I to realize full decoupling or optimal FD. Observer (9.3) estimates the state vector $x(kT+\tau)$ and, based on it, reconstructs the outputs over one period $y(kT+\tau), y(kT+\tau+1), \cdots, y(kT+\tau+T-1)$. Both the state vector $\hat{\tilde{x}}_\tau(k)$ of observer (9.3) and the residual signal $r_\tau(k)$ are updated every T time instants.

In fault detection, the detection delay should be as small as possible. Therefore, it is advantageous if a residual signal can be obtained at each time instant. To this aim, a bank of LTI residual generators (9.3) can be used, each of which is designed for $\tau = 0, 1, \cdots, T-1$, respectively [51]. This scheme is characterized by a simple design but needs much online computational efforts.

In the next, we shall introduce two alternative ways to realize a simpler periodic implementation.

The first way is to transform the weighting matrix W_τ into a lower block triangular matrix in the form of

$$W_\tau = \begin{bmatrix} W_{\tau,1,1} & O & \cdots & O \\ W_{\tau,2,1} & W_{\tau,2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ W_{\tau,T,1} & \cdots & W_{\tau,T,T-1} & W_{\tau,T,T} \end{bmatrix} \tag{9.4}$$

so that the causality constraint is satisfied. If this is achieved, then for a given integer τ, the residual generator (9.3) can be implemented as

$$\begin{aligned}\hat{\tilde{x}}_\tau(k+1) &= \tilde{A}_\tau \hat{\tilde{x}}_\tau(k) + \tilde{B}_\tau \tilde{u}_\tau(k)\\ &\quad + L_\tau \left(\begin{bmatrix} y(kT+\tau) \\ y(kT+\tau+1) \\ \vdots \\ y(kT+\tau+T-1) \end{bmatrix} - \begin{bmatrix} \hat{y}(kT+\tau) \\ \hat{y}(kT+\tau+1) \\ \vdots \\ \hat{y}(kT+\tau+T-1) \end{bmatrix} \right)\\ r(kT+\tau+j) &= \begin{bmatrix} W_{\tau,j+1,1} & \cdots & W_{\tau,j+1,j+1} \end{bmatrix}\\ &\quad \times \left(\begin{bmatrix} y(kT+\tau) \\ \vdots \\ y(kT+\tau+j) \end{bmatrix} - \begin{bmatrix} \hat{y}(kT+\tau) \\ \vdots \\ \hat{y}(kT+\tau+j) \end{bmatrix} \right)\\ \hat{y}(kT+\tau+j) &= C(\tau+j)\Psi(\tau+j,\tau)\hat{\tilde{x}}_\tau(k)\\ &\quad + \begin{bmatrix} \tilde{D}_{\tau,j+1,1} & \cdots & \tilde{D}_{\tau,j+1,j+1} \end{bmatrix} \begin{bmatrix} u(kT+\tau) \\ \vdots \\ u(kT+\tau+j) \end{bmatrix}\\ j &= 0, 1, \cdots, T-1\end{aligned} \tag{9.5}$$

In this case, the state estimation is still updated at every T time instants, but at each time instant $kT+\tau+j$, $j = 0, 1, \cdots, T-1$, a residual signal $r(kT+\tau+j)$ is calculated from the control inputs and the measured outputs available up to the time instant $kT+\tau+j$.

The second way aims to explore the possibility of realizing (9.3) in the form of

$$\begin{aligned}\hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) + L(k)(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C(k)\hat{x}(k) + D(k)u(k) \\ r(k) &= W(k)(y(k) - \hat{y}(k))\end{aligned} \tag{9.6}$$

where $L(k)$ and $W(k)$ are T-periodically time-varying matrices. Compared with (9.5), both the state estimation and the residual signal in (9.6) will be updated at each time instant.

For this purpose, we give the following theorem.

Theorem 9.1 Given the periodic system (9.1), the residual generators (9.3) and (9.6), let $r_\tau(k)$ denote the residual signal in (9.3) and $r_{lift}(k)$ denote the lifting of the residual signal $r(k)$ in (9.6), i.e.

$$r_{lift}(k) = \begin{bmatrix} r(kT+\tau) \\ r(kT+\tau+1) \\ \vdots \\ r((k+1)T+\tau-1) \end{bmatrix}$$

Then $r_{lift}(k) = r_\tau(k)$, if $\hat{x}(\tau) = \hat{\tilde{x}}_\tau(0)$ and

$$L_\tau = \begin{bmatrix} L_{\tau,0} \cdots L_{\tau,T-2} \; L_{\tau,T-1} \end{bmatrix} \tag{9.7}$$

$$W_\tau = \begin{bmatrix} W(\tau) & O & \cdots & O \\ W_{\tau,2,1} & W(\tau+1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ W_{\tau,T,1} & \cdots & W_{\tau,T,T-1} & W(\tau+T-1) \end{bmatrix} \tag{9.8}$$

where

$$\begin{aligned} L_{\tau,0} &= \Psi_{err}(\tau+T, \tau+1)L(\tau), \cdots \\ L_{\tau,T-2} &= \Psi_{err}(\tau+T, \tau+T-1)L(\tau+T-2) \\ L_{\tau,T-1} &= L(\tau+T-1) \\ W_{\tau,i,j} &= -W(\tau+i-1)C(\tau+i-1)\Psi_{err}(\tau+i-1, \tau+j)L(\tau+j-1), i > j \\ \Psi_{err}(j,i) &= \begin{cases} I, & \text{if } j = i \\ (A(j-1) - L(j-1)C(j-1)) \cdots (A(i) - L(i)C(i)), & \text{if } j > i \end{cases} \end{aligned}$$

Proof: Over the time interval $[kT+\tau, (k+1)T+\tau)$, $\hat{x}$ and r got by the residual generator (9.6) evolve like

$$\begin{aligned} \hat{x}((k+1)T+\tau) &= \tilde{A}_\tau \hat{x}(kT+\tau) + \tilde{B}_\tau \tilde{u}_\tau(k) + L_{lift}\left(\tilde{y}_\tau(k) - \hat{y}_{lift}(k)\right) \\ \hat{y}_{lift}(k) &= \tilde{C}_\tau \hat{x}(kT+\tau) + \tilde{D}_\tau \tilde{u}_\tau(k) + Q\left(\tilde{y}_\tau(k) - \hat{y}_{lift}(k)\right) \\ r_{lift}(k) &= W_{lift}\left(\tilde{y}_\tau(k) - \hat{y}_{lift}(k)\right) \end{aligned} \tag{9.9}$$

where

$$\hat{y}_{lift}(k) = \begin{bmatrix} \hat{y}(kT+\tau) \\ \hat{y}(kT+\tau+1) \\ \vdots \\ \hat{y}((k+1)T+\tau-1) \end{bmatrix}$$

$$L_{lift} = \begin{bmatrix} \Psi(\tau+T,\tau+1)L(\tau) & \Psi(\tau+T,\tau+2)L(\tau+1) \cdots & L(\tau+T-1) \end{bmatrix}$$

$$W_{lift} = \begin{bmatrix} W & O & \cdots & O \\ O & W & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & W \end{bmatrix}$$

$$Q = \begin{bmatrix} O & O & \cdots & O \\ C(\tau+1)L(\tau) & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ Q_{T,1} & \cdots & C(\tau+T-1)L(\tau+T-2) & O \end{bmatrix}$$

$$Q_{T,1} = C(\tau+T-1)\Psi(\tau+T-1,\tau+1)L(\tau)$$

It can be verified that

$$L_\tau(Q+I) = L_{lift}, \; W_\tau(Q+I) = W_{lift} \tag{9.10}$$

If $\hat{x}(\tau) = \hat{\tilde{x}}_\tau(0)$, then according to (9.9) there is

$$\hat{\tilde{y}}_\tau(0) = \hat{y}_{lift}(0) - Q\left(\tilde{y}_\tau(0) - \hat{y}_{lift}(0)\right)$$

It follows

$$\begin{aligned} r_\tau(0) &= W_\tau(\tilde{y}_\tau(0) - \hat{\tilde{y}}_\tau(0)) \\ &= W_\tau(I+Q)\left(\tilde{y}_\tau(0) - \hat{y}_{lift}(0)\right) \\ &= W_{lift}\left(\tilde{y}_\tau(0) - \hat{y}_{lift}(0)\right) \\ &= r_{lift}(0) \\ \hat{\tilde{x}}_\tau(1) &= \tilde{A}_\tau\hat{\tilde{x}}_\tau(0) + \tilde{B}_\tau\tilde{u}_\tau(0) + L_\tau(\tilde{y}_\tau(0) - \hat{\tilde{y}}_\tau(0)) \\ &= \tilde{A}_\tau\hat{x}(\tau) + \tilde{B}_\tau\tilde{u}_\tau(0) + L_\tau(I+Q)\left(\tilde{y}_\tau(0) - \hat{y}_{lift}(0)\right) \\ &= \tilde{A}_\tau\hat{x}(\tau) + \tilde{B}_\tau\tilde{u}_\tau(0) + L_{lift}\left(\tilde{y}_\tau(0) - \hat{y}_{lift}(0)\right) \\ &= \hat{x}(T+\tau) \end{aligned}$$

In this way, it can be shown that $\hat{\tilde{y}}_\tau(k) = \hat{y}_{lift}(k) - Q\left(\tilde{y}_\tau(k) - \hat{y}_{lift}(k)\right)$, $\hat{\tilde{x}}_\tau(k) = \hat{x}(kT+\tau)$ and $r_{lift}(k) = r_\tau(k), \forall k$. □

Theorem 9.1 provides another possibility of transforming the LTI residual generator (9.3) into the periodic observer based residual generator (9.6). The periodic observer matrix $L(k)$ and the periodic weighting matrix $W(k)$ in (9.6) can be recovered from the constant matrices L_τ and W_τ in (9.3) according to (9.7) and (9.8) as follows.

Algorithm 9.1 Transformation of the LTI residual generator (9.3) into the periodic observer based residual generator (9.6):

- Transform the matrix W_τ into a lower triangular matrix.
- Partition the matrix L_τ into T blocks and the matrix W_τ into $T \times T$ blocks with compatible dimensions, as shown in (9.7) and (9.8)
- Let $L(\tau+T-1) = L_{\tau,T-1}$ and $W(\tau+i) = W_{\tau,i+1,i+1}$ for $i = 0, \cdots, T-1$.
- Solve the following equation to get $L(\tau+j)$

$$\begin{bmatrix} \Psi_{err}(\tau+T,\tau+T-1) \\ -W(\tau+j+1)C(\tau+j+1) \\ \vdots \\ -W(\tau+T-1)C(\tau+T-1)\Psi_{err}(\tau+T-1,\tau+j+1) \end{bmatrix} L(\tau+j)$$

$$= \begin{bmatrix} L_{\tau,j} \\ W_{\tau,j+2,j+1} \\ \vdots \\ W_{\tau,T,j+1} \end{bmatrix}$$

in sequence for $j = T-2, T-1, \cdots, 0$. If there is no solution, then (9.3) can not be implemented as (9.6).

In this section, we have shown how to design an observer based LTI residual generator (9.3) for the lifted system (9.2) and then implement the residual generator as a periodic residual generator (9.5) or (9.6). No matter which periodic implementation is used, it is always necessary to first transform the matrix W_τ into a lower triangular matrix. How to do such a transformation, will be discussed later in Section 9.3.

9.2 Parity relation based system design and implementation

Similarly, a parity relation based LTI residual generator can be built as follows

$$r_\tau(k) = V_{\tau,s}\left(\tilde{y}_{\tau,k,s} - \tilde{H}_{u,s}\tilde{u}_{\tau,k,s}\right) \tag{9.11}$$

$$\tilde{y}_{\tau,k,s} = \begin{bmatrix} \tilde{y}_\tau(k-s) \\ \tilde{y}_\tau(k-s+1) \\ \vdots \\ \tilde{y}_\tau(k) \end{bmatrix}, \ \tilde{u}_{\tau,k,s} = \begin{bmatrix} \tilde{u}_\tau(k-s) \\ \tilde{u}_\tau(k-s+1) \\ \vdots \\ \tilde{u}_\tau(k) \end{bmatrix}$$

$$\tilde{H}_{u,s} = \begin{bmatrix} \tilde{D}_\tau & O & \cdots & O \\ \tilde{C}_\tau\tilde{B}_\tau & \tilde{D}_\tau & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ \tilde{C}_\tau\tilde{A}_\tau^{s-1}\tilde{B}_\tau & \cdots & \tilde{C}_\tau\tilde{B}_\tau & \tilde{D}_\tau \end{bmatrix}$$

where $V_{\tau,s} = \begin{bmatrix} V_{\tau,s,0} & V_{\tau,s,1} & \cdots & V_{\tau,s,s-1} & V_{\tau,s,s} \end{bmatrix}$, $V_{\tau,s,l} \in \mathbf{R}^{n_r \times mT}$, $l = 0, 1, \cdots, s$, is a constant parity matrix and satisfies

$$V_{\tau,s} \begin{bmatrix} \tilde{C}_\tau \\ \tilde{C}_\tau\tilde{A}_\tau \\ \vdots \\ \tilde{C}_\tau\tilde{A}_\tau^s \end{bmatrix} = 0$$

It can be designed for the lifted system (9.2) using the methods introduced in Part I.

To get a residual signal at each time instant, we can use a bank of parity relation based residual generators, each one for $\tau = 0, 1, \cdots, T-1$, respectively. Alternatively, we can also impose a structural constraint on parity matrix $V_{\tau,s}$ as

$$V_{\tau,s,s} = \begin{bmatrix} (V_{\tau,s,s})_{1,1} & O & \cdots & O \\ (V_{\tau,s,s})_{2,1} & (V_{\tau,s,s})_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ (V_{\tau,s,s})_{T,1} & \cdots & (V_{\tau,s,s})_{T,T-1} & (V_{\tau,s,s})_{T,T} \end{bmatrix} \tag{9.12}$$

$$(V_{\tau,s,s})_{i,j} \in \mathbf{R}^{\rho_i \times m}, \ \sum_{i=1}^{T} \rho_i = n_r$$
$$i = 1, \cdots, T, \ j = 1, \cdots, T$$

i.e., the last block in $V_{\tau,s}$ is a lower triangular matrix. Then, for a fixed τ, the residual generator (9.11) can be implemented in such a way that only the control inputs and the measured outputs available up to the time instant $kT + \tau + j$ is needed for the calculation of $r(kT + \tau + j)$, $j = 0, 1, \cdots, T - 1$. Partition $V_{\tau,s,l}$, $l = 0, 1, \cdots, s - 1$, as follows

$$V_{\tau,s,l} = \begin{bmatrix} (V_{\tau,s,l})_1 \\ (V_{\tau,s,l})_2 \\ \vdots \\ (V_{\tau,s,l})_T \end{bmatrix}, \ (V_{\tau,s,l})_i \in \mathbf{R}^{\rho_i \times mT}, \ i = 0, 1, \cdots, s - 1$$

The periodic implementation of the residual generator (9.11) is

$$\begin{aligned} & r(kT + \tau + j) \\ & = \begin{bmatrix} (V_{\tau,s,0})_{j+1} & \cdots & (V_{\tau,s,s-1})_{j+1} \end{bmatrix} \\ & \times \left(\begin{bmatrix} \tilde{y}_\tau(k-s) \\ \tilde{y}_\tau(k-s+1) \\ \vdots \\ \tilde{y}_\tau(k-1) \end{bmatrix} - \tilde{H}_{u,s,I} \begin{bmatrix} \tilde{u}_\tau(k-s) \\ \tilde{u}_\tau(k-s+1) \\ \vdots \\ \tilde{u}_\tau(k-1) \end{bmatrix} \right) \\ & + \begin{bmatrix} (V_{\tau,s,s})_{j+1,1} & \cdots & (V_{\tau,s,s})_{j+1,j+1} \end{bmatrix} \\ & \times \left(\begin{bmatrix} y(kT+\tau) \\ y(kT+\tau+1) \\ \vdots \\ y(kT+\tau+j) \end{bmatrix} - \tilde{D}_{\tau,j} \begin{bmatrix} u(kT+\tau) \\ u(kT+\tau+1) \\ \vdots \\ u(kT+\tau+j) \end{bmatrix} \right. \\ & \left. - \begin{bmatrix} \tilde{C}_{\tau,j} \tilde{A}_\tau^{s-1} \tilde{B}_\tau & \cdots & \cdots & \tilde{C}_{\tau,j} \tilde{B}_\tau \end{bmatrix} \begin{bmatrix} \tilde{u}_\tau(k-s) \\ \tilde{u}_\tau(k-s+1) \\ \vdots \\ \tilde{u}_\tau(k-1) \end{bmatrix} \right) \end{aligned} \tag{9.13}$$

where

$$\tilde{H}_{u,s,I} = \begin{bmatrix} \tilde{D}_\tau & O & \cdots & O \\ \tilde{C}_\tau \tilde{B}_\tau & \tilde{D}_\tau & \ddots & \vdots \\ \vdots & \cdots & \ddots & O \\ \tilde{C}_\tau \tilde{A}_\tau^{s-2} \tilde{B}_\tau & \cdots & \tilde{C}_\tau \tilde{B}_\tau & \tilde{D}_\tau \end{bmatrix}$$

$$\tilde{D}_{\tau,j} = \begin{bmatrix} \tilde{D}_{\tau,1,1} & O & \cdots & O \\ \tilde{D}_{\tau,2,1} & \tilde{D}_{\tau,2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ \tilde{D}_{\tau,j+1,1} & \cdots & \tilde{D}_{\tau,j+1,j} & \tilde{D}_{\tau,j+1,j+1} \end{bmatrix}$$

$$\tilde{C}_{\tau,j} = \begin{bmatrix} C(\tau) \\ C(\tau+1)\Psi(\tau+1,\tau) \\ \vdots \\ C(\tau+j)\Psi(\tau+j,\tau) \end{bmatrix}$$

9.3 Computational aspects

In this section, we shall show how to transform W_τ and $V_{\tau,s}$, respectively, into the form of (9.4) and (9.12) to realize a periodic implementation.

Recall from Chapter 5 that in observer based residual generators, optimization problems (5.103)-(5.105) are solved by (5.106), the optimal weighting matrix is $W_{opt} = Q_d W_d$, where Q_d is an arbitrary unitary matrix of compatible dimensions, W_d is the left inverse of a full column rank matrix H_d satisfying

$$H_d H_d^T = C X_d C^T + F_d F_d^T \tag{9.14}$$

and X_d is obtained by solving a DTARS. That means, left multiplying the optimal weighting matrix with a unitary matrix will not change the optimality. Notice that $H_d H_d^T$ is a real symmetric matrix. If $H_d H_d^T$ is positive definite, then the Cholesky factorization (Matlab function *chol*) can be used to calculate the matrix H_d. H_d is obtained as a lower block triangular matrix. Correspondingly, its left inverse W_d is also a lower block triangular matrix and satisfies (9.4). In case that $H_d H_d^T$ is only positive semidefinite and $rank H_d H_d^T = n_H$ ($n_H < m$), then the matrices H_d and W_d can be calculated as follows:

- Do the SVD
$$C X_d C^T + F_d F_d^T = U_H \begin{bmatrix} \Sigma_H & O \\ O & O \end{bmatrix} U_H^T \tag{9.15}$$
where $U_H \in \mathbf{R}^{m \times m}$ is a unitary matrix and $\Sigma_H \in \mathbf{R}^{n_H \times n_H}$ a diagonal matrix.
- Partition U_H as
$$U_H = \begin{bmatrix} U_{H1} & U_{H2} \end{bmatrix}, \; U_{H1} \in \mathbf{R}^{m \times n_H}, \; U_{H2} \in \mathbf{R}^{m \times (m-n_H)} \tag{9.16}$$
- Calculate
$$H_d = U_{H1} \Sigma_H^{\frac{1}{2}} \in \mathbf{R}^{m \times n_H}, \; W_d = \Sigma_H^{-\frac{1}{2}} U_{H1}^T \in \mathbf{R}^{n_H \times m} \tag{9.17}$$

In general cases, to transform the matrix W_d into the form of (9.4), the QR decomposition can be applied. The standard function *qr* in Matlab decomposes a given matrix into the product of a unitary matrix and an upper triangular matrix. Therefore, to apply this Matlab function, some elementary column and row transformations are necessary. Such a procedure is provided below.

Algorithm 9.2 Transformation of the matrix W_d into a lower block triangular matrix into the form of (9.4):

- Calculate
$$\bar{W}_d = W_d V_1 \in \mathbf{R}^{n_H \times m} \tag{9.18}$$
where $V_1 \in \mathbf{R}^{m \times m}$ has on its anti-diagonal only elements 1, i.e.
$$V_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \diagup & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
- Do an QR decomposition using the Matlab function *qr*
$$\bar{W}_d = Q_W R_W, \; Q_W \in \mathbf{R}^{n_H \times n_H}, \; R_W \in \mathbf{R}^{n_H \times m} \tag{9.19}$$
where Q_W is a unitary matrix, R_W is an upper triangular matrix,
- Let
$$Q_d = V_2 Q_W^T \in \mathbf{R}^{n_H \times n_H} \tag{9.20}$$
where $V_2 \in \mathbf{R}^{n_H \times n_H}$ has a structure similar to V_1 and has on its anti-diagonal only elements 1,
$$V_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \diagup & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
The matrix Q_d obtained above is a unitary matrix and leads to
$$Q_d W_d = V_2 Q_W^T \bar{W}_d V_1^{-1} = V_2 Q_W^T Q_W R_W V_1^{-1} = V_2 R_W V_1^{-1}$$

Due to the special structure of V_1, V_2, R_W and the relation $V_1^{-1} = V_1$, it can be seen that $Q_d W_d$ has a lower triangular structure and satisfies (9.4).

In case that the optimal solution is given by (5.119), a similar procedure can be applied.

In the case of optimal parity relation based residual generators, it is shown by (5.22) and (5.27) that left multiplying the optimal solution $V_{\tau,s}$ by a unitary matrix will also not change the optimality. It is also easy to see that, if $V_{\tau,s}$ realizes a full decoupling, i.e.
$$V_{\tau,s} \left[\tilde{H}_{u,s} \; \tilde{H}_{d,s} \right] = 0$$
then multiplying $V_{\tau,s}$ from the left side with a unitary matrix will keep the property of full decoupling. Hence, no matter it is a full decoupling or optimal design, the resulting parity matrix $V_{\tau,s}$ can be brought into a form satisfying (9.12) using the above procedure described in Algorithm 9.2 as well.

In the next, we shall give two examples.

Example 9.1 Consider the periodic system (9.1) with period $T = 2$ and
$$A(0) = \begin{bmatrix} 1 & 1 \\ 0 & 0.5 \end{bmatrix}, \; A(1) = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix}, \; B(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B(1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$
$$E_d(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; E_d(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E_f(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; E_f(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C(0) = C(1) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D(0) = D(1) = 0$$
$$F_d(0) = 1, \; F_d(1) = 0, \; F_f(0) = F_f(1) = 1$$

Design a periodic residual generator using the observer based approach.

Let $\tau = 0$. The lifted reformulation of (9.1) is
$$\tilde{A}_0 = \begin{bmatrix} 0.5 & 0.75 \\ 0 & 0.5 \end{bmatrix}, \; \tilde{B}_0 = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}, \; \tilde{E}_{d,0} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \; \tilde{E}_{f,0} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$
$$\tilde{C}_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; \tilde{D}_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; \tilde{F}_{d,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; \tilde{F}_{f,0} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{9.21}$$

According to Theorem 5.10, for the lifted system an observer based residual generator that is optimal in the sense of (5.103)-(5.105) is given by (9.3) with

$$L_0 = \begin{bmatrix} -0.25 & 0.75 \\ -0.5 & 0.5 \end{bmatrix}, W_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

As W_0 satisfies (9.4), the residual generator can be implemented as a periodic system in the form of (9.5) as

$$\begin{aligned}
\hat{\bar{x}}_0(2(k+1)) &= \begin{bmatrix} 0.5 & 0.75 \\ 0 & 0.5 \end{bmatrix} \hat{\bar{x}}_0(2k) + \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(2k) \\ u(2k+1) \end{bmatrix} \\
&+ \begin{bmatrix} -0.25 & 0.75 \\ -0.5 & 0.5 \end{bmatrix} \left(\begin{bmatrix} y(2k) \\ y(2k+1) \end{bmatrix} - \begin{bmatrix} \hat{y}(2k) \\ \hat{y}(2k+1) \end{bmatrix} \right) \\
\hat{y}(2k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\bar{x}}_0(2k) \\
r(2k) &= y(2k) - \hat{y}(2k) \\
\hat{y}(2k+1) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \hat{\bar{x}}_0(2k) + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u(2k) \\ u(2k+1) \end{bmatrix} \\
r(2k+1) &= \begin{bmatrix} -1 & 1 \end{bmatrix} \left(\begin{bmatrix} y(2k) \\ y(2k+1) \end{bmatrix} - \begin{bmatrix} \hat{y}(2k) \\ \hat{y}(2k+1) \end{bmatrix} \right)
\end{aligned} \tag{9.22}$$

In this example, the residual generator can also be implemented in the form of (9.6). In the first step, let

$$L(1) = \begin{bmatrix} 0.75 \\ 0.5 \end{bmatrix}, \; W(0) = 1, \; W(1) = 1$$

Solving the equation

$$\begin{aligned}
&\begin{bmatrix} \Psi_{err}(2,1) \\ -W(1)C(1) \end{bmatrix} L(0) = \begin{bmatrix} A(1) - L(1)C(1) \\ -W(1)C(1) \end{bmatrix} L(0) \\
&= \begin{bmatrix} -0.25 & 0.5 \\ -0.5 & 1 \\ -1 & 0 \end{bmatrix} L(0) = \begin{bmatrix} -0.25 \\ -0.5 \\ -1 \end{bmatrix}
\end{aligned}$$

we get

$$L(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, the periodic residual generator can be implemented as

$$\begin{aligned}
\hat{x}(2k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 0.5 \end{bmatrix} \hat{x}(2k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(2k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (y(2k) - \hat{y}(2k)) \\
\hat{x}(2(k+1)) &= \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix} \hat{x}(2k+1) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u(2k+1) \\
&+ \begin{bmatrix} 0.75 \\ 0.5 \end{bmatrix} (y(2k+1) - \hat{y}(2k+1)) \\
\hat{y}(2k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(2k) \\
r(2k) &= y(2k) - \hat{y}(2k) \\
\hat{y}(2k+1) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(2k+1) \\
r(2k+1) &= y(2k+1) - \hat{y}(2k+1)
\end{aligned} \tag{9.23}$$

Example 9.2 Consider the same periodic system (9.1) as defined in Example 9.1. Design a periodic residual generator using the parity space approach.

Let $s = 1$. According to Theorem 5.2, a parity relation based residual generator (9.11) that is optimal in the sense of (5.15)-(5.17) can be obtained as

$$V_{0,s} = \begin{bmatrix} -0.4343 & 0.8896 & -0.4974 & -0.4132 \\ 0.3520 & -0.1454 & -1.3238 & 0.9106 \end{bmatrix}$$

The last block of $V_{0,s}$ is

$$V_{0,s,s} = \begin{bmatrix} -0.4974 & -0.4132 \\ -1.3238 & 0.9106 \end{bmatrix}$$

Let

$$V_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; V_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The QR decomposition of $V_{0,s,s}V_1$ is

$$V_{0,s,s}V_1 = Q_W R_W = \begin{bmatrix} -0.4132 & -0.4974 \\ 0.9106 & -1.3238 \end{bmatrix}$$

$$Q_W = \begin{bmatrix} -0.4132 & 0.9106 \\ 0.9106 & 0.4132 \end{bmatrix}, \; R_W = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

Finally, $V_{0,s,s}$ can be transformed through the transformation matrix

$$Q_d = V_2 Q_W^T = \begin{bmatrix} 0.9106 & 0.4132 \\ -0.4132 & 0.9106 \end{bmatrix}$$

into a matrix satisfying (9.12) as follows

$$Q_d V_{0,s,s} = \begin{bmatrix} -0.25 & 0.75 & -1 & 0 \\ 0.5 & -0.5 & -1 & 1 \end{bmatrix}$$

The optimal parity relation based residual generator can be implemented as a periodic system as

$$\begin{aligned} r(2k) &= \begin{bmatrix} -0.25 & 0.75 \end{bmatrix} \left(\begin{bmatrix} y(2k-2) \\ y(2k-1) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(2k-2) \\ u(2k-1) \end{bmatrix} \right) \\ &\quad - \left(y(2k) - \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u(2k-2) \\ u(2k-1) \end{bmatrix} \right) \\ r(2k+1) &= \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} \left(\begin{bmatrix} y(2k-2) \\ y(2k-1) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(2k-2) \\ u(2k-1) \end{bmatrix} \right) \\ &\quad + \begin{bmatrix} -1 & 1 \end{bmatrix} \times \left(\begin{bmatrix} y(2k) \\ y(2k+1) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(2k) \\ u(2k+1) \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u(2k-2) \\ u(2k-1) \end{bmatrix} \right) \end{aligned} \tag{9.24}$$

Example 9.3 In this example, we shall design the optimal observer based residual generator for the following periodic system with period $T = 3$ [89], [179]

$$A(k) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_{1k} & a_{2k} & a_{3k} & a_{4k} & a_{5k} \end{bmatrix}, B(k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, E_d(k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_{2k} \\ b_{1k} & 0 \end{bmatrix}$$

$$E_f(k) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ g_{1k} \end{bmatrix}, \; C(k) = \begin{bmatrix} 0 & 0 & 0 & c_{1k} & c_{2k} \\ 0.5 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9.25}$$

where $a_{1k} = \sin(2\pi(k+T)/T + 0.9)$, $a_{2k} = \sin(2\pi(k+T)/T + 0.7)$, $a_{3k} = 2\sin(2\pi(k+T)/T - 0.5)$, $a_{4k} = \sin(2\pi(k+T)/T - 0.1)$, $a_{5k} = \sin(2\pi(k+T)/T + 0.3)$, $b_{1k} = \cos(2\pi(k+T)/T)$, $b_{2k} = 2\cos(2\pi(k+T)/T + 0.5)$, $c_{1k} = 2\cos(2\pi(k+T)/T + 0.2)$, $c_{2k} = \cos(2\pi(k+T)/T + 0.1)$, $g_{1k} = 0.5\sin(2\pi(k+T)/T + 0.4)$.

For the lifted reformulation of (9.1) ($\tau = 0$), an observer based residual generator that maximizes simultaneously the H_∞/H_∞, H_-/H_∞ and H_i/H_∞ indices is given by (9.3) with [29], [172]

$$L_0 = \begin{bmatrix} 0.0186 & 0.0336 & 0.0014 & 0.0353 & -0.0007 & -3.0039 \\ -0.0395 & -0.4447 & -0.7738 & -0.3423 & 0.1775 & -0.1816 \\ -0.0356 & -0.1817 & 0.4220 & -0.3347 & -1.4615 & 3.2661 \\ 0.0551 & 0.2811 & -0.6530 & 0.5179 & -0.1717 & -5.0532 \\ -0.0021 & 0.3031 & 1.0980 & 0.0881 & 1.1698 & 3.5624 \end{bmatrix}$$

$$W_0 = \begin{bmatrix} 0.9081 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0565 & 43.2694 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.1529 & -2.0414 & 0.3988 & 0.0000 & 0.0000 & 0.0000 \\ 0.6328 & -3.5730 & -0.0224 & 6.8563 & 0.0000 & 0.0000 \\ 0.3867 & 2.1100 & -0.2466 & 1.1677 & 0.7443 & 0.0000 \\ 1.6966 & 7.1507 & 0.1630 & 2.0807 & -0.1299 & 11.8429 \end{bmatrix}$$

W_0 already satisfies (9.8). According to Theorem 1, we get

$$L(2) = \begin{bmatrix} -0.0007 & -3.0039 \\ 0.1775 & -0.1816 \\ -1.4615 & 3.2661 \\ -0.1717 & -5.0532 \\ 1.1698 & 3.5624 \end{bmatrix}, \ W(0) = \begin{bmatrix} 0.9081 & 0.0000 \\ -0.0565 & 43.2694 \end{bmatrix}$$

$$W(1) = \begin{bmatrix} 0.3988 & 0.0000 \\ -0.0224 & 6.8563 \end{bmatrix}, \ W(2) = \begin{bmatrix} 0.7443 & 0.0000 \\ -0.1299 & 11.8429 \end{bmatrix}$$

Solving the equation

$$\begin{bmatrix} \Psi_{err}(3,2) \\ -W(2)C(2) \end{bmatrix} L(1) = \begin{bmatrix} A(2) - L(2)C(2) \\ -W(2)C(2) \end{bmatrix} L(1) = \begin{bmatrix} 0.0014 & 0.0353 \\ -0.7738 & -0.3423 \\ 0.4220 & -0.3347 \\ -0.6530 & 0.5179 \\ 1.0980 & 0.0881 \\ -0.2466 & 1.1677 \\ 0.1630 & 2.0807 \end{bmatrix},$$

we get

$$L(1) = \begin{bmatrix} -0.0203 & -0.3858 \\ 0.0316 & 0.6158 \\ -0.7132 & -0.5857 \\ -0.0953 & 1.3282 \\ -0.6587 & 1.7620 \end{bmatrix}$$

Solving the equation

$$\begin{bmatrix} \Psi_{err}(3,1) \\ -W(1)C(1) \\ -W(2)C(2)\Psi_{err}(2,1) \end{bmatrix} L(0)$$

$$= \begin{bmatrix} (A(2)-L(2)C(2))(A(1)-L(1)C(1)) \\ -W(1)C(1) \\ -W(2)C(2)(A(1)-L(1)C(1)) \end{bmatrix} L(0)$$

$$= \begin{bmatrix} 0.0186 & 0.0336 \\ -0.0395 & -0.4447 \\ -0.0356 & -0.1817 \\ 0.0551 & 0.2811 \\ -0.0021 & 0.3031 \\ 0.1529 & -2.0414 \\ 0.6328 & -3.5730 \\ 0.3867 & 2.1100 \\ 1.6966 & 7.1507 \end{bmatrix}$$

yields

$$L(0) = \begin{bmatrix} -0.1871 & 1.0757 \\ -0.2541 & -1.5810 \\ 0.3967 & 2.4354 \\ 0.2236 & -4.7977 \\ 0.1495 & 2.1146 \end{bmatrix}$$

9.4 Conclusion

This chapter discusses lifting based FD approaches for the linear discrete-time periodic (LDP) systems. The design procedure includes three steps. At first, the periodic system is lifted into a discrete LTI system. Then, a discrete LTI residual generator is designed based on the lifted LTI reformulation, using the parity space approach or the observer based approach. At last, the discrete LTI residual generator is transformed into a periodic residual generator to reduce detection delay. The key is to recover the parameters of the periodic residual generators from the parameters of the LTI residual generators [188]. An QR based algorithm is provided for such a transformation. In the case of observer based approaches, an analytic transformation is derived to directly recover the parameters of the periodic observer based residual generators from the parameters of the LTI observer based residual generators. With the approaches introduced in this chapter, many existing optimization and decoupling methods for the FD of the discrete LTI systems can be transferred easily to the LDP systems.

10

Periodic design

In the last chapter, it is shown that residual generators can be designed for periodic system (9.1) by first lifting the periodic system into an LTI system. In this chapter, we shall extend the parity space approach and the observer based approach directly to periodic systems.

10.1 Periodic parity space approach

The extension of the parity space approach to periodic systems is straightforward. Recall that the essence of the parity space approach is to derive the so-called parity relations. At time instant k, consider the input-output relation of the periodic system (9.1) during the moving horizon $[k-s,k]$, where s is an integer and represents the length of the horizon. A parity relation is obtained as

$$y_s(k) = H_{o,s}(k)x(k-s) + H_{u,s}(k)u_s(k) + H_{d,s}(k)d_s(k) + H_{f,s}(k)f_s(k) \tag{10.1}$$

where the vectors $u_s(k), d_s(k), f_s(k)$ and $y_s(k)$ contain the input and output sequences of the periodic system (9.1) over the moving horizon and are defined in the same way as (2.4), and the matrices $H_{o,s}(k), H_{u,s}(k), H_{d,s}(k), H_{f,s}(k)$ are as follows,

$$H_{o,s}(k) = \begin{bmatrix} C(k-s) \\ C(k-s+1)A(k-s) \\ \vdots \\ C(k)\Psi(k,k-s+1)A(k-s) \end{bmatrix} \tag{10.2}$$

$$H_{u,s}(k) = \begin{bmatrix} D(k-s) & O & \cdots & O \\ C(k-s+1)B(k-s) & D(k-s+1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C(k)\Psi(k,k-s+1)B(k-s) & \cdots & C(k)B(k-1) & D(k) \end{bmatrix}$$

$$H_{d,s}(k) = \begin{bmatrix} F_d(k-s) & O & \cdots & O \\ C(k-s+1)E_d(k-s) & F_f(k-s+1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C(k)\Psi(k,k-s+1)E_d(k-s) & \cdots & C(k)E_d(k-1) & F_d(k) \end{bmatrix}$$

$$H_{f,s}(k) = \begin{bmatrix} F_f(k-s) & O & \cdots & O \\ C(k-s+1)E_f(k-s) & F_d(k-s+1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C(k)\Psi(k,k-s+1)E_f(k-s) & \cdots & C(k)E_f(k-1) & F_f(k) \end{bmatrix}.$$

Due to the periodicity of system matrices, in parity relation (10.1) the matrices $H_{o,s}(k), H_{u,s}(k), H_{d,s}(k)$ and $H_{f,s}(k)$ are periodic functions with respect to k.

Based on the parity relation (10.1), a residual generator can be constructed as

$$r(k) = v_s(k)(y_s(k) - H_{u,s}(k)u_s(k)) \tag{10.3}$$

where $r \in \mathbf{R}$ is the so-called residual signal, and the design parameter $v_s(k)$ is a T-periodic vector called parity vector that satisfies $v_s(k)H_{o,s}(k) = 0$. If a periodic parity matrix is used in the residual generation, then the residual generator is written as

$$r(k) = V_s(k)(y_s(k) - H_{u,s}(k)u_s(k)) \tag{10.4}$$

where $V_s(k)$ denotes the periodic parity matrix, $V_s(k)H_{o,s}(k) = 0$.

10.2 Periodic observer based approach

To the aim of fault detection, a linear discrete-time periodic residual generator can be constructed as

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) + L(k)(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C(k)\hat{x}(k) + D(k)u(k) \\ r(k) &= W(k)(y(k) - \hat{y}(k)) \end{aligned} \tag{10.5}$$

where $r \in \mathbf{R}^{n_r}$ is the residual signal, $L(k)$ and $W(k)$ are periodic matrices of period T to be determined, $L(k)$ denotes the periodic observer gain matrix, $W(k)$ denotes the periodic weighting matrix. The dynamics of the periodic observer based residual generator (10.5) is governed by

$$\begin{aligned} e(k+1) &= (A(k) - L(k)C(k))e(k) + (E_d(k) - L(k)F_d(k))d(k) \\ &\quad + (E_f(k) - L(k)F_f(k))f(k) \end{aligned} \tag{10.6}$$

$$r(k) = W(k)(C(k)e(k) + F_d(k)d(k) + F_f(k)f(k)) \tag{10.7}$$

where $e(k) = x(k) - \hat{x}(k)$.

Similar as in LTI systems, the residual generators can also be built based on a periodic functional observer as

$$\begin{aligned} z(k+1) &= G(k)z(k) + H(k)u(k) + L(k)y(k) \\ r(k) &= w(k)z(k) + q(k)u(k) + p(k)y(k) \end{aligned} \tag{10.8}$$

with $z \in \mathbf{R}^s$. The design parameters are the T-periodic matrices $G(k), H(k), L(k), w(k), q(k)$ and $p(k)$. Let $e(k) = z(k) - T(k)x(k)$. If $G(k)$ is stable and the following equations

$$T(k+1)A(k) - G(k)T(k) = L(k)C(k) \tag{10.9}$$

$$w(k)T(k) + p(k)C(k) = 0 \tag{10.10}$$

$$H(k) = T(k+1)B(k) - L(k)D(k) \tag{10.11}$$

$$q(k) = -p(k)D(k) \tag{10.12}$$

hold for any k, then the dynamics of residual generator (10.8) is governed by

$$\begin{aligned} e(k+1) &= G(k)e(k) + (L(k)F_d(k) - T(k+1)E_d(k))d(k) \\ &\quad + (L(k)F_f(k) - T(k+1)E_f(k))f(k), \\ r(k) &= w(k)e(k) + p(k)F_d(k)d(k) + p(k)F_f(k)f(k) \end{aligned} \tag{10.13}$$

and meets the basic requirement that $\forall u$, $\lim_{k\to\infty} r(k) = 0$, if $d = 0, f = 0$. The equations (10.9)-(10.12) are an extension of the well-known Luenberger condition in discrete-time periodic systems. A numerically stable algorithm based on the use of the periodic Schur form is proposed by [149] to solve the periodic Sylvester equations of the form $\hat{T}(k+1)\hat{A}(k) - \hat{B}(k)\hat{T}(k) = \hat{C}(k)$, in which $\hat{T}(k)$ is unknown and $\hat{A}(k), \hat{B}(k), \hat{C}(k)$ are given. However, in general it is not an easy task to solve the equations (10.9)-(10.10).

10.3 Relation between periodic parity space and periodic observer

Inspired by the fact that in the LTI case there is a one to one relationship between observer based and parity relation based residual generators [30], Theorem 10.1 below shows the construction of a periodic observer from a periodic parity vector [183].

Theorem 10.1 Assume that a periodic vector

$$v_s(k) = \left[v_{s,0}(k)\ v_{s,1}(k)\ \cdots\ v_{s,s}(k) \right]$$

satisfies $v_s(k)H_{o,s}(k) = 0$. Then the equations (10.9)-(10.10) are solved by

$$\begin{aligned}
T(k) &= \begin{bmatrix} v_{s,1}(k+s-1) & \cdots & v_{s,s-1}(k+s-1) & v_{s,s}(k+s-1) \\ v_{s,2}(k+s-2) & \cdots & v_{s,s}(k+s-2) & 0 \\ \vdots & & & \vdots \\ v_{s,s}(k) & 0 & \cdots & 0 \end{bmatrix} \\
&\times \begin{bmatrix} C(k) \\ C(k+1)A(k) \\ \vdots \\ C(k+s-1)A(k+s-2)\cdots A(k) \end{bmatrix} \\
G(k) &= \begin{bmatrix} 0 & 0 & \cdots & 0 & g_1(k) \\ 1 & 0 & \ddots & \vdots & g_2(k) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & 0 & g_{s-1}(k) \\ 0 & \cdots & 0 & 1 & g_s(k) \end{bmatrix} \\
L(k) &= -\begin{bmatrix} v_{s,0}(k+s) \\ v_{s,1}(k+s-1) \\ \vdots \\ v_{s,s-1}(k+1) \end{bmatrix} - \begin{bmatrix} g_1(k) \\ g_2(k) \\ \vdots \\ g_s(k) \end{bmatrix} v_{s,s}(k) \\
w(k) &= \left[0\ 0\ \cdots\ 0\ {-1} \right] \\
p(k) &= v_{s,s}(k)
\end{aligned} \tag{10.14}$$

where the periodic scalars $g_1(k), \cdots, g_s(k)$ appearing in the matrices $G(k), L(k)$ are free parameters and should be selected in such a way that all characteristic multipliers, i.e., the eigenvalues of $G(T-1)\cdots G(1)G(0)$, are inside the open unit disk of the complex plane.

Proof: Note that $v_s(k)H_{o,s}(k) = 0$ can be expanded as

$$\left[v_{s,0}(k)\ v_{s,1}(k)\ \cdots\ v_{s,s}(k) \right] \begin{bmatrix} C(k-s) \\ C(k-s+1)A(k-s) \\ \vdots \\ C(k)A(k-1)\cdots A(k-s+1)A(k-s) \end{bmatrix} = 0$$

Hence, the first row of $T(k+1)A(k) - G(k)T(k)$ equals the first row of $L(k)C(k)$. It is straightforward to show that the other rows of $T(k+1)A(k) - G(k)T(k)$ are identical with those of $L(k)C(k)$, respectively, if $G(k), T(k)$ and $L(k)$ are selected as (10.14). Therefore, equation (10.9) holds. Since $w(k)T(k) = -v_{s,s}(k)C(k)$ and $p(k) = v_{s,s}(k)$, equation (10.10) holds. □

Remark 10.1 We would like to point out that an alternative way to derive solution (10.14) is to exploit the isomorphism between periodic systems and LTI systems with the help of the cyclic time-invariant representation.

Theorem 10.1 reveals that, given a periodically time-varying vector belonging to the parity space, a periodic observer based residual generator satisfying (10.9)-(10.10) can be readily constructed according to (10.14). To ensure the stability of the residual dynamics (10.13), a simple choice of $g_j(k)$ is $g_j(k) \equiv 0$, $j = 1, \cdots, s$. In this case, all characteristic multipliers will be placed at the origin and the residual signals obtained by residual generators (10.3) and (10.8) are identical. In general, no matter what $g_j(k)$ is, residual generator (10.8) can always be rewritten as

$$\begin{aligned} z(k+1) &= \bar{G}(k)z(k) + \bar{H}(k)u(k) + \bar{L}(k)y(k) - g(k)r(k) \\ r(k) &= w(k)z(k) + q(k)u(k) + p(k)y(k) \end{aligned} \tag{10.15}$$

and residual dynamics (10.13) can be expressed by

$$\begin{aligned} e(k+1) &= \bar{G}(k)e(k) + (\bar{L}(k)F_d(k) - T(k+1)E_d(k))d(k) \\ &\quad +(\bar{L}(k)F_f(k) - T(k+1)E_f(k))f(k) - g(k)r(k) \\ r(k) &= w(k)e(k) + p(k)F_d(k)d(k) + p(k)F_f(k)f(k) \end{aligned} \tag{10.16}$$

where

$$\bar{G}(k) = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \bar{L}(k) = -\begin{bmatrix} v_{s,0}(k+s) \\ v_{s,1}(k+s-1) \\ \vdots \\ v_{s,s-1}(k+1) \end{bmatrix}, g(k) = \begin{bmatrix} g_1(k) \\ g_2(k) \\ \vdots \\ g_s(k) \end{bmatrix}$$

$$\bar{H}(k) = T(k+1)B(k) - \bar{L}(k)D(k)$$

Note that $\bar{G}(k), \bar{L}(k)$ and $\bar{H}(k)$ are independent of column vector $g(k)$. Hence, $g(k)$ can be interpreted as the gain of the implicit feedback in observer based residual generator (10.8). Moreover, it can be shown that (10.15) and (10.16) are, respectively, equivalent to

$$\begin{aligned} r(k) &= v_s(k)\left(y_s(k) - H_{u,s}(k)u_s(k)\right) + g_s(k-1)r(k-1) \\ &\quad +g_{s-1}(k-2)r(k-2) + \cdots + g_1(k-s)r(k-s) \qquad (10.17) \\ &= v_s(k)(H_{d,s}(k)d_s(k) + H_{f,s}(k)f_s(k)) + g_s(k-1)r(k-1) \\ &\quad +g_{s-1}(k-2)r(k-2) + \cdots + g_1(k-s)r(k-s) \qquad (10.18) \end{aligned}$$

It means that $g(k) \neq 0$ will lead to a closed-loop structured implementation. The freedom provided by $g(k)$ could be used to meet additional specifications on the residual dynamics, for instance, to modulate the frequency domain behaviour of the residual generator [164].

Based on Theorem 10.1, the equations (10.9)-(10.12) can be solved by first solving algebraic equations $v_s(k)H_{o,s}(k) = 0$ for $v_s(k)$ over one period, then making use of (10.14) to get a solution to equations (10.9)-(10.10), and finally computing $H(k)$ and $q(k)$ by (10.11)-(10.12).

On the other side, a periodic observer based residual generator can also be related to a periodic parity vector.

Theorem 10.2 Assume that a periodic observer-based residual generator (10.8) satisfying the equations (10.9)-(10.12) with $G(k), L(k), w(k)$ of the form

$$G(k) = \begin{bmatrix} 0 & \cdots & 0 & g_1(k) \\ 1 & \ddots & \vdots & g_2(k) \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & g_s(k) \end{bmatrix}, L(k) = \begin{bmatrix} l_1(k) \\ l_2(k) \\ \vdots \\ l_s(k) \end{bmatrix}, w(k) = \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix} \tag{10.19}$$

is given. Then the row vector

$$\begin{aligned}
v_s(k) &= \left[v_{s,0}(k) \; v_{s,1}(k) \cdots \;\; v_{s,s-1}(k) \;\; v_{s,s}(k) \right] \qquad (10.20)\\
v_{s,0}(k) &= l_1(k-s) + g_1(k-s)p(k-s)\\
v_{s,1}(k) &= l_2(k-s+1) + g_2(k-s+1)p(k-s+1)\\
&\vdots\\
v_{s,s-1}(k) &= l_s(k-1) + g_s(k-1)p(k-1)\\
v_{s,s}(k) &= -p(k)
\end{aligned}$$

is a periodic parity vector satisfying $v_s(k)H_{o,s}(k) = 0$.

Proof: Let

$$\begin{aligned}
\alpha_{s,0}(k) &= \left(l_1(k-s) + g_1(k-s)p(k-s)\right) C(k-s), \;\; \cdots\\
\alpha_{s,s-1}(k) &= \left(l_s(k-1) + g_s(k-1)p(k-1)\right) C(k-1)\Psi(k-1,k-s)\\
\alpha_{s,s}(k) &= -p(k)C(k)\Psi(k,k-s)\\
w_i(k) &= w(k)\bar{G}(k-1)\cdots\bar{G}(k-i), \quad i = 1,\cdots,s-1
\end{aligned}$$

Considering (10.9)-(10.10), we have

$$\alpha_{s,s}(k) = w(k)\left(G(k-1)T(k-1) + L(k-1)C(k-1)\right)\Psi(k-1,k-s)$$

Note that

$$\begin{aligned}
w(k)L(k-1) &= -l_s(k-1)\\
w(k)(k-1) &= w(k)\bar{G}(k-1) + g_s(k-1)w(k-1)\\
p(k-1)C(k-1) &= -w(k-1)T(k-1)
\end{aligned}$$

There is

$$\alpha_{s,s-1}(k) + \alpha_{s,s}(k) = w_1(k)T(k-1)\Psi(k-1,k-s)$$

Repeating the above derivation, we get

$$\sum_{j=s-i}^{s} \alpha_{s,j}(k) = w_i(k)T(k-i)\Psi(k-i-1,k-s)$$

Since $w_{s-1}(k) = [-1 \; 0 \cdots \; 0]$, there is

$$\begin{aligned}
v_s(k)H_{o,s}(k) &= \sum_{j=0}^{s} \alpha_{s,j}(k) = \alpha_{s,0}(k) + w_{s-1}(k)T(k-s+1)A(k-s)\\
&= \alpha_{s,0}(k) + w_{s-1}(k)\left(G(k-s)T(k-s) + L(k-s)C(k-s)\right)\\
&= 0
\end{aligned}$$

The theorem is thus proven. □

10.4 Disturbance decoupling

In this section, we shall consider the full decoupling problem [183]. The aim is to design a residual generator, so that the residual r satisfies
(i) $\lim_{k\to\infty} r(k) = 0$, if $f = 0$ and no matter what the control inputs and the disturbances are;
(ii) $r(k) \neq 0$ if $f_i(k) \neq 0$, $i = 1,\cdots,n_f$.

As the parity space approach treats each time instant independently, the full decoupling problem can be easily solved. If $v_s(k)$ can be selected in such a way that

$$v_s(k)\begin{bmatrix} H_{o,s}(k) & H_{d,s}(k) \end{bmatrix} = 0,\ v_s(k)H_{f,s}(k) \neq 0 \tag{10.21}$$

holds for any k, then

$$r(k) = v_s(k)H_{f,s}(k)f_s(k).$$

The residual will be influenced neither by the initial state $x(k-s)$ nor by the disturbance vector d or the control input vector u. As a result, a full decoupling is realized and each individual fault can be detected.

Note that (10.21) is a set of *independent* linear equations and can be easily solved. This means a periodic parity relation based full decoupling residual generator can be simply designed by solving (10.21) for $v_s(k)$. Moreover, as $H_{o,s}(k), H_{d,s}(k)$ and $H_{f,s}(k)$ are periodic, (10.21) only needs to be solved over one period.

Theorem 10.3 For the periodic system (9.1), a full decoupling residual generator (10.3) exists if and only if the following rank condition is satisfied

$$rank\begin{bmatrix} H_{o,s}(k)\ H_{d,s}(k)\ H_{f,s}(k) \end{bmatrix} > rank\begin{bmatrix} H_{o,s}(k)\ H_{d,s}(k) \end{bmatrix} \tag{10.22}$$

for $k = 0, 1, \cdots, T-1$.

In the sequel, we shall discuss how to realize the full decoupling with the periodic observer based residual generator (10.8). If, besides (10.9)-(10.12), the following conditions

$$p(k)F_d(k) = 0, \tag{10.23}$$
$$T(k+1)E_d(k) - L(k)F_d(k) = 0, \tag{10.24}$$
$$\begin{bmatrix} T(k+1)E_f(k) - L(k)F_f(k) & p(k)F_f(k) \end{bmatrix} \neq 0,\ i = 1, \cdots, n_f \tag{10.25}$$

are also fulfilled by $L(k), T(k)$ and $p(k)$, then the dynamics of residual generator (10.13) satisfies the conditions (i)-(ii) and a full decoupling can be achieved. To solve (10.9)-(10.12) and (10.23)-(10.25) simultaneously, the following theorem is given.

Theorem 10.4 Assume that the periodic vector

$$v_s(k) = \begin{bmatrix} v_{s,0}(k)\ v_{s,1}(k) \cdots v_{s,s}(k) \end{bmatrix}$$

satisfies (10.21). Then $G(k), L(k), T(k), w(k)$ and $p(k)$ given by (10.14) satisfy (10.9)-(10.10) and (10.23)-(10.25) simultaneously.

Proof: In view of $v_s(k)H_{d,s}(k) = 0$, multiplying $v_s(k)$ with each column of $H_{d,s}(k)$ yields

$$v_{s,s}(k)F_d(k) = 0, \tag{10.26}$$

$$\sum_{l=j+1}^{s} v_{s,l}(k)C(k-s+l)\Psi(k-s+l, k-s+j+1)E_d(k-s+j)$$
$$+v_{s,j}(k)F_d(k-s+j) = 0, \quad j = 0, 1, \cdots, s-1 \tag{10.27}$$

where the state transition matrix Ψ is given by (8.4). As $p(k) = v_{s,s}(k)$, the equation (10.23) follows immediately from (10.26). The equation (10.27) can be rewritten as

$$\sum_{l=j+1}^{s} v_{s,l}(k+s-j)C(k+l-j)\Psi(k+l-j, k+1)E_d(k)$$
$$+v_{s,j}(k+s-j)F_d(k) = 0$$

by substituting k with $k+s-j$. Thus, (10.24) holds. In a similar manner it can be proven that $v_s(k)H_{f,s}(k) \neq 0$ ensures (10.25). □

Theorem 10.4 states that if the periodic parity vector $v_s(k)$ realizes a full decoupling, so does the periodic observer based residual generator (10.8) with coefficients (10.14). Thus a periodic observer based full decoupling residual generator can be obtained from a periodic full decoupling parity vector.

The order of the periodic observer is equal to the order of the parity relation s. It is also interesting to notice that the matrices $L(k)$ and $T(k)$ of the periodic observer-based residual generator (7.3) at *each* time are related to the periodic parity vector *over one period*.

In summary, the proposed procedure of designing a periodic observer based full decoupling residual generator in the form of (10.8) is as follows:

- Set the value of s and construct the matrices $H_{o,s}(k), H_{d,s}(k)$, $H_{f,s}(k)$ by (10.2).
- Solve (10.21) for the periodic row vector $v_s(k)$ over a period.
- Partition $v_s(k)$ as $v_s(k) = [v_{s,0}(k)\ v_{s,1}(k) \cdots v_{s,s}(k)]$ with $v_{s,j}(k) \in \mathbf{R}^{1\times m}$, $j = 0, 1, \cdots, s$.
- get $G(k), L(k), T(k), w(k), p(k)$ by (10.14) with $g_1(k), \cdots,\ g_s(k)$ ensuring the stability of the residual dynamics.
- Compute $H(k)$ and $q(k)$ from (10.11)-(10.12).

Similarly, a full decoupling parity vector can be obtained from a full decoupling observer based residual generator.

Theorem 10.5 Given the periodic system (9.1) and the full decoupling observer based residual generator (10.8). If $G(k), L(k), w(k)$ can be brought into the form of (10.19), then the vector given by (10.20) satisfies (10.21) and also realizes a full decoupling.

To illustrate the proposed design procedures, we shall look at the following example.

Example 10.1 Consider a periodic system of period $T = 2$ described by (9.1) with [51]

$$A(0) = \begin{bmatrix} 0.25 & 0.25 & 0.1 & -0.1 \\ 0.5 & 0.1 & 0.1 & 0.5 \\ 0.5 & -0.2 & 0.2 & 0.25 \\ 0.1 & 0 & 0.25 & 0.1 \end{bmatrix},\ A(1) = \begin{bmatrix} 0.1 & 0.2 & 0.1 & -0.1 \\ -0.1 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0.1 & 0.25 \\ 0 & 0.1 & 0.1 & 0.25 \end{bmatrix}$$

$$C(0) = \begin{bmatrix} 0.25 & 0.1 & 0.2 & 0.1 \\ -0.1 & 0.5 & 0.2 & 0.5 \\ 0.25 & 0.5 & -0.1 & 0.1 \end{bmatrix},\ C(1) = \begin{bmatrix} 0.1 & 0.25 & 0.1 & -0.1 \\ 0.25 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.25 & -0.2 & 0.5 \end{bmatrix}$$

$$B(0) = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.1 \\ 0.25 \end{bmatrix},\ B(1) = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.1 \\ 0.5 \end{bmatrix}$$

$$E_d(0) = \begin{bmatrix} 1.3 \\ 1.8 \\ 1.6 \\ 0.32 \end{bmatrix},\ E_d(1) = \begin{bmatrix} 3.2 \\ 2 \\ -1 \\ -2 \end{bmatrix}$$

$$E_f(0) = \begin{bmatrix} 0.1 \\ -1 \\ 0.2 \\ 0.1 \end{bmatrix},\ E_f(1) = \begin{bmatrix} 0.1 \\ -1 \\ 0.2 \\ 0.1 \end{bmatrix}$$

$$D(0) = O,\ D(1) = O$$

$$F_d(0) = O,\ F_d(1) = O,\ F_f(0) = O,\ F_f(1) = O \tag{10.28}$$

Let $s = 1$. We obtain the matrices $H_{o,s}(k), H_{u,s}(k), H_{d,s}(k)$,$H_{f,s}(k)$ by (10.2). As for any k,

$$rank \begin{bmatrix} H_{o,s}(k)\ H_{d,s}(k)\ H_{f,s}(k) \end{bmatrix} = 6$$

$$rank \begin{bmatrix} H_{o,s}(k)\ \ H_{d,s}(k) \end{bmatrix} = 5$$

the condition (10.22) is satisfied. To decouple the residual from the unknown disturbances, we then solve (10.21) for $v_s(k)$, $k = 0, 1$, respectively. As a result, the periodic parity relation based full decoupling residual generator is

$$r(k) = v_s(k) \left(\begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} - H_{u,s}(k) \begin{bmatrix} u(k-1) \\ u(k) \end{bmatrix} \right) \tag{10.29}$$

where

$$v_s(0) = [0.3535\ 0.2589\ 0.1962\ -0.8290\ -0.1421\ 0.2491]$$
$$v_s(1) = [-0.0631\ -0.1348\ 0.0314\ 0.2316\ -0.5703\ 0.7733]$$

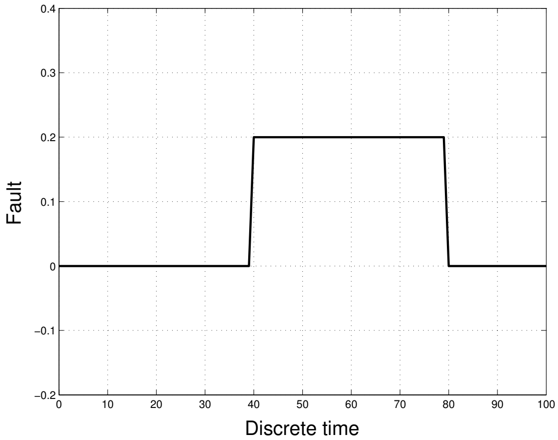

Fig. 10.1 Fault signal

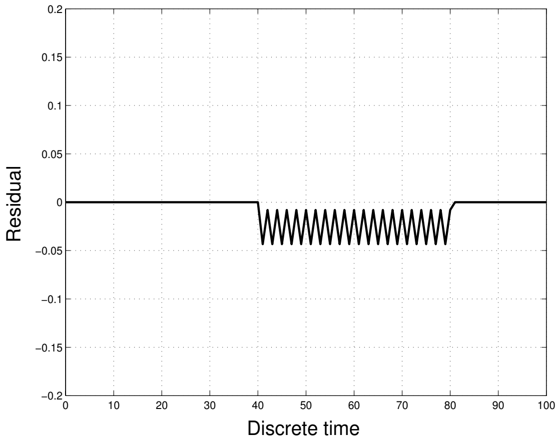

Fig. 10.2 Residual signal generated by periodic parity relation based residual generator (10.29)

In the simulation, it is assumed that the control input is a step signal (step time at 0) of amplitude 1, the disturbance $d(k) = \sin(0.01\pi k)$, and the fault appears at the $40th$ discrete time as illustrated in Fig. 10.1. The residual signal obtained by the residual generator (10.29) is shown in Fig. 10.2. It can be seen that the residual signal r is not influenced by u, d and changes only if $f \neq 0$. This means the residual generator (10.29) has achieved a full decoupling.

Now let $g(0) = -0.2$, $g(1) = -0.3$. From the periodic full decoupling parity vector got above, a periodic observer based full decoupling residual generator can be readily obtained as

$$\begin{aligned} z(k+1) &= G(k)z(k) + H(k)u(k) + L(k)y(k) \\ r(k) &= w(k)z(k) + q(k)u(k) + p(k)y(k) \end{aligned} \tag{10.30}$$

with

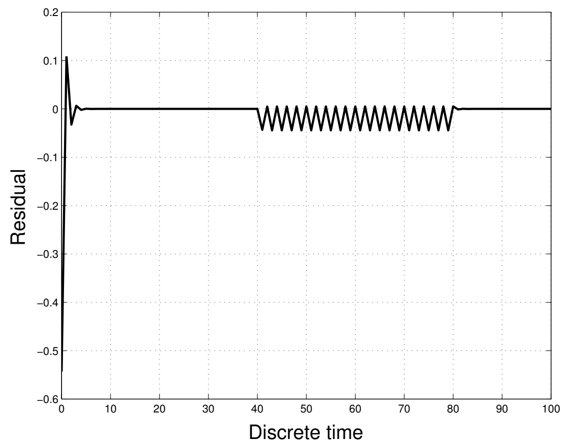

Fig. 10.3 Residual signal generated by periodic observer based residual generator (10.30)

$$
\begin{aligned}
G(0) &= -0.2,\ G(1) = -0.3 \\
H(0) &= 0.0504,\ H(1) = -0.1142 \\
L(0) &= \begin{bmatrix} -0.1027 & 0.1064 & 0.0184 \end{bmatrix} \\
L(1) &= \begin{bmatrix} -0.2840 & -0.4300 & 0.0358 \end{bmatrix} \\
p(0) &= \begin{bmatrix} -0.8290 & -0.1421 & 0.2491 \end{bmatrix} \\
p(1) &= \begin{bmatrix} 0.2316 & -0.5703 & 0.7733 \end{bmatrix} \\
w(0) &= -1,\ w(1) = -1 \\
q(0) &= 0,\ q(1) = 0
\end{aligned} \tag{10.31}
$$

It is worth noticing that the periodic observer (10.30) is only of first order. By changing the value of s, the order of the periodic observer could be adjusted. Under the same simulation conditions as before, the residual signal obtained by the periodic observer based residual generator (10.30) is presented in Fig. 10.3. As can be seen, the influence of initial estimation error disappears after several time points and the residual signal is not influenced by u, d. Hence, the residual generator (10.30) also allows a full decoupling and a reliable detection of the fault.

10.5 Optimization of residual generators

In this section, we shall discuss, if a full decoupling is not achievable, how to design the optimal residual generators for the periodic system (9.1). The main objective of the optimal design is to enhance the robustness of the FD system to the unknown disturbances without loss of the sensitivity to the faults.

In case that the full decoupling condition (10.22) is not satisfied, optimization problems similar to (5.15)-(5.17) are formulated as

$$\max_{\substack{V_s(k)\\ V_s(k)H_{o,s}(k)=0}} J_{LTP,PS,\infty/\infty} = \max_{\substack{V_s(k)\\ V_s(k)H_{o,s}(k)=0}} \frac{\bar{\sigma}^2(V_s(k)H_{f,s}(k))}{\bar{\sigma}^2(V_s(k)H_{d,s}(k))} \tag{10.32}$$

$$\max_{\substack{V_s(k)\\ V_s(k)H_{o,s}(k)=0}} J_{LTP,PS,-/\infty} = \max_{\substack{V_s(k)\\ V_s(k)H_{o,s}(k)=0}} \frac{\underline{\sigma}^2(V_s(k)H_{f,s}(k))}{\bar{\sigma}^2(V_s(k)H_{d,s}(k))} \tag{10.33}$$

$$\max_{\substack{V_s(k)\\ V_s(k)H_{o,s}(k)=0}} J_{LTP,PS,i/\infty} = \max_{\substack{V_s(k)\\ V_s(k)H_{o,s}(k)=0}} \frac{\sigma_i^2(V_s(k)H_{f,s}(k))}{\bar{\sigma}^2(V_s(k)H_{d,s}(k))} \tag{10.34}$$

which are solved over one period to get the optimal periodic parity matrix $V_s(k)$. Because the parity space approach handles each time instant independently and there is no stability problem, the solutions of problems (10.32)-(10.34) at each time instant are independent of each other and can be obtained following the procedures introduced in Section 5.1.

If the order s of the parity relation (10.1) is an integer multiple of the period T, then the periodic parity space approach is equivalent with a bank of residual generators (9.11). In comparison, the periodic parity space approach provides more flexibility. The order of the parity relation s doesn't need to be related to the period T. Moreover, s may take different values at different time instants. In this case, the threshold for the residual evaluation may need to be chosen differently at different time.

In the following, an approach is proposed to design the optimal periodic observer based residual generators (10.5) for the periodic system (9.1). The optimal design problem is formulated as [179]

$$\max_{L(k),W(k)} J_{LTP,OBS,\infty/\infty} = \max_{L(k),W(k)} \frac{\sup_{d=0,\ f\in l_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in l_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}} \tag{10.35}$$

$$\max_{L(k),W(k)} J_{LTP,OBS,-/\infty} = \max_{L(k),W(k)} \frac{\inf_{d=0,\ f\in l_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in l_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}} \tag{10.36}$$

The solutions of optimization problems (10.35)-(10.36) are derived by solving an equivalent optimization problem for the cyclically lifted LTI systems first and then recover the periodic matrices $L(k)$ and $W(k)$.

Theorem 10.6 Given the periodic system (9.1), assume that the associated sub-system $(A(k), E_d(k), C(k), F_d(k))$ is detectable and has no transmission zeros on the unit circle, no unreachable characteristic multipliers on the unit circle and no unobservable zero characteristic multipliers at any time. With $Q_d(k)$ an arbitrary periodic unitary matrix, $(X_d(k), L_d(k))$ the T-periodic stabilizing solution of the discrete-time periodic Riccati system (DPRS)

$$\begin{bmatrix} A(k)X_d(k)A^T(k) - X_d(k+1) + E_d(k)E_d^T(k) & A(k)X_d(k)C^T(k) + E_d(k)F_d^T(k) \\ C(k)X_d(k)A^T(k) + F_d(k)E_d^T(k) & C(k)X_d(k)C^T(k) + F_d(k)F_d^T(k) \end{bmatrix} \times \begin{bmatrix} I \\ L_d(k) \end{bmatrix} = 0, \tag{10.37}$$

$H_d(k)$ a T-periodic full column rank matrix satisfying

$$H_d(k)H_d^T(k) = C(k)X_d(k)C^T(k) + F_d(k)F_d^T(k), \tag{10.38}$$

and $W_d(k)$ a T-periodic matrix satisfying

$$W_d(k)H_d(k) = I, \tag{10.39}$$

the optimization problems (10.35) and (10.36) are simultaneously solved by

$$L_{opt}(k) = -L_d^T(k),\ W_{opt}(k) = Q_d(k)W_d(k). \tag{10.40}$$

Proof: The cyclic reformulation of the periodic system (10.7) is [8]

$$\begin{aligned} \bar{e}_\tau(k+1) &= (\bar{A}_\tau - \bar{L}_\tau\bar{C}_\tau)\bar{e}_\tau(k) + (\bar{E}_{d\tau} - \bar{L}_\tau\bar{F}_{d\tau})\bar{d}_\tau(k) + (\bar{E}_{f\tau} - \bar{L}_\tau\bar{F}_{f\tau})\bar{f}_\tau(k) \\ \bar{r}_\tau(k) &= \bar{W}_\tau(\bar{C}_\tau\bar{e}_\tau(k) + \bar{F}_{d\tau}\bar{d}_\tau(k) + \bar{F}_{f\tau}\bar{f}_\tau(k)) \end{aligned} \tag{10.41}$$

where the signals $\bar{e}_\tau, \bar{d}_\tau, \bar{f}_\tau, \bar{r}_\tau$ are defined as (8.9), the matrices $\bar{A}_\tau, \bar{E}_{d\tau}, \bar{E}_{f\tau}, \bar{L}_\tau, \bar{C}_\tau, \bar{W}_\tau, \bar{F}_{d\tau}, \bar{F}_{f\tau}$ defined as (8.11). According to [8], $\forall\tau$,

$$\sup_{f=0,d\in l_2-\{0\}} \frac{\|r\|_2}{\|d\|_2} = \left\|\bar{W}_\tau \bar{G}_{d\tau}(z)\right\|_\infty$$

$$\sup_{d=0,\ f\in l_2-\{0\}} \frac{\|r\|_2}{\|f\|_2} = \left\|\bar{W}_\tau \bar{G}_{f\tau}(z)\right\|_\infty \tag{10.42}$$

where

$$\bar{G}_{d\tau}(z) = \bar{F}_{d\tau} + \bar{C}_\tau(zI - \bar{A}_\tau + \bar{L}_\tau \bar{C}_\tau)^{-1} \left(\bar{E}_{d\tau} - \bar{L}_{d\tau}\bar{F}_{d\tau}\right)$$

$$\bar{G}_{f\tau}(z) = \bar{F}_{f\tau} + \bar{C}_\tau(zI - \bar{A}_\tau + \bar{L}_\tau \bar{C}_\tau)^{-1}(\bar{E}_{f\tau} - \bar{L}_{d\tau}\bar{F}_{f\tau}) \tag{10.43}$$

Thus, the optimization problems (10.35) and (10.36) are, respectively, equivalent to

$$\max_{L(k),W(k)} J_{LTP,OBS,\infty/\infty} = \max_{\bar{L}_\tau,\bar{W}_\tau} \frac{\left\|\bar{W}_\tau \bar{G}_{d\tau}(z)\right\|_\infty}{\left\|\bar{W}_\tau \bar{G}_{f\tau}(z)\right\|_\infty} \tag{10.44}$$

$$\max_{L(k),W(k)} J_{LTP,OBS,-/\infty} = \max_{\bar{L}_\tau,\bar{W}_\tau} \frac{\left\|\bar{W}_\tau \bar{G}_{d\tau}(z)\right\|_-}{\left\|\bar{W}_\tau \bar{G}_{f\tau}(z)\right\|_\infty} \tag{10.45}$$

Because the cyclic reformulation preserves the structural properties of periodic systems, ($\bar{A}_\tau$, $\bar{E}_{d\tau}$, $\bar{C}_\tau, \bar{F}_{d\tau}$) has no transmission zeros on the unit circle [25], $(\bar{A}_\tau, \bar{C}_\tau)$ is detectable and has no unobservable modes at the origin, $(\bar{A}_\tau, \bar{E}_{d\tau})$ has no unreachable modes on the unit circle [8]. In view of Theorem 5.11, the unified optimal solution to the optimization problems (10.44)-(10.45) is

$$\bar{L}_{\tau,opt} = -\bar{L}_{d\tau}^T,\ \bar{W}_{\tau,opt} = Q_{d\tau}\bar{W}_{d\tau} \tag{10.46}$$

where $Q_{d\tau}$ can be any constant unitary matrix. The matrix $\bar{W}_{d\tau}$ is determined by

$$\bar{W}_{d\tau}\bar{H}_{d\tau} = I, \quad \bar{H}_{d\tau}\bar{H}_{d\tau}^T = \bar{C}_\tau \bar{X}_{d\tau}\bar{C}_\tau^T + \bar{F}_{d\tau}\bar{F}_{d\tau}^T \tag{10.47}$$

and $(\bar{X}_{d\tau}, \bar{L}_{d\tau})$ is the stabilizing solution to the DTARS

$$\begin{bmatrix} \bar{A}_\tau \bar{X}_{d\tau}\bar{A}_\tau^T - \bar{X}_{d\tau} + \bar{E}_{d\tau}\bar{E}_{d\tau}^T & \bar{A}_\tau \bar{X}_{d\tau}\bar{C}_\tau^T + \bar{E}_{d\tau}\bar{F}_{d\tau}^T \\ \bar{C}_\tau \bar{X}_{d\tau}\bar{A}_\tau^T + \bar{F}_{d\tau}\bar{E}_{d\tau}^T & \bar{C}_\tau \bar{X}_{d\tau}\bar{C}_\tau^T + \bar{F}_{d\tau}\bar{F}_{d\tau}^T \end{bmatrix} \begin{bmatrix} I \\ \bar{L}_{d\tau} \end{bmatrix} = 0 \tag{10.48}$$

Due to the special structure of matrices $\bar{A}_\tau, \bar{C}_\tau, \bar{E}_{d\tau}, \bar{F}_{d\tau}$, it is clear that matrices $\bar{X}_{d\tau}$ and $\bar{L}_{d\tau}$ in the equation (10.48) should have the following structure

$$\bar{X}_{d\tau} = diag\{X_d(\tau), \cdots, X_d(\tau+T-2), X_d(\tau+T-1)\} \tag{10.49}$$

$$\bar{L}_{d\tau} = \begin{bmatrix} O & L_d(\tau) \cdots & O \\ \vdots & \vdots \ \ddots & \vdots \\ O & O \ \cdots & L_d(\tau+T-2) \\ L_d(\tau+T-1) & O \ \cdots & O \end{bmatrix} \tag{10.50}$$

By selecting $Q_{d\tau}$ to have a block diagonal structure and substituting (8.11), (10.49), (10.50) into (10.46)-(10.48), it turns out that the optimal periodic matrices $L_{opt}(k)$, $W_{opt}(k)$ are given by (10.40)-(10.37). □

It should be emphasized that besides the observer gain matrices $L(k)$, the theorem provides also an analytic way to the optimal selection of the weighting matrices $W(k)$, which also play an important role in improving the performance index $J_{LTP,OBS}(L(k), W(k))$. The assumptions of the theorem can be verified by checking the corresponding structural properties of the cyclic reformulation or the lifted reformulation of $(A(k), E_d(k), C(k), F_d(k))$, due to the correspondence between the structural properties of periodic systems and cyclic/lifted reformulations.

Based on the theorem, the optimal design of the periodic residual generator (10.5) for the periodic system (9.1) can be carried out as follows:

- solve the DPRS (10.37) for the T-periodic stabilizing solution $(X_d(k), L_d(k))$,
- find the periodic full column rank matrices $H_d(k)$ satisfying (10.38),
- determine $W_d(k)$ according to (10.39),
- get the optimal observer gain matrices $L_{opt}(k)$ and weighting matrices $W_{opt}(k)$ by (10.40),
- construct the periodic residual generator according to (10.5).

The main computation of the optimal design consists in finding the T-periodic stabilizing solution $(X_d(k), L_d(k))$ of the DPRS (10.37). As will be shown in the next section, it is finally reduced to solving one DTARS.

The dynamics of the residual delivered by (10.5) with the parameters (10.40) is governed by

$$\begin{aligned} e_{opt}(k+1) &= A_{L_{opt}}(k)e_{opt}(k) + E_{d,L_{opt}}(k)d(k) + E_{f,L_{opt}}(k)f(k) \\ r_{opt}(k) &= W_{opt}(k)(C(k)e_{opt}(k) + F_{d,L_{opt}}(k)d(k) + F_{f,L_{opt}}(k)f(k)) \\ A_{L_{opt}}(k) &= A(k) - L_{opt}(k)C(k) \\ E_{d,L_{opt}}(k) &= E_d(k) - L_{opt}(k)F_d(k) \\ E_{f,L_{opt}}(k) &= E_f(k) - L_{opt}(k)F_f(k) \end{aligned} \tag{10.51}$$

It represents the best compromise between the sensitivity to the faults and the robustness to the disturbances in the sense of (10.35) and (10.36).

The optimal residual generator can also be realized in the form of an observer followed by a periodic post filter as

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) + L(k)(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C(k)\hat{x}(k) + D(k)u(k) \\ x_R(k+1) &= A_R(k)x_R(k) + B_R(k)(y(k) - \hat{y}(k)) \\ r(k) &= C_R(k)x_R(k) + D_R(k)(y(k) - \hat{y}(k)) \end{aligned} \tag{10.52}$$

where $L(k)$ is any periodic gain matrix that stabilizes $A(k) - L(k)C(k)$, the parameters of the periodic post filter are given by

$$\begin{aligned} A_R(k) &= A(k) + L_d^T(k)C(k) \\ B_R(k) &= L_d^T(k) + L(k) \\ C_R(k) &= Q_d(k)W_d(k)C(k) \\ D_R(k) &= Q_d(k)W_d(k) \end{aligned} \tag{10.53}$$

Though the order of the residual generator (10.52) is increased, its dynamics is the same with (10.51).

10.6 Discrete-time periodic Riccati system (DPRS)

As shown in Theorem 10.6, the discrete-time periodic Riccati system (10.37) plays an important role for the optimal design. This section will give two approaches to solve the DPRS.

As can be seen from the proof of Theorem 10.6, one approach is to compute the stabilizing solution $(\bar{X}_{d\tau}, \bar{L}_{d\tau})$ of the DTARS and then partition $\bar{X}_{d\tau}, \bar{L}_{d\tau}$ according to (10.49)-(10.50).

- Build the cyclic reformulation $(\bar{A}_\tau, \bar{E}_{d\tau}, \bar{C}_\tau, \bar{F}_{d\tau})$ of the periodic system $(A(k), E_d(k), C(k), F_d(k))$ for some initial time τ [8].
- Find the stabilizing solution $(\bar{X}_{d\tau}, \bar{L}_{d\tau})$ to the DTARS (10.48). To this aim, a numerically sound algorithm is available (see [80] and the references therein). The existence of the solution is guaranteed if the assumptions of Theorem 10.6 are satisfied. In many cases, (10.48) reduces to the well-known DTARE which can be solved by the Matlab function *dare*.
- Partition $\bar{X}_{d\tau}, \bar{L}_{d\tau}$ according to (10.49)-(10.50) to get $X_d(k), L_d(k)$ for $k = \tau, \tau+1, \cdots, \tau+T-1$.

Let $(X_d(k), L_d(k))$ be the stabilizing solution of (10.37). Assume that $(\tilde{A}_\epsilon, \tilde{E}_{d\epsilon}, \tilde{C}_\epsilon, \tilde{F}_{d\epsilon})$ is the lifted reformulation of the periodic system $(A(k), E_d(k), C(k), F_d(k))$ for some initial time ϵ. Let

$$\tilde{X}_d = X_d(\epsilon),\ \tilde{L}_d = \begin{bmatrix} \tilde{L}_{d,0} \\ \tilde{L}_{d,1} \\ \vdots \\ \tilde{L}_{d,T-1} \end{bmatrix} \tag{10.54}$$

where

$$\begin{aligned}
\tilde{L}_{d,T-1} &= L(\epsilon + T - 1) \\
\tilde{L}_{d,T-2} &= L(\epsilon + T - 2)(A^T(\epsilon + T - 1) + C^T(\epsilon + T - 1)\tilde{L}_{d,T-1}) \\
&\vdots \\
\tilde{L}_{d,i} &= L(\epsilon + i)\left(\Phi^T(\epsilon + T, k + i + 1) + \right. \\
&\left. \begin{bmatrix} C(\epsilon + i + 1) \\ C(\epsilon + i + 2)\Phi^T(\epsilon + i + 2, \epsilon + i + 1) \\ \vdots \\ C(\epsilon + T - 1)\Phi^T(\epsilon + T - 1, \epsilon + i + 1) \end{bmatrix}^T \begin{bmatrix} \tilde{L}_{d,i+1}(k) \\ \tilde{L}_{d,i+2}(k) \\ \vdots \\ \tilde{L}_{d,T-1}(k) \end{bmatrix} \right) \\
i &= 0, 1, \cdots, T - 2
\end{aligned}$$

Then it can be proven that $(\tilde{X}_d, \tilde{L}_d)$ given by (10.54) is the stabilizing solution to the DTARS

$$\begin{bmatrix} \tilde{A}_\epsilon \tilde{X}_d \tilde{A}_\epsilon^T - \tilde{X}_d + \tilde{E}_{d\epsilon}\tilde{E}_{d\epsilon}^T & \tilde{A}_\epsilon \tilde{X}_d \tilde{C}_\epsilon^T + \tilde{E}_{d\epsilon}\tilde{F}_{d\epsilon}^T \\ \tilde{C}_\epsilon \tilde{X}_d \tilde{A}_\epsilon^T + \tilde{F}_{d\epsilon}\tilde{E}_{d\epsilon}^T & \tilde{C}_\epsilon \tilde{X}_d \tilde{C}_\epsilon^T + \tilde{F}_{d\epsilon}\tilde{F}_{d\epsilon}^T \end{bmatrix} \begin{bmatrix} I \\ \tilde{L}_d \end{bmatrix} = 0 \tag{10.55}$$

Therefore, (10.37) can also be solved as follows.

- Build the lifted reformulation $(\tilde{A}_\epsilon, \tilde{E}_{d\epsilon}, \tilde{C}_\epsilon, \tilde{F}_{d\epsilon})$ of the periodic system $(A(k), E_d(k), C(k), F_d(k))$ for some initial time ϵ [8, 12].
- Find the stabilizing solution $(\tilde{X}_d, \tilde{L}_d)$ to the DTARS

$$\begin{bmatrix} \tilde{A}_\epsilon \tilde{X}_d \tilde{A}_\epsilon^T - \tilde{X}_d + \tilde{E}_{d\epsilon}\tilde{E}_{d\epsilon}^T & \tilde{A}_\epsilon \tilde{X}_d \tilde{C}_\epsilon^T + \tilde{E}_{d\epsilon}\tilde{F}_{d\epsilon}^T \\ \tilde{C}_\epsilon \tilde{X}_d \tilde{A}_\epsilon^T + \tilde{F}_{d\epsilon}\tilde{E}_{d\epsilon}^T & \tilde{C}_\epsilon \tilde{X}_d \tilde{C}_\epsilon^T + \tilde{F}_{d\epsilon}\tilde{F}_{d\epsilon}^T \end{bmatrix} \begin{bmatrix} I \\ \tilde{L}_d \end{bmatrix} = 0 \tag{10.56}$$

- Set $X_d(\epsilon) = \tilde{X}_d$ and substitute $X_d(\epsilon)$ into (10.37) to compute $X_d(k), L_d(k)$ for $k = \epsilon, \epsilon+1, \cdots, \epsilon+T-1$.

In our experience, the lifted reformulation based solution procedure shows better numerical property than the cyclic reformulation based solution procedure.

To illustrate the proposed optimal observer based design approach, we consider the following example.

Example 10.2 Design the optimal residual generator for the periodic system (9.1) with period $T = 3$ and

$$A(k) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_{1k} & a_{2k} & a_{3k} & a_{4k} & a_{5k} \end{bmatrix}, \ B(k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_d(k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_{2k} \\ b_{1k} & 0 \end{bmatrix}, \ E_f(k) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ g_{1k} \end{bmatrix}$$

$$C(k) = \begin{bmatrix} 0 & 0 & 0 & c_{1k} & c_{2k} \\ 0.5 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{10.57}$$

where

$$\begin{aligned} a_{1k} &= \sin(2\pi(k+T)/T + 0.9), \ a_{2k} = \sin(2\pi(k+T)/T + 0.7) \\ a_{3k} &= 2\sin(2\pi(k+T)/T - 0.5), \ a_{4k} = \sin(2\pi(k+T)/T - 0.1) \\ a_{5k} &= \sin(2\pi(k+T)/T + 0.3), \ b_{1k} = \cos(2\pi(k+T)/T) \\ b_{2k} &= 2\cos(2\pi(k+T)/T + 0.5)\,, \ c_{1k} = 2\cos(2\pi(k+T)/T + 0.2) \\ c_{2k} &= \cos(2\pi(k+T)/T + 0.1), \ g_{1k} = 0.5\sin(2\pi(k+T)/T + 0.4) \end{aligned}$$

The system (10.57) is an extension of the model used in [89] and satisfies the assumptions of Theorem 10.6. Applying the optimal design procedure yields

$$L_{opt}(0) = \begin{bmatrix} -0.19 & 1.08 \\ -0.25 & -1.58 \\ 0.40 & 2.44 \\ 0.22 & -4.80 \\ 0.15 & 2.11 \end{bmatrix}, \ W_{opt}(0) = \begin{bmatrix} 0.91 & 0 \\ -0.06 & 43.27 \end{bmatrix}$$

$$L_{opt}(1) = \begin{bmatrix} -0.02 & -0.39 \\ 0.03 & 0.62 \\ -0.71 & -0.59 \\ -0.10 & 1.33 \\ -0.66 & 1.76 \end{bmatrix}, \ W_{opt}(1) = \begin{bmatrix} 0.40 & 0 \\ -0.02 & 6.86 \end{bmatrix}$$

$$L_{opt}(2) = \begin{bmatrix} -0.00 & -3.00 \\ 0.18 & -0.18 \\ -1.46 & 3.27 \\ -0.17 & -5.05 \\ 1.17 & 3.56 \end{bmatrix}, \ W_{opt}(2) = \begin{bmatrix} 0.74 & 0 \\ -0.13 & 11.84 \end{bmatrix}$$

To get a better understanding of the optimal solution, the above residual generator is compared with

- a residual generator which solves the optimization problem

$$\min_{L(k)} \sup_{f=0, d \in l_2 - \{0\}} \frac{\|y - \hat{y}\|_2}{\|d\|_2}$$

  and is optimally robust against the unknown disturbances. The observer gain matrices are obtained by iteratively solving a set of linear matrix inequalities [13], which finally yields

$$L_{\inf}(0) = \begin{bmatrix} -0.21 & 1.14 \\ -0.25 & -1.59 \\ 0.39 & 2.44 \\ 0.23 & -4.81 \\ 0.14 & 2.16 \end{bmatrix}, \; L_{\inf}(1) = \begin{bmatrix} -0.02 & -0.35 \\ 0.03 & 0.56 \\ -0.71 & -0.58 \\ -0.10 & 1.32 \\ -0.66 & 1.66 \end{bmatrix}$$

$$L_{\inf}(2) = \begin{bmatrix} -0.00 & -2.99 \\ 0.19 & -0.21 \\ -1.46 & 3.24 \\ -0.18 & -5.01 \\ 1.16 & 3.56 \end{bmatrix}$$

- and a residual generator which assigns the characteristic multipliers of the residual dynamics at 0.35, 0.45, 0.6, 0.7, 0.8. It is designed by using the algorithm introduced in Section 8.5 and the observer gain matrices are

$$L_{place}(0) = \begin{bmatrix} -0.04 & 1.04 \\ -0.09 & -1.13 \\ 0.39 & 2.51 \\ -0.16 & -5.69 \\ 0.89 & -2.18 \end{bmatrix}, \; L_{place}(1) = \begin{bmatrix} -0.15 & 0.17 \\ 0.41 & -0.04 \\ 0.11 & -5.87 \\ -1.54 & 8.55 \\ -0.85 & 4.35 \end{bmatrix}$$

$$L_{place}(2) = \begin{bmatrix} 0.04 & -0.43 \\ 0.65 & -0.10 \\ -1.25 & -1.67 \\ -0.55 & 0.30 \\ 0.59 & -1.47 \end{bmatrix}$$

As the selection of weighting matrices $W(k)$ can not be incorporated in the latter two designs, the weighting matrices are simply chosen as $W(k) = W_I(k) = I$.

The performance indices $J_{LTP,OBS,\infty/\infty}(L(k), W(k))$ achieved by each approach are, respectively,

$$J_{LTP,OBS,\infty/\infty}(L_{opt}(k), W_{opt}(k)) = 0.9924$$
$$J_{LTP,OBS,\infty/\infty}(L_{\inf}(k), W_I(k)) = 0.3124$$
$$J_{LTP,OBS,\infty/\infty}(L_{place}(k), W_I(k)) = 0.1589$$

It can be seen that the performance index $J_{LTP,OBS,\infty/\infty}(L_{opt}(k), W_{opt}(k))$ got by the optimal design is much bigger than the others.

During the simulation, assume that the control inputs are step signals (step time at 0) of amplitude 1 and 0.5 respectively, the unknown disturbances are discrete-time signals taking value randomly from a normal distribution with mean 0 and standard deviation 0.01, the fault appears at the 601th discrete time as a step function of amplitude 0.06. The residual $r = [r_1 \; r_2]^T$ is evaluated by $\|r\|_{ev} = \left(\sum_{i=k-\rho+1}^{k} r^T(i) r(i) \right)^{\frac{1}{2}}$ with $\rho = 100$ being the length of the evaluation window. A fault is detected if $\|r\|_{ev}$ surpasses threshold $J_{th} = \sup_{f=0} \|r\|_{ev}$. Fig. 10.4 shows both the residual signals and the evaluation results. The fault was detected by the optimal residual generator at the 608th discrete time, while it was detected by the other two residual generators at the 613th and the 623th discrete time, respectively. It demonstrates that the proposed optimal design achieves a quicker fault detection and thus a better FD performance.

10.7 Conclusion

Direct approaches to design the FD systems for the linear discrete periodic (LDP) systems are addressed in this chapter. Because the parity space approach handles each time instant independently,

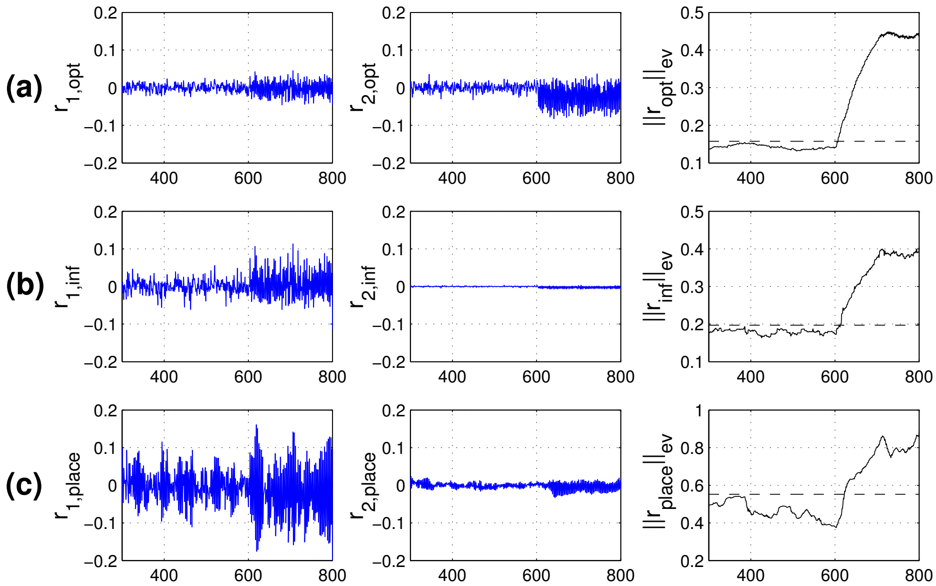

Fig. 10.4 The residual signal (r_1: left column, r_2: middle column) and the evaluated residual signal (right column, $\|r\|_{ev}$: solid line, J_{th}: dashed line) got by residual generator with parameters (a) $L_{opt}(k), W_{opt}(k)$, (b) $L_{\inf}(k), W_I(k)$, (c) $L_{place}(k)\,, W_I(k)$.

both the full decoupling problem and the optimal design problem can be rather easily solved by the periodic parity space approach. It needs only to solve a set of *independent* linear algebraic equations or optimization problems. Due to the close relationship between the periodic parity vectors and the periodic observer based residual generators, a periodic observer based full decoupling residual generator can be readily obtained. The optimal design of the periodic observer based residual generators is formulated as ratio-type optimization problems, whose solution is shown to be closely related to a difference periodic Riccati system. Readers are referred to [150, 110] for other full decoupling approaches and [54] for a fault estimation algorithm.

11

Uncertain periodic systems

In this chapter, we consider the FD problem of uncertain periodic systems described by [191]

$$\begin{aligned} x(k+1) &= A_{\triangle}(k)x(k) + B_{\Delta}(k)u(k) + E_{d\Delta}(k)d(k) + E_{f\Delta}(k)f(k) \\ y(k) &= C(k)x(k) + D(k)u(k) + F_d(k)d(k) + F_f(k)f(k) \end{aligned} \tag{11.1}$$

with

$$\begin{aligned} &[\, A_{\triangle}(k)\ B_{\Delta}(k)\ E_{d\Delta}(k)\ E_{f\Delta}(k) \,] \\ &= [\, A(k)\ B(k)\ E_d(k)\ E_f(k) \,] + [\, \Delta A(k)\ \Delta B(k)\ \Delta E_d(k)\ \Delta E_f(k) \,] \end{aligned} \tag{11.2}$$

where $A(k), B(k), C(k), D(k), E_d(k), E_f(k), F_d(k), F_f(k)$ are known T-periodic real matrices of appropriate dimensions, $\Delta A(k), \Delta B(k), \Delta E_d(k), \Delta E_f(k)$ are unknown real matrices representing polytopic uncertainties described by

$$\begin{aligned} &\left[\, \Delta A(k)\ \Delta B(k)\ \Delta E_d(k)\ \Delta E_f(k) \,\right] \\ &= \sum_{i=1}^{p} \lambda_i(k) \left[\, A_i(k)\ B_i(k)\ E_{di}(k)\ E_{fi}(k) \,\right] \end{aligned} \tag{11.3}$$

where $A_i(k), B_i(k), E_{di}(k), E_{fi}(k)$, $i = 1, 2, \cdots, p$, are known T-periodic real matrices, $\lambda_i(k)$, $i = 1, 2, \cdots, p$ are unknown quantities but satisfy

$$\lambda_i(k) \geqslant 0, \quad \sum_{i=1}^{p} \lambda_i(k) = 1$$

In the subsequent discussion, it is assumed that $A_{\triangle}(k)$ is stable, $(A(k), C(k))$ is detectable. For the sake of brevity, it is assumed that system (11.1) has no uncertainties in the output equation. This assumption can be easily removed by a straightforward extension.

The main purpose is to design an optimal residual generator for uncertain periodic systems (11.1), which is sensitive to the faults and robust to the unknown disturbances as well as the model uncertainties.

11.1 Problem formulation

To the aim of FD, a periodic post filter based residual generator is constructed as

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) + L(k)r_b(k) \\ \hat{y}(k) &= C(k)\hat{x}(k) + D(k)u(k) \\ r_b(k) &= y(k) - \hat{y}(k) \\ x_R(k+1) &= A_R(k)x_R(k) + B_R(k)r_b(k) \\ r(k) &= C_R(k)x_R(k) + D_R(k)r_b(k) \end{aligned} \tag{11.4}$$

where r_b represents the output estimation error, $r \in \mathbf{R}^{n_r}$ is the residual signal, $x_R \in \mathbf{R}^{n_R}$ is the state vector of the periodic post-filter, $L(k)$ is the T-periodic observer gain matrix, $A_R(k)$, $B_R(k), C_R(k), D_R(k)$ are T-periodic real matrices.

Recall that in the nominal case the residual dynamics (10.51) represents the best compromise between the sensitivity to the faults and the robustness to the disturbances in the sense of (10.35) and (10.36). To keep the optimality in the presence of the uncertainties (11.3), the design parameters of the residual generator (11.4) can be selected in such a way that the residual r obtained by the periodic residual generator (11.4) is in the near of $r_{opt}(k)$. Let

$$\xi(k) = r(k) - r_{opt}(k)$$

which represents the difference between the residual r generated by (11.4) and the optimal residual dynamics r_{opt}. Because the dynamics of $L(k)$ can always be compensated by the post filter, $L(k)$ can be arbitrarily selected, as long as $A(k) - L(k)C(k)$ is stable. Thus, the FD problem of the uncertain periodic systems (11.1) is formulated as to determine $L(k)$ and $A_R(k), B_R(k), C_R(k), D_R(k)$, such that [191]

$$r(k) \to r_{opt}(k), \text{ i.e. } \xi(k) \to 0$$

11.2 Design of the optimal periodic post-filter

In this section, we shall show how to formulate the FD problem as an H_∞ problem and, based on it, derive the optimal periodic post filter.

Define

$$X_p(k) = \begin{bmatrix} x(k) \\ e(k) \\ e_{opt}(k) \end{bmatrix}, \ e(k) = x(k) - \hat{x}(k)$$
$$\delta(k) = \begin{bmatrix} f(k) \\ d(k) \\ u(k) \end{bmatrix}$$

where $e(k)$ is the state estimation error, $X_p(k)$ the extended state vector and $\delta(k)$ the extended disturbance vector. Then, from (11.1), (11.4), and (10.51) we get

$$\begin{aligned} X_p(k+1) &= A_p(k)X_p(k) + B_p(k)\delta(k) \\ \xi(k) &= C_{1p}(k)X_p(k) + D_{1p}(k)\delta(k) + r(k) \\ r_b(k) &= C_{2p}(k)X_p(k) + D_{2p}(k)\delta(k) \end{aligned} \tag{11.5}$$

where

$$A_p(k) = A_{po}(k) + \varDelta A_p(k),\ B_p(k) = B_{po}(k) + \varDelta B_p(k)$$
$$A_{po}(k) = \begin{bmatrix} A(k) & O & O \\ O & A_L(k) & O \\ O & O & A_{L_{opt}}(k) \end{bmatrix}, \varDelta A_p(k) = \begin{bmatrix} \varDelta A(k) & O & O \\ \varDelta A(k) & O & O \\ O & O & O \end{bmatrix}$$
$$B_{po}(k) = \begin{bmatrix} E_f(k) & E_d(k) & B(k) \\ E_{f,L}(k) & E_{d,L}(k) & O \\ E_{f,L_{opt}}(k) & E_{d,L_{opt}}(k) & O \end{bmatrix}$$
$$\varDelta B_p(k) = \begin{bmatrix} \varDelta E_f(k) & \varDelta E_d(k) & \varDelta B(k) \\ \varDelta E_f(k) & \varDelta E_d(k) & \varDelta B(k) \\ O & O & O \end{bmatrix}$$
$$C_{1p}(k) = \begin{bmatrix} O & O & -W_{opt}(k)C(k) \end{bmatrix}$$
$$D_{1p}(k) = \begin{bmatrix} -W_{opt}(k)F_f(k) & -W_{opt}(k)F_d(k) & O \end{bmatrix}$$
$$C_{2p}(k) = \begin{bmatrix} O & C(k) & O \end{bmatrix}$$
$$D_{2p}(k) = \begin{bmatrix} F_f(k) & F_d(k) & O \end{bmatrix}$$
$$A_L(k) = A(k) - L(k)C(k)$$
$$E_{d,L}(k) = E_d(k) - L(k)F_d(k), \quad E_{f,L}(k) = E_f(k) - L(k)F_f(k)$$

It can be seen that the error dynamics is internally stable, as long as system (11.1) is stable and $L(k)$ is so selected that the characteristic multipliers of $A(k) - L(k)C(k)$ belong to the open unit disk.

Denote

$$X_e(k) = \begin{bmatrix} X_p(k) \\ x_R(k) \end{bmatrix},\ \varTheta_R(k) = \begin{bmatrix} A_R(k) & B_R(k) \\ C_R(k) & D_R(k) \end{bmatrix}$$

The dynamics of the whole system is governed by

$$\begin{aligned} X_e(k+1) &= A_e(k)X_e(k) + B_e(k)\delta(k) \\ \xi(k) &= C_e(k)X_e(k) + D_e(k)\delta(k) \end{aligned} \tag{11.6}$$

where

$$\begin{aligned} A_e(k) &= A_{eo}(k) + \varDelta A_e(k) + \tilde{B}\varTheta_R(k)\tilde{C}(k) \\ B_e(k) &= B_{eo}(k) + \varDelta B_e(k) + \tilde{B}\varTheta_R(k)\tilde{D}_{21}(k) \\ C_e(k) &= C_{eo}(k) + \tilde{D}_{12}\varTheta_R(k)\tilde{C}(k) \\ D_e(k) &= D_{1p}(k) + \tilde{D}_{12}\varTheta_R(k)\tilde{D}_{21}(k) \\ A_{eo}(k) &= \begin{bmatrix} A_{po}(k) & O \\ O & O \end{bmatrix},\ \varDelta A_e(k) = \begin{bmatrix} \varDelta A_p(k) & O \\ O & O \end{bmatrix} \\ B_{eo}(k) &= \begin{bmatrix} B_{po}(k) \\ O \end{bmatrix},\ \varDelta B_e(k) = \begin{bmatrix} \varDelta B_p(k) \\ O \end{bmatrix} \\ C_{eo}(k) &= \begin{bmatrix} C_{1p}(k) & O \end{bmatrix} \\ \tilde{B} &= \begin{bmatrix} O & O \\ I & O \end{bmatrix},\ \tilde{C}(k) = \begin{bmatrix} O & I \\ C_{2p}(k) & O \end{bmatrix} \\ \tilde{D}_{21}(k) &= \begin{bmatrix} O \\ D_{2p}(k) \end{bmatrix},\ \tilde{D}_{12} = \begin{bmatrix} O & I \end{bmatrix} \end{aligned} \tag{11.7}$$

The optimal design problem can thus be reformulated as

$$\min_{\varTheta_R(k)} \alpha \tag{11.8}$$

$$\sup_{\delta \in l_2, \delta \neq o} \frac{\|\xi\|_2}{\|\delta\|_2} < \alpha \tag{11.9}$$

According to Theorem 8.1, (11.9) holds if and only if there exists a T-periodic symmetric positive-definite matrix $P_e(k)$ (i.e. $\forall k$, $P_e(k) = P_e^T(k) > 0$, $P_e(k) = P_e(k+T)$) such that for $k = 0, 1, \cdots, T-1$,

$$\begin{bmatrix} -P_e^{-1}(k+1) & O & A_e(k) & B_e(k) \\ O & -I & C_e(k) & D_e(k) \\ A_e^T(k) & C_e^T(k) & -P_e(k) & O \\ B_e^T(k) & D_e^T(k) & O & -\alpha^2 I \end{bmatrix} < 0 \tag{11.10}$$

The uncertainty $\Delta A_e(k)$ and $\Delta B_e(k)$ can be expressed as

$$\Delta A_e(k) = \sum_{i=1}^{p} \lambda_i(k) A_{ei}(k),\ \Delta B_e(k) = \sum_{i=1}^{p} \lambda_i(k) B_{ei}(k)$$

$$A_{ei}(k) = \begin{bmatrix} A_i(k) & O & O & O \\ A_i(k) & O & O & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix}$$

$$B_{ei}(k) = \begin{bmatrix} E_{fi}(k) & E_{di}(k) & B_i(k) \\ E_{fi}(k) & E_{di}(k) & B_i(k) \\ O & O & O \\ O & O & O \end{bmatrix}$$

$$i = 1, 2, \cdots, p$$

By taking (11.7) into account, (11.10) holds if and only if there exists a T-periodic matrix $P_e(k) = P_e^T(k) > 0$, such that for $i = 1, 2, \cdots, p$ and $k = 0, 1, \cdots, T-1$, the

$$\begin{bmatrix} -P_e^{-1}(k+1) & O & \Psi_{13}(k) & \Psi_{14}(k) \\ O & -I & C_e(k) & D_e(k) \\ \Psi_{13}^T(k) & C_e^T(k) & -P_e(k) & O \\ \Psi_{14}^T(k) & D_e^T(k) & O & -\alpha^2 I \end{bmatrix} < 0$$

$$\Psi_{13}(k) = A_{eo}(k) + A_{ei}(k) + \tilde{B}\Theta_R(k)\tilde{C}(k)$$

$$\Psi_{14}(k) = B_{eo}(k) + B_{ei}(k) + \tilde{B}\Theta_R(k)\tilde{D}_{21}(k) \tag{11.11}$$

Applying the Schur Lemma, (11.11) can be re-written into

$$(\Pi_i(k) + \Gamma(k)\Theta_R(k)\Lambda(k))^T\, \Omega(k+1)\, (\Pi_i(k) + \Gamma(k)\Theta_R(k)\Lambda(k)) < \Theta(k)$$

$$i = 1, 2, \cdots, p;\ k = 0, 1, \cdots, T-1 \tag{11.12}$$

where

$$\Pi_i(k) = \begin{bmatrix} A_{eo}(k) + A_{ei}(k) & B_{eo}(k) + B_{ei}(k) \\ C_{eo}(k) & D_{1p}(k) \end{bmatrix}$$

$$\Gamma(k) = \begin{bmatrix} \tilde{B} \\ \tilde{D}_{12} \end{bmatrix},\ \Lambda(k) = \begin{bmatrix} \tilde{C}(k) & \tilde{D}_{21}(k) \end{bmatrix}$$

$$\Omega(k+1) = \begin{bmatrix} P_e(k+1) & O \\ O & I \end{bmatrix},\ \Theta(k) = \begin{bmatrix} P_e(k) & O \\ O & \alpha^2 I \end{bmatrix}$$

There exists a matrix $\Theta_R(k)$ so that (11.12) holds, if and only if [142]

$$\left(\Lambda^T(k)\right)^{\perp} (\Theta(k) - \Pi_i^T(k)\Omega(k+1)\Pi_i(k)) \left(\left(\Lambda^T(k)\right)^{\perp}\right)^T > 0$$

$$\Gamma^{\perp}(k)(\Omega^{-1}(k+1) - \Pi_i(k)\Theta^{-1}(k)\Pi_i^T(k)) \left(\Gamma^{\perp}(k)\right)^T > 0$$

$$i = 1, 2, \cdots, p, \quad k = 0, 1, \cdots, T-1 \tag{11.13}$$

where the superscript $\perp$ denotes the base of the left null space of the matrix.

Assume

$$P_e(k) = \begin{bmatrix} S(k) & Y(k) \\ Y^T(k) & \hat{S}(k) \end{bmatrix}, P_e^{-1}(k) = \begin{bmatrix} Q(k) & Z(k) \\ Z^T(k) & \hat{Q}(k) \end{bmatrix} \quad (11.14)$$

By substituting (11.7), (11.13) turns out to be

$$Q(k+1) - \tilde{A}_i(k)Q(k)\tilde{A}_i^T(k) - \alpha^{-2}\tilde{B}_i(k)\tilde{B}_i^T(k) > 0 \quad (11.15)$$

$$\begin{bmatrix} C_{2p}^T(k) \\ D_{2p}^T(k) \end{bmatrix}^{\perp} \begin{bmatrix} \digamma_{11}(k) & \digamma_{12}(k) \\ \digamma_{12}^T(k) & \digamma_{22}(k) \end{bmatrix} \begin{bmatrix} C_{2p}^T(k) \\ D_{2p}^T(k) \end{bmatrix}^{\perp T} < 0 \quad (11.16)$$

where

$$\begin{aligned}
\digamma_{11}(k) &= \tilde{A}_i^T(k)S(k+1)\tilde{A}_i(k) - S(k) + C_{1p}^T(k)C_{1p}(k) \\
\digamma_{12}(k) &= \tilde{A}_i^T(k)S(k+1)\ \tilde{B}_i(k) + C_{1p}^T(k)D_{1p}(k) \\
\digamma_{22}(k) &= \tilde{B}_i^T(k)S(k+1)\tilde{B}_i(k) - \alpha^2 I + D_{1p}^T(k)D_{1p}(k) \\
\tilde{A}_i(k) &= A_{po}(k) + A_{pi}(k),\ \tilde{B}_i(k) = B_{po}(k) + B_{pi}(k)
\end{aligned} \quad (11.17)$$

It is known that (11.14) holds, if and only if [200]

$$S(k) \geqslant Q^{-1}(k), rank(Q^{-1}(k) - S(k)) \leqslant n_R \quad (11.18)$$

As a result, we get the following theorem.

Theorem 11.1 Given system (11.6) and $\gamma > 0$. There exists a post-filter (11.4) of order n_R so that the characteristic multipliers of $A_e(k)$ belong to the open unit disk and (11.9) holds, if there exist T-periodic symmetric positive-definite matrices $S(k)$ and $Q(k)$ so that (11.15)-(11.18) hold for $i = 1, 2, \cdots, p$, $k = 0, 1, \cdots, T-1$.

The optimal solution to the optimization problem (11.8) can thus be obtained as follows. Iteratively solve (11.15)-(11.18) till the minimal α is found. Using the resulting $S(k)$ and $Q(k)$, calculate full column rank matrices $Z(k), Y(k), k = 0, 1, \cdots, T-1$, satisfying

$$S(k)Q(k) + Y(k)Z^T(k) = I$$

Then solve

$$\begin{bmatrix} S(k) & I \\ Y^T(k) & O \end{bmatrix} = P_e(k) \begin{bmatrix} I & Q(k) \\ O & Z^T(k) \end{bmatrix} \quad (11.19)$$

for $P_e(k)$. Finally we get the optimal post-filter parameter set $\Theta_R(k)$, $k = 0, 1, \cdots, T-1$, by solving (11.11).

11.3 Conclusion

This chapter addresses the FD problem of the uncertain linear discrete-time periodic systems with polytopic uncertainties. The periodic post filter is designed in such a way that the residual dynamics of the uncertain system can approximate the best residual dynamics of the nominal system [191]. The approach proposed involves periodic linear matrix inequality (LMI) technique, whose computational aspect is explored by [13].

12

Identification of periodic residual generator

In the last three chapters, it has been shown how to design the FD systems, provided that the model of the periodic system is available. In case that the model of the periodic system is unknown, usually the FD system design is carried out in two steps:

- at first identify the parameters of the system, and then
- design a residual generator using the available approaches.

The subspace identification technique has been developed since the nineties, which delivers a state space model from the process input and output data [57, 120, 151]. This kind of identification technique don't need to parameterize the model set *a priori* or to solve nonlinear parametric optimization problems.

In this chapter, an approach will be presented for the design of residual generators for unknown periodic systems with dynamics

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + E_f(k)f(k) \\ y(k) &= C(k)x(k) + D(k)u(k) + F_f(k)f(k) \end{aligned} \tag{12.1}$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^{n_u}$ the control input vector, $y \in \mathbf{R}^m$ the measured output vector, and $f \in \mathbf{R}^{n_f}$ the vector of faults to be detected, $A(k), B(k), C(k), D(k), E_f(k), F_f(k)$ are unknown real bounded periodic matrices of period T and with appropriate dimensions. The proposed approach condenses the two steps in the conventional design into one step. The parameters of the residual generator will be identified directly from data without identifying the model of the periodic system.

This work follows the line of [41, 55, 56, 114]. In [55], some analog between subspace based system identification and model predictive controller is discovered. Based on it, the identification and the controller design are combined. It is shown in [56] that LQG controller design can also be integrated with the identification step. Ding et al. [41, 167] point out that an integrated identification and fault detection system design is possible for linear time-invariant systems. More recently, identification and model reduction are combined to get a balanced reduced order model directly from data [114]. Here we shall extend the result in [41, 167] to unknown periodic system (12.1).

The problem to be solved in this chapter is formulated as: Given a set of data of the outputs $y(k)$ and the inputs $u(k)$ of unknown periodic system (12.1), find a periodic parity relation based residual generator in the form of (10.3) or a periodic observer based residual generator (10.8).

12.1 Identification of periodic parity relation based residual generator

From discrete time k_1 to discrete time k_2, the input-output relation of periodic system (12.1) can be expressed as

$$Y_{N,i}(k) = H_{o,i}X_{N,i}(k) + H_{u,i}U_{N,i}(k) \tag{12.2}$$

where

$$\begin{aligned}
Y_{N,i}(k) &= \begin{bmatrix} y_s(k) \; y_s(k+T) \cdots \; y_s(k+NT) \end{bmatrix} \\
U_{N,i}(k) &= \begin{bmatrix} u_s(k) \; u_s(k+T) \cdots \; u_s(k+NT) \end{bmatrix} \\
X_{N,i}(k) &= \begin{bmatrix} x(k-s) \cdots \; x(k+NT-s) \end{bmatrix} \\
y_s(k) &= \begin{bmatrix} y(k-s) \\ y(k-s+1) \\ \vdots \\ y(k) \end{bmatrix}, \; u_s(k) = \begin{bmatrix} u(k-s) \\ u(k-s+1) \\ \vdots \\ u(k) \end{bmatrix} \\
H_{o,i} &= \begin{bmatrix} C_i \\ C_{i+1}A_i \\ \vdots \\ C_{i+s}A_{i+s-1}\cdots A_{i+1}A_i \end{bmatrix} \\
H_{u,i} &= \begin{bmatrix} D_i & O & \cdots & O \\ C_{i+1}B_i & D_{i+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ C_{i+s}A_{i+s-1}\cdots A_{i+1}B_i & \cdots & C_{i+s}B_{i+s-1} & D_{i+s} \end{bmatrix}
\end{aligned} \tag{12.3}$$

where $k = jT + i + s$, $k \geqslant k_1 + s$ and N is so selected that $k_1 + s + (N+1)T \leqslant k_2$.

Equation (12.2) can be re-written into

$$\begin{bmatrix} Y_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix} = \begin{bmatrix} H_{o,i} & H_{u,i} \\ O & I \end{bmatrix} \begin{bmatrix} X_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix}$$

Under the assumption that the matrix $\begin{bmatrix} X_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix}$ is of full row rank, the left null space of matrix $\begin{bmatrix} Y_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix}$ will be identical with that of $\begin{bmatrix} H_{o,i} & H_{u,i} \\ O & I \end{bmatrix}$. The left null space of $\begin{bmatrix} Y_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix}$ can be figured out through a singular value decomposition (SVD). Assume that the SVD of $\begin{bmatrix} Y_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix}$ is

$$\begin{bmatrix} Y_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix} = U_i \begin{bmatrix} S_i & O \\ O & O \end{bmatrix} W_i^T \tag{12.4}$$

where U_i, W_i are orthogonal matrices, S_i is a diagonal matrix containing the nonzero singular values, $S_i \in \mathbf{R}^{l \times l}$. Partition U_i as

$$U_i = \begin{bmatrix} U_{i1} \; U_{i2} \end{bmatrix}, U_{i1} \in \mathbf{R}^{(m+n_u)(s+1)\times l}, \; U_{i2} \in \mathbf{R}^{(m+n_u)(s+1)\times((m+n_u)(s+1)-l)}$$

then U_{i2}^T is a basis of the left null space of $\begin{bmatrix} Y_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix}$.

As a result,

$$U_{i2}^T \begin{bmatrix} H_{o,i} & H_{u,i} \\ O & I \end{bmatrix} = 0 \tag{12.5}$$

Further partition U_{i2}^T into

$$U_{i2}^T = \begin{bmatrix} U_{i\Sigma 1} \; U_{i\Sigma 2} \end{bmatrix} \tag{12.6}$$

Substituting (12.6) into (12.5) yields

$$\begin{bmatrix} U_{i\Sigma 1} \; U_{i\Sigma 2} \end{bmatrix} \begin{bmatrix} H_{o,i} & H_{u,i} \\ O & I \end{bmatrix} = 0$$

i.e.

$$\begin{aligned} U_{i\Sigma 1}H_{o,i} &= 0 \\ U_{i\Sigma 1}H_{u,i} &= -U_{i\Sigma 2} \end{aligned} \tag{12.7}$$

Let α_i be a nonzero row vector of compatible dimensions. Then

$$\begin{aligned} v_i &= \alpha_i U_{i\Sigma 1} \\ \rho_i &= -\alpha_i U_{i\Sigma 2} \end{aligned} \tag{12.8}$$

give a vector v_i satisfying $v_i H_{o,i} = 0$ and the corresponding vector $\rho_i = v_i H_{u,i}$.

The value of vectors v_i and ρ_i over one period can be obtained by repeating the above process for $i = 0, 1, \cdots, T-1$.

Algorithm 12.1 Design of a periodic parity relation based residual generator in the form of

$$r(k) = v_i y_s(k) - \rho_i u_s(k) \tag{12.9}$$

for the unknown periodic system (12.1) from the input and output data $y(k_1), u(k_1), y(k_1+1), u(k_1+1), \cdots, y(k_2), u(k_2)$:

Step 1: Set the value of s, $s \gg n$.
Step 2: Build matrices $Y_{N,i}(k), U_{N,i}(k)$ with $k = k_1 + s$.
Step 3: Do the SVD of $\begin{bmatrix} Y_{N,i}(k) \\ U_{N,i}(k) \end{bmatrix}$ as (12.4) to get matrix U_{i2}.
Step 4: Partition U_{i2}^T as $\begin{bmatrix} U_{i\Sigma 1} \; U_{i\Sigma 2} \end{bmatrix}$.
Step 5: Compute vectors v_i and ρ_i by (12.8) with an arbitrarily selected nonzero vector α_i.
Step 6: Let $k = k+1$ and go back to step 2, till $k = k_1 + s + T$.

There is some freedom in selecting parameter α. It can be used, for instance, to reduce the order of parity relation (12.9), as shown later by the example.

When the system is affected by noise, the lower diagonal block in the SVD (12.4) may not be exactly zero block. In this case, the number of dominant singular values determines the dimension of matrix U_{i2}.

12.2 Identification of periodic observer based residual generator

According to Chapter 10, the matrices $G(k), L(k), w(k), p(k)$ of a periodic observer based residual generator in the form of (10.8) can be determined from the periodic parity vector $v(k)$. However, the matrices $H(k), q(k)$ can not be computed by (10.11)-(10.12), because the model of the periodic system (12.1) is unknown. Therefore, the key problem in identifying a periodic observer based residual generator is to find out a way to get the matrices $H(k)$ and $q(k)$.

Theorem 12.1 Assume that $v(k)H_{o,s}(k) = 0$, $v(k) = \begin{bmatrix} v_0(k) \; v_1(k) \; \cdots \; v_s(k) \end{bmatrix}$, and

$$\rho(k) = v(k)H_{u,s}(k) = \begin{bmatrix} \rho_0(k) \; \rho_1(k) \; \cdots \; \rho_s(k) \end{bmatrix} \tag{12.10}$$

where $\rho_0(k) \in \mathbf{R}^{1\times n_u}$. Then

$$T(k) = \begin{bmatrix} v_1(k-1) & \cdots & v_{s-1}(k-1) & v_s(k-1) \\ v_2(k-2) & \cdots & v_s(k-2) & 0 \\ \vdots & & \vdots & \vdots \\ v_s(k-s) & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} C(k) \\ C(k+1)A(k) \\ \vdots \\ C(k+s-1)A(k+s-2)\cdots A(k) \end{bmatrix} \quad (12.11)$$

$$G(k) = \begin{bmatrix} 0 & \cdots & 0 & g_1(k) \\ 1 & \ddots & \vdots & g_2(k) \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & g_s(k) \end{bmatrix} \quad (12.12)$$

$$L(k) = -\begin{bmatrix} v_0(k) \\ v_1(k-1) \\ \vdots \\ v_{s-1}(k-s+1) \end{bmatrix} - \begin{bmatrix} g_1(k) \\ g_2(k) \\ \vdots \\ g_s(k) \end{bmatrix} v_s(k-s) \quad (12.13)$$

$$w(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \quad (12.14)$$

$$p(k) = v_s(k-s) \quad (12.15)$$

$$q(k) = -\rho_s(k-s) \quad (12.16)$$

$$H(k) = \begin{bmatrix} \rho_0(k) \\ \rho_1(k-1) \\ \vdots \\ \rho_{s-1}(k-s+1) \end{bmatrix} + \begin{bmatrix} g_1(k) \\ g_2(k) \\ \vdots \\ g_s(k) \end{bmatrix} \rho_s(k-s) \quad (12.17)$$

Proof: Similar to the derivation of Theorem 10.1, (12.11)-(12.15) can be obtained from

$$\begin{bmatrix} v_0(k) & v_1(k) & \cdots & v_s(k) \end{bmatrix} \begin{bmatrix} C(k) \\ C(k+1)A(k) \\ \vdots \\ C(k+s)A(k+s-1)\cdots A(k+1)A(k) \end{bmatrix} = 0$$

According to (10.11)-(10.12),

$$q(k) = -p(k)D(k) = -v_s(k-s)D(k)$$

and

$$\begin{aligned} H(k) &= T(k+1)B(k) - L(k)D(k) \\ &= \begin{bmatrix} v_1(k) & \cdots & v_s(k) \\ \vdots & & \vdots \\ v_s(k-s+1) & \cdots & 0 \end{bmatrix} \begin{bmatrix} C(k+1) \\ \vdots \\ C(k+s)A(k+s-1)\cdots A(k+1) \end{bmatrix} B(k) \\ &+ \begin{bmatrix} v_0(k) \\ \vdots \\ v_{s-1}(k-s+1) \end{bmatrix} D(k) + \begin{bmatrix} g_1(k) \\ \vdots \\ g_s(k) \end{bmatrix} v_s(k-s)D(k) \end{aligned}$$

Notice that

$$\begin{aligned} &\begin{bmatrix} \rho_0(k) & \rho_1(k) & \cdots & \rho_s(k) \end{bmatrix} \\ &= \begin{bmatrix} v_0(k) & v_1(k) & \cdots & v_s(k) \end{bmatrix} \begin{bmatrix} D(k) & O & \cdots & O \\ C(k+1)B(k) & D(k+1) & \ddots & \vdots \\ \vdots & & \ddots & O \\ C(k+s)\cdots A(k+1)B(k) & \cdots & & D(k+s) \end{bmatrix} \end{aligned}$$

which means

$$\rho_0(k) = \begin{bmatrix} v_1(k) \cdots v_s(k) \end{bmatrix} \begin{bmatrix} C(k+1)B(k) \\ \vdots \\ C(k+s)\cdots A(k+1)B(k) \end{bmatrix} + v_0(k)D(k)$$

$$\rho_1(k) = \begin{bmatrix} v_2(k) \cdots v_s(k) \end{bmatrix} \begin{bmatrix} C(k+2)B(k+1) \\ \vdots \\ C(k+s)\cdots A(k+2)B(k+1) \end{bmatrix} + v_1(k)D(k+1)$$

$$\vdots$$

$$\rho_s(k) = v_s(k)D(k+s)$$

Therefore, (12.16) and (12.17) can be obtained. □

Theorem 12.1 shows that a periodic observer based residual generator in the form of (10.8) can be directly calculated based on vectors $v(k)$ and $\rho(k)$. Based on it, the following algorithm is obtained.

Algorithm 12.2 Design of a periodic observer based residual generator in the form of

$$\begin{aligned} z(k+1) &= G(k)z(k) + H(k)u(k) + L(k)y(k) \\ r(k) &= w(k)z(k) + q(k)u(k) + p(k)y(k) \end{aligned} \tag{12.18}$$

for the unknown periodic system (12.1) from the input and output data $y(k_1), u(k_1), y(k_1+1), u(k_1+1), \cdots, y(k_2), u(k_2)$:

- Compute the T-periodic vectors $v(k)$ and $\rho(k)$ according to Algorithm 12.1.
- Determine $g_1(k), \cdots, g_s(k)$, $k = 0, 1, \cdots, T-1$, so that all the eigenvalues of $G(T-1)G(T-2)\cdots G(0)$ are inside the unit circle.
- Compute $G(k), L(k), H(k), w(k), p(k), q(k)$ according to (12.12)-(12.17).

The free selectability of parameters $g_1(k), \cdots, g_s(k)$, $k = 0, 1, \cdots, T-1$, which influence matrices $G(k), L(k), H(k)$, represents the increased design freedom provided by the periodic observer based residual generator.

We would like to emphasize that both in Algorithm 12.1 and in Algorithm 12.2 it is not necessary to identify the model of the periodic system. Instead of that, the parameters of the residual generators are directly obtained from the results of the identification.

12.3 Example

In this section, an example is given to illustrate the proposed design procedures.

Example 12.1 Given the data of inputs and outputs of a periodic system of period $T = 2$ from discrete time 100 to discrete time 800. Indeed the data are generated by a system in the form of (12.1) with $d = 0$ and parameters given by (10.28).

Because instead of the parameters (10.28) only the input and output data of the periodic system are available, we apply Algorithm 12.1 to design a periodic parity relation based residual generator.

Let $s = 5$. The value of N is chosen as $N = 346$. First the matrices $Y_{N,0} \in \mathbf{R}^{18\times 347}$, $Y_{N,1} \in \mathbf{R}^{18\times 347}$, $U_{N,0} \in \mathbf{R}^{6\times 347}$ and $U_{N,1} \in \mathbf{R}^{6\times 347}$ are built as (12.3). Do the SVD of $\begin{bmatrix} Y_{N,0} \\ U_{N,0} \end{bmatrix}$ and $\begin{bmatrix} Y_{N,1} \\ U_{N,1} \end{bmatrix}$, respectively, to get matrix $U_{02} \in \mathbf{R}^{24\times 14}$, $U_{12} \in \mathbf{R}^{24\times 14}$. Partition U_{02}^T and U_{12}^T, respectively, into $U_{0\Sigma 1} \in \mathbf{R}^{14\times 18}$, $U_{0\Sigma 2} \in \mathbf{R}^{14\times 6}$, $U_{1\Sigma 1} \in \mathbf{R}^{14\times 18}$, $U_{1\Sigma 2} \in \mathbf{R}^{14\times 6}$. To reduce the order of the residual generator, we select

$$\alpha_0 = \big[-0.3777\ 0.0987\ -0.0006\ -0.0143\ -0.0016\ 0.0941$$
$$0.2319\ 0.0371\ -0.1127\ 0.0827\ -0.2224\ 0.4108\ 0.1706\ 0.7187\big]$$
$$\alpha_1 = \big[0.5280\ 0.0672\ 0.3390\ 0.0579\ -0.0396\ 0.0378$$
$$0.3086\ -0.0762\ 0.4591\ -0.4185\ -0.1874\ -0.1229\ -0.1720\ 0.1692\big]$$

with the help of QR decomposition $U_{0\Sigma 1} = U_{0Q}U_{0R}$, $U_{1\Sigma 1} = U_{1Q}U_{1R}$, which yields

$$\bar{v}_0 = \alpha_0 U_{0\Sigma 1} = \big[-0.3896\ 0.0476\ 0.0025\ -0.4158\ 0.8114\ 0.0000\ O_{1\times 12}\big]$$
$$\bar{v}_1 = \alpha_1 U_{1\Sigma 1} = \big[0.3981\ 0.1720\ 0.1526\ -0.8802\ 0.0789\ -0.0000\ O_{1\times 12}\big]$$
$$\bar{\rho}_0 = -\alpha_0 U_{0\Sigma 2} = \big[0.1211\ 0.0000\ O_{1\times 4}\big]$$
$$\bar{\rho}_1 = -\alpha_1 U_{1\Sigma 2} = \big[-0.0874\ -0.0000\ O_{1\times 4}\big]$$

The order of the periodic relation based residual generator can thus be reduced to $s_r = 1$ and

$$v_0 = \big[-0.3896\ 0.0476\ 0.0025\ -0.4158\ 0.8114\ 0.0000\big]$$
$$v_1 = \big[0.3981\ 0.1720\ 0.1526\ -0.8802\ 0.0789\ -0.0000\big]$$
$$\rho_0 = \big[0.1211\ 0.0000\big]$$
$$\rho_1 = \big[-0.0874\ -0.0000\big]$$

As a result, the periodic parity relation based residual generator is

$$r(k) = \begin{cases} v_0 \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} - \rho_0 \begin{bmatrix} u(k-1) \\ u(k) \end{bmatrix}, & \text{if } k = 2j+1 \\ \\ v_1 \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} - \rho_1 \begin{bmatrix} u(k-1) \\ u(k) \end{bmatrix}, & \text{if } k = 2j+2 \end{cases} \tag{12.19}$$

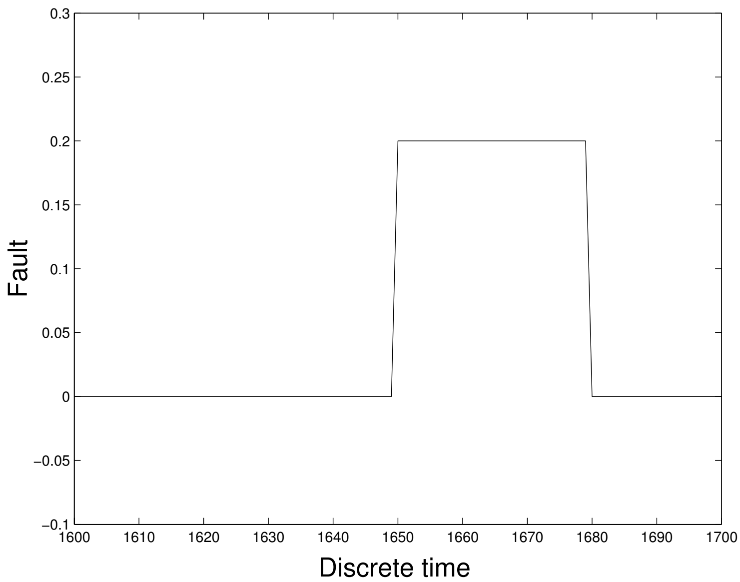

Fig. 12.1 The fault signal

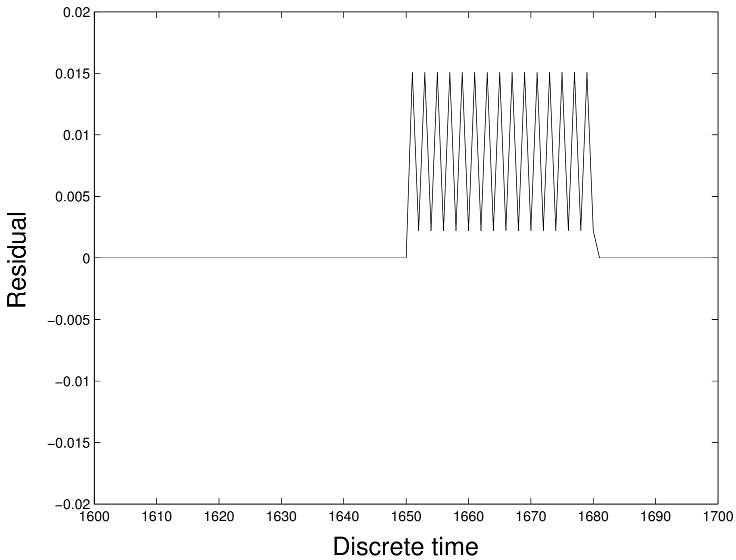

Fig. 12.2 The residual signal generated by the periodic parity relation based residual generator

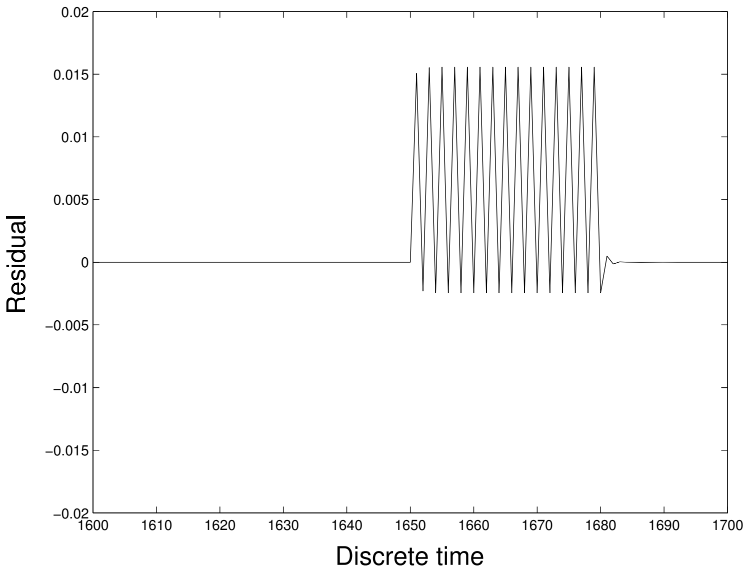

Fig. 12.3 The residual signal generated by the periodic observer based residual generator

To design a periodic observer based residual generator, set the values of $g_1(0), g_1(1)$ as $g_1(0) = -0.2$, $g_1(1) = -0.3$. Based on Algorithm 12.2, the resulting residual generator is in the form of (12.18) with parameters

$$\begin{aligned}
&G(0) = -0.2,\ G(1) = -0.3,\ H(0) = 0.1211,\ H(1) = -0.0874\\
&L(0) = \begin{bmatrix} 0.2136\ -0.0318\ -0.0025 \end{bmatrix}\\
&L(1) = \begin{bmatrix} -0.5228\ 0.0714\ -0.1526 \end{bmatrix}\\
&w(0) = -1,\ w(1) = -1,\ q(0) = 0,\ q(1) = 0\\
&p(0) = \begin{bmatrix} -0.8802\ 0.0789\ -0.0000 \end{bmatrix}\\
&p(1) = \begin{bmatrix} -0.4158\ 0.8114\ 0.0000 \end{bmatrix}
\end{aligned} \tag{12.20}$$

In the simulation, it is assumed that the control input u is a step signal with step time at 0 and amplitude 1, the fault appears at the $1650th$ discrete time as shown in Fig. 12.1. The residual signal got by the periodic parity relation based residual generator (12.19) is shown in Fig. 12.2. It is seen that the residual signal r deviates from zero only if $f \neq 0$ and thus ensures a successful fault detection. Fig. 12.3 demonstrates the residual signal obtained by the periodic observer based residual generator (12.20). Similarly, the fault is quickly detected.

12.4 Conclusion

In this chapter, a fault detection approach has been proposed for linear discrete-time periodic systems, whose models are not available. The basic idea is to identify the parameters of residual generators directly from input and output data. Both periodic parity relation based residual generators and periodic observer based residual generators can be designed in this way [168]. An example is given to illustrate the design procedures.

Part III

FD of Discrete Linear Time-Varying Systems

13

FD of discrete linear time-varying systems

Strictly speaking, many practical systems are time-varying, i.e. the system matrices changes with the time. In this chapter, we shall consider the FD problem of discrete linear time-varying (LTV) systems described by [193]

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + E_d(k)d(k) + E_f(k)f(k) \\ y(k) &= C(k)x(k) + D(k)u(k) + F_d(k)d(k) + F_f(k)f(k) \end{aligned} \tag{13.1}$$

where $A(k), B(k), C(k), D(k), E_d(k), E_f(k), F_d(k)$ and $F_f(k)$ are known, real bounded but time dependent matrix functions of compatible dimensions. We assume that $(A(k), C(k))$ is detectable.

From the viewpoint of fault detection, the extension of the parity space approach to the LTV systems is rather straightforward, because, as mentioned before, the parity space approach treats each time instant independently. To extend the observer based FD technique to the LTV systems, the concept of operators need to be introduced so that the robustness and the sensitivity can be defined accordingly.

13.1 Extension of the parity space approach

The parity relation of the LTV system can be expressed as

$$y_s(k) = H_{o,s}(k)x(k-s) + H_{u,s}(k)u_s(k) + H_{d,s}(k)d_s(k) + H_{f,s}(k)f_s(k) \tag{13.2}$$

where the vectors $u_s(k), d_s(k), f_s(k), y_s(k)$ and the matrices $H_{o,s}(k), H_{u,s}(k)$, $H_{d,s}(k), H_{f,s}(k)$ have the same definitions as those in (10.1). However, compared with (10.1), the main difference is that in (13.2) the matrices $H_{o,s}(k), H_{u,s}(k)$, $H_{d,s}(k)$ and $H_{f,s}(k)$ are time-varying functions, but not necessarily periodic with respect to k.

Based on the parity relation (13.2), a residual generator can be constructed as

$$r(k) = V_s(k)(y_s(k) - H_{u,s}(k)u_s(k)) \tag{13.3}$$

where $V_s(k)$ denotes the periodic parity matrix, $V_s(k)H_{o,s}(k) = 0$. The residual dynamics is governed by

$$r(k) = V_s(k)(H_{d,s}(k)d_s(k) + H_{f,s}(k)f_s(k)) \tag{13.4}$$

The most distinguished feature of the parity space approach is that each time instant can be treated independently. If at time k, the rank condition

$$rank \begin{bmatrix} H_{o,s}(k) & H_{d,s}(k) & H_{f,s}(k) \end{bmatrix} > rank \begin{bmatrix} H_{o,s}(k) & H_{d,s}(k) \end{bmatrix} \tag{13.5}$$

is satisfied, then a parity matrix $V_s(k)$ that achieves a full decoupling at time k can be obtained by solving

$$V_s(k)\begin{bmatrix} H_{o,s}(k) & H_{d,s}(k) \end{bmatrix} = 0,\ V_s(k)H_{f,s}(k) \neq 0 \tag{13.6}$$

If at time k the condition (13.5) is not satisfied, then optimization problems similar to (10.32)-(10.34) can be formulated and solved to get the optimal parity matrix $V_s(k)$. Therefore, in summary, in the LTV system the decoupling problem or the optimization problem need to be solved at each time instant.

13.2 Extension of the observer based approach

For the LTV system (13.1) an observer based residual generator can be constructed as

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) + L(k)(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C(k)\hat{x}(k) + D(k)u(k) \\ r(k) &= W(k)(y(k) - \hat{y}(k)) \end{aligned} \tag{13.7}$$

where $L(k)$ is the observer gain matrix function, $W(k)$ the weighting matrix function. Define $e(k) = x(k) - \hat{x}(k)$. The residual dynamics is governed by

$$\begin{aligned} e(k+1) &= A(k)e(k) + (E_d(k) - L(k)F_d(k))\, d(k) + (E_f(k) - L(k)F_f(k))\, f(k) \\ r(k) &= W(k)\left(C(k)e(k) + F_d(k)d(k) + F_f(k)f(k)\right) \end{aligned} \tag{13.8}$$

The time-varying matrices $L(k)$ and $W(k)$ should be selected so that the residual dynamics is stable, robust against the unknown disturbances and sensitive to the faults.

At first we introduce two operators Π_{rd} and Π_{rf} to describe the mapping from the unknown disturbances and the faults to the residual, respectively. According to the functional analysis, the norms of the operators defined by

$$\begin{aligned} \|\Pi_{rd}\| &= \sup_{f=0,d\neq 0} \frac{\|r\|_2}{\|d\|_2} \\ \|\Pi_{rf}\| &= \sup_{d=0,f\neq 0} \frac{\|r\|_2}{\|f\|_2} \end{aligned} \tag{13.9}$$

describe the maximal influence of the unknown disturbances and the faults on the residual signal. The problem of finding the optimal trade-off between the robustness of the residual generator to the unknown disturbances and the sensitivity of the residual generator to the faults can thus be formulated as the following optimization problem

$$\max_{L(k),W(k)} J = \max_{L(k),W(k)} \frac{\|\Pi_{rf}\|}{\|\Pi_{rd}\|} \tag{13.10}$$

To solve the optimization problem (13.10) we shall first introduce several concepts of discrete LTV systems. For the definition of stability, stabilizability and detectability of discrete LTV systems, please refer to the book [71].

Definition 13.1 A stable discrete LTV system

$$\Pi\ :\ u \to y, \quad x(k+1) = A(k)x(k) + B(k)u(k) \tag{13.11}$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

is said to be co-inner, if $\Pi\Pi^* = I$ holds, where Π^* is the adjoint operator of Π associated with the following discrete LTV system

$$\Pi^*\ :\ w \to v, \quad z(k+1) = -A^T(k)z(k) - C^T(k)w(k) \tag{13.12}$$
$$v(k) = B^T(k)z(k) + D^T(k)w(k)$$

To simplify the notation, we write system (13.11) in some places also as $\Pi = (A(k), B(k), C(k), D(k))$. If the system (13.11) has no dynamics, then it is denoted simply by $\Pi = (D(k))$.

Definition 13.2 A stable discrete LTV system (13.11) is co-outer, if its left inverse system is also stable.

Lemma 13.1 [71] A stable discrete LTV system $\Pi = (A(k), B(k), C(k), D(k))$ is co-inner, if there is a symmetric positive-semidefinite (SPS) matrix function $Q(k)$, i.e. $\forall k$, $Q(k) = Q^T(k) \geq 0$, so that

$$A(k)X(k)C^T(k) + B(k)D^T(k) = 0 \tag{13.13}$$

$$C(k)X(k)C^T(k) + D(k)D^T(k) = I \tag{13.14}$$

$$A(k)X(k)A^T(k) + B(k)B^T(k) = X(k+1) \tag{13.15}$$

Lemma 13.2 [71] Given a stable discrete LTV system $\Pi = (A(k), B(k), C(k), D(k))$, there is the following co-inner-outer factorization

$$\Pi = \Pi_o \Pi_i \tag{13.16}$$

$$\Pi_o = (A(k), -\bar{L}(k)\bar{W}^{-1}(k), -C(k), \bar{W}^{-1}(k)) \tag{13.17}$$

$$\Pi_i = (A(k) - \bar{L}(k)C(k), B(k) - \bar{L}(k)D(k), \bar{W}(k)C(k), \bar{W}(k)D(k)) \tag{13.18}$$

where Π_o is co-outer and Π_i is co-inner,

$$\begin{aligned} \bar{L}(k) &= (A(k)X(k)C^T(k) + B(k)D^T(k))(C(k)X(k)C^T(k) + D(k)D^T(k))^{-1} \\ \bar{W}(k) &= (C(k)X(k)C^T(k) + D(k)D^T(k))^{-1/2} \end{aligned} \tag{13.19}$$

and $X(k)$ is the stabilizing SPS solution of the following Riccati difference equation

$$\begin{aligned} &A(k)X(k)A^T(k) - X(k+1) + B(k)B^T(k) - (A(k)X(k)C^T(k) + B(k)D^T(k)) \\ &\times (C(k)X(k)C^T(k) + D(k)D^T(k))^{-1}(C(k)X(k)A^T(k) + D(k)B^T(k)) = 0 \end{aligned} \tag{13.20}$$

The Riccati difference equation (13.20) has a stabilizing SPS solution, if $(A(k), C(k))$ is detectable and $(A(k), B(k))$ is stabilizable [118, 71, 79]. The stabilizability of $(A(k), B(k))$ is a sufficient, but not a necessary condition.

The operators Π_{rd} and Π_{rf} can be re-written as the series connection of several subsystems

$$\Pi_{rd} = \Pi_W \Pi_L \Pi_{yd},\ \Pi_{rf} = \Pi_W \Pi_L \Pi_{yf} \tag{13.21}$$

where

$$\begin{aligned} &\Pi_W = (W(k)), \quad \Pi_L = (A(k) - L(k)C(k), L(k), -C(k), I) \\ &\Pi_{yd} = (A(k), E_d(k), C(k), F_d(k)), \quad \Pi_{yf} = (A(k), E_f(k), C(k), F_f(k)) \end{aligned} \tag{13.22}$$

Correspondingly, the optimization problem (13.10) can be re-written as

$$\max_{L(k), W(k)} J = \max_{L(k), W(k)} \frac{\|\Pi_W \Pi_L \Pi_{yf}\|}{\|\Pi_W \Pi_L \Pi_{yd}\|} \tag{13.23}$$

To solve the optimization problem (13.23), in the first step we assume that the observer gain matrix function $L(k)$ is fixed and $A(k) - L(k)C(k)$ is stable and consider the following sub-optimization problem

$$\max_{\Pi_R} J_o = \max_{\Pi_R} \frac{\|\Pi_R \Pi_L \Pi_{yf}\|}{\|\Pi_R \Pi_L \Pi_{yd}\|} \tag{13.24}$$

where Π_R is a stable discrete LTV system.

Theorem 13.1 Given a discrete LTV system (13.1) and a stabilizing observer gain matrix function $L(k)$, where $(A(k), C(k))$ is detectable and $(A(k), E_d(k))$ is stabilizable. Then (13.24) is solved by

$$\varPi_R = (A(k) - L_o(k)C(k), L(k) - L_o(k), QW_o(k)C(k), QW_o(k)) \tag{13.25}$$

where Q is a unitary matrix,

$$\begin{aligned} L_o(k) &= \left(A(k)X(k)C^T(k) + E_d(k)F_d^T(k)\right)\left(C(k)X(k)C^T(k) + F_d(k)F_d^T(k)\right)^{-1} \\ W_o(k) &= \left(C(k)X(k)C^T(k) + F_d(k)F_d^T(k)\right)^{-1/2} \end{aligned} \tag{13.26}$$

and $X(k)$ is the stabilizing SPS solution of the following Riccati difference equation

$$\begin{aligned} X(k+1) = {} & A(k)X(k)A^T(k) + E_d(k)E_d^T(k) - \left(A(k)X(k)C^T(k) + E_d(k)F_d^T(k)\right) \\ & \times \left(C(k)X(k)C^T(k) + F_d(k)F_d^T(k)\right)^{-1}\left(C(k)X(k)A^T(k) + F_d(k)E_d^T(k)\right) \end{aligned} \tag{13.27}$$

Proof: According to Lemma 13.2, $\varPi_L \varPi_{yd} = (A(k) - L(k)C(k), E_d(k) - L(k)F_d(k), C(k), F_d(k))$ can be factorized as

$$\varPi_L \varPi_{yd} = \varPi_{do} \varPi_{di}$$

where $\varPi_{di}$ is co-inner and satisfies $\varPi_{di}\varPi_{di}^* = I$, $\varPi_{do}$ is co-outer and its stable left inverse is

$$\varPi_{do}^{-1} = (A(k) - L(k)C(k) - \bar{L}(k)C(k), -\bar{L}(k), \bar{W}(k)C(k), \bar{W}(k))$$

with

$$\begin{aligned} \bar{L}(k) &= ((A(k) - L(k)C(k))X(k)C^T(k) + (E_d(k) - L(k)F_d(k))F_d^T(k)) \\ & \quad \times (C(k)X(k)C^T(k) + F_d(k)F_d^T(k))^{-1} \\ &= (A(k)X(k)C^T(k) + E_d(k)F_d^T(k))(C(k)X(k)C^T(k) + F_d(k)F_d^T(k))^{-1} - L(k) \\ \bar{W}(k) &= (C(k)X(k)C^T(k) + F_d(k)F_d^T(k))^{-1/2} \end{aligned} \tag{13.28}$$

and $X(k)$ is the stabilizing SPS solution of the Riccati difference equation

$$\begin{aligned} & (A(k) - L(k)C(k))X(k)(A(k) - L(k)C(k))^T - X(k+1) \\ & +(E_d(k) - L(k)F_d(k))(E_d(k) - L(k)F_d(k))^T \\ & -((A(k) - L(k)C(k))X(k)C^T(k) + (E_d(k) - L(k)F_d(k))F_d^T(k)) \\ & \times(C(k)X(k)C^T(k) + F_d(k)F_d^T(k))^{-1} \\ & \times\left(C(k)X(k)(A(k) - L(k)C(k))^T + F_d(k)(E_d(k) - L(k)F_d(k))^T\right) \\ & = 0 \end{aligned} \tag{13.29}$$

As $\varPi_R \varPi_L \varPi_{yd}$ is a bounded operator in the Hilbert space, there is

$$\|\varPi_R \varPi_L \varPi_{yd}\| = \|\varPi_R \varPi_{do} \varPi_{di}\| = \|\varPi_R \varPi_{do} \varPi_{di} \varPi_{di}^* \varPi_{do}^* \varPi_R^*\|^{\frac{1}{2}} = \|\varPi_R \varPi_{do}\|.$$

Substituting $\varPi_R$ by $\varPi_R = \bar{\varPi}_R \varPi_{do}^{-1}$ yields

$$J_o = \frac{\|\varPi_R \varPi_L \varPi_{yf}\|}{\|\varPi_R \varPi_{do}\|} = \frac{\left\|\bar{\varPi}_R \varPi_{do}^{-1} \varPi_L \varPi_{yf}\right\|}{\left\|\bar{\varPi}_R \varPi_{do}^{-1} \varPi_{do}\right\|} = \frac{\left\|\bar{\varPi}_R \varPi_{do}^{-1} \varPi_L \varPi_{yf}\right\|}{\left\|\bar{\varPi}_R\right\|} \tag{13.30}$$

Because $\left\|\bar{\varPi}_R \varPi_{do}^{-1} \varPi_L \varPi_{yf}\right\| \le \left\|\bar{\varPi}_R\right\| \left\|\varPi_{do}^{-1} \varPi_L \varPi_{yf}\right\|$, it gives

$$J_o \le \left\|\varPi_{do}^{-1} \varPi_L \varPi_{yf}\right\| \tag{13.31}$$

Similarly as in the LTI case, selecting $\bar{\varPi}_R$ as a unitary matrix will make the equation in (13.31) hold. Hence, $\varPi_R = Q\varPi_{do}^{-1}$ is the optimal solution to the optimization problem (13.24). By the definitions in (13.26), $\bar{L}(k)$ and $\bar{W}(k)$ can be re-written as $\bar{L}(k) = L_o(k) - L(k), \bar{W}(k) = W_o(k)$, the Riccati difference equation (13.29) reduces to (13.27) and the optimal solution $\Pi_R = Q\Pi_{do}^{-1}$ can be expressed as (13.25). □

Theorem 13.2 Given a discrete linear LTV system (13.1), where $(A(k), C(k))$ is detectable, $(A(k), E_d(k))$ is stabilizable. Then the optimization problem (13.23) is solved by

$$L_{opt}(k) = L_o(k), \quad W_{opt}(k) = QW_o(k) \tag{13.32}$$

where Q is a unitary matrix, $L_o(k)$ and $W_o(k)$ are given by (13.26)-(13.27).

Proof: It can be seen from Theorem 13.1 that the optimal solution to (13.24) Π_R given in (13.25) comes down to a matrix function $QW_o(k)$, if we select $L(k) = L_o(k)$, i.e.

$$\frac{\|\Pi_R \Pi_{L_o} \Pi_{yf}\|}{\|\Pi_R \Pi_{L_o} \Pi_{yd}\|} \,|_{\Pi_R=(QW_o(k))} = \max_{\Pi_R} \frac{\|\Pi_R \Pi_{L_o} \Pi_{yf}\|}{\|\Pi_R \Pi_{L_o} \Pi_{yd}\|}$$

For any arbitrarily stabilizing observer gain matrix $L(k)$, there exists always a stable and invertible LTV system $\Pi_{trans} = (A(k) - L_o(k)C(k), L(k) - L_o(k), C(k), I)$ such that

$$\Pi_{L_o} = \Pi_{trans} \Pi_L$$

Hence, there is

$$\frac{\|\Pi_R \Pi_{L_o} \Pi_{yf}\|}{\|\Pi_R \Pi_{L_o} \Pi_{yd}\|} \,|_{\Pi_R=QW_o(k)} = \max_{\Pi_R} \frac{\|\Pi_R \Pi_{trans} \Pi_L \Pi_{yf}\|}{\|\Pi_R \Pi_{trans} \Pi_L \Pi_{yd}\|} = \max_{\hat{\Pi}_R} \frac{\left\|\hat{\Pi}_R \Pi_L \Pi_{yf}\right\|}{\left\|\hat{\Pi}_R \Pi_L \Pi_{yd}\right\|}$$

where $\hat{\Pi}_R = \Pi_R \Pi_{trans}$. As the weighting matrix is only a special case of a dynamic system $\hat{\Pi}_R$,

$$\max_{\hat{\Pi}_R} \frac{\left\|\hat{\Pi}_R \Pi_L \Pi_{yf}\right\|}{\left\|\hat{\Pi}_R \Pi_L \Pi_{yd}\right\|} \geq \max_{W(k)} \frac{\|\Pi_W \Pi_L \Pi_{yf}\|}{\|\Pi_W \Pi_L \Pi_{yd}\|} \tag{13.33}$$

Because

$$\frac{\|\Pi_R \Pi_{L_o} \Pi_{yf}\|}{\|\Pi_R \Pi_{L_o} \Pi_{yd}\|} \,|_{\Pi_R=QW_o(k)} \geq \max_{W(k)} \frac{\|\Pi_W \Pi_L \Pi_{yf}\|}{\|\Pi_W \Pi_L \Pi_{yd}\|}$$

holds for any stabilizing observer gain matrix $L(k)$, we come to the conclusion that

$$\frac{\|\Pi_R \Pi_{L_o} \Pi_{yf}\|}{\|\Pi_R \Pi_{L_o} \Pi_{yd}\|} \,|_{\Pi_R=QW_o(k)} \geq \max_{L(k),W(k)} \frac{\|\Pi_W \Pi_L \Pi_{yf}\|}{\|\Pi_W \Pi_L \Pi_{yd}\|}$$

i.e. $L_{opt}(k) = L_o(k)$ and $W_{opt}(k) = QW_o(k)$ solve the optimization problem (13.23). □

The design of the observer based residual generator (13.7) for the LTV system (13.1) can be summarized as follows.

- Solve the Riccati difference equation (13.27).
- Determine the matrix function $L_o(k), W_o(k)$ according to (13.26).
- Select a unitary matrix Q and get $L_{opt}(k), W_{opt}(k)$ by (13.32).
- Build the optimal residual generator in the form of (13.7).

The main computation in the design is the solution of the Riccati difference equation (13.27).

13.3 Conclusion

In this chapter, the extension of the parity space approach and the observer based approach to the linear time-varying (LTV) systems is briefly presented.

The derivation of the parity space approach is, though straightforward, included for the sake of completeness. In applications the parity space approach can be easily implemented. At *each time instant* either a decoupling or an optimization problem needs to be solved.

In the observer based approach, the operators are introduced to analyze the residual dynamics and formulate the optimization problem [193]. The co-inner-outer factorization of the LTV system plays an important role for the solution of the optimization problem. It is shown that the optimal observer based residual generator is obtained by solving a Riccati difference equation.

Part IV

FD of Sampled-Data Systems

14

FD of single-rate sampled-data (SSD) systems

The study on FD problems of sampled-data (SD) systems has been motivated by the digital implementation of controllers and monitoring systems [174, 176, 178, 180, 187]. Figure 14.1 sketches a typical application of an FD system in a control system. The process under consideration is a continuous-time process. Both the controller and the FD system are discrete-time systems which are implemented on a computer system or on an embedded microprocessor. The sensor output signals are discretized by the A/D converters and then fed to the controller as well as to the FD system. The D/A converters convert the discrete-time control input signals into continuous-time signals. Since both continuous-time and discrete-time signals exist in the system, the system design should be indeed considered from the viewpoint of an SD system [20, 132]. The intersample behavior is the main factor that should be considered in developing discrete-time FD systems for SD systems. In this chapter, we shall focus on single-rate sampled-data (SSD) systems.

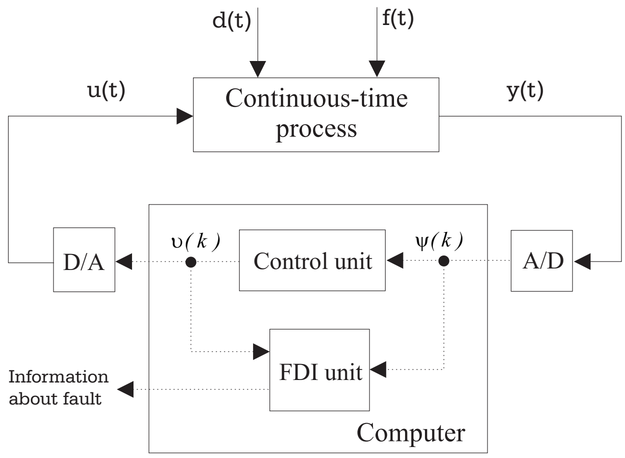

Fig. 14.1 Schematic description of the application of an FDI system in a process control system

14.1 System description

Assume that in the SD systems the plant is a continuous LTI process represented by

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t) + E_{dc} d(t) + E_{fc} f(t) \\ y(t) &= C x(t) \end{aligned} \tag{14.1}$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^{n_u}$ the vector of control signals, $y \in \mathbf{R}^m$ the vector of process outputs, $d \in \mathbf{R}^{n_d}$ the vector of unknown disturbances, $f \in \mathbf{R}^{n_f}$ the vector of faults to be detected, A_c, B_c, E_{dc}, E_{fc} and C are known constant matrices of appropriate dimensions. In *single-rate sampled-data (SSD)* systems, the A/D converter and the D/A converter are, respectively, described by

$$\psi(k) = y(kh) \tag{14.2}$$

$$u(t) = \upsilon(k), \ kh \leqslant t < (k+1)h \tag{14.3}$$

where h is the sampling period, $\psi \in \mathbf{R}^m$ is the sampled process output signal, $\upsilon \in \mathbf{R}^p$ is the discrete-time control input sequence delivered by the controller program. If there is no model uncertainty in (14.1), then the influence of the control input signal on the measured output signal can be easily figured out and compensated in residual generation, no matter which kind of D/A converter is adopted. Therefore, for the sake of simplicity, it is assumed in (14.3) that the D/A converter is a zero-order hold. To reduce the effect of measurement noise, a low-pass filter is usually added to the plant output $y(t)$ before it is discretized. On the assumption that model (14.1)-(14.3) describes the whole system dynamics, including the dynamics of the low-pass filter, we suppose, without loss of generality, that the plant model under consideration is strictly proper.

At the sampling instants, the dynamics of the SSD system described by (14.1)-(14.3) can be described by

$$\begin{aligned} x(k+1) &= Ax(k) + B\upsilon(k) + \bar{d}(k) + \bar{f}(k) \\ \psi(k) &= Cx(k) \end{aligned} \tag{14.4}$$

where

$$x(k) = x(kh), \ A = e^{A_c h}, \ B = \int_0^h e^{A_c t} B_c dt \tag{14.5}$$

$$\bar{d}(k) = \int_0^h e^{A_c(h-\tau)} E_{dc} d(kh+\tau) d\tau, \ \bar{f}(k) = \int_0^h e^{A_c(h-\tau)} E_{fc} f(kh+\tau) d\tau$$

It is worth noticing that in SD systems there is significant difference between $u(t)$ and $d(t), f(t)$. Due to the D/A converter (14.3), $u(t)$ is a piecewise constant signal. The influence of $u(t)$ on $y(t)$ is exactly known from the information of $\upsilon(k)$ and can thus be completely compensated in residual generation. In comparison, $d(t)$ and $f(t)$ are unknown signals and can take an arbitrary form between two sampling instants. Hence, the key is to study the influence of continuous-time signals $d(t)$ and $f(t)$ on the discrete-time sampled output signals $\psi(k)$ and residual signals $r(k)$.

14.2 Indirect FD approaches

Conventionally, an FD system can be designed for the SSD system by *indirect approaches*, i.e.

- *analog design and SD implementation*, or
- *discrete-time design based on the discretization of the process model.*

In the first kind of indirect design approaches, a continuous-time decoupling or optimal residual generator will be designed for the continuous-time process (14.1) at first. Then, the resulting residual generator is discretized, as the measurement information of the output $y(t)$ is only available at discrete sampling instants. However, as $y(t)$ is usually not constant during the sampling interval, the discretization of the residual generator will subject to approximation error. Hence, though the performance index is optimal in terms of the continuous-time residual generators, little can be said about the optimality of the discrete-time residual generator obtained after the discretization.

In the second kind of indirect design approaches, the continuous-time process model (14.1) will be discretized at first into

$$\begin{aligned} x(k+1) &= Ax(k) + Bv(k) + E_{dd}d(k) + E_{fd}f(k) \\ \psi(k) &= Cx(k) \end{aligned} \tag{14.6}$$

where $\bar{d}(k)$ and $\bar{f}(k)$ in (14.4) are approximated by

$$\begin{aligned} \bar{d}(k) &\approx E_{dd}d(k),\ E_{dd} = \int_0^h e^{A_c t} E_{dc} dt \\ \bar{f}(k) &\approx E_{fd}f(k),\ E_{fd} = \int_0^h e^{A_c t} E_{fc} dt \end{aligned} \tag{14.7}$$

Then, based on the discrete-time model (14.6), a discrete-time residual generator is designed using the approaches introduced in Part I. The approximation error in the model (14.6) would be negligible, if $d(t)$ and $f(t)$ are piecewise constant or vary slowly with the time. However, if there is not any prior knowledge of $d(t)$ and $f(t)$, the approximation error may strongly influence the FD performance.

Example 14.1 Given an SD system described by (14.1)-(14.3) with sampling period $h = 1s$ and

$$\begin{aligned} A_c &= \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix},\ B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ E_{dc} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \\ E_{fc} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{14.8}$$

Indirect design "analog design and SD implementation"

Based on the continuous-time process model (14.1), a continuous-time residual generator can be designed as

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} z(t) + \begin{bmatrix} -0.02 \\ 0.001 \end{bmatrix} u(t) + \begin{bmatrix} -0.20 & 0.97 \\ -0.19 & -0.03 \end{bmatrix} y(t) \\ r(t) &= \begin{bmatrix} 0 & -1 \end{bmatrix} z(t) + \begin{bmatrix} -0.01 & 0.001 \end{bmatrix} y(t) \end{aligned} \tag{14.9}$$

It can be verified

$$G_{rd}(s) = 0, G_{rf}(s) \neq 0$$

where $G_{rd}(s)$ and $G_{rf}(s)$ denote, respectively, the transfer function matrices from $d(t)$ and $f(t)$ to the residual $r(t)$. That means (14.9) realizes a full decoupling for (14.1). In the next step, we discretize (14.9) with $h = 1s$ and obtain

$$\begin{aligned} z(k+1) &= \begin{bmatrix} 0.66 & -0.53 \\ 0.53 & 0.13 \end{bmatrix} z(k) + \begin{bmatrix} -0.02 \\ -0.006 \end{bmatrix} u(k) + \begin{bmatrix} -0.11 & 0.86 \\ -0.17 & 0.31 \end{bmatrix} y(k) \\ r(k) &= \begin{bmatrix} 0 & -1 \end{bmatrix} z(k) + \begin{bmatrix} -0.01 & 0.001 \end{bmatrix} y(k) \end{aligned} \tag{14.10}$$

Indirect design "discrete-time design based on the discretized process model"

Discretizing the process model (14.1) with $h = 1s$ yields a discrete-time model in the form of (14.6) with

$$A = \begin{bmatrix} 0.37 & 1.16 \\ 0 & 0.14 \end{bmatrix},\ B = \begin{bmatrix} 1.00 \\ 0.43 \end{bmatrix},\ E_d = \begin{bmatrix} 1.06 \\ 0.43 \end{bmatrix},\ E_f = \begin{bmatrix} 1.00 \\ 0.43 \end{bmatrix} \tag{14.11}$$

Based on (14.6), a discrete-time residual generator can be designed as

$$\begin{aligned} z(k+1) &= \begin{bmatrix} 0 & -0.24 \\ 1 & 1 \end{bmatrix} z(k) + \begin{bmatrix} 0.0049 \\ 0.0210 \end{bmatrix} u(k) + \begin{bmatrix} -0.1080 & 0.1321 \\ 0.2869 & -1.2812 \end{bmatrix} y(k) \\ r(k) &= \begin{bmatrix} 0 & -1 \end{bmatrix} z(k) + \begin{bmatrix} -0.3323 & 0.8165 \end{bmatrix} y(k) \end{aligned} \tag{14.12}$$

14.3 Direct FD approaches

Motivated by the development of sampled-data control, in the last years the FD problems of the SSD systems have been studied from the viewpoint of *direct design* to take into account the intersample behavior and eliminate the approximation made during the design. The basic idea is to introduce operators to describe the influence of the continuous-time disturbance and fault signals on the discrete-time residual signals [174, 176, 178, 180]. Based on it, optimization problems will be defined and solved.

14.3.1 Parity relation based FD scheme for SSD systems

The output of the SSD system over the moving horizon $[k-s,k]$ can be expressed as

$$\psi_s(k) = H_{o,s}x(k-s) + H_{\upsilon,s}\upsilon_s(k) + H_s\left(\bar{d}_s(k) + \bar{f}_s(k)\right) \tag{14.13}$$

where $H_{o,s}$ is the same as in (2.4),

$$\psi_s(k) = \begin{bmatrix} \psi(k-s) \\ \psi(k-s+1) \\ \vdots \\ \psi(k) \end{bmatrix},\ \upsilon_s(k) = \begin{bmatrix} \upsilon(k-s) \\ \upsilon(k-s+1) \\ \vdots \\ \upsilon(k) \end{bmatrix}$$
$$\bar{d}_s(k) = \begin{bmatrix} \bar{d}(k-s) \\ \bar{d}(k-s+1) \\ \vdots \\ \bar{d}(k) \end{bmatrix},\ \bar{f}_s(k) = \begin{bmatrix} \bar{f}(k-s) \\ \bar{f}(k-s+1) \\ \vdots \\ \bar{f}(k) \end{bmatrix}$$
$$H_{\upsilon,s} = \begin{bmatrix} O & O & \cdots & O \\ CB & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}B & \cdots & CB & O \end{bmatrix},\ H_s = \begin{bmatrix} O & O & \cdots & O \\ C & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1} & \cdots & C & O \end{bmatrix} \tag{14.14}$$

Hence, a parity relation based residual generator

$$r(k) = V_s(\psi_s(k) - H_{\upsilon,s}\upsilon_s(k)) \tag{14.15}$$

can be used for residual generation, where $V_sH_{o,s} = 0$. The residual dynamics is governed by

$$r(k) = V_sH_s\left(\bar{d}_s(k) + \bar{f}_s(k)\right) \tag{14.16}$$

To describe the intersample behavior, for a continuous-time signal $\delta(t)$ with δ standing for d and f, an operator is defined as follows

$$\Psi^\delta\delta_{k,s}(t) = \begin{bmatrix} \bar{\delta}(k-s) \\ \bar{\delta}(k-s+1) \\ \vdots \\ \bar{\delta}(k) \end{bmatrix} = \begin{bmatrix} \int_0^h e^{A_c(h-\tau)}E_{\delta c}\delta((k-s)h+\tau)d\tau \\ \int_0^h e^{A_c(h-\tau)}E_{\delta c}\delta((k-s+1)h+\tau)d\tau \\ \vdots \\ \int_0^h e^{A_c(h-\tau)}E_{\delta c}\delta(kh+\tau)d\tau \end{bmatrix} \tag{14.17}$$

The inner products on the input signal space and the output signal space are defined, respectively, by

$$\begin{aligned} \left\langle \delta_{k,s}(t), \gamma_{k,s}(t)\right\rangle &= \sum_{i=0}^{s}\int_0^h \delta^T((k-s+i)h+\tau)\gamma((k-s+i)h+\tau)d\tau \\ \left\langle \beta_s(k), \eta_s(k)\right\rangle &= \beta_s^T(k)\eta_s(k) \end{aligned} \tag{14.18}$$

With the help of the above operators, residual dynamics (14.16) can be re-written as

$$r(k) = V_s H_s (\Psi^d d_{k,s}(t) + \Psi^f f_{k,s}(t)) \tag{14.19}$$

The influence of the continuous-time signal $\delta(t)$ over the time interval $[(k-s)h, (k+1)h)$ on the discrete-time residual signal $r(k)$ is measured by

$$\sup_{\delta \in \mathcal{L}_{2,[(k-s)h,(k+1)h)}} \frac{r^T(k)r(k)}{\|\delta\|^2_{2,[(k-s)h,(k+1)h)}} = \bar{\lambda}(V_s H_s \Psi^\delta \left(\Psi^\delta\right)^* H_s^T V_s^T)$$

$$\inf_{\delta \in \mathcal{L}_{2,[(k-s)h,(k+1)h)}} \frac{r^T(k)r(k)}{\|\delta\|^2_{2,[(k-s)h,(k+1)h)}} = \underline{\lambda}(V_s H_s \Psi^\delta \left(\Psi^\delta\right)^* H_s^T V_s^T)$$

where $\left(\Psi^\delta\right)^*$ denotes the adjoint of the operator Ψ^δ. The optimization problems are thus formulated as

$$\max_{V_s, V_s H_{o,s}=0} J_{SSD,PS,\infty/\infty}(V_s) = \max_{V_s, V_s H_{o,s}=0} \frac{\bar{\lambda}(V_s H_s \Psi^f \left(\Psi^f\right)^* H_s^T V_s^T)}{\bar{\lambda}(V_s H_s \Psi^d \left(\Psi^d\right)^* H_s^T V_s^T)} \tag{14.20}$$

$$\max_{V_s, V_s H_{o,s}=0} J_{SSD,PS,-/\infty}(V_s) = \max_{V_s, V_s H_{o,s}=0} \frac{\underline{\lambda}(V_s H_s \Psi^f \left(\Psi^f\right)^* H_s^T V_s^T)}{\bar{\lambda}(V_s H_s \Psi^d \left(\Psi^d\right)^* H_s^T V_s^T)} \tag{14.21}$$

$$\max_{V_s, V_s H_{o,s}=0} J_{SSD,PS,i/\infty}(V_s) = \max_{V_s, V_s H_{o,s}=0} \frac{\lambda_i(V_s H_s \Psi^f \left(\Psi^f\right)^* H_s^T V_s^T)}{\bar{\lambda}(V_s H_s \Psi^d \left(\Psi^d\right)^* H_s^T V_s^T)} \tag{14.22}$$

To solve optimization problems (14.20)-(14.22), it is necessary to get an analytical expression of $\Psi^\delta \left(\Psi^\delta\right)^*$. For this purpose, consider the following equation [138]

$$\left\langle \Psi^\delta \delta_{k,s}(t), \beta_s(k) \right\rangle = \left\langle \delta_{k,s}(t), \left(\Psi^\delta\right)^* \beta_s(k) \right\rangle$$

from which $\left(\Psi^\delta\right)^*$ can be determined uniquely, where

$$\beta_s(k) = \begin{bmatrix} \beta_{s,0}(k) \\ \beta_{s,1}(k) \\ \vdots \\ \beta_{s,s}(k) \end{bmatrix}$$

is a vector of compatible dimensions. From

$$\begin{aligned} &\left\langle \Psi^\delta \delta_{k,s}(t), \beta_s(k) \right\rangle \\ &= \left(\Psi^\delta \delta_{k,s}(t)\right)^T \beta_s(k) \\ &= \sum_{j=0}^{s} \left(\int_0^h e^{A_c(h-\tau)} E_{\delta c} \delta((k-j)h+\tau) d\tau \right)^T \beta_{s,s-j}(k) \\ &= \sum_{j=0}^{s} \int_{(k-j)h}^{(k-j+1)h} \delta^T(t) E_{\delta c}^T e^{A_c^T((k-j+1)h-\tau)} \beta_{s,s-j}(k) d\tau \end{aligned}$$

we obtain that

$$\left(\Psi^\delta\right)^* \beta_s(k) = \begin{bmatrix} E_{\delta c}^T e^{A_c^T((k-s+1)h-\tau)} \beta_{s,0}(k), \text{ if } (k-s)h \le t < (k-s+1)h \\ E_{\delta c}^T e^{A_c^T((k-s+2)h-\tau)} \beta_{s,1}(k), \text{ if } (k-s+1)h \le t < (k-s+2)h \\ \vdots \\ E_{\delta c}^T e^{A_c^T((k+1)h-\tau)} \beta_{s,s}(k), \text{ if } kh \le t < (k+1)h \end{bmatrix}$$

Therefore, $\Psi^{\delta}\left(\Psi^{\delta}\right)^{*}$ is obtained as

$$\Psi^{\delta}\left(\Psi^{\delta}\right)^{*}=\begin{bmatrix}\bar{E}_{\delta}\bar{E}_{\delta}^{T} & O & \cdots & O\\ O & \bar{E}_{\delta}\bar{E}_{\delta}^{T} & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ O & \cdots & O & \bar{E}_{\delta}\bar{E}_{\delta}^{T}\end{bmatrix} \tag{14.23}$$

$$\bar{E}_{\delta}\bar{E}_{\delta}^{T}=\int_{0}^{h} e^{A_{c}\tau}E_{\delta c}E_{\delta c}^{T}e^{A_{c}^{T}\tau}d\tau \tag{14.24}$$

Due to (14.24), optimization problems (14.20)-(14.22) can be transformed into some equivalent optimization problems

$$\max_{V_s,\ V_sH_{o,s}=0} J_{SSD,PS,\infty/\infty}(V_s)=\max_{V_s,\ V_sH_{o,s}=0}\frac{\bar{\sigma}^{2}(V_s\bar{H}_{f,s})}{\bar{\sigma}^{2}(V_s\bar{H}_{d,s})} \tag{14.25}$$

$$\max_{V_s,\ V_sH_{o,s}=0} J_{SSD,PS,-/\infty}(V_s)=\max_{V_s,\ V_sH_{o,s}=0}\frac{\underline{\sigma}^{2}(V_s\bar{H}_{f,s})}{\bar{\sigma}^{2}(V_s\bar{H}_{d,s})} \tag{14.26}$$

$$\max_{V_s,\ V_sH_{o,s}=0} J_{SSD,PS,i/\infty}(V_s)=\max_{V_s,\ V_sH_{o,s}=0}\frac{\sigma_i^{2}(V_s\bar{H}_{f,s})}{\bar{\sigma}^{2}(V_s\bar{H}_{d,s})} \tag{14.27}$$

where $\bar{H}_{f,s}$ and $\bar{H}_{d,s}$ are built based on $(A,\bar{E}_f,C,O)$ and $(A,\bar{E}_d,C,O)$ in a way similar to $H_{v,s}$ in (14.14) as

$$\bar{H}_{d,s}=\begin{bmatrix}O & O & \cdots & O\\ C\bar{E}_d & O & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ CA^{s-1}\bar{E}_d & \cdots & C\bar{E}_d & O\end{bmatrix}$$

$$\bar{H}_{f,s}=\begin{bmatrix}O & O & \cdots & O\\ C\bar{E}_f & O & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ CA^{s-1}\bar{E}_f & \cdots & C\bar{E}_f & O\end{bmatrix} \tag{14.28}$$

The equivalent optimization problems (14.25)-(14.27) are of the standard form and can be solved with the approaches introduced in Chapter 5.

Theorem 14.1 Given an SD system described by (14.1)-(14.3) with sampling time h and a residual generator (14.15). Let $A,B,\bar{E}_d,\bar{E}_f,H_{o,s},\bar{H}_{d,s}$ and $\bar{H}_{f,s}$ be given by (14.5), (14.24), (2.4) and (14.28), respectively, and N_{basis} be the basis matrix of the left null space of $H_{o,s}$. Assume that the singular value decomposition of $N_{basis}\bar{H}_{d,s}$ is

$$N_{basis}\bar{H}_{d,s}=U\begin{bmatrix}S & O\end{bmatrix}V^{T} \tag{14.29}$$

where U and V are unitary matrices. Then the optimization problems (14.20)-(14.22) are solved simultaneously by

$$V_{s,opt}=\bar{P}_s S^{-1}U^{T}N_{basis} \tag{14.30}$$

where $\bar{P}_s$ is any unitary matrix of compatible dimensions.

Theorem 14.2 Under the same assumptions as stated in Theorem 14.1, assume that $N_{basis}\bar{H}_{f,s}$ is of full column rank. Then the optimization problem (14.21) is also solved by

$$V_{s,-/\infty,opt}=\bar{Q}_s\left(N_{basis}\bar{H}_{f,s}\right)^{-1}N_{basis} \tag{14.31}$$

where $\bar{Q}_s$ is any unitary matrix of compatible dimensions, $\left(N_{basis}\bar{H}_{f,s}\right)^{-1}$ is the left inverse of $N_{basis}\bar{H}_{f,s}$.

Example 14.2 For the SD system given in Example 14.1, design an optimal discrete-time residual generator using the direct parity space approach.

According to (14.24),

$$\bar{E}_d = \begin{bmatrix} 1.1076 & 0 \\ 0.3505 & 0.3501 \end{bmatrix}, \ \bar{E}_f = \begin{bmatrix} 1.0521 & 0 \\ 0.3389 & 0.3613 \end{bmatrix}$$

Let $s = 2$. The optimal parity matrix $V_{s,opt}$ solving (14.20)-(14.22) can be obtained by

$$V_{s,opt} = \begin{bmatrix} 0.0646 & 0.3282 & 0.0642 & -0.1306 & -0.6521 & -0.1980 \\ -0.2688 & -0.8576 & 0.8568 & 0.5172 & -0.3427 & -0.4441 \\ 0.2652 & 0.6043 & -0.9600 & 1.2698 & 0.6501 & -2.2022 \\ 0.2723 & 0.5788 & -0.4397 & 2.7940 & -0.8167 & 1.7532 \end{bmatrix}$$

The optimal performance indices are

$$J_{SSD,PS,\infty/\infty,opt} = 1.0672$$
$$J_{SSD,PS,-/\infty,opt} = 0.9007$$

According to (14.31), the optimization problem (14.21) is also solved by

$$V_{s,-/\infty,opt} = \begin{bmatrix} 0.0659 & 0.3384 & -0.1473 & 0.0692 & -0.6751 & -0.2079 \\ -0.2780 & -0.8898 & 0.8840 & 0.5556 & -0.3487 & -0.5034 \\ 0.2712 & 0.6356 & -0.9760 & 1.1728 & 0.6491 & -2.1426 \\ 0.2732 & 0.5911 & -0.4349 & 2.7592 & -0.8363 & 1.6650 \end{bmatrix} \tag{14.32}$$

As $V_{s,-/\infty,opt}\bar{H}_{f,s}\bar{H}_{f,s}^T V_{s,-/\infty,opt}^T = I$,

$$\begin{aligned} & J_{SSD,PS,\infty/\infty}(V_{s,-/\infty,opt}) \\ = & J_{SSD,PS,-/\infty}(V_{s,-/\infty,opt}) \\ = & J_{SSD,PS,i/\infty}(V_{s,-/\infty,opt}) = 0.9007 \end{aligned}$$

14.3.2 Post filter based FD scheme for SSD systems

In the frequency domain, the SSD system (14.1)-(14.3) can be equivalently described by

$$y(s) = G_{uc}(s)u(s) + G_{dc}(s)d(s) + G_{fc}(s)f(s) \tag{14.33}$$

$$\psi(e^{j\omega h}) = \frac{1}{h}\sum_{k=-\infty}^{+\infty} y(j\omega + jk\omega_s) \tag{14.34}$$

$$u(j\omega) = h\phi(j\omega)v(e^{j\omega h}),\ \phi(j\omega) = e^{-j\omega\frac{h}{2}}\frac{\sin\omega\frac{h}{2}}{\omega\frac{h}{2}} \tag{14.35}$$

where $\omega_s = \frac{2\pi}{h}$ is the sampling frequency, $G_{uc}(s), G_{dc}(s)$ and $G_{fc}(s)$ denote, respectively, the transfer function matrix from $u(t), d(t)$ and $f(t)$ to the output $y(t)$ in the process model,

$$G_{uc}(s) = C(sI - A_c)^{-1}B_c$$
$$G_{dc}(s) = C(sI - A_c)^{-1}E_{dc}$$
$$G_{uc}(s) = C(sI - A_c)^{-1}E_{fc}$$

Let $\omega_k = \omega + k\omega_s$. Substituting (14.33), (14.35) into (14.34) yields

$$\psi(e^{j\omega h}) = \sum_{k=-\infty}^{+\infty} G_{uc}(j\omega_k)\phi(j\omega_k)v(e^{j\omega h}) + \frac{1}{h}\sum_{k=-\infty}^{+\infty}(G_{dc}(j\omega_k)d(j\omega_k) + G_{fc}(j\omega_k)f(j\omega_k)) \tag{14.36}$$

It is well-known that [20]

$$\sum_{k=-\infty}^{+\infty} G_{uc}(j\omega_k)\phi(j\omega_k) = G_{ud}(e^{j\omega h})$$

where $G_{ud}(e^{j\omega h})$ is the frequency response of the step-invariant transformation of $G_{uc}(s)$. So (14.36) can be re-written into

$$\psi(e^{j\omega h}) - G_{ud}(e^{j\omega h})v(e^{j\omega h}) = \frac{1}{h}\sum_{k=-\infty}^{+\infty}(G_{dc}(j\omega_k)d(j\omega_k) + G_{fc}(j\omega_k)f(j\omega_k)) \tag{14.37}$$

The left side of (14.37) is determined by the available information of the sampled outputs and the discrete-time control inputs, while its right side describes the influence of the unknown disturbances and faults. Thus, based on (14.37), a residual generator can be constructed as

$$r(z) = Q(z)(\psi(z) - G_{ud}(z)v(z)) \tag{14.38}$$

where $Q(z) \in \mathbf{RH}_\infty$ and will be determined as follows. Let $M_u(z)$ and $N_u(z)$ be left coprime $\mathbf{RH}_\infty$-matrices and $G_{ud}(z) = M_u^{-1}(z)N_u(z)$. In order to ensure the stability of the residual generator, we select

$$Q(z) = R(z)M_u(z) \tag{14.39}$$

where the so-called post-filter $R(z) \in \mathbf{RH}_\infty$ is arbitrarily selectable. As a result, we have a stable residual generator

$$r(z) = R(z)(M_u(z)\psi(z) - N_u(z)v(z)) \tag{14.40}$$

It follows from (14.37) that the dynamics of the residual generator is governed by

$$r(e^{j\omega h}) = R(e^{j\omega h})M_u(e^{j\omega h})(\psi_d(e^{j\omega h}) + \psi_f(e^{j\omega h})) \tag{14.41}$$

$$\psi_d(e^{j\omega h}) = \frac{1}{h}\sum_{k=-\infty}^{+\infty} G_{dc}(j\omega_k)d(j\omega_k),\ \psi_f(e^{j\omega h}) = \sum_{k=-\infty}^{+\infty} G_{fc}(j\omega_k)f(j\omega_k)$$

For our purpose, an operator Γ^δ is now introduced, which maps a vector $\delta(j\omega) \in \mathcal{L}_2(j\mathbf{R}, \mathbf{C}^{n_\delta})$, the continuous-time Fourier transform of a continuous-time signal $\delta(t)$ to $\beta(e^{j\omega h}) \in \mathcal{L}_2(\mathbf{\Omega}, \mathbf{C}^m)$, which is equivalent to the discrete-time Fourier transform of a certain discrete-time signal with $\mathbf{\Omega}$ denoting the unit circle in the complex plane, i.e.

$$\Gamma^\delta \delta(j\omega) = \frac{1}{h}\sum_{k=-\infty}^{+\infty} G_{\delta c}(j\omega_k)\delta(j\omega_k)$$

Further, define the inner products on $\mathcal{L}_2(j\mathbf{R}, \mathbf{C}^{n_\delta})$ and on $\mathcal{L}_2(\mathbf{\Omega}, \mathbf{C}^m)$, respectively, by

$$< \delta(j\omega), \gamma(j\omega) >= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta^*(j\omega)\gamma(j\omega)d\omega$$

$$< \beta(e^{j\omega h}), \chi(e^{j\omega h}) >= \frac{h}{2\pi}\int_0^{\omega_s} \beta^*(e^{j\omega h})\chi(e^{j\omega h})d\omega$$

Using operator Γ^{δ}, (14.41) can be expressed as

$$r(e^{j\omega h}) = R(e^{j\omega h})M_u(e^{j\omega h})(\Gamma^d d(j\omega) + \Gamma^f f(j\omega)) \tag{14.42}$$

Aiming at a trade-off between the sensitivity of r to f and simultaneously its robustness to d, the optimal design problem of the post-filter $R(z)$ can be formulated in a way similar to the performance indices (5.50)-(5.52) and (5.69) as

$$\begin{aligned}
&\max_{R(z)} J_{SSD,FRE,\infty/\infty}(R(z)) \qquad (14.43)\\
&= \max_{R(z)} \frac{\sup_{d=0,\ f\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}}\\
&= \max_{R(z)} \frac{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^f \left(\Gamma^f\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h}))}{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^d \left(\Gamma^d\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h}))}\\
&\max_{R(z)} J_{SSD,FRE,-/\infty}(R(z)) \qquad (14.44)\\
&= \max_{R(z)} \frac{\inf_{d=0,\ f\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}}\\
&= \max_{R(z)} \frac{\inf_{\omega\in[0,\omega_s]} \underline{\lambda}^{1/2}(R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^f \left(\Gamma^f\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h}))}{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^d \left(\Gamma^d\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h}))}\\
&\max_{R(z)} J_{SSD,FRE,i/\infty}(R(z)) \qquad (14.45)\\
&= \max_{R(z)} \frac{\lambda_i^{1/2}(R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^f \left(\Gamma^f\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h}))}{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^d \left(\Gamma^d\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h}))}\\
&\max_{R(z)\in\mathbf{RH}_\infty^{1\times m}} J_{SSD,FRE,2/2}(R(z)) \qquad (14.46)\\
&= \max_{R(z)\in\mathbf{RH}_\infty^{1\times m}} \frac{\int_0^{\omega_s} R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^f \left(\Gamma^f\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h})d\omega}{\int_0^{\omega_s} R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^d \left(\Gamma^d\right)^* M_u^*(e^{j\omega h})\, R^*(e^{j\omega h})d\omega}
\end{aligned}$$

where $\left(\Gamma^f\right)^*, \left(\Gamma^d\right)^*$ denote the adjoint of Γ^f and Γ^d, respectively, $\lambda_i(\cdot)$ denotes any eigenvalue of the matrix.

It is evident that, in order to solve the optimization problems (14.43)-(14.46), we have to first study $\Gamma^d \left(\Gamma^d\right)^*$ and $\Gamma^f \left(\Gamma^f\right)^*$. To this end, consider again the following equation [138]

$$\left\langle \Gamma^{\delta}\delta(j\omega), \beta(e^{j\omega h})\right\rangle = \left\langle \delta(j\omega), \left(\Gamma^{\delta}\right)^* \beta(e^{j\omega h})\right\rangle$$

Since

$$\begin{aligned}
&< \Gamma^{\delta}\delta(j\omega), \beta(e^{j\omega h}) >\\
&= \frac{h}{2\pi}\int_0^{\omega_s}\left(\frac{1}{h}\sum_{k=-\infty}^{\infty} G_{\delta c}(j\omega + jk\omega_s)\delta(j\omega + jk\omega_s)\right)^* \beta(e^{j\omega h})d\omega\\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int_0^{\omega_s} \delta^*(j\omega + jk\omega_s)G_{\delta c}^*(j\omega + jk\omega_s)\beta(e^{j\omega h})d\omega\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta^*(j\omega)G_{\delta c}^*(j\omega)\beta(e^{j\omega h})d\omega\\
&= < \delta(j\omega), \left(\Gamma^{\delta}\right)^* \beta(e^{j\omega h}) >
\end{aligned}$$

it turns out

$$\left(\Gamma^{\delta}\right)^{*}\beta(e^{j\omega h}) = G_{\delta c}^{*}(j\omega)\beta(e^{j\omega h})$$

$$\Gamma^{\delta}\left(\Gamma^{\delta}\right)^{*} = \frac{1}{h}\sum_{k=-\infty}^{+\infty} G_{\delta c}(j\omega + jk\omega_s)G_{\delta c}^{*}(j\omega + jk\omega_s)$$

Lemma 14.2 [178] Assume that $G_{\delta c}(j\omega)G_{\delta c}^{*}(j\omega)$ can be expressed as

$$G_{\delta c}(j\omega)G_{\delta c}^{*}(j\omega) = \sum_{i=1}^{l}\sum_{q=1}^{\nu_i}\tilde{P}_{iq}(\omega) \tag{14.47}$$

$$\tilde{P}_{iq}(\omega) = \frac{P_{iq}}{(j\omega - \lambda_i)^q} + \frac{(-1)^q P_{iq}^T}{(j\omega + \lambda_i)^q} \tag{14.48}$$

$$\lambda_i = \begin{cases} \sigma_i, \text{ if } Re(\sigma_i) < 0 \\ -\sigma_i, \text{ if } Re(\sigma_i) \geq 0 \end{cases} \tag{14.49}$$

where $\sigma_1, \cdots, \sigma_l$ are the poles of $G_{\delta c}(s)$ with multiplicity $\nu_1, \cdots, \nu_l$ respectively and the coefficient matrices $P_{iq} \in \mathbf{C}^{m\times m}$ can be calculated by known techniques. Then $\Gamma^{\delta}\left(\Gamma^{\delta}\right)^{*}$ is derived to be

$$\begin{aligned}\Gamma^{\delta}\left(\Gamma^{\delta}\right)^{*} &= \frac{1}{h}\sum_{k=-\infty}^{+\infty} G_{\delta c}(j\omega + jk\omega_s)G_{\delta c}^{*}(j\omega + jk\omega_s) \\ &= \sum_{i=1}^{l}\sum_{q=1}^{\nu_i}\frac{j^{q-1}}{(q-1)!}(P_{iq}\frac{\partial^{q-1}}{\partial\omega^{q-1}}(\frac{1}{1-e^{(\lambda_i - j\omega)h}}) \\ &+ (-1)^{q-1}P_{iq}^T\frac{\partial^{q-1}}{\partial\omega^{q-1}}(\frac{e^{(\lambda_i + j\omega)h}}{1-e^{(\lambda_i + j\omega)h}}))\end{aligned} \tag{14.50}$$

Lemma 14.2 provides a closed expression of $\Gamma^{\delta}\left(\Gamma^{\delta}\right)^{*}$ derived using the partial fractional expression approach [178]. It also helps the understanding of the optimization problems (14.43)-(14.46).

Lemma 14.3 [70, 192] Assume that $(A_c, E_{\delta c}, C, O)$ is a state space realization of $G_{\delta c}(s)$. Then $\Gamma^{\delta}\left(\Gamma^{\delta}\right)^{*}$ can be factorized as

$$\Gamma^{\delta}\left(\Gamma^{\delta}\right)^{*} = \bar{G}_{\delta}(e^{j\omega h})\bar{G}_{\delta}^{*}(e^{j\omega h}) \tag{14.51}$$

where

$$\bar{G}_{\delta}(z) = C(zI - A)^{-1}\bar{E}_{\delta} \tag{14.52}$$

with A given by (14.5) and $\bar{E}_{\delta}$ by (14.24).

Thus, the optimization problems (14.43)-(14.46) are equivalent to

$$\max_{R(z)} J_{SSD,FRE,\infty/\infty}(R(z)) = \max_{R(z)}\frac{\|R(z)M_u(z)\bar{G}_f(z)\|_{\infty}}{\|R(z)M_u(z)\bar{G}_d(z)\|_{\infty}} \tag{14.53}$$

$$\max_{R(z)} J_{SSD,FRE,-/\infty}(R(z)) = \max_{R(z)}\frac{\|R(z)M_u(z)\bar{G}_f(z)\|_{-}}{\|R(z)M_u(z)\bar{G}_d(z)\|_{\infty}} \tag{14.54}$$

$$\max_{R(z)} J_{SSD,FRE,i/\infty}(R(z)) = \max_{R(z)}\frac{\sigma_i(R(e^{j\omega h})M_u(e^{j\omega h})\bar{G}_f(e^{j\omega h}))}{\|R(z)M_u(z)\bar{G}_d(z)\|_{\infty}} \tag{14.55}$$

$$\begin{aligned}&\max_{R(z)} J_{SSD,FRE,2/2}(R(z)) \\ &= \max_{R(z)}\frac{\int_0^{\omega_s} R(e^{j\omega h})M_u(e^{j\omega h})\bar{G}_f(e^{j\omega h})\bar{G}_f^{*}(e^{j\omega h})M_u^{*}(e^{j\omega h})R^{*}(e^{j\omega h})d\omega}{\int_0^{\omega_s} R(e^{j\omega h})\,M_u(e^{j\omega h})\bar{G}_d(e^{j\omega h})\bar{G}_d^{*}(e^{j\omega h})M_u^{*}(e^{j\omega h})R^{*}(e^{j\omega h})d\omega}\end{aligned} \tag{14.56}$$

Theorem 14.3 Given an SD system described by (14.1)-(14.3) with sampling time h and a residual generator (14.40). Assume that $(M_u(z), N_u(z))$ is a left coprime factorization of $G_{ud}(z)$ and

$$M_u(z)\bar{G}_d(z) = \bar{G}_{do}(z)\bar{G}_{di}(z) \tag{14.57}$$

is a co-inner-outer factorization of $M_u(z)\bar{G}_d(z)$, where

$$G_{ud}(z) = C(zI - A)^{-1}B, \ \bar{G}_d(z) = C(zI - A)^{-1}\bar{E}_d$$
$$A = e^{A_c h}, \ B = \int_0^h e^{A_c t}B_c dt, \ \bar{E}_d\bar{E}_d^T = \int_0^h e^{A_c\tau}E_{dc}E_{dc}^T e^{A_c^T\tau}d\tau$$

$\bar{G}_{do}(z)$ is the co-outer and $\mathbf{RH}_\infty$-left-invertible, and $\bar{G}_{di}(z)$ is the co-inner satisfying

$$\bar{G}_{di}(e^{j\omega h})\bar{G}_{di}^*(e^{j\omega h}) = I, \ \forall\omega \in [0, \omega_s]$$

Let Q_d be an arbitrary unitary matrix. Then residual generator (14.40) with

$$R_{opt}(z) = Q_d\bar{G}_{do}^{-1}(z) \tag{14.58}$$

solves simultaneously the $H_\infty/H_\infty, H_-/H_\infty$ and H_i/H_∞ optimization problems defined by (14.43)-(14.45).

Theorem 14.4 Given an SD system described by (14.1)-(14.3) with sampling time h and a residual generator (14.40). Assume that

$$M_u(z)\bar{G}_f(z) = \bar{G}_{fo}(z)\bar{G}_{fi}(z) \tag{14.59}$$

is a co-inner-outer factorization of $M_u(z)\bar{G}_f(z)$, where

$$\bar{G}_f(z) = C(zI - A)^{-1}\bar{E}_f, \ \bar{E}_f\bar{E}_f^T = \int_0^h e^{A_c\tau}E_{fc}E_{fc}^T e^{A_c^T\tau}d\tau$$

$\bar{G}_{fo}(z)$ is the co-outer and $\mathbf{RH}_\infty$-left-invertible, and $\bar{G}_{fi}(z)$ is the co-inner satisfying

$$\bar{G}_{fi}(e^{j\omega h})\bar{G}_{fi}^*(e^{j\omega h}) = I, \ \forall\omega \in [0, \omega_s]$$

Let Q_f be an arbitrary unitary matrix. Then residual generator (14.40) with

$$R_{opt,-/\infty}(z) = Q_f\bar{G}_{fo}^{-1}(z) \tag{14.60}$$

solves the H_-/H_∞ optimization problems defined by (14.44).

Theorem 14.5 For the SD system given in Theorem 14.3, the optimal solution to the optimization problem (14.46) is

$$\begin{aligned} R_{opt,2/2}(z) &= f_{\omega_0}(z)v_{\max}(z) \\ J_{SSD,FRE,2/2,opt} &= \sup_\omega \lambda_{\max}(\omega) = \lambda_{\max}(\omega_0) \end{aligned} \tag{14.61}$$

where $v_{\max}(z)$ is obtained by solving the generalized eigenvalue-eigenvector problem

$$v_{\max}(e^{j\omega h})(M_u(e^{j\omega h})\Gamma^f\left(\Gamma^f\right)^* M_u^*(e^{j\omega h}) - \lambda_{\max}(\omega)M_u(e^{j\omega h})\Gamma^d\left(\Gamma^d\right)^* M_u^*(e^{j\omega h})) = 0 \tag{14.62}$$

$\Gamma^f\left(\Gamma^f\right)^*$ and $\Gamma^d\left(\Gamma^d\right)^*$ can be calculated by (14.50) or (14.51), and $f_{\omega_0}(z)$ is an ideal discrete-time frequency-selective filter with selective frequency at ω_0.

14.3.3 Observer based FD scheme for SSD systems

Considering the dynamics of the SSD system described by (14.4), it is easy to construct an observer based residual generator as

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bv(k) + L(\psi(k) - \hat{\psi}(k)) \\ \hat{\psi}(k) &= C\hat{x}(k) \\ r(k) &= W(\psi(k) - \hat{\psi}(k))\end{aligned} \tag{14.63}$$

where L stabilizes $A - LC$. Using operator Γ^{δ} introduced in the last subsection, the dynamics of residual generator (14.63) can be expressed in the frequency domain as

$$r(e^{j\omega h}) = WM_u(e^{j\omega h})(\Gamma^d d(j\omega) + \Gamma^f f(j\omega)) \tag{14.64}$$

where

$$M_u(e^{j\omega h}) = I - C(e^{j\omega h}I - A + LC)^{-1}L$$

As a result, the optimal design problems of the observer based residual generators are formulated as

$$\max_{L,W} J_{SSD,OBS,\infty/\infty}(L,W) = \max_{L,W} \frac{\sup_{d=0,\ f\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}}$$

$$= \max_{L,W} \frac{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(WM_u(e^{j\omega h})\Gamma^f \left(\Gamma^f\right)^* M_u^*(e^{j\omega h})W^T)}{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(WM_u(e^{j\omega h})\Gamma^d \left(\Gamma^d\right)^* M_u^*(e^{j\omega h})W^T)} \tag{14.65}$$

$$\max_{L,W} J_{SSD,OBS,-/\infty}(L,W) = \max_{L,W} \frac{\inf_{d=0,\ f\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}}$$

$$= \max_{L,W} \frac{\inf_{\omega\in[0,\omega_s]} \underline{\lambda}^{1/2}(WM_u(e^{j\omega h})\Gamma^f \left(\Gamma^f\right)^* M_u^*(e^{j\omega h})W^T)}{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(WM_u(e^{j\omega h})\Gamma^d \left(\Gamma^d\right)^* M_u^*(e^{j\omega h})W^T)} \tag{14.66}$$

$$\max_{L,W} J_{SSD,OBS,i/\infty}(L,W)$$

$$= \max_{L,W} \frac{\lambda_i^{1/2}(WM_u(e^{j\omega h})\Gamma^f \left(\Gamma^f\right)^* M_u^*(e^{j\omega h})W^T)}{\sup_{\omega\in[0,\omega_s]} \bar{\lambda}^{1/2}(WM_u(e^{j\omega h})\Gamma^d \left(\Gamma^d\right)^* M_u^*(e^{j\omega h})W^T)} \tag{14.67}$$

Due to (14.51), the $H_\infty/H_\infty, H_-/H_\infty$ and H_i/H_∞ design problems of the SSD system are equivalent, respectively, to that of a discrete LTI system and can be obtained by solving equivalent optimization problems [180, 176]

$$\max_{L,W} J_{SSD,OBS,\infty/\infty}(L,W) = \max_{L,W} \frac{\|\bar{G}_{rf}(z)\|_\infty}{\|\bar{G}_{rd}(z)\|_\infty} \tag{14.68}$$

$$\max_{L,W} J_{SSD,OBS,-/\infty}(L,W) = \max_{L,W} \frac{\|\bar{G}_{rf}(z)\|_-}{\|\bar{G}_{rd}(z)\|_\infty} \tag{14.69}$$

$$\max_{L,W} J_{SSD,OBS,i/\infty}(L,W) = \max_{L,W} \frac{\sigma_i(\bar{G}_{rf}(z))}{\|\bar{G}_{rd}(z)\|_\infty} \tag{14.70}$$

where

$$\begin{aligned}\bar{G}_{rf}(z) &= WM_u(z)\bar{G}_f(z) = C(zI - A + LC)^{-1}\bar{E}_f \\ \bar{G}_{rd}(z) &= WM_u(z)\bar{G}_d(z) = C(zI - A + LC)^{-1}\bar{E}_d\end{aligned}$$

By applying the techniques in Chapter 5 to solve the equivalent optimization problems (14.68)-(14.70), the following theorems are obtained.

Theorem 14.6 Given an SD system described by (14.1)-(14.3) with sampling time h and an observer-based residual generator (14.63). Let $A, \bar{E}_d$ and $\bar{E}_f$ be given by (14.5) and (14.24), respectively. Then the optimal observer parameters L and W that solves (14.65)-(14.67) are given by

$$L_{opt} = -L_d^T, \; W_{opt} = Q_d W_d \tag{14.71}$$

where Q_d is an arbitrary unitary matrix, W_d is the left inverse of a full column rank matrix H_d satisfying $H_d H_d^T = C X_d C^T$, (X_d, L_d) is the stabilizing solution to the DTARS

$$\begin{bmatrix} AX_dA^T - X_d + \bar{E}_d\bar{E}_d^T & AX_dC^T \\ CX_dA^T & CX_dC^T \end{bmatrix} \begin{bmatrix} I \\ L_d \end{bmatrix} = 0 \tag{14.72}$$

Theorem 14.7 Given an SD system described by (14.1)-(14.3) with sampling time h and an observer-based residual generator (14.63). Let $A, \bar{E}_d$ and $\bar{E}_f$ be given by (14.5) and (14.24), respectively. Then L and W given by

$$L_{opt,-/\infty} = -L_f^T, \; W_{opt,-/\infty} = Q_f W_f \tag{14.73}$$

solves also optimization problem (14.66), where Q_f is an arbitrary unitary matrix, W_f is the left inverse of a full column rank matrix H_f satisfying $H_f H_f^T = C X_f C^T$, (X_f, L_f) is the stabilizing solution to the DTARS

$$\begin{bmatrix} AX_fA^T - X_f + \bar{E}_f\bar{E}_f^T & AX_fC^T \\ CX_fA^T & CX_fC^T \end{bmatrix} \begin{bmatrix} I \\ L_f \end{bmatrix} = 0 \tag{14.74}$$

Example 14.3 To illustrate the proposed design procedures, in this section, we consider a sampled-data system (14.1)-(14.3) with the continuous-time plant described by

$$A_c = \begin{bmatrix} -5.5 & -6.5 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; E_{dc} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$E_{fc} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 4.5 & 2 \end{bmatrix} \tag{14.75}$$

and the sampling period $h = 0.3s$. The sampling frequency is $\omega_s = \frac{2\pi}{h} = 20.9440$.

According to (14.5) and (14.24), it can be easily computed that

$$A = \begin{bmatrix} 0.0880 & -0.9192 & -0.2664 \\ 0.1332 & 0.8208 & -0.0533 \\ 0.0266 & 0.2797 & 0.9939 \end{bmatrix}, \; B = \begin{bmatrix} -0.2325 \\ 0.2737 \\ 0.3429 \end{bmatrix}$$
$$\bar{E}_d = \begin{bmatrix} 0.2819 & 0 & 0 \\ 0.0315 & 0.0425 & 0 \\ 0.0027 & 0.0064 & 0.0018 \end{bmatrix}, \; \bar{E}_f = \begin{bmatrix} 0.4571 & 0 & 0 \\ -0.4295 & 0.2586 & 0 \\ -0.6878 & 0.0372 & 0.4672 \end{bmatrix}$$

Solving the DTARS (14.72), we get

$$X_d = \begin{bmatrix} 0.0794 & 0.0089 & 0.0008 \\ 0.0089 & 0.0028 & 0.0004 \\ 0.0008 & 0.0004 & 0.0001 \end{bmatrix}, \; L_d = \begin{bmatrix} -0.1381 & 0.0159 & 0.0333 \\ 0.1911 & -0.1682 & -0.0875 \end{bmatrix}$$

As a result, the optimal residual generator is obtained to be

$$\hat{x}(k+1) = A\hat{x}(k) + Bv(k) + L_{opt}(\psi(k) - \hat{\psi}(k)), \;\; \hat{\psi}(k) = C\hat{x}(k)$$
$$r(k) = W_{opt}(\psi(k) - \hat{\psi}(k)) \tag{14.76}$$

where

$$L_{opt} = \begin{bmatrix} 0.1381 & -0.1911 \\ -0.0159 & 0.1682 \\ -0.0333 & 0.0875 \end{bmatrix}, W_{opt} = \begin{bmatrix} 3.2767 & -2.2489 \\ -2.2489 & 4.0970 \end{bmatrix}$$

Recall that (14.76) is optimal in the sense of the $H_\infty/H_\infty, H_-/H_\infty$ and H_i/H_∞ for all i and $\omega \in [0, \omega_s]$. In our example, $i = 1, 2$. As

$$\left\| W_{opt}C(zI - A + L_{opt}C)^{-1}\bar{E}_d \right\|_\infty = 1$$

The optimal performance indices are

$$\begin{aligned}
J_{SSD,OBS,\infty/\infty,opt} &= J_{SSD,OBS,\infty/\infty}(L_{opt}, W_{opt}) \\
&= \max_\omega \bar{\sigma}(W_{opt}C(e^{j\omega h}I - A + L_{opt}C)^{-1}\bar{E}_f) \\
&= \left\| W_{opt}C(zI - A + L_{opt}C)^{-1}\bar{E}_f \right\|_\infty \\
&= 35.374 \\
J_{SSD,OBS,-/\infty,opt} &= J_{SSD,OBS,-/\infty}(L_{opt}, W_{opt}) \\
&= \min_\omega \underline{\sigma}(W_{opt}C(e^{j\omega h}I - A + L_{opt}C)^{-1}\bar{E}_f) \\
&= \min_\omega \sigma_2(W_{opt}C(e^{j\omega h}I - A + L_{opt}C)^{-1}\bar{E}_f) \\
&= 0.9193 \\
J_{SSD,OBS,i/\infty,opt} &= J_{SSD,OBS,i/\infty}(L_{opt}, W_{opt}) \\
&= \sigma_i(W_{opt}C(e^{j\omega h}I - A + L_{opt}C)^{-1}\bar{E}_f), \; i = 1, 2
\end{aligned}$$

According to Theorem 14.7, an alternative solution to the H_-/H_∞ optimal design problem can be obtained by solving the DTARS (14.75), which yields

$$X_f = \begin{bmatrix} 0.2343 & -0.2132 & -0.3105 \\ -0.2132 & 0.2612 & 0.2875 \\ -0.3105 & 0.2875 & 0.7241 \end{bmatrix}, \; L_f = \begin{bmatrix} 0.0122 & -0.1457 & 0.3218 \\ 0.2177 & -0.1968 & -0.0364 \end{bmatrix}$$

The corresponding H_-/H_∞ optimal residual generator is obtained as

$$\begin{aligned}
\hat{x}(k+1) = A\hat{x}(k) + Bv(k) + L_{opt,-/\infty}(\psi(k) - \hat{\psi}(k)), \quad \hat{\psi}(k) = C\hat{x}(k) \\
r(k) = W_{opt,-/\infty}(\psi(k) - \hat{\psi}(k))
\end{aligned} \tag{14.77}$$

where

$$L_{opt,-/\infty} = \begin{bmatrix} -0.0122 & -0.2177 \\ 0.1457 & 0.1968 \\ -0.3218 & 0.0364 \end{bmatrix}, W_{opt,-/\infty} = \begin{bmatrix} 0.9468 & 0.5336 \\ 0.5336 & 0.6561 \end{bmatrix}$$

As

$$\sigma_i(W_{opt,-/\infty}C(e^{j\omega h}I - A + L_{opt,-/\infty}C)^{-1}\bar{E}_f) \equiv 1, \, i = 1, 2, \; \forall \omega$$

the performance indices are, respectively,

$$\begin{aligned}
& J_{SSD,OBS,\infty/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) \\
&= J_{SSD,OBS,-/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) \\
&= J_{SSD,OBS,i/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) \\
&= \frac{1}{\left\| W_{opt,-/\infty}C(zI - A + L_{opt,-/\infty}C)^{-1}\bar{E}_d \right\|_\infty} \\
&= 0.9193 = J_{SSD,OBS,-/\infty,opt}
\end{aligned}$$

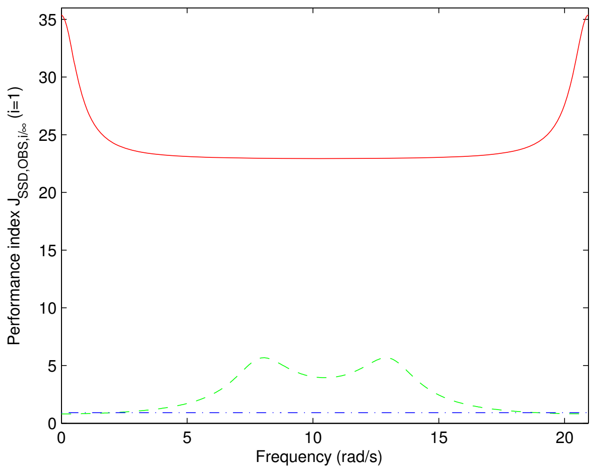

Fig. 14.2 Performance indices $J_{SSD,OBS,i/\infty}(L_{opt},W_{opt})$ (solid line), $J_{SSD,OBS,i/\infty}(L_{opt,-/\infty},W_{opt,-/\infty})$ (dash-dotted line) and $J_{SSD,OBS,i/\infty}(L_{comp},W_{comp})$ (dashed line) for $i=1$

To get a better understanding of the optimal solution, we compare the performance indices achieved by (14.76) and (14.77) with the performance indices

$$\begin{aligned}
&J_{SSD,OBS,\infty/\infty}(L_{comp},W_{comp})\\
&=\frac{\max_{\omega}\bar{\sigma}(W_{comp}C(e^{j\omega h}I-A+L_{comp}C)^{-1}\bar{E}_f)}{\left\|W_{comp}C(zI-A+L_{comp}C)^{-1}\bar{E}_d\right\|_{\infty}}\\
&J_{SSD,OBS,-/\infty}(L_{comp},W_{comp})\\
&=\frac{\min_{\omega}\underline{\sigma}(W_{comp}C(e^{j\omega h}I-A+L_{comp}C)^{-1}\bar{E}_f)}{\left\|W_{comp}C(zI-A+L_{comp}C)^{-1}\bar{E}_d\right\|_{\infty}}\\
&J_{SSD,OBS,i/\infty}(L_{comp},W_{comp})\\
&=\frac{\sigma_i(W_{comp}C(e^{j\omega h}I-A+L_{comp}C)^{-1}\bar{E}_f)}{\left\|W_{comp}C(zI-A+L_{comp}C)^{-1}\bar{E}_d\right\|_{\infty}},\ i=1,2
\end{aligned}$$

achieved by another residual generator

$$\begin{aligned}
\hat{x}(k+1)&=A\hat{x}(k)+Bv(k)+L_{comp}(\psi(k)-\hat{\psi}(k)),\ \hat{\psi}(k)=C\hat{x}(k)\\
r(k)&=W_{comp}(\psi(k)-\hat{\psi}(k))\\
L_{comp}&=\begin{bmatrix}-0.2971 & -0.1469\\ 0.6835 & 0.0899\\ -1.1007 & 0.3690\end{bmatrix},W_{comp}=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}
\end{aligned}$$

where W_{comp} is selected randomly and L_{comp} is obtained by placing the poles of the residual generator at $-0.5+0.5j$, $-0.5-0.5j$, 0.3.

Fig. 14.2 and Fig. 14.3 show, respectively, the values of $J_{SSD,OBS,i/\infty}(L_{opt},\ W_{opt})$ (solid line), $J_{SSD,OBS,i/\infty}(L_{opt,-/\infty},W_{opt,-/\infty})$ (dash-dotted line) and $J_{SSD,OBS,i/\infty}(L_{comp},W_{comp})$ (dashed line) for $i=1$ and $i=2$ as ω changes from 0 to ω_s.

For $i=1$,

$$max_{\omega\in[0,\omega_s)}J_{SSD,OBS,i/\infty}(L_{opt},W_{opt})=35.374$$

Therefore, the optimal H_∞/H_∞ performance index is

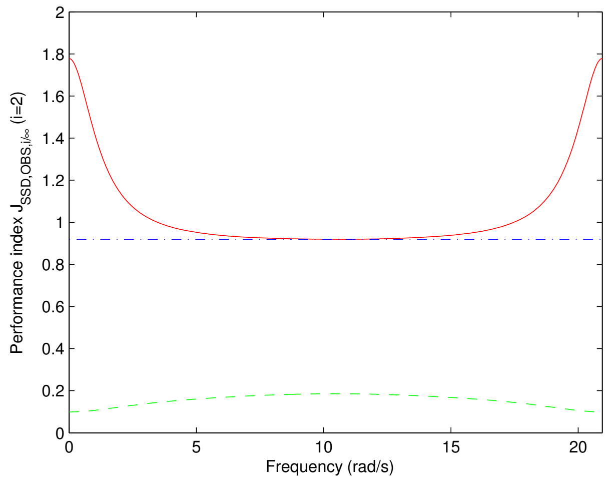

Fig. 14.3 Performance indices $J_{SSD,OBS,i/\infty}(L_{opt}, W_{opt})$ (solid line), $J_{SSD,OBS,i/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty})$ (dash-dotted line) and $J_{SSD,OBS,i/\infty}(L_{comp}, W_{comp})$ (dashed line) for $i = 2$

$$\begin{aligned} J_{SSD,OBS,\infty/\infty,opt} &= \max_{L,W} \frac{\sup_{\omega\in[0,\omega_s)} \sigma_{\max}(WM_u(e^{j\omega h})\bar{G}_f(e^{j\omega h}))}{\sup_{f=0,d\neq 0} \frac{\|r\|_2}{\|d\|_2}} \\ &= \max_{L,W} \frac{\sup_{d=0,f\neq 0} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,d\neq 0} \frac{\|r\|_2}{\|d\|_2}} = 35.374 \end{aligned}$$

and

$$\begin{aligned} \max_{\omega\in[0,\omega_s)} J_{SSD,OBS,i/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) &= 0.9193 < J_{SSD,OBS,\infty/\infty,opt} \\ \max_{\omega\in[0,\omega_s)} J_{SSD,OBS,i/\infty}(L_{comp}, W_{comp}) &= 5.6775 < J_{SSD,OBS,\infty/\infty,opt} \end{aligned}$$

For $i = 2$,

$$\begin{aligned} \min_{\omega\in[0,\omega_s)} J_{SSD,OBS,i/\infty}(L_{opt}, W_{opt}) &= 0.9193 \\ J_{SSD,OBS,i/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) &\equiv 0.9193, \forall\omega \in [0,\omega_s) \end{aligned}$$

The optimal H_-/H_∞ performance index is

$$\begin{aligned} J_{SSD,OBS,-/\infty,opt} &= \max_{L,W} \frac{\inf_{\omega\in[0,\omega_s)} \sigma_{\min}(WM_u(e^{j\omega h})\bar{G}_f(e^{j\omega h}))}{\sup_{f=0,d\neq 0} \frac{\|r\|_2}{\|d\|_2}} \\ &= \max_{L,W} \frac{\inf_{d=0,f\neq 0} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,d\neq 0} \frac{\|r\|_2}{\|d\|_2}} = 0.9193 \end{aligned}$$

and

$$\max_{\omega\in[0,\omega_s)} J_{SSD,OBS,-/\infty}(L_{comp}, W_{comp}) = 0.0991 < J_{SSD,OBS,-/\infty,opt}$$

For both $i = 1$ and $i = 2$

$$\begin{aligned} J_{SSD,OBS,i/\infty}(L_{opt,-/\infty}, W_{opt,-/\infty}) &\leq J_{SSD,OBS,i/\infty}(L_{opt}, W_{opt}) \\ J_{SSD,OBS,i/\infty}(L_{comp}, W_{comp}) &< J_{SSD,OBS,i/\infty}(L_{opt}, W_{opt}) \end{aligned}$$

at *each* frequency $\omega \in [0, \omega_s)$, which agree with the theoretical results.

14.4 Full decoupling

In the framework of the parity space approach, recall that a full decoupling of the residual signal $r(k)$ from unknown inputs $d(t)$ is achievable if and only if there exists V_s such that [174]

$$\begin{cases} V_s H_s \Psi^d d_{k,s}(t) = 0, \ \forall d(t) \\ V_s H_s \Psi^f f_{k,s}(t) \neq 0, \ \text{ if } f(t) \neq 0 \end{cases} \tag{14.78}$$

which is equivalent to

$$V_s \bar{H}_{d,s} = 0, \ V_s \bar{H}_{f,s} \neq 0 \tag{14.79}$$

There is a nonzero solution to (14.79) if and only if

$$rank \ \begin{bmatrix} H_{o,s} \ \bar{H}_{d,s} \ \bar{H}_{f,s} \end{bmatrix} > rank \ \begin{bmatrix} H_{o,s} \ \bar{H}_{d,s} \end{bmatrix} \tag{14.80}$$

Then we get the following theorem on the condition of full decoupling.

Theorem 14.8 Given an SD system described by (14.1), (14.2)-(14.3), there exists a linear discrete-time residual generator (14.15) fully decoupled from $d(t)$ if and only if (14.80) holds, where $H_{o,s}, \bar{H}_{d,s}$ and $\bar{H}_{f,s}$ are given, respectively, by (2.4) and (14.28).

Lemma 14.4 Given an SD system described by (14.1), (14.2)-(14.3). If (A_c, E_{dc}) is controllable, then there doesn't exist a linear discrete-time residual generator (14.15) fully decoupled from $d(t)$.

Proof: If (A_c, E_{dc}) is controllable, then

$$rank \int_0^h e^{A_c \tau} E_{dc} E_{dc}^T e^{A_c^T \tau} d\tau = n$$

It leads to $rank \bar{E}_d = rank \bar{E}_d \bar{E}_d^T = n$. Therefore, in this case a full decoupling from the unknown disturbances would be impossible. □

In the frequency domain [178], a residual signal $r(z)$ is said to be fully decoupled from the unknown disturbances $d(s)$, if there exists a post-filter $R(z)$ such that

$$r(e^{j\omega h}) \equiv R(e^{j\omega h}) M_u(e^{j\omega h}) \Gamma^f f(j\omega) \tag{14.81}$$

i.e.

$$\begin{cases} R(z) M_u(z) \Gamma^d d = 0, \ \forall d \\ R(z) M_u(z) \Gamma^f f \neq 0, \ \text{ if } f \neq 0 \end{cases} \tag{14.82}$$

From the definition of the norm we know that this is the case iff

$$\begin{aligned} \left\| R(z) M_u(z) \Gamma^d \right\| &= \left\| R(z) M_u(z) \Gamma^d \left(\Gamma^d\right)^* M_u^*(z) \, R^*(z) \right\|^{1/2} = 0 \\ \left\| R(z) M_u(z) \Gamma^f \right\| &= \left\| R(z) M_u(z) \Gamma^f \left(\Gamma^f\right)^* M_u^*(z) \, R^*(z) \right\|^{1/2} \neq 0 \end{aligned} \tag{14.83}$$

(14.83) is again equivalent to

$$\begin{aligned} R(z) M_u(z) \Gamma^d \left(\Gamma^d\right)^* M_u^*(z) R^*(z) = 0 \\ R(z) M_u(z) \Gamma^f \left(\Gamma^f\right)^* M_u^*(z) R^*(z) \neq 0 \end{aligned} \tag{14.84}$$

Since $M_u(z)$ is a full rank square matrix, (14.84) holds iff

$$rank \left[\Gamma^d \left(\Gamma^d\right)^* \ \ \Gamma^f \left(\Gamma^f\right)^* \right] > rank \ \Gamma^d \left(\Gamma^d\right)^* \tag{14.85}$$

A necessary condition for (14.85) to hold is

$$rank \Gamma^d \left(\Gamma^d\right)^* < m, \ \forall \omega \tag{14.86}$$

i.e. $\Gamma^d \left(\Gamma^d\right)^*$ is not of full rank.

Theorem 14.9 Given an SD system described by (14.1), (14.2)-(14.3), there exists a linear discrete-time residual generator (14.40) fully decoupled from $d(t)$ if and only if (14.85) holds, where $\Gamma^d \left(\Gamma^d\right)^*$ and $\Gamma^f \left(\Gamma^f\right)^*$ can be obtained by (14.50) or (14.51).

However, if the H_2/H_2 optimal design (see (14.46)) is considered and for some ω_Q,

$$\det \Gamma^d \left(\Gamma^d\right)^* = 0, \ \det \Gamma^f \left(\Gamma^f\right)^* \neq 0 \tag{14.87}$$

This means $\lambda = \infty$ is a singular value of (14.62) for $\omega = \omega_Q$. Therefore, selecting the post-filter as $R(z) = f_{\omega_Q}(z)v(z)$ with an ideal frequency selector $f_{\omega_Q}(z)$ gives

$$J_{SSD,FRE,2/2,opt} = \sup_{\omega} \lambda_{\max}(\omega) = \lambda_{\max}(\omega_0) = \infty$$

It is evident that condition (14.87) is much weaker than (14.85), thus such kind of decoupling is easier to achieve than the one described by (14.85). We call a full decoupling satisfying (14.87) a *weak* full decoupling. However, since in practice we can only approximate an ideal frequency selector, $R(z)M_u(z)\Gamma_{G_d}$ will not be exactly zero and thus we have

$$r(e^{j\omega h}) \approx R(e^{j\omega h})M_u(e^{j\omega h})\Gamma^f f(j\omega)$$

Sampling introduces structural change into the original continuous-time systems. The following theorem describes the influence of sampling on full decoupling.

Theorem 14.10 Compared with the original continuous-time system described by (14.1), the full decoupling becomes more difficult in the SD system described by (14.1), (14.2)-(14.3).

Proof: Notice that

$$rank\, \Gamma^d \left(\Gamma^d\right)^* \geq rank G_{dc}(s)$$
$$rank \bar{E}_d = rank \bar{E}_d \bar{E}_d^T = rank \int_0^h e^{A_c\tau} E_{dc} E_{dc}^T e^{A_c^T \tau} d\tau \geq rank E_{dc} \tag{14.88}$$

It shows that after sampling the dimension of the influence space of the unknown disturbances may increase. Compare the full decoupling conditions (14.80) and (14.85), respectively, with (4.29) and (4.27). We can see that it is more difficult to satisfy (14.80) and (14.85). Hence, the existence of the sampling effect will make a full decoupling from the unknown disturbances more difficult than in the original continuous-time systems, no matter which residual generation approach is adopted. □

14.5 Conclusion

In this paper, problems related to fault detection in the single-rate sampled-data (SSD) systems have been studied. To take into account the intersample behaviour, operators are introduced for the analysis of the SSD systems from the viewpoint of FD. With the help of the introduced operators, the influence of continuous-time disturbances and faults on the discrete-time residual can be quantitatively analyzed without any approximation. Based on it, direct design approaches of optimal fault detection systems are developed. It is shown that the optimization problems of the SSD system are equivalent to that of a discrete LTI system [84, 174, 176, 178, 180]. Through the analysis of the full decoupling conditions, it is shown that the full decoupling becomes more difficult in the SSD systems than in the original continuous-time systems.

It is worth pointing out that the FD problem of the SSD systems can also be solved using the parametric transfer function (PTF) theory developed by Rosenwasser and Lampe [132], as shown in [187].

15

FD of general sampled-data systems

In practice, it happens often that the A/D and D/A converters are working at different sampling rates (see [3, 93, 98, 101, 116, 128] and the references therein), as shown in Fig. 15.1. In this chapter, we shall consider the FD problem of the SD systems with various sampling mechanisms [177, 175, 170, 172].

15.1 System description

Assume that the continuous-time process is still described by (14.1). In *multirate sampled-data (MSD)* systems, the A/D converters and the D/A converters working at different sampling rates are modelled, respectively, by

$$\psi_l(k^l) = y_l(k^l T_{y,l}),\ l = 1, 2, \cdots, m;\ k^l = 0, 1, 2, \cdots \tag{15.1}$$

$$u_j(t) = \upsilon_j(k^j),\ k^j T_{u,j} \le t < (k^j + 1)T_{u,j} \tag{15.2}$$

$$j = 1, 2, \cdots, n_u;\ k^j = 0, 1, 2, \cdots$$

where $T_{y,l}$ and $T_{u,j}$ denote, respectively, the sampling periods of the A/D converter in the l-th output channel and the D/A converter in the j-th input channel.

A more general class of sampled-data systems are *non-uniformly sampled-data (NSD)* systems, where the sampling instants may be multirate, asynchronous and non-equidistantly distributed, i.e.,

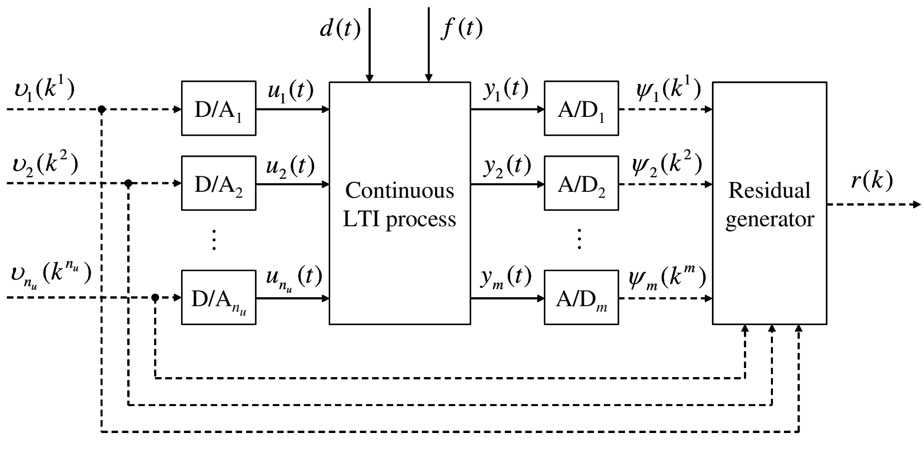

Fig. 15.1 Structure diagram of MSD and NSD systems

$$\psi_l(k^l) = y_l(t_{y,k^l}),\ l = 1, 2, \cdots, m;\ k^l = 0, 1, 2, \cdots \tag{15.3}$$

$$u_j(t) = \upsilon_j(k^j),\ t_{u,k^j} \le t < t_{u,k^j+1},\ \ j = 1, 2, \cdots, n_u;\ k^j = 0, 1, 2, \cdots \tag{15.4}$$

where t_{y,k^l} represents the sampling instants in the l-th output channel and t_{u,k^j} the time instants at which the j-th control input is updated. It is worth mentioning that a special kind of NSD systems, where the sampling instants are non-equidistant spaced but periodic, has been studied in the literature rather intensively [105, 104, 99, 141].

In the following, we shall first consider the FD problem of the NSD systems and then that of the MSD systems.

15.2 FD of NSD systems

15.2.1 Reformulation of system model

From the FD viewpoint, in NSD systems only those time instants with sampled outputs are of interest [170, 172]. Denote with $\{t_k\}$ the sequence of time instants at which one or more sampled outputs are available, $t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$. Let $\bar{\psi}(k)$ represent the vector of sampled output signals at time instant t_k. The dimension of $\bar{\psi}(k)$ is time-varying and upper bounded by m. Let $x(k) = x(t_k)$. For the purpose of FD, the NSD system described by (14.1), (15.3) and (15.4) can be equivalently re-modelled as

$$\begin{aligned} x(k+1) &= A(k)x(k) + \bar{u}(k) + \bar{d}(k) + \bar{f}(k) \\ \bar{\psi}(k) &= C(k)x(k) \end{aligned} \tag{15.5}$$

where

$$A(k) = e^{A_c(t_{k+1}-t_k)},\ \bar{u}(k) = \int_{t_k}^{t_{k+1}} e^{A_c(t_{k+1}-t)} B_c u(t)dt, \tag{15.6}$$

$$\bar{d}(k) = \int_{t_k}^{t_{k+1}} e^{A_c(t_{k+1}-t)} E_{dc} d(t)dt,\quad \bar{f}(k) = \int_{t_k}^{t_{k+1}} e^{A_c(t_{k+1}-t)} E_{fc} f(t)dt.$$

The new description considers the transition of system dynamics only at the time instants with sampled outputs. The terms $\bar{d}(k)$ and $\bar{f}(k)$ characterize, respectively, the influence of the disturbances and the faults on the sampled outputs. Recalling (15.4), the input term $\bar{u}(k)$ related to the control inputs $\upsilon_1, \cdots, \upsilon_{n_u}$ can be written into the form of

$$\bar{u}(k) = B(k)\bar{\upsilon}(k) \tag{15.7}$$

where $\bar{\upsilon}(k)$ is the control input vector that works during the time interval $[t_k, t_{k+1})$, and $B(k)$ is a time-varying matrix. The calculation of $B(k)$ is straightforward, though it can be computationally complicated, and will be shown later in an example. The influence of the control input vector on the sampled outputs is exactly known and can be easily compensated. The matrices $A(k)$ and $C(k)$ are time-varying matrices, as $t_{k+1} - t_k$ is time-varying with respect to time k. FD systems for the NSD system can be designed based on the reformulated time-varying system model (15.5).

15.2.2 Parity relation based FD scheme for NSD systems

The input-output relationship of (15.5) over the moving horizon $[t_{k-s}, t_k]$ is

$$\tilde{\psi}(k) = H_{o,s}(k)x(k-s) + H_s(k)(\tilde{u}(k) + \tilde{d}(k) + \tilde{f}(k)) \tag{15.8}$$

where s is an integer denoting the length of the moving horizon, $\tilde{\psi}(k), \tilde{u}(k), \tilde{d}(k)$ and $\tilde{f}(k)$ are stacked vectors based on $\bar{\psi}(j), \bar{u}(j), \bar{d}(j)$ and $\bar{f}(j)$, $j = k-s, \cdots, k$, respectively,

$$\tilde{\psi}(k) = \begin{bmatrix} \bar{\psi}(k-s) \\ \bar{\psi}(k-s+1) \\ \vdots \\ \bar{\psi}(k) \end{bmatrix}, \ \tilde{u}(k) = \begin{bmatrix} \bar{u}(k-s) \\ \bar{u}(k-s+1) \\ \vdots \\ \bar{u}(k) \end{bmatrix}$$

$$\tilde{d}(k) = \begin{bmatrix} \bar{d}(k-s) \\ \bar{d}(k-s+1) \\ \vdots \\ \bar{d}(k) \end{bmatrix}, \ \tilde{f}(k) = \begin{bmatrix} \bar{f}(k-s) \\ \bar{f}(k-s+1) \\ \vdots \\ \bar{f}(k) \end{bmatrix}$$

$$H_{o,s}(k) = \begin{bmatrix} C(k-s) \\ C(k-s+1)A(k-s) \\ \vdots \\ C(k)A(k-1)A(k-2)\cdots A(k-s) \end{bmatrix} \tag{15.9}$$

$$H_s(k) = \begin{bmatrix} O & \cdots & O \\ C(k-s+1) & \ddots & \vdots \\ \vdots & \ddots & O \\ C(k)A(k-1)\cdots A(k-s+1) & \cdots & C(k) \end{bmatrix}$$

Build a parity relation based residual generator as

$$r(k) = V(k)(\tilde{\psi}(k) - H_s(k)\tilde{u}(k)), \quad V(k)H_{o,s}(k) = 0 \tag{15.10}$$

where $V(k)$ is a time-varying parity matrix (or vector). The dynamics of residual generator (15.10) is governed by

$$r(k) = V(k)H_s(k)(\tilde{d}(k) + \tilde{f}(k)) \tag{15.11}$$

$$= V(k)H_s(k)(\Psi_k^d d_k(t) + \Psi_k^f f_k(t)) \tag{15.12}$$

where Ψ_k^δ (δ standing for d or f) is a linear time-varying operator defined by

$$\Psi_k^\delta \ : \quad \mathcal{L}_{2,[t_{k-s},t_k)}(\mathbf{R}, \mathbf{R}^{n_\delta}) \to l_2(\mathbf{Z}, \mathbf{R}^{sn}),$$

$$\tilde{\delta}(k) = \Psi_k^\delta \delta_k(t) = \begin{bmatrix} \int_{t_{k-s}}^{t_{k-s+1}} e^{A_c(t_{k-s+1}-t)} E_{\delta c}\delta(t)dt \\ \int_{t_{k-s+1}}^{t_{k-s+2}} e^{A_c(t_{k-s+2}-t)} E_{\delta c}\delta(t)dt \\ \vdots \\ \int_{t_{k-1}}^{t_k} e^{A_c(t_k-t)} E_{\delta c}\delta(t)dt \end{bmatrix}.$$

In order to improve the performance of residual generator (15.10) using the freedom provided by $V(k)$, the optimization problem is defined as

$$\begin{aligned}
&\max_{V(k),\ V(k)H_{o,s}(k)=0} J_{NSD,PS,\infty/\infty}(V(k)) \\
&= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\sup_{d=0,f\neq 0} \frac{\|r(k)\|_E^2}{\|f\|_{2,[t_{k-s},t_k]}^2}}{\sup_{f=0,d\neq 0} \frac{\|r(k)\|_E^2}{\|d\|_{2,[t_{k-s},t_k]}^2}} \\
&= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\bar{\lambda}\left(V(k)H_s(k)\Psi_k^f(\Psi_k^f)^* H_s^T(k)V^T(k)\right)}{\bar{\lambda}\left(V(k)H_s(k)\Psi_k^d(\Psi_k^d)^* H_s^T(k)V^T(k)\right)}
\end{aligned} \tag{15.13}$$

$$\begin{aligned}
&\max_{V(k),\ V(k)H_{o,s}(k)=0} J_{NSD,PS,-/\infty}(V(k)) \\
&= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\inf_{d=0,f\neq 0} \frac{\|r(k)\|_E^2}{\|f\|_{2,[t_{k-s},t_k]}^2}}{\sup_{f=0,d\neq 0} \frac{\|r(k)\|_E^2}{\|d\|_{2,[t_{k-s},t_k]}^2}} \\
&= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\underline{\lambda}\left(V(k)H_s(k)\Psi_k^f(\Psi_k^f)^* H_s^T(k)V^T(k)\right)}{\bar{\lambda}\left(V(k)H_s(k)\Psi_k^d(\Psi_k^d)^* H_s^T(k)V^T(k)\right)}
\end{aligned} \tag{15.14}$$

$$\begin{aligned}
&\max_{V(k),\ V(k)H_{o,s}(k)=0} J_{NSD,PS,i/\infty}(V(k)) \\
&= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\lambda_i\left(V(k)H_s(k)\Psi_k^f(\Psi_k^f)^* H_s^T(k)V^T(k)\right)}{\bar{\lambda}\left(V(k)H_s(k)\Psi_k^d(\Psi_k^d)^* H_s^T(k)V^T(k)\right)}
\end{aligned} \tag{15.15}$$

Consider the equation

$$\left\langle \Psi_k^\delta \delta_k(t), \beta(k) \right\rangle = \left\langle \delta_k(t), \left(\Psi_k^\delta\right)^* \beta(k) \right\rangle, \quad \forall \beta(k) \in \mathbf{R}^{sn}$$

where $\delta_k(t) = \delta(t)$, $t_{k-s} \le t < t_k$, $\delta_k(t) \in \mathcal{L}_{2,[t_{k-s},t_k)}$, $\beta(k)$ is partitioned as

$$\beta(k) = \begin{bmatrix} \beta_1(k) \\ \beta_2(k) \\ \vdots \\ \beta_s(k) \end{bmatrix}, \quad \beta_j(k) \in \mathbf{R}^n,\ j = 1, 2, \cdots, s.$$

As

$$\begin{aligned}
\left\langle \Psi_k^\delta \delta_k(t), \beta(k) \right\rangle &= (\Psi_k^\delta \delta_k(t))^T \beta(k) \\
&= \begin{bmatrix} \int_{t_{k-s}}^{t_{k-s+1}} e^{A_c(t_{k-s+1}-t)} E_{\delta c} \delta_k(t) dt \\ \int_{t_{k-s+1}}^{t_{k-s+2}} e^{A_c(t_{k-s+2}-t)} E_{\delta c} \delta_k(t) dt \\ \vdots \\ \int_{t_{k-1}}^{t_k} e^{A_c(t_k-t)} E_{\delta c} \delta_k(t) dt \end{bmatrix}^T \begin{bmatrix} \beta_1(k) \\ \beta_2(k) \\ \vdots \\ \beta_s(k) \end{bmatrix} \\
&= \sum_{j=1}^{s} \int_{t_{k-j}}^{t_{k-j+1}} \delta_k^T(t) E_{\delta c}^T e^{A_c^T(t_{k-j+1}-t)} \beta_{s-j+1}(k) dt,
\end{aligned}$$

it yields

$$\left(\Psi_k^\delta\right)^* \beta(k) = \begin{cases} E_{\delta c}^T e^{A_c^T(t_{k-s+1}-t)} \beta_1(k), \text{if } t_{k-s} \le t < t_{k-s+1} \\ \vdots \\ E_{\delta c}^T e^{A_c^T(t_k-t)} \beta_s(k), \text{if } t_{k-1} \le t < t_k \end{cases}$$

Finally, we have

$$\Psi_k^\delta \left(\Psi_k^\delta\right)^* \beta(k) = \bar{\Psi}_k^\delta \beta(k), \quad \forall \beta(k) \in \mathbf{R}^{sn}$$

where

$$\begin{aligned}\bar{\Psi}_k^\delta &= diag\left\{\int_{t_{k-s}}^{t_{k-s+1}} e^{A_c(t_{k-s+1}-t)} E_{\delta c} E_{\delta c}^T e^{A_c^T(t_{k-s+1}-t)} dt,\right.\\ &\qquad \left.\cdots, \int_{t_{k-1}}^{t_k} e^{A_c(t_k-t)} E_{\delta c} E_{\delta c}^T e^{A_c^T(t_k-t)} dt\right\}\\ &= diag\left\{\int_0^{t_{k-s+1}-t_{k-s}} e^{A_c t} E_{\delta c} E_{\delta c}^T e^{A_c^T t} dt,\right.\\ &\qquad \left.\cdots, \int_0^{t_k-t_{k-1}} e^{A_c t} E_{\delta c} E_{\delta c}^T e^{A_c^T t} dt\right\}\end{aligned}$$

Therefore,

$$\Psi_k^\delta \left(\Psi_k^\delta\right)^* = \bar{\Psi}_k^\delta = \begin{bmatrix} \Psi_{k,s} & O & \cdots & O \\ O & \Psi_{k,s-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & \Psi_{k,1} \end{bmatrix} \tag{15.16}$$

$$\Psi_{k,j} = \int_0^{t_{k-j+1}-t_{k-j}} e^{A_c t} E_{\delta c} E_{\delta c}^T e^{A_c^T t} dt, \quad j = 1, \cdots, s$$

Optimization problems (15.13)-(15.15) are equivalent, respectively, to

$$\begin{aligned}&\max_{V(k),\ V(k)H_{o,s}(k)=0} J_{NSD,PS,\infty/\infty}(V(k))\\ &= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\bar{\lambda}\left(V(k)H_s(k)\bar{\Psi}_k^f H_s^T(k)V^T(k)\right)}{\bar{\lambda}\left(V(k)H_s(k)\bar{\Psi}_k^d H_s^T(k)V^T(k)\right)}\end{aligned} \tag{15.17}$$

$$\begin{aligned}&\max_{V(k),\ V(k)H_{o,s}(k)=0} J_{NSD,PS,-/\infty}(V(k))\\ &= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\underline{\lambda}\left(V(k)H_s(k)\bar{\Psi}_k^f H_s^T(k)V^T(k)\right)}{\bar{\lambda}\left(V(k)H_s(k)\bar{\Psi}_k^d H_s^T(k)V^T(k)\right)}\end{aligned} \tag{15.18}$$

$$\begin{aligned}&\max_{V(k),\ V(k)H_{o,s}(k)=0} J_{NSD,PS,i/\infty}(V(k))\\ &= \max_{V(k),\ V(k)H_{o,s}(k)=0} \frac{\lambda_i\left(V(k)H_s(k)\bar{\Psi}_k^f H_s^T(k)V^T(k)\right)}{\bar{\lambda}\left(V(k)H_s(k)\bar{\Psi}_k^d H_s^T(k)V^T(k)\right)}\end{aligned} \tag{15.19}$$

Then, for any given k, optimization problem (15.17)-(15.19) can be solved by the approaches introduced in Chapter 5.1.

In summary, a parity relation based fast rate residual generator in the form of (15.10) can be designed for NSD systems described by (14.1), (15.3)-(15.4) as below:

- Determine the sequence of time instants t_k and the matrix $C(k)$.
- Calculate the matrix $A(k)$ by (15.6).
- Calculate the matrices $H_{o,s}(k), H_s(k)$ as (15.9) and the matrices $\bar{\Psi}_k^d, \bar{\Psi}_k^f$ according to (15.16).
- Solve the optimization problems (15.17)-(15.19) for $V(k)$.
- Calculate $\tilde{u}(k)$ and build residual generator as (15.10).

15.2.3 Observer based FD scheme for NSD systems

For the aim of fault detection, a fast rate time-varying observer-based residual generator can be constructed as

$$\begin{aligned}\hat{x}(k+1) &= A(k)\hat{x}(k) + \bar{u}(k) + L(k)(\bar{\psi}(k) - \hat{\psi}(k)) \\ r(k) &= W(k)(\bar{\psi}(k) - \hat{\psi}(k)),\ \hat{\psi}(k) = C(k)\hat{x}(k)\end{aligned} \tag{15.20}$$

where the gain matrix $L(k)$ and the weighting matrix $W(k)$ are time-varying matrices to be determined. The dimensions of $L(k)$ and $W(k)$ may change with the number of the available sampled output signals. Define the state estimation error as $e(k) = x(k) - \hat{x}(k)$. The dynamics of residual generator (15.20) is governed by

$$\begin{aligned}e(k+1) &= (A(k) - L(k)C(k))e(k) + \bar{d}(k) + \bar{f}(k) \\ r(k) &= W(k)C(k)e(k)\end{aligned} \tag{15.21}$$

Introduce linear time-varying operators Γ_k^δ (δ standing for d or f)

$$\begin{aligned}\Gamma_k^\delta &: \quad \mathcal{L}_{2,[t_k,t_{k+1})}(\mathbf{R}, \mathbf{R}^{n_\delta}) \to l_2(\mathbf{Z}, \mathbf{R}^n) \\ \Gamma_k^\delta \delta_k(t) &= \int_{t_k}^{t_{k+1}} e^{A_c(t_{k+1}-t)} E_{\delta c}\delta(t)dt.\end{aligned} \tag{15.22}$$

The residual dynamics (15.21) can be re-written as

$$\begin{aligned}e(k+1) &= (A(k) - L(k)C(k))e(k) + \Gamma_k^d d_k(t) + \Gamma_k^f f_k(t) \\ r(k) &= W(k)C(k)e(k)\end{aligned} \tag{15.23}$$

To enhance the robustness of the FD system to the unknown disturbances without loss of the sensitivity to the faults, the design problem is formulated as

$$\begin{aligned}&\max_{L(k),W(k)} J_{NSD,OBS,\infty/\infty}(L(k), W(k)) \\ &= \max_{L(k),W(k)} \frac{\sup_{d=0,\ f\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}} \\ &= \max_{L(k),W(k)} \frac{\sup_{d=0,\ f_k\in l_2-\{0\}} \frac{\|r\|_2}{\|f_k\|_2}}{\sup_{f=0,\ d_k\in l_2-\{0\}} \frac{\|r\|_2}{\|d_k\|_2}}\end{aligned} \tag{15.24}$$

$$\begin{aligned}&\max_{L(k),W(k)} J_{NSD,OBS,-/\infty}(L(k), W(k)) \\ &= \max_{L(k),W(k)} \frac{\inf_{d=0,\ f\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|f\|_2}}{\sup_{f=0,\ d\in\mathcal{L}_2-\{0\}} \frac{\|r\|_2}{\|d\|_2}} \\ &= \max_{L(k),W(k)} \frac{\inf_{d=0,\ f_k\in l_2-\{0\}} \frac{\|r\|_2}{\|f_k\|_2}}{\sup_{f=0,\ d_k\in l_2-\{0\}} \frac{\|r\|_2}{\|d_k\|_2}}\end{aligned} \tag{15.25}$$

Using the technique as introduced in Chapter 14, it can be found out that for the operator Γ_k^δ and its adjoint $\left(\Gamma_k^\delta\right)^*$ there is

$$\Gamma_k^\delta \left(\Gamma_k^\delta\right)^* = \bar{\Gamma}_k^\delta = \int_0^{t_{k+1}-t_k} e^{A_c t} E_{\delta c} E_{\delta c}^T e^{A_c^T t} dt. \tag{15.26}$$

Hence, in the sense of the optimization problems (15.24)-(15.25), the residual dynamics is equivalent to a linear discrete time-varying system

$$\begin{aligned} e(k+1) &= (A(k) - L(k)C(k))e(k) + \bar{E}_d(k)d_{eq}(k) + \bar{E}_f(k)f_{eq}(k) \\ r(k) &= W(k)C(k)e(k) \end{aligned} \tag{15.27}$$

where the l_2-norms of $d_{eq}(k)$ and $f_{eq}(k)$ in (15.27) have the same upper bounds, respectively, with the $\mathcal{L}_2$-norms of $d(t)$ and $f(t)$ in (14.1), the matrices $\bar{E}_d(k)$ and $\bar{E}_f(k)$ are time-varying matrices reflecting the sampling effect and satisfy

$$\bar{E}_\delta(k)\bar{E}_\delta^T(k) = \bar{\Gamma}_k^\delta = \int_0^{t_{k+1}-t_k} e^{A_c t} E_{\delta c} E_{\delta c}^T e^{A_c^T t} dt \tag{15.28}$$

as

$$\sup_{f=0,\ d_k \in l_2 - \{0\}} \frac{\|r\|_2}{\|d_k\|_2} = \sup_{f=0,\ d_{eq} \in l_2 - \{0\}} \frac{\|r\|_2}{\|d_{eq}\|_2},$$
$$\sup_{d=0,\ f_k \in l_2 - \{0\}} \frac{\|r\|_2}{\|f_k\|_2} = \sup_{d=0,\ f_{eq} \in l_2 - \{0\}} \frac{\|r\|_2}{\|f_{eq}\|_2}$$

Then, the optimization problems (15.24)-(15.25) can be solved with the FD approaches introduced in Chapter 13 for the linear time-varying systems. The opimal solution is

$$L(k) = -L_d^T(k), \quad W(k) = M^+(k) \tag{15.29}$$

where $M^+(k)$ denotes the left inverse of the matrix $M(k)$, $M(k)$ is a full column rank matrix satisfying $M(k)M^T(k) = C(k)X_d(k)C^T(k)$, and $(X_d(k), L_d(k))$ is the stabilizing solution of the Riccati difference system (DRS)

$$\begin{bmatrix} A(k)X_d(k)A^T(k) - X_d(k+1) + \bar{\Gamma}_k^d & A(k)X_d(k)C^T(k) \\ C(k)X_d(k)A^T(k) & C(k)X_d(k)C^T(k) \end{bmatrix} \begin{bmatrix} I \\ L_d(k) \end{bmatrix} = 0. \tag{15.30}$$

In summary, an observer based fast rate residual generator in the form of (15.20) can be designed for the NSD systems described by (14.1), (15.3)-(15.4) as below:

- Determine the sequence of time instants with sampled outputs t_k and the matrix $C(k)$.
- Calculate the matrix $A(k)$ by (15.6), determine the matrix $B(k)$.
- Calculate the matrix $\bar{\Gamma}_k^d$ and $\bar{\Gamma}_k^f$ according to (15.28).
- Solve the DRS (15.30) for the stabilizing solution $(X_d(k), L_d(k))$.
- Compute the optimal observer gain matrix $L(k)$ and the weighting matrix $W(k)$ by (15.29).

In general NSD systems, the key point of this design is to guarantee the stability of $A(k) - L(k)C(k)$, which is not a trivial task. However, in a special kind of NSD systems, where the sampling instants are non-equidistant spaced but periodic, as well as in the MSD systems that will be discussed in the coming section, the DRS (15.30) will reduce to a DPRS and its solution can be easily obtained.

15.3 FD of MSD systems

The MSD system described by (14.1), (15.1)-(15.2) is a special case of NSD systems. It is in nature a periodic system. The *system period*, denoted by T, is the least common multiple of the sampling periods $T_{y,l},\ l = 1, 2, \cdots, m$ and $T_{u,j},\ j = 1, 2, \cdots, n_u$. The maximal common multiplier of the sampling periods is often called *base period*, denoted by h. The FD problem of the MSD systems can be handled along different lines.

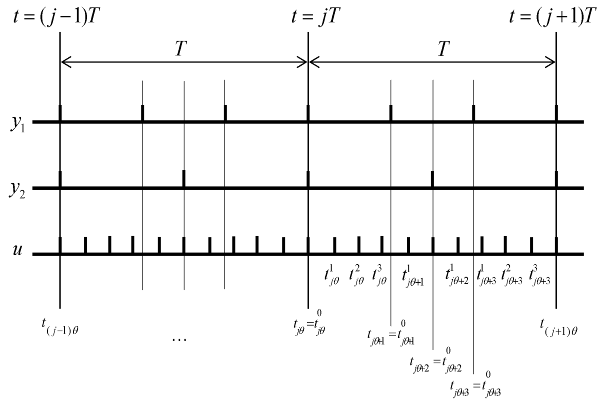

Fig. 15.2 Sampling strategy in Example 15.1 ($T_{y,1} = 1s, T_{y,2} = 1.5s, T_u = 0.3s$, system period $T = 3s$, sequence period $\theta = 4$)

15.3.1 Design based on reformulated periodic model

The idea of re-ordering the sampling instants and the design procedures proposed in the last section can be directly applied to the FD of the MSD systems. For the MSD systems, $t_{k+1} - t_k$ in the reformulated model (15.5) is periodically time-varying with respect to time k. The period of the sequence $\{t_{k+1} - t_k\}$ is called the *sequence period*, denoted later by θ. The sequence period θ and the system period T are related by $T = t_{(j+1)\theta+i} - t_{j\theta+i}$, $\forall i, j$. Therefore, in the MSD systems, $A(k), C(k), \bar{E}_d(k), \bar{E}_f(k)$ reduce to periodically time-varying matrices and the designs are considerably simplified. In consequence, for the MSD systems

- if the parity space approach is used, then the time-varying parity matrix $V(k)$ needs only to be calculated over one period,
- if the observer based approach is adopted, then the observer gain matrix $L(k)$ needs to guarantee that the characteristic multipliers of $A(k) - L(k)C(k)$ are located inside the unit circle.

In principle, after the MSD system is reformulated as a periodic system, all the FD methods introduced in Chapter 9 and 10 can be applied to it as well.

In the following, we shall illustrate it with an example [170].

Example 15.1 Consider an MSD system with a continuous-time plant described by (14.1) with

$$A_c = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix},\ B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ E_{dc} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix},$$
$$E_{fc} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The sampling periods of the A/D and D/A converters are, respectively, $T_{y,1} = 1s, T_{y,2} = 1.5s, T_u = 0.3s$.

The system period is $T = 3s$. Over each system period $[jT, (j+1)T)$, one or more outputs are sampled at time instants jT, $jT + 1, jT + 1.5, jT + 2$, as shown in Fig. 15.2. The sequence period is $\theta = 4$, i.e. $A_{(j+1)\theta} = A_{j\theta}$, $C_{(j+1)\theta} = C_{j\theta}$, etc. The sampled output vectors are

$$\bar{\psi}(j\theta) = \begin{bmatrix} y_1(t_{j\theta}) \\ y_2(t_{j\theta}) \end{bmatrix},\ \bar{\psi}(j\theta+1) = y_1(t_{j\theta+1})$$
$$\bar{\psi}(j\theta+2) = y_2(t_{j\theta+2}),\ \bar{\psi}(j\theta+3) = y_1(t_{j\theta+3})$$

The control input vector are, respectively,

$$\bar{v}(j\theta) = \begin{bmatrix} u(t_{j\theta}^0) \\ u(t_{j\theta}^1) \\ u(t_{j\theta}^2) \\ u(t_{j\theta}^3) \end{bmatrix}, \bar{v}(j\theta+1) = \begin{bmatrix} u(t_{j\theta+1}^0) \\ u(t_{j\theta+1}^1) \end{bmatrix}$$

$$\bar{v}(j\theta+2) = \begin{bmatrix} u(t_{j\theta+2}^0) \\ u(t_{j\theta+2}^1) \end{bmatrix}, \bar{v}(j\theta+3) = \begin{bmatrix} u(t_{j\theta+3}^0) \\ u(t_{j\theta+3}^1) \\ u(t_{j\theta+3}^2) \\ u(t_{j\theta+3}^3) \end{bmatrix}$$

which can be determined from the control input sequences $v_1, \cdots, v_{n_u}$. The matrices $A(k)$ and $C(k)$ in the reformulated model (15.5) are, respectively,

$$\begin{aligned} A(j\theta) &= e^{A_c(t_{j\theta+1}-t_{j\theta})} = e^{A_c} = \begin{bmatrix} 0.3679 & 1.1627 \\ 0 & 0.1353 \end{bmatrix}, \\ A(j\theta+1) &= e^{0.5A_c} = \begin{bmatrix} 0.6065 & 1.1933 \\ 0 & 0.3679 \end{bmatrix}, \\ A(j\theta+2) &= e^{0.5A_c} = \begin{bmatrix} 0.6065 & 1.1933 \\ 0 & 0.3679 \end{bmatrix}, \\ A(j\theta+3) &= e^{A_c} = \begin{bmatrix} 0.3679 & 1.1627 \\ 0 & 0.1353 \end{bmatrix}, \\ C(j\theta) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C(j\theta+2) = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\ C(j\theta+1) &= C(j\theta+3) = \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned}$$

Denote with p_k the number of control input updates happening between $[t_k, t_{k+1})$. As

$$\bar{u}(k) = \int_{t_k}^{t_{k+1}} e^{A_c(t_{k+1}-t)} B_c u(t)dt = \sum_{l=0}^{p_k-1} \int_{t_{k+1}-t_k^{l+1}}^{t_{k+1}-t_k^l} e^{A_c t} B_c dt u(t_k^l)$$

the matrix $B(k)$ is determined by

$$\begin{aligned} B(j\theta) &= \begin{bmatrix} \int_{t_{j\theta+1}-t_{j\theta}^1}^{t_{j\theta+1}-t_{j\theta}^0} e^{A_c\sigma} B_c d\sigma & \int_{t_{j\theta+1}-t_{j\theta}^2}^{t_{j\theta+1}-t_{j\theta}^1} e^{A_c\sigma} B_c d\sigma & \int_{t_{j\theta+1}-t_{j\theta}^3}^{t_{j\theta+1}-t_{j\theta}^2} e^{A_c\sigma} B_c d\sigma & \int_0^{t_{j\theta+1}-t_{j\theta}^3} e^{A_c\sigma} B_c d\sigma \end{bmatrix} \\ &= \begin{bmatrix} 0.3654 & 0.3618 & 0.2491 & 0.0226 \\ 0.0556 & 0.1014 & 0.1847 & 0.0906 \end{bmatrix} \\ B(j\theta+1) &= \begin{bmatrix} 0.2191 & 0.1679 \\ 0.0905 & 0.2256 \end{bmatrix} \\ B(j\theta+2) &= \begin{bmatrix} 0.3049 & 0.0821 \\ 0.1512 & 0.1648 \end{bmatrix} \\ B(j\theta+3) &= \begin{bmatrix} 0.1185 & 0.3715 & 0.3410 & 0.1679 \\ 0.0150 & 0.0679 & 0.1238 & 0.2256 \end{bmatrix} \end{aligned}$$

Parity relation based residual generator

Let $s = 3$. Calculate the matrices $H_{o,s}(k), H_{u,s}(k)$ and $H_s(k)$ by (15.9) and $\bar{\Psi}_k^d, \bar{\Psi}_k^f$ are, respectively,

$$
\begin{aligned}
\bar{\Psi}^{\delta}_{j\theta} &= diag\left\{\rho_\delta(0.5), \rho_\delta(0.5), \rho_\delta(1)\right\} \\
\bar{\Psi}^{\delta}_{j\theta+1} &= diag\left\{\rho_\delta(0.5), \rho_\delta(1), \rho_\delta(1)\right\} \\
\bar{\Psi}^{\delta}_{j\theta+2} &= diag\left\{\rho_\delta(1), \rho_\delta(1), \rho_\delta(0.5)\right\} \\
\bar{\Psi}^{\delta}_{j\theta+3} &= diag\left\{\rho_\delta(1), \rho_\delta(0.5), \rho_\delta(0.5)\right\} \\
\rho_\delta(\Delta T) &= \int_0^{\Delta T} e^{A_c t} E_{\delta c} E_{\delta c}^T e^{A_c^T t} dt, \ \delta = d, f
\end{aligned}
$$

The optimal parity relation based residual generator is

$$
r(k) = v(k) \left(\begin{bmatrix} \bar{\psi}(k-3) \\ \bar{\psi}(k-2) \\ \bar{\psi}(k-1) \\ \bar{\psi}(k) \end{bmatrix} - H_{u,s}(k) \begin{bmatrix} \bar{v}(k-3) \\ \bar{v}(k-2) \\ \bar{v}(k-1) \\ \bar{v}(k) \end{bmatrix} \right) \tag{15.31}
$$

with

$$
\begin{aligned}
v(j\theta) &= \begin{bmatrix} -0.0375 & -0.2524 & 0.2004 & -0.2678 & 0.9072 \end{bmatrix} \\
v(j\theta+1) &= \begin{bmatrix} -0.0631\ v - 0.0954 & 0.2592 & -0.9590 & 0 \end{bmatrix} \\
v(j\theta+2) &= \begin{bmatrix} -0.0425 & 0.1155 & -0.9924 & 0 & 0 \end{bmatrix} \\
v(j\theta+3) &= \begin{bmatrix} 0.0188 & 0.0341 & 0.0043 & 0.9878 & -0.1508 \end{bmatrix} \\
J_0 = 0.9348,\ & J_1 = 0.9405,\ J_2 = 0.9676,\ J_3 = 0.9821.
\end{aligned}
$$

Observer-based residual generator
Calculate $\bar{\Gamma}^d_k$ as

$$
\begin{aligned}
\bar{\Gamma}^d_{j\theta} = \bar{\Gamma}^d_{j\theta+3} &= \int_0^1 e^{A_c t} E_{dc} E_{dc}^T e^{A_c^T t} dt = \begin{bmatrix} -1.1023 & -0.1091 \\ -0.3833 & 0.3138 \end{bmatrix} \\
\bar{\Gamma}^d_{j\theta+1} = \bar{\Gamma}^d_{j\theta+2} &= \int_0^{0.5} e^{A_c t} E_{dc} E_{dc}^T e^{A_c^T t} dt = \begin{bmatrix} -0.6330 & -0.1320 \\ -0.4203 & 0.1987 \end{bmatrix}
\end{aligned}
$$

Solving the DPRS (15.30) yields the periodic stabilizing solution

$$
\begin{aligned}
X_d(j\theta) &= \begin{bmatrix} 1.3635 & 0.4042 \\ 0.4042 & 0.2473 \end{bmatrix} \\
X_d(j\theta+1) &= \begin{bmatrix} 1.2269 & 0.3883 \\ 0.3883 & 0.2454 \end{bmatrix} \\
X_d(j\theta+2) &= \begin{bmatrix} 0.5926 & 0.2936 \\ 0.2936 & 0.2328 \end{bmatrix} \\
X_d(j\theta+3) &= \begin{bmatrix} 0.4998 & 0.2398 \\ 0.2398 & 0.2162 \end{bmatrix}
\end{aligned}
$$

Finally, by calculating (15.29), we get a periodic observer based residual generator described by

$$
\begin{aligned}
\hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)\bar{v}(k) + L(k)(\bar{\psi}(k) - \hat{\psi}(k)) \\
r(k) &= W(k)(\bar{\psi}(k) - \hat{\psi}(k)),\ \hat{\psi}(k) = C(k)\hat{x}(k)
\end{aligned} \tag{15.32}
$$

with the optimal observer gain matrix $L(k)$ and the weighting matrix $W(k)$ as below

$$L(j\theta) = \begin{bmatrix} 0.3679 & 1.1627 \\ 0 & 0.1353 \end{bmatrix}, \; L(j\theta+1) = \begin{bmatrix} 0.9841 \\ 0.1164 \end{bmatrix}$$
$$L(j\theta+2) = \begin{bmatrix} 1.9585 \\ 0.3679 \end{bmatrix}, \; L(j\theta+3) = \begin{bmatrix} 0.9258 \\ 0.0649 \end{bmatrix}$$
$$W(j\theta) = \begin{bmatrix} 0.8564 & 0 \\ -0.8302 & 2.8008 \end{bmatrix}, \; W(j\theta+1) = 0.9028$$
$$W(j\theta+2) = 2.0728, \; W(j\theta+3) = 1.4145$$

15.3.2 Lifting based design

In this subsection, we would also like to introduce another approach to the FD of the MSD systems [177]. The basic idea of this approach is to get the input-output relations of the MSD systems at the base periods at first and then downsample them according to different sampling periods to get the parity relation of the MSD systems.

Let

$$\vartheta = T/h, \; \underline{\alpha}_j = T_{u,j}/h, \; \bar{\alpha}_j = T/T_{u,j}, \qquad (15.33)$$
$$\underline{\beta}_l = T_{y,l}/h, \; \bar{\beta}_l = T/T_{y,l}, \; \beta = \sum_{l=1}^{m} \bar{\beta}_l$$

for $j = 1, 2, \cdots, n_u$, $l = 1, 2, \cdots, m$. $\bar{\beta}_l$ represents the number of sampling points of the l-th output over a system period and β the total number of the sampling points of all output signals over a system period.

At each base period, the dynamics of the continuous-time process (14.1) is described by

$$x((k+1)h) = Ax(kh) + Bu(kh) + \bar{d}(k) + \bar{f}(k), \quad y(kh) = Cx(kh) \qquad (15.34)$$

where $A, B, \bar{d}(k)$ and $\bar{f}(k)$ are the same as given by (14.5). During the moving horizon $[kT - sh, kT]$, a group of input-output equations can be obtained as

$$y_s(k\vartheta h) = H_{o,s}x((k\vartheta - s)h) + H_{u,s}u_s(k\vartheta h) + H_s\left(\bar{d}_s(k\vartheta) + \bar{f}_s(k\vartheta)\right) \qquad (15.35)$$

where

$$y_s(k\vartheta h) = \begin{bmatrix} y((k\vartheta - s)h) \\ y((k\vartheta - s + 1)h) \\ \vdots \\ y(k\vartheta h) \end{bmatrix}, \; u_s(k\vartheta h) = \begin{bmatrix} u((k\vartheta - s)h) \\ u((k\vartheta - s + 1)h) \\ \vdots \\ u(k\vartheta h) \end{bmatrix}$$
$$\bar{d}_s(k\vartheta) = \begin{bmatrix} \bar{d}(k\vartheta - s) \\ \bar{d}(k\vartheta - s + 1) \\ \vdots \\ \bar{d}(k\vartheta) \end{bmatrix}, \; \bar{f}_s(k\vartheta) = \begin{bmatrix} \bar{f}(k\vartheta - s) \\ \bar{f}(k\vartheta - s + 1) \\ \vdots \\ \bar{f}(k\vartheta) \end{bmatrix}$$
$$H_{o,s} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^s \end{bmatrix}, \; H_{u,s} = \begin{bmatrix} O & O & \cdots & O \\ CB & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}B & \cdots & CB & O \end{bmatrix}$$
$$H_s = \begin{bmatrix} O & O & \cdots & O \\ C & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1} & \cdots & C & O \end{bmatrix} \qquad (15.36)$$

Using operators Ψ^d and Ψ^f introduced in (14.17), (15.35) can be re-written as

$$\begin{aligned} y_s(k\vartheta h) = & H_{o,s}x((k\vartheta - s)h) + H_{u,s}u_s(k\vartheta h) \\ & + H_s\left(\Psi^d d_{k\vartheta,s}(t) + \Psi^f f_{k\vartheta,s}(t)\right) \end{aligned} \tag{15.37}$$

However, due to the different sampling rates, not all the components in the vector $y_s(k\vartheta h)$ are available. To pick out the components in the vector $y_s(k\vartheta h)$ with available sampled values, define some subscript sets as

$$\Omega_i = \{\, l \mid 1 \leqslant l \leqslant m,\, l \in \mathbf{N},\, (s-i)/\underline{\beta}_l \in \mathbf{Z}\} \tag{15.38}$$

for $i = 0, 1, \cdots, s$. The set Ω_i indicates which process outputs have been sampled at the time instant $(k\vartheta - s + i)h$. Apparently, Ω_i is independent of the value of k. If Ω_i is empty, then in (15.37) those equations relating to $y((k\vartheta - s + i)h)$ should be left out completely. Assume that set Ω_i has a total of μ_i components which are denoted as $\rho_{i,1}, \rho_{i,2}, \cdots, \rho_{i,\mu_i}$ in ascending order, i.e. $1 \leqslant \rho_{i,1} < \rho_{i,2} < \cdots < \rho_{i,\mu_i} \leqslant m$. Corresponding to it, define a matrix $N_S \in \mathbf{R}^{\mu\times(s+1)m}$ with $\mu = \sum_{i=0}^{s}\mu_i$ as follows to describe the sampling mechanism. The components in the matrix N_S are either 1 or 0. Only one component can be 1 in each row of N_S.

Based on the sets Ω_i, define

$$y_{\Omega_i}((k\vartheta - s + i)h) = \begin{bmatrix} y_{\rho_{i,1}}((k\vartheta - s + i)h) \\ y_{\rho_{i,2}}((k\vartheta - s + i)h) \\ \vdots \\ y_{\rho_{i,\mu_i}}((k\vartheta - s + i)h) \end{bmatrix} \tag{15.39}$$

where $i = 0, 1, \cdots, s$, and y_l denotes the l-th process output.

Thus those equations in (15.37) with available sampled values can be picked out and form a new group of equations as

$$\hat{y}_s(k\vartheta h) = N_S\left(H_{o,s}x((k\vartheta - s)h) + H_{u,s}u_s(k\vartheta h) + H_s\left(\bar{d}_s(k\vartheta) + \bar{f}_s(k\vartheta)\right)\right) \tag{15.40}$$

where

$$\hat{y}_s(k\vartheta h) = \begin{bmatrix} y_{\Omega_0}((k\vartheta - s)h) \\ y_{\Omega_1}((k\vartheta - s + 1)h) \\ \vdots \\ y_{\Omega_s}(k\vartheta h) \end{bmatrix},\ \hat{H}_{o,s} = N_S H_{o,s} \tag{15.41}$$

$$\hat{H}_{u,s} = N_S H_{u,s},\ \hat{H}_s = N_S H_s$$

In the next, the vectors $\hat{y}_s(k\vartheta h)$ and $u_s(k\vartheta h)$ in (15.40) will be expressed with the available information ψ_l $(l = 1, \cdots, m)$ and υ_j $(j = 1, \cdots, n_u)$.

According to (15.1), there is

$$y_{\rho_{i,j}}((k\vartheta - s + i)h) = \psi_{\rho_{i,j}}((k\vartheta - s + i)/\underline{\beta}_{\rho_{i,j}}),\ j = 1, 2, \cdots, \mu_i \tag{15.42}$$

Therefore, $y_{\Omega_i}((k\vartheta - s + i)h)$ can be expressed in terms of the available sampled values $\psi_l(k^l)$ as

$$y_{\Omega_i}((k\vartheta - s + i)h) = \begin{bmatrix} \psi_{\rho_{i,1}}((k\vartheta - s + i)/\underline{\beta}_{\rho_{i,1}}) \\ \psi_{\rho_{i,2}}((k\vartheta - s + i)/\underline{\beta}_{\rho_{i,2}}) \\ \vdots \\ \psi_{\rho_{i,\mu_i}}((k\vartheta - s + i)/\underline{\beta}_{\rho_{i,\mu_i}}) \end{bmatrix} \tag{15.43}$$

Denote the vector on the right side of (15.43) as ψ_{Ω_i} and define

$$\hat{\psi}_s(k) = \begin{bmatrix} \psi_{\Omega_0} \\ \psi_{\Omega_1} \\ \vdots \\ \psi_{\Omega_s} \end{bmatrix} \tag{15.44}$$

From (15.41) and (15.43), it can be concluded that

$$\hat{y}_s(k\vartheta h) = \hat{\psi}_s(k) \tag{15.45}$$

According to (15.2) there is

$$u((k\vartheta - s + i)h) = \begin{bmatrix} u_1((k\vartheta - s + i)h) \\ u_2((k\vartheta - s + i)h) \\ \vdots \\ u_{n_u}((k\vartheta - s + i)h) \end{bmatrix} = \begin{bmatrix} \upsilon_1(k_i^1) \\ \upsilon_2(k_i^2) \\ \vdots \\ \upsilon_{n_u}(k_i^{n_u}) \end{bmatrix} \tag{15.46}$$

$$k_i^j \in ((k\vartheta - s + i)/\underline{\alpha}_j - 1, (k\vartheta - s + i)/\underline{\alpha}_j],\ k_i^j \in \mathbf{Z},\ \ j = 1, 2, \cdots, n_u$$

Denote the vector on the right side of (15.46) as $\hat{\upsilon}_i$ and define

$$\hat{\upsilon}_s(k) = \begin{bmatrix} \hat{\upsilon}_0 \\ \hat{\upsilon}_1 \\ \vdots \\ \hat{\upsilon}_s \end{bmatrix} \tag{15.47}$$

there is

$$u_s(k\vartheta h) = \hat{\upsilon}_s(k) \tag{15.48}$$

Based on (15.45) and (15.48), (15.40) reduces to

$$\hat{\psi}_s(k) = N_S H_{o,s} x((k\vartheta - s)h) + N_S H_{u,s} \hat{\upsilon}_s(k) + N_S H_s \left(\Psi^d d_{k\vartheta,s}(t) + \Psi^f f_{k\vartheta,s}(t) \right) \tag{15.49}$$

A parity relation based residual generator can thus be constructed from (15.49) as

$$\hat{r}(k) = \hat{V}_s(\hat{\psi}_s(k) - N_S H_{u,s} \hat{\upsilon}_s(k)) \tag{15.50}$$

where $\hat{r}(k) \in \mathbf{R}^{n_r}, \hat{V}_s \in \mathbf{R}^{n_r \times \mu}$ is the parity vector which satisfies

$$\hat{V}_s N_S H_{o,s} = 0 \tag{15.51}$$

The dynamics of residual generator (15.50) is governed by

$$\hat{r}(k) = \hat{V}_s N_S H_s \left(\Psi^d d_{k\vartheta,s}(t) + \Psi^f f_{k\vartheta,s}(t) \right) \tag{15.52}$$

Similar to the discussion in the last chapter, the optimal design of the parity space based residual generator for the MSD system can be formulated as the following optimization problem

$$\max_{\hat{V}_s, \hat{V}_s N_S H_{o,s}=0} J_{MSD,PS,\infty/\infty} = \max_{\hat{V}_s, \hat{V}_s N_S H_{o,s}=0} \frac{\bar{\lambda}\left(\hat{V}_s N_S H_s \Psi^f \left(\Psi^f\right)^* H_s^T N_s^T \hat{V}_s^T\right)}{\bar{\lambda}\left(\hat{V}_s N_S H_s \Psi^d \left(\Psi^d\right)^* H_s^T N_s^T \hat{V}_s^T\right)} \tag{15.53}$$

$$\max_{\hat{V}_s, \hat{V}_s N_S H_{o,s}=0} J_{MSD,PS,-/\infty} = \max_{\hat{V}_s, \hat{V}_s N_S H_{o,s}=0} \frac{\underline{\lambda}\left(\hat{V}_s N_S H_s \Psi^f \left(\Psi^f\right)^* H_s^T N_s^T \hat{V}_s^T\right)}{\bar{\lambda}\left(\hat{V}_s N_S H_s \Psi^d \left(\Psi^d\right)^* H_s^T N_s^T \hat{V}_s^T\right)} \tag{15.54}$$

$$\max_{\hat{V}_s, \hat{V}_s N_S H_{o,s}=0} J_{MSD,PS,i/\infty} = \max_{\hat{V}_s, \hat{V}_s N_S H_{o,s}=0} \frac{\lambda_i\left(\hat{V}_s N_S H_s \Psi^f \left(\Psi^f\right)^* H_s^T N_s^T \hat{V}_s^T\right)}{\bar{\lambda}\left(\hat{V}_s N_S H_s \Psi^d \left(\Psi^d\right)^* H_s^T N_s^T \hat{V}_s^T\right)} \tag{15.55}$$

The optimal solution to the optimization problem (15.53)-(15.55) can then be obtained by solving its equivalent problem

$$\max_{\hat{V}_s, \hat{V}_s \hat{H}_{o,s}=0} J_{MSD,PS,\infty/\infty} = \max_{\hat{V}_s, \hat{V}_s \hat{H}_{o,s}=0} \frac{\bar{\lambda}\left(\hat{V}_s \hat{H}_{f,s} \hat{H}_{f,s}^T \hat{V}_s^T\right)}{\bar{\lambda}\left(\hat{V}_s \hat{H}_{d,s} \hat{H}_{d,s}^T \hat{V}_s^T\right)} \tag{15.56}$$

$$\max_{\hat{V}_s, \hat{V}_s \hat{H}_{o,s}=0} J_{MSD,PS,-/\infty} = \max_{\hat{V}_s, \hat{V}_s \hat{H}_{o,s}=0} \frac{\underline{\lambda}\left(\hat{V}_s \hat{H}_{f,s} \hat{H}_{f,s}^T \hat{V}_s^T\right)}{\bar{\lambda}\left(\hat{V}_s \hat{H}_{d,s} \hat{H}_{d,s}^T \hat{V}_s^T\right)} \tag{15.57}$$

$$\max_{\hat{V}_s, \hat{V}_s \hat{H}_{o,s}=0} J_{MSD,PS,i/\infty} = \max_{\hat{V}_s, \hat{V}_s \hat{H}_{o,s}=0} \frac{\lambda_i\left(\hat{V}_s \hat{H}_{f,s} \hat{H}_{f,s}^T \hat{V}_s^T\right)}{\bar{\lambda}\left(\hat{V}_s \hat{H}_{d,s} \hat{H}_{d,s}^T \hat{V}_s^T\right)} \tag{15.58}$$

where

$$\hat{H}_{o,s} = N_S \begin{bmatrix} C \\ CA \\ \vdots \\ CA^s \end{bmatrix}$$

$$\hat{H}_{d,s} = N_S \begin{bmatrix} O & O & \cdots & O \\ C\bar{E}_d & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}\bar{E}_d & \cdots & C\bar{E}_d & O \end{bmatrix}$$

$$\hat{H}_{f,s} = N_S \begin{bmatrix} O & O & \cdots & O \\ C\bar{E}_f & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{s-1}\bar{E}_f & \cdots & C\bar{E}_f & O \end{bmatrix} \tag{15.59}$$

Algorithm 15.3 Optimal design of residual generator for MSD systems described by (14.1), (15.1) and (15.2):

- Set the value of s.
- Compute T and h.

- Determine $\vartheta, \underline{\alpha}_j, \bar{\alpha}_j, \underline{\beta}_l, \bar{\beta}_l$ according to (15.33).
- Compute $A, B, \bar{E}_d$ and $\bar{E}_f$ according to (14.5) and (14.24).
- Determine the sets Ω_i according to (15.38) for $i = 0, 1, \cdots, s$ and then the matrices N_S.
- Determine the matrices $\hat{H}_{o,s}, \hat{H}_{d,s}, \hat{H}_{f,s}$ by (15.59).
- Solve the optimization problems (15.56)-(15.58) using Theorem 5.1-5.3.

Example 15.2 Given an MSD system where the continuous LTI process (14.1) is given by

$$\dot{x}(t) = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$
$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) \tag{15.60}$$

and the periods of the A/D and D/A converters are respectively $T_{y,1} = 0.5s$, $T_{y,2} = 1s$, $T_u = 0.5s$.

Assume that there is no time delays, i.e. $\tau_{y,1} = 0$, $\tau_{y,2} = 0$, $\tau_u = 0$. Then, $T = 1s$ and $h = 0.5s$. According to (15.33), there is $\vartheta = 2$, $\underline{\alpha}_1 = 1, \bar{\alpha}_1 = 2, \underline{\beta}_1 = 1, \bar{\beta}_1 = 2, \underline{\beta}_2 = 2, \bar{\beta}_2 = 1$. The matrices $A, B, \bar{E}_d$ and $\bar{E}_f$ are obtained as

$$A = \begin{bmatrix} 0.61 & 1.19 \\ 0 & 0.37 \end{bmatrix}, B = \begin{bmatrix} 0.39 \\ 0.32 \end{bmatrix}, \bar{E}_d = \begin{bmatrix} 0.65 & 0 \\ 0.37 & 0.28 \end{bmatrix}, \bar{E}_f = \begin{bmatrix} 0.60 & 0 \\ 0.36 & 0.30 \end{bmatrix} \tag{15.61}$$

Set $s = 3$. Then the subscript sets are

$$\Omega_0 = \{1\}, \Omega_1 = \{1, 2\}, \Omega_2 = \{1\}, \Omega_3 = \{1, 2\}$$

Correspondingly, N_S is a 6×8 matrix as follows

$$N_S = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

Based on it, we get

$$\hat{H}_{0,s} = N_S \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}, \quad \hat{H}_{u,s} = N_S \begin{bmatrix} O_{2\times 1} & O_{2\times 1} & O_{2\times 1} & O_{2\times 1} \\ CB & O_{2\times 1} & O_{2\times 1} & O_{2\times 1} \\ CAB & CB & O_{2\times 1} & O_{2\times 1} \\ CA^2B & CAB & CB & O_{2\times 1} \end{bmatrix}$$
$$\hat{H}_{d,s} = N_S \begin{bmatrix} O_{2\times 2} & O_{2\times 2} & O_{2\times 2} & O_{2\times 2} \\ C\bar{E}_d & O_{2\times 2} & O_{2\times 2} & O_{2\times 2} \\ CA\bar{E}_d & C\bar{E}_d & O_{2\times 2} & O_{2\times 2} \\ CA^2\bar{E}_d & CA\bar{E}_d & C\bar{E}_d & O_{2\times 2} \end{bmatrix}$$
$$\hat{H}_{f,s} = N_S \begin{bmatrix} O_{2\times 2} & O_{2\times 2} & O_{2\times 2} & O_{2\times 2} \\ C\bar{E}_f & O_{2\times 2} & O_{2\times 2} & O_{2\times 2} \\ CA\bar{E}_f & C\bar{E}_f & O_{2\times 2} & O_{2\times 2} \\ CA^2\bar{E}_f & CA\bar{E}_f & C\bar{E}_f & O_{2\times 2} \end{bmatrix}$$

- According to Theorem 5.1, solve the eigenvalue-eigenvector problem and get the optimal parity vector
$$\hat{v}_{s,opt} = \begin{bmatrix} 0 & 0.24 & 0.18 & -1.43 & 1.71 & -3.43 \end{bmatrix}$$
which is optimal in the sense of (15.53).

- Construct the residual generator by (15.50) as

$$\hat{r}(k) = \hat{v}_{s,opt}(\hat{\psi}_s(k) - \hat{H}_{u,s}\hat{v}_s(k)) \tag{15.62}$$

where

$$\hat{\psi}_s(k) = \begin{bmatrix} \psi_1(2k-3) \\ \psi_1(2k-2) \\ \psi_2(k-1) \\ \psi_1(2k-1) \\ \psi_1(2k) \\ \psi_2(k) \end{bmatrix}, \ \hat{v}_s(k) = \begin{bmatrix} \upsilon(2k-3) \\ \upsilon(2k-2) \\ \upsilon(2k-1) \\ \upsilon(2k) \end{bmatrix}$$

The approach presented in this section can be extended to handle the MSD systems with *time delays*. To describe the time delays, the multirate A/D and D/A converters (15.1) and (15.2) are extended, respectively, to

$$\psi_l(k^l) = y_l(k^l T_{y,l} - \tau_{y,l}), \ l = 1, 2, \cdots, m; \ k^l = 0, 1, 2, \cdots \tag{15.63}$$

$$u_j(t) = \upsilon_j(k^j), \ k^j T_{u,j} + \tau_{u,j} \le t < (k^j + 1)T_{u,j} + \tau_{u,j} \tag{15.64}$$

$$j = 1, 2, \cdots, n_u; \ \ k^j = 0, 1, 2, \cdots$$

where $\tau_{y,l}$ and $\tau_{u,j}$ denote the corresponding time delays in each input and output channels respectively. Define

$$\sigma_j = \tau_{u,j}/h, \ \varepsilon_l = \tau_{y,l}/h, \ j = 1, 2, \cdots, n_u; \ l = 1, 2, \cdots, m \tag{15.65}$$

where h is the base period. Furthermore, define

$$\begin{aligned} \varepsilon_{\max} &= \max\{\varepsilon_1, \varepsilon_2 \cdots, \varepsilon_m\}, \quad k_{st} = k\vartheta - s - \varepsilon_{\max} \\ \varepsilon_{\min} &= \min\{\varepsilon_1, \varepsilon_2 \cdots, \varepsilon_m\}, \quad k_{end} = k\vartheta - \varepsilon_{\min} \\ \delta_s &= k_{end} - k_{st} \end{aligned} \tag{15.66}$$

and extend the subscript sets (15.38) to

$$\begin{aligned} \Omega_i = \Big\{ & \ l \mid 1 \leqslant l \leqslant m, \, l \in \mathbf{N}, \, (s + \varepsilon_{\max} - i - \varepsilon_l)/\underline{\beta}_l \in \mathbf{Z} \\ & \text{and } \varepsilon_{\max} - i \leqslant \varepsilon_l \leqslant s + \varepsilon_{\max} - i \} \end{aligned} \tag{15.67}$$

for $i = 0, 1, \cdots, \delta_s$. The set Ω_i indicates which process outputs have been sampled at the time instant $(k_{st}+i)h$ and can also be received by the computer during the period from $(k\vartheta - s)h$ to $k\vartheta h$ although the presence of the time delays. The equations (15.42) and (15.46) are extended, respectively, to

$$y_{\rho_{i,j}}((k_{st} + i)h) = \psi_{\rho_{i,j}}((k_{st} + i + \varepsilon_{\rho_{i,j}})/\underline{\beta}_{\rho_{i,j}}), \ j = 1, 2, \cdots, \mu_i \tag{15.68}$$

$$u((k_{st}+i)h) = \begin{bmatrix} u_1((k_{st}+i)h) \\ u_2((k_{st}+i)h) \\ \vdots \\ u_{n_u}((k_{st}+i)h) \end{bmatrix} = \begin{bmatrix} \upsilon_1(k_i^1) \\ \upsilon_2(k_i^2) \\ \vdots \\ \upsilon_{n_u}(k_i^{n_u}) \end{bmatrix} \tag{15.69}$$

$$k_i^j \in ((k_{st} + i - \sigma_j)/\underline{\alpha}_j - 1, (k_{st} + i - \sigma_j)/\underline{\alpha}_j], \ k_i^j \in \mathbf{Z}, \ \ j = 1, 2, \cdots, n_u$$

The other steps of the design procedure are the same as the delay-free case. In the following, we shall use an example to show it briefly.

Example 15.3 Consider an MSD system with the same continuous LTI process (15.60). The periods of the A/D and D/A converters are respectively $T_{y,1} = 0.5s, T_{y,2} = 1s, T_u = 0.5s$. Assume that the time delays in the inputs and the outputs are, respectively, $\tau_{y,1} = 0.5s, \tau_{y,2} = 2s, \tau_u = 1s$. Design a discrete-time residual generator for such a system.

- Still, $T = 1s, h = 0.5s$, $\vartheta = 2, \underline{\alpha}_1 = 1, \bar{\alpha}_1 = 2, \underline{\beta}_1 = 1, \bar{\beta}_1 = 2, \underline{\beta}_2 = 2, \bar{\beta}_2 = 1$. According to (15.65), there is $\sigma_1 = 2$, $\varepsilon_1 = 1$, $\varepsilon_2 = 4$.
- The matrices $A, B, \bar{E}_d, \bar{E}_f$ are the same as in (15.61).
- Set $s = 4$. Then $k_{st} = 2k - 8$, $k_{end} = 2k - 1$, $\delta_s = 7$. The subscript sets are got from (15.67) as

$$\Omega_0 = \Omega_2 = \{2\},\ \Omega_1 = \{\},\ \Omega_3 = \Omega_5 = \Omega_6 = \Omega_7 = \{1\}, \Omega_4 = \{1, 2\}$$

- Correspondingly, the matrix N_S that represents the sampling mechanism is a 8×16 matrix as follows

$$N_S = \begin{bmatrix} 0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \end{bmatrix}$$

- Compute $\hat{H}_{o,\delta_s}, \hat{H}_{u,\delta_s}, \hat{H}_{d,\delta_s}$ and $\hat{H}_{f,\delta_s}$

$$\hat{H}_{0,\delta_s} = N_S \begin{bmatrix} C \\ CA \\ \vdots \\ CA^7 \end{bmatrix}, \hat{H}_{u,\delta_s} = N_S \begin{bmatrix} O & O & \cdots & O \\ CB & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^6B & \cdots & CB & O \end{bmatrix}$$

$$\hat{H}_{d,\delta_s} = N_S \begin{bmatrix} O & O & \cdots & O \\ C\bar{E}_d & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^6\bar{E}_d & \cdots & C\bar{E}_d & O \end{bmatrix}, \hat{H}_{f,\delta_s} = N_S \begin{bmatrix} O & O & \cdots & O \\ C\bar{E}_f & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^6\bar{E}_f & \cdots & C\bar{E}_f & O \end{bmatrix}$$

- According to Theorem 5.1, solve the eigenvalue-eigenvector problem and get the optimal parity vector

$$\hat{v}_{\delta_s,opt} = \begin{bmatrix} 0 & -0.20 & -0.90 & 1.49 & -3.31 & 0 & 0 & 0 \end{bmatrix}$$

which achieves the optimal trade-off between the robustness and the sensitivity.
- Construct the residual generator according to (15.50) as

$$\hat{r}(k) = \hat{v}_{\delta_s,opt}(\hat{\psi}_{\delta_s}(k) - \hat{H}_{u,\delta_s}\hat{\upsilon}_{\delta_s}(k)) \tag{15.70}$$

where

$$\hat{\psi}_{\delta_s}(k) = \begin{bmatrix} \psi_2(k-2) \\ \psi_2(k-1) \\ \psi_1(2k-4) \\ \psi_1(2k-3) \\ \psi_2(k) \\ \psi_1(2k-2) \\ \psi_1(2k-1) \\ \psi_1(2k) \end{bmatrix},\ \hat{\upsilon}_{\delta_s}(k) = \begin{bmatrix} \upsilon(2k-10) \\ \upsilon(2k-9) \\ \vdots \\ \upsilon(2k-3) \end{bmatrix}$$

15.4 Concluding remarks

In this chapter, the FD problems of the NSD and the MSD systems are considered. The basic idea of the FD approach to the NSD systems introduced in Section 15.2 is to re-model the NSD systems as time-varying systems and then apply the time-varying system theory to design the FD systems. It is motivated by the fact that, for the purpose of fault detection, only at the time instants with at least one sampled plant output available it is possible to generate a residual signal that can reflect the system operating state. This viewpoint is especially helpful for the FD of the MSD systems [170]. Following it, the MSD systems can be re-formulated as periodically time-varying systems. The FD problems can then be solved with the approaches introduced in Chapter 9-10, as shown in Section 15.3.1. The intersample behavior is taken into account by introducing operators.

Concerning the FD of the NSD systems, there is few work in the literature. For the development of the FD approaches for the MSD systems during the last years, the readers are referred to [52, 53, 50, 86, 87, 153, 175, 177, 170, 198]. Viswanadham and Minto (1990) have made the first efforts to the FDI of a special kind of MSD systems, in which all the control inputs are updated at a single slow rate while the process outputs are sampled at different fast rates [153]. In the derivation, it is assumed that the supervised process has neither model uncertainties nor unknown disturbances acting on it. Going a step further, Fadali *et al.* have extended the observer based FDI scheme and the parity-space approach to another kind of MSD systems with a single fast control input updating rate and different slow process output sampling rates for the special case that the unknown inputs can be perfectly decoupled from the residuals [52, 53, 50]. In these studies, the intersample behaviour and its influence on the FDI performance have not been taken into consideration. To take into account both the intersample behaviour and the multirate nature, a *direct design* approach for the MSD systems is proposed in [177]. *Aiming at a fast fault detection*, a periodically time-varying observer-based residual generator has been presented in [175], whose basic idea is to look at the system dynamics at each base period instead of over a system period. The problem of fast rate residual generation is further pursued by [86, 87, 198], which design a bank of residual generators with appropriate post-filtering to cope with the causality constraints. However, the calculation needed for designing such post-filtering terms is rather complicated. Compared with [175], the method proposed in [170, 172] considerably reduces the frequency of updating the observer parameters during one system period and gives a unified solution to the MSD systems and the NSD systems.

16

Influence of sampling period

Sampling period is a key parameter in sampled-data systems. The smaller it is, the more online information we have about the system to be supervised. On the other hand, it also leads to increased transmission and computational loads. In this chapter, we shall study the influence of sampling period on the FD performance [173, 171, 185].

16.1 Optimal FD performance in the parity space approach

Assume that the sampling period is increased from h to $\mathbf{h} = \rho h$, where ρ is a natural number. It is well-known that the performance index of the parity space approach depends on the size of the moving horizon s [47]. With the increase of s, the optimal performance index $J_{SSD,PS,\infty/\infty,opt}$ will be non-decreasing [47]. To discuss quantitatively the influence of the sampling period on the FD performance $J_{SSD,PS,\infty/\infty,opt}$, we assume that, the size of the moving horizon in the continuous domain is constant, i.e. $\mathbf{sh} = sh$. In this section, we shall use bold letters to indicate the matrices related with the sampling period $\mathbf{h}$.

At each time instant $\mathbf{t} = \mathbf{kh}$, $\mathbf{k} = 0, 1, \cdots$, the residual signal is generated by

$$\mathbf{r}(\mathbf{k}) = \mathbf{V_s} \left(\begin{bmatrix} y(\mathbf{k}-\mathbf{s}) \\ \vdots \\ y(\mathbf{k}-1) \\ y(\mathbf{k}) \end{bmatrix} - \mathbf{H}_{u,\mathbf{s}} \begin{bmatrix} u(\mathbf{k}-\mathbf{s}) \\ \vdots \\ u(\mathbf{k}-1) \\ u(\mathbf{k}) \end{bmatrix} \right) \tag{16.1}$$

Following the same principle, the parity vector $\mathbf{V_s}$ is optimized by

$$\mathbf{V}_{\mathbf{s},opt} = \arg \max_{\mathbf{V_s}, \mathbf{V_s}\mathbf{H}_{o,\mathbf{s}}=0} J_{SSD,PS,\infty/\infty,\mathbf{h}}(\mathbf{V_s}) \tag{16.2}$$

$$J_{SSD,PS,\infty/\infty,\mathbf{h}}(\mathbf{V_s}) = \frac{\bar{\sigma}^2(\mathbf{V_s}\bar{\mathbf{H}}_{f,\mathbf{s}})}{\bar{\sigma}^2(\mathbf{V_s}\bar{\mathbf{H}}_{d,\mathbf{s}})} \tag{16.3}$$

where in the subscript of the performance index an additional term is added to indicate the sampling period,

$$\mathbf{H}_{o,\mathbf{s}} = \begin{bmatrix} C \\ C(e^{A_c\mathbf{h}}) \\ C(e^{A_c\mathbf{h}})^2 \\ \vdots \\ C(e^{A_c\mathbf{h}})^{\mathbf{s}} \end{bmatrix}, \ \mathbf{B} = \int_0^{\mathbf{h}} e^{A_c t} B_c dt$$

$$\mathbf{H}_{u,\mathbf{s}} = \begin{bmatrix} O & O & O & \cdots & O \\ C\mathbf{B} & O & \ddots & \ddots & \vdots \\ Ce^{A_c\mathbf{h}}\mathbf{B} & C\mathbf{B} & O & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ C(e^{A_c\mathbf{h}})^{\mathbf{s}-1}\mathbf{B} & C(e^{A_c\mathbf{h}})^{\mathbf{s}-2}\mathbf{B} & \cdots & C\mathbf{B} & O \end{bmatrix}$$

$$\bar{\mathbf{H}}_{d,\mathbf{s}} = \begin{bmatrix} O & O & O & \cdots & O \\ C\bar{\mathbf{E}}_d & O & \ddots & \ddots & \vdots \\ Ce^{A_c\mathbf{h}}\bar{\mathbf{E}}_d & C\bar{\mathbf{E}}_d & O & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ C(e^{A_c\mathbf{h}})^{\mathbf{s}-1}\bar{\mathbf{E}}_d & C(e^{A_c\mathbf{h}})^{\mathbf{s}-2}\bar{\mathbf{E}}_d & \cdots & C\bar{\mathbf{E}}_d & O \end{bmatrix}$$

$$\bar{\mathbf{E}}_d\bar{\mathbf{E}}_d^T = \int_0^{\mathbf{h}} e^{A_c\tau} E_{dc} E_{dc}^T e^{A_c^T\tau} d\tau, \ \bar{\mathbf{H}}_{f,\mathbf{s}} \text{ similar to } \bar{\mathbf{H}}_{d,\mathbf{s}}$$

Notice that

$$\mathbf{H}_{o,\mathbf{s}} = \begin{bmatrix} C \\ C(e^{A_c h})^{\rho} \\ C(e^{A_c h})^{2\rho} \\ \vdots \\ C(e^{A_c h})^{\mathbf{s}\rho} \end{bmatrix}$$

As matrix $\mathbf{H}_{o,\mathbf{s}}$ is indeed composed of some equidistant rows of matrix $H_{o,s}$, it can be re-written into

$$\mathbf{H}_{o,\mathbf{s}} = N_\rho H_{o,s} \tag{16.4}$$

where N_ρ is a full row rank matrix constructed as

$$N_\rho = \begin{bmatrix} I & O & \cdots & O & O & \cdots & O \\ O & \cdots & O & I & O & \cdots & O \\ & & & \cdots & & & \\ O & \cdots & O & O & \cdots & O & I \end{bmatrix}$$

Denote the basis matrix of the left null space of $\mathbf{H}_{o,\mathbf{s}}$ by $\mathbf{N}_{basis,\mathbf{s}}$ and that of $H_{o,s}$ by $N_{basis,s}$. From

$$\mathbf{N}_{basis,\mathbf{s}}\mathbf{H}_{o,\mathbf{s}} = \mathbf{N}_{basis,\mathbf{s}} N_\rho H_{o,s} = 0 \tag{16.5}$$

It can be seen that $\mathbf{N}_{basis,\mathbf{s}} N_\rho$ lies in the left null space of matrix $H_{o,s}$ and is a linear combination of the basis matrix $N_{basis,s}$. Therefore, a full row rank matrix P can be found, such that

$$\mathbf{N}_{basis,\mathbf{s}} N_\rho = P N_{basis,s} \tag{16.6}$$

As

$$\begin{aligned}
\bar{\mathbf{E}}_d\bar{\mathbf{E}}_d^T &= \sum_{l=1}^{\rho}\int_{(l-1)h}^{lh} e^{A_c\tau}E_{dc}E_{dc}^T e^{A_c^T\tau}d\tau \\
&= \sum_{l=1}^{\rho}\int_{0}^{h} e^{A_c((l-1)h+\xi)}E_{dc}E_{dc}^T e^{A_c^T((l-1)h+\xi)}d\xi \\
&= \sum_{l=1}^{\rho} e^{(l-1)A_ch}\left(\int_{0}^{h} e^{A_c\xi}E_{dc}E_{dc}^T e^{A_c^T\xi}d\xi\right)e^{(l-1)A_c^Th} \\
&= \sum_{l=1}^{\rho} e^{(l-1)A_ch}\bar{E}_d\bar{E}_d^T e^{(l-1)A_c^Th}
\end{aligned} \tag{16.7}$$

The (i,j)-block of the matrix $\bar{\mathbf{H}}_{d,\mathbf{s}}\bar{\mathbf{H}}_{d,\mathbf{s}}^T$, where $i=2,\cdots,s+1$, $j=2,\cdots,s+1$, $i\geq j$, is

$$(\bar{\mathbf{H}}_{d,\mathbf{s}}\bar{\mathbf{H}}_{d,\mathbf{s}}^T)_{ij} = \sum_{q=0}^{j-2} C(e^{A_c\mathbf{h}})^{i-2-q}\bar{\mathbf{E}}_d\bar{\mathbf{E}}_d^T(e^{A_c^T\mathbf{h}})^{j-2-q}C^T \tag{16.8}$$

Substituting $\mathbf{h}=\rho h$ and (16.7) into (16.8) yields

$$\begin{aligned}
&(\bar{\mathbf{H}}_{d,\mathbf{s}}\bar{\mathbf{H}}_{d,\mathbf{s}}^T)_{ij} \\
&= \sum_{q=0}^{j-2}\sum_{l=1}^{\rho} C(e^{A_ch})^{\rho(i-2)-q\rho+l-1}\bar{E}_d\bar{E}_d^T(e^{A_c^Th})^{\rho(j-2)-q\rho+l-1}C^T \\
&= \sum_{p=0}^{\rho(j-1)-1} C(e^{A_ch})^{\rho(i-j)+p}\bar{E}_d\bar{E}_d^T(e^{A_c^Th})^{p}C^T = \beta\gamma^T
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \left[\, C(e^{A_ch})^{\rho(i-1)-1}\bar{E}_d \cdots C(e^{A_ch})^{\rho(i-j)}\bar{E}_d \,\right] \\
\gamma &= \left[\, C(e^{A_ch})^{\rho(j-1)-1}\bar{E}_d \cdots C\bar{E}_d \,\right]
\end{aligned}$$

It can then be verified that

$$\bar{\mathbf{H}}_{d,\mathbf{s}}\bar{\mathbf{H}}_{d,\mathbf{s}}^T = N_\rho\bar{H}_{d,s}\bar{H}_{d,s}^TN_\rho^T \tag{16.9}$$

Based on (16.6) and (16.9), the following Theorem can be obtained.

Theorem 16.1 Given the SD system described by (14.1)-(14.3) and an arbitrary positive integer $\rho\geq 2$. Let $J_{SSD,PS,\infty/\infty,h,opt}$ and $J_{SSD,PS,\infty/\infty,\rho h,opt}$ denote, respectively, the optimal performance index defined by (14.20) achievable under sampling period h (size of moving horizon is s) and ρh (size of moving horizon is s/ρ) in the framework of the parity space approach. Then, $J_{SSD,PS,\infty/\infty,h,opt}\geq J_{SSD,PS,\infty/\infty,\rho h,opt}$.

Proof: As $\mathbf{N}_{basis,\mathbf{s}}$ is the basis matrix of the left null space of $\mathbf{H}_{o,\mathbf{s}}$, the parity vector $\mathbf{V}_\mathbf{s}$ can be substituted by $\mathbf{V}_\mathbf{s}=\mathbf{P}_\mathbf{s}\mathbf{N}_{basis,\mathbf{s}}$, where $\mathbf{P}_\mathbf{s}$ is a vector of compatible dimensions. The optimization problem (16.3) can be equivalently re-written as

$$J_{SSD,PS,\infty/\infty,\mathbf{h},opt} = \max_{\mathbf{P}_\mathbf{s}}\frac{\bar{\lambda}\left(\mathbf{P}_\mathbf{s}\mathbf{N}_{basis,\mathbf{s}}\bar{\mathbf{H}}_{f,\mathbf{s}}\bar{\mathbf{H}}_{f,\mathbf{s}}^T\mathbf{N}_{basis,\mathbf{s}}^T\mathbf{P}_\mathbf{s}^T\right)}{\bar{\lambda}\left(\mathbf{P}_\mathbf{s}\mathbf{N}_{basis,\mathbf{s}}\bar{\mathbf{H}}_{d,\mathbf{s}}\bar{\mathbf{H}}_{d,\mathbf{s}}^T\mathbf{N}_{basis,\mathbf{s}}^T\mathbf{P}_\mathbf{s}^T\right)}$$

Based on (16.6) and (16.9), we have

$$J_{SSD,PS,\infty/\infty,\mathbf{h},opt} = \max_{\mathbf{P_s}} \frac{\bar{\lambda}(\mathbf{P_s N}_{basis,\mathbf{s}} N_\rho \bar{H}_{f,s} \bar{H}_{f,s}^T N_\rho^T \mathbf{N}_{basis,\mathbf{s}}^T \mathbf{P_s}^T)}{\bar{\lambda}(\mathbf{P_s N}_{basis,\mathbf{s}} N_\rho \bar{H}_{d,s} \bar{H}_{d,s}^T N_\rho^T \mathbf{N}_{basis,\mathbf{s}}^T \mathbf{P_s}^T)}$$
$$= \max_{\mathbf{P_s}} \frac{\bar{\lambda}(\mathbf{P_s} P N_{basis,s} \bar{H}_{f,s} \bar{H}_{f,s}^T N_{basis,s}^T P^T \mathbf{P_s}^T)}{\bar{\lambda}(\mathbf{P_s} P N_{basis,s} \bar{H}_{d,s} \bar{H}_{d,s}^T N_{basis,s}^T P^T \mathbf{P_s}^T)}$$

Since the matrix P is of full row rank, the following inequality holds

$$J_{SSD,PS,\infty/\infty,\mathbf{h},opt} \leq \max_{P_s} \frac{\bar{\lambda}\left(P_s N_{basis,s} \bar{H}_{f,s} \bar{H}_{f,s}^T N_{basis,s}^T P_s^T\right)}{\bar{\lambda}\left(P_s N_{basis,s} \bar{H}_{d,s} \bar{H}_{d,s}^T N_{basis,s}^T P_s^T\right)}$$
$$= J_{SSD,PS,\infty/\infty,h,opt}$$

The conclusion in Theorem 16.1 is thus proven. □

Theorem 16.1 shows that with the increase of the sampling period the FD performance $J_{SSD,PS,\infty/\infty,opt}$ will decrease.

Example 16.1 Consider the SD system described by (14.1)-(14.3) with

$$\begin{aligned} G_u(s) &= (A_c, B_c, C, O) = 0 \\ G_d(s) &= (A_c, E_{dc}, C, O) = \frac{8}{s^2 + 19s + 9} \\ G_f(s) &= (A_c, E_{fc}, C, O) = \frac{1}{s^2 + 4s + 6} \end{aligned} \tag{16.10}$$

The optimal performance index $J_{SSD,PS,\infty/\infty,h,opt}$ with respect to different sampling period h is shown in Table 16.1. It can be seen that

$$J_{SSD,PS,\infty/\infty,h_i,opt} \geq J_{SSD,PS,\infty/\infty,\rho_i h_i,opt},\ i = 1, 2, 3$$

for $h_1 = 0.1s$, $\rho_1 = 2, 3, 5, 6$, $h_2 = 0.2s$, $\rho_2 = 3$, $h_3 = 0.3s$, $\rho_3 = 2$. It is consistent with the conclusion of Theorem 16.1.

h	0.1	0.2	0.3	0.5	0.6
s	30	15	10	6	5
$J_{SSD,PS,\infty/\infty,h,opt}$	0.2921	0.2871	0.2764	0.2360	0.2189

Table 16.1 The optimal index $J_{SSD,PS,\infty/\infty,h,opt}$ with respect to different sampling periods h in Example 16.1

16.2 Optimal H_2/H_2 performance

In this section, we continue to discuss the influence of sampling period on the optimal H_2/H_2 performance. Due to the close relationship between the optimal H_2/H_2 index $J_{SSD,FRE,2/2,opt}$ and the optimal index $J_{SSD,PS,opt}$ in the parity space approach [195], the following theorem is readily obtained.

Theorem 16.2 Given the SD system described by (14.1)-(14.3) and an arbitrary positive integer $\rho \geq 2$. Let $J_{SSD,FRE,2/2,h,opt}$ and $J_{SSD,FRE,2/2,\rho h,opt}$ denote, respectively, the optimal performance index defined by (14.46) achievable under sampling period h and ρh in the framework of the post-filter based approach. Then, $J_{SSD,FRE,2/2,h,opt} \geq J_{SSD,FRE,2/2,\rho h,opt}$.

Proof: Recall the limiting property of $J_{SSD,PS,h,opt}$ in case of $s \to \infty$, i.e. $J_{SSD,FRE,2/2,opt}$ and $J_{SSD,PS,opt}$ are related by

$$J_{SSD,FRE,2/2,h,opt} = \lim_{s\to\infty} J_{SSD,PS,h,opt}$$
$$J_{SSD,FRE,2/2,\rho h,opt} = \lim_{s\to\infty} J_{SSD,PS,\rho h,opt} \tag{16.11}$$

According to Theorem 16.1, there is

$$\lim_{s\to\infty} J_{SSD,PS,h,opt} \geq \lim_{s\to\infty} J_{SSD,PS,\rho h,opt}$$

It follows that

$$J_{SSD,FRE,2/2,h,opt} \geq J_{SSD,FRE,2/2,\rho h,opt}$$

Example 16.2 For the SD system given in Example 16.1, compare the optimal H_2/H_2 index.

The optimal H_2/H_2 index $J_{SSD,FRE,2/2,h,opt}$ is obtained by solving (14.62) and (14.61). Table 16.2 shows the value of $J_{SSD,FRE,2/2,h,opt}$ under different sampling periods.

h	0.1	0.2	0.3	0.4	0.5
$J_{SSD,FRE,2/2,h,opt}$	0.3540	0.3529	0.3494	0.3431	0.3346
h	0.6	0.7	0.8	0.9	1.0
$J_{SSD,FRE,2/2,h,opt}$	0.3250	0.31562	0.3078	0.3029	0.3021
h	1.1	1.2	1.3	1.4	1.5
$J_{SSD,FRE,2/2,h,opt}$	0.3069	0.3151	0.3177	0.31561	0.3099
h	1.6	1.7	1.8	1.9	2.0
$J_{SSD,FRE,2/2,h,opt}$	0.3017	0.2920	0.2814	0.2699	0.2839

Table 16.2 The optimal H_2/H_2 index $J_{SSD,FRE,2/2,h,opt}$ with respect to different sampling periods h in Example 16.2

As can be seen, $J_{SSD,FRE,2/2,h_i,opt} \geq J_{SSD,FRE,2/2,\rho_i h_i,opt}, i = 1, \cdots, 10$, ρ_i is a positive integer, for $h_1 = 0.1s, 2 \leq \rho_1 \leq 20$, $h_2 = 0.2s, 2 \leq \rho_2 \leq 10$, $h_3 = 0.3s, 2 \leq \rho_3 \leq 6$, etc.

16.3 Optimal H_∞/H_∞ performance

The problem to be addressed in this section is: How will the optimal H_∞/H_∞ index $J_{SSD,OBS,\infty/\infty,opt}$ change with respect to the sampling period h? Compared with the parity space approach, the discussion in the observer based case is much more complex [185].

To answer the above question, the key of the analysis is to find a way to connect systems with different sampling periods. For this purpose, we shall first consider an alternative scheme of generating a residual signal for SD system (14.1)-(14.3). In this section, bold letters will be used to indicate the matrices related with the lifting operation.

16.3.1 An alternative scheme of residual generation

Let $\rho \in N$. Note that the dynamics of SD system (14.1)-(14.3) during $[k\rho h, (k+1)\rho h)$ can be described by

$$\mathbf{x}_\rho(k+1) = \mathbf{A}_\rho \mathbf{x}_\rho(k) + \mathbf{B}_\rho \boldsymbol{\upsilon}_\rho(k) + \int_{k\rho h}^{(k+1)\rho h} e^{A_c\tau}\left(E_{dc}d(\tau) + E_{fc}f(\tau)\right) d\tau$$
$$\boldsymbol{\psi}_\rho(k) = \mathbf{C}_\rho \mathbf{x}_\rho(k) + \mathbf{D}_\rho \boldsymbol{\upsilon}_\rho(k) \tag{16.12}$$

where $\mathbf{x}_\rho(k) = x(k\rho h)$, $\boldsymbol{\upsilon}_\rho(k)$ and $\boldsymbol{\psi}_\rho(k)$ are the lifting of $\upsilon(k)$ and $\psi(k)$, i.e.

$$\boldsymbol{\upsilon}_\rho(k) = \begin{bmatrix} \upsilon(k\rho) \\ \upsilon(k\rho+1) \\ \vdots \\ \upsilon(k\rho+\rho-1) \end{bmatrix}, \boldsymbol{\psi}_\rho(k) = \begin{bmatrix} \psi(k\rho) \\ \psi(k\rho+1) \\ \vdots \\ \psi(k\rho+\rho-1) \end{bmatrix}$$

$(\mathbf{A}_\rho, \mathbf{B}_\rho, \mathbf{C}_\rho, \mathbf{D}_\rho)$ is the ρ-step lifting of (A, B, C, O),

$$\mathbf{A}_\rho = A^\rho, \ \mathbf{B}_\rho = \begin{bmatrix} A^{\rho-1}B \ A^{\rho-2}B \cdots \ B \end{bmatrix} \tag{16.13}$$
$$\mathbf{C}_\rho = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\rho-1} \end{bmatrix}, \mathbf{D}_\rho = \begin{bmatrix} O & O & \cdots & O \\ CB & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{\rho-2}B & \cdots & CB & O \end{bmatrix}$$

Note that $\mathbf{B}_\rho, \mathbf{C}_\rho$ and $\mathbf{D}_\rho$ can be re-written as

$$\mathbf{B}_\rho = \begin{bmatrix} A\mathbf{B}_{\rho-1} \ B \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\rho-1}B \ \mathbf{B}_{\rho-1} \end{bmatrix}$$
$$\mathbf{C}_\rho = \begin{bmatrix} \mathbf{C}_{\rho-1} \\ C\mathbf{A}_{\rho-1} \end{bmatrix} = \begin{bmatrix} C \\ \mathbf{C}_{\rho-1}A \end{bmatrix}$$
$$\mathbf{D}_\rho = \begin{bmatrix} \mathbf{D}_{\rho-1} & O \\ C\mathbf{B}_{\rho-1} & O \end{bmatrix} = \begin{bmatrix} O & O \\ \mathbf{C}_{\rho-1}B & \mathbf{D}_{\rho-1} \end{bmatrix} \tag{16.14}$$

One of the main properties of the lifting is that it preserves the l_2-norm of the signal and thus the $\mathbf{H}_\infty$-norm of the system [8, 20]. The lifting technique is used here to bridge systems with different sampling periods.

Thus, a residual signal r_o can be generated by the following residual generator with constant free parameters $\mathbf{L}_\rho$ and $\mathbf{W}_\rho$

$$\begin{aligned} \tilde{\mathbf{x}}_\rho(k+1) &= \mathbf{A}_\rho \tilde{\mathbf{x}}_\rho(k) + \mathbf{B}_\rho \boldsymbol{\upsilon}_\rho(k) + \mathbf{L}_\rho(\boldsymbol{\psi}_\rho(k) - \tilde{\boldsymbol{\psi}}_\rho(k)) \\ \tilde{\boldsymbol{\psi}}_\rho(k) &= \mathbf{C}_\rho \tilde{\mathbf{x}}_\rho(k) + \mathbf{D}_\rho \boldsymbol{\upsilon}_\rho(k) \\ r_o(k) &= \mathbf{W}_\rho(\boldsymbol{\psi}_\rho(k) - \tilde{\boldsymbol{\psi}}_\rho(k)) \end{aligned} \tag{16.15}$$

Different from r generated by (14.63), the residual signal r_o is calculated at $t = k\rho h$, $k = 1, 2, \cdots$, based on the output samples $y(((k-1)\rho + j)h)$, $j = 0, \cdots, \rho - 1$. The norms of the operators from f and d to r_o are, respectively, equivalent to

$$\begin{aligned} \|\mathbf{\Gamma}_{r_o f}\| &= \|\bar{\mathbf{G}}_{r_o f}\|_\infty \\ \|\mathbf{\Gamma}_{r_o d}\| &= \|\bar{\mathbf{G}}_{r_o d}\|_\infty \\ \bar{\mathbf{G}}_{r_o f} &= (\mathbf{A}_\rho - \mathbf{L}_{\boldsymbol{\rho}}\mathbf{C}_\rho, \bar{\mathbf{E}}_{f,\rho} - \mathbf{L}_{\boldsymbol{\rho}}\bar{\mathbf{F}}_{f,\rho}, \mathbf{W}_{\boldsymbol{\rho}}\mathbf{C}_\rho, \mathbf{W}_{\boldsymbol{\rho}}\bar{\mathbf{F}}_{f,\rho}) \\ \bar{\mathbf{G}}_{r_o d} &= (\mathbf{A}_\rho - \mathbf{L}_{\boldsymbol{\rho}}\mathbf{C}_\rho, \bar{\mathbf{E}}_{d,\rho} - \mathbf{L}_{\boldsymbol{\rho}}\bar{\mathbf{F}}_{d,\rho}, \mathbf{W}_{\boldsymbol{\rho}}\mathbf{C}_\rho, \mathbf{W}_{\boldsymbol{\rho}}\bar{\mathbf{F}}_{d,\rho}) \end{aligned} \tag{16.16}$$

where

$$\bar{\mathbf{F}}_{f,\rho} = \begin{bmatrix} O & O & \cdots & O \\ C\bar{E}_f & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{\rho-2}\bar{E}_f & \cdots & C\bar{E}_f & O \end{bmatrix}, \ \bar{\mathbf{F}}_{d,\rho} = \begin{bmatrix} O & O & \cdots & O \\ C\bar{E}_d & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{\rho-2}\bar{E}_d & \cdots & C\bar{E}_d & O \end{bmatrix}$$

Lemma 16.1 Given SD system described by (14.1)-(14.3) and an arbitrary positive integer $\rho \geq 2$. Then the residual signal r_o generated by (16.15) is identical with the ρ-step lifting of the residual signal r generated by (14.63), if $\tilde{\mathbf{x}}_\rho(0) = \hat{x}(0)$ and

$$\mathbf{L}_\rho = \left[(A-LC)^{\rho-1}L \cdots (A-LC)L \;\; L \right]$$

$$\mathbf{W}_\rho = \begin{bmatrix} W & O & \cdots & O \\ -WCL & W & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ -WC(A-LC)^{\rho-2}L & \cdots & -WCL & W \end{bmatrix} \tag{16.17}$$

Proof: Over time interval $[k\rho h, (k+1)\rho h)$, $\hat{\psi}$ and r got by residual generator (14.63) evolve like

$$\hat{x}((k+1)\rho) = \mathbf{A}_\rho \hat{x}(k\rho) + \mathbf{B}_\rho \boldsymbol{v}_\rho(k) + \hat{\mathbf{L}}_\rho \left(\boldsymbol{\psi}_\rho(k) - \hat{\boldsymbol{\psi}}_\rho(k) \right) \tag{16.18}$$

$$\hat{\boldsymbol{\psi}}_\rho(k) = \mathbf{C}_\rho \hat{x}(k\rho) + \mathbf{D}_\rho \boldsymbol{v}_\rho(k) + Q \left(\boldsymbol{\psi}_\rho(k) - \hat{\boldsymbol{\psi}}_\rho(k) \right)$$

$$\mathbf{r}_\rho(k) = \hat{\mathbf{W}}_\rho \left(\boldsymbol{\psi}_\rho(k) - \hat{\boldsymbol{\psi}}_\rho(k) \right)$$

where

$$\hat{\boldsymbol{\psi}}_\rho(k) = \begin{bmatrix} \hat{\psi}(k\rho) \\ \hat{\psi}(k\rho+1) \\ \vdots \\ \hat{\psi}(k\rho+\rho-1) \end{bmatrix}, \; \mathbf{r}_\rho(k) = \begin{bmatrix} r(k\rho) \\ r(k\rho+1) \\ \vdots \\ r(k\rho+\rho-1) \end{bmatrix}$$

$$Q = \begin{bmatrix} O & O & \cdots & O \\ CL & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{\rho-2}L & \cdots & CL & O \end{bmatrix}$$

$$\hat{\mathbf{L}}_\rho = \left[A^{\rho-1}L \; A^{\rho-2}L \cdots L \right], \; \hat{\mathbf{W}}_\rho = \begin{bmatrix} W & O & \cdots & O \\ O & W & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & W \end{bmatrix}$$

It can be easily verified that $\mathbf{L}_\rho, \mathbf{W}_\rho$ given by (16.17) are related to $\hat{\mathbf{L}}_\rho, \hat{\mathbf{W}}_\rho$ in (16.18) by

$$\mathbf{L}_{\boldsymbol{\rho}}(Q+I) = \hat{\mathbf{L}}_\rho, \mathbf{W}_{\boldsymbol{\rho}}(Q+I) = \hat{\mathbf{W}}_\rho$$

If $\tilde{\mathbf{x}}_\rho(0) = \hat{x}(0)$, then

$$\hat{\boldsymbol{\psi}}_\rho(0) = \tilde{\boldsymbol{\psi}}_\rho(0) + Q \left(\boldsymbol{\psi}_\rho(0) - \hat{\boldsymbol{\psi}}_\rho(0) \right)$$

It leads to

$$\begin{aligned} \boldsymbol{\psi}_\rho(0) - \tilde{\boldsymbol{\psi}}_\rho(0) &= (I+Q) \left(\boldsymbol{\psi}_\rho(0) - \hat{\boldsymbol{\psi}}_\rho(0) \right) \\ &\Rightarrow \; r_o(0) = \mathbf{r}_\rho(0), \; \tilde{\mathbf{x}}_\rho(1) = \hat{x}(\rho) \end{aligned}$$

Repeating the above derivation yields $r_o(k) = \mathbf{r}_\rho(k)$, $\forall k$. □

Lemma 16.1 shows that any residual signal that can be obtained by residual generator (14.63) with constant parameters L, W can also be achieved, though with a time lag, by residual generator (16.15). Therefore, using residual generator (16.15) will cause no loss in the H_∞/H_∞ optimal FD performance.

Lemma 16.2 In a more general case, suppose that residual generator (14.63) has ρ-periodically time-varying gain matrix $L(k)$ and weighting matrix $W(k)$, i.e., $L(k\rho+j) = L(j)$, $W(k\rho+j) = W(j)$, $\forall j = 0, 1, \cdots, \rho-1$, $k = 0, 1, 2, \cdots$. Let

$$\Psi(j,i) = \begin{cases} I, & \text{if } j = i \\ (A - L(j)C)(A - L(j-1)C)\cdots(A - L(i+1)C), & \text{if } j > i \end{cases}$$

Then $r_o(k) = \mathbf{r}_\rho(k)$, if $\tilde{\mathbf{x}}_\rho(0) = \hat{x}(0)$ and

$$\mathbf{L}_\rho = \begin{bmatrix} \Psi(\rho-1,0)L(0) \cdots \Psi(\rho-1,\rho-2)L(\rho-2) & L(\rho-1) \end{bmatrix}$$

$$\mathbf{W}_\rho = \begin{bmatrix} W(0) & O & \cdots & O \\ -W(1)CL(0) & W(1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ -W(\rho-1)C\Psi(\rho-2,0)L(0) & \cdots & -W(\rho-1)CL(\rho-2) & W(\rho-1) \end{bmatrix}$$

Proof: The proof of Lemma 16.2 is similar to that of Lemma 16.1 and thus omitted. □

From Lemma 16.2 we see that residual generator (16.15) can indeed represent a more general class of residual generators, namely, those in the form of (14.63) but with periodic gain matrix and weighting matrix. The question now is, compared with (14.63), whether (16.15) can achieve a better H_∞/H_∞ index due to such additional freedom.

Lemma 16.3 Given SD system (14.1)-(14.3), an arbitrary positive integer $\rho \geq 2$, residual generators (14.63) and (16.15). Assume that $A, \bar{E}_d, \bar{E}_f, X_d, H_d, L_{opt}, W_{opt}$ are given by (14.5), (14.24) and (14.71)-(14.72). Let

$$\mathbf{L}_{\rho,opt} = \begin{bmatrix} (A - L_{opt}C)^{\rho-1}L_{opt} \cdots (A - L_{opt}C)L_{opt} & L_{opt} \end{bmatrix} \tag{16.19}$$

$$\mathbf{W}_{\rho,opt} = \begin{bmatrix} W_{opt} & O & \cdots & O \\ -W_{opt}CL_{opt} & W_{opt} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ -W_{opt}C(A - L_{opt}C)^{\rho-2}L_{opt} & \cdots & -W_{opt}CL_{opt} & W_{opt} \end{bmatrix}$$

$$\mathbf{H}_{d,\rho} = \begin{bmatrix} H_d & O & \cdots & O \\ CL_{opt}H_d & H_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{\rho-2}L_{opt}H_d & \cdots & CL_{opt}H_d & H_d \end{bmatrix}$$

Then:

(i) $\mathbf{A}_\rho - \mathbf{L}_{\rho,opt}\mathbf{C}_\rho = (A - L_{opt}C)^\rho$ and $\mathbf{X}_{d,\rho} = X_d$, $\mathbf{L}_{d,\rho} = -(\mathbf{L}_{\rho,opt})^T$ is the stabilizing solution to DTARS

$$\begin{bmatrix} \mathbf{A}_\rho\mathbf{X}_{d,\rho}\mathbf{A}_\rho^T - \mathbf{X}_{d,\rho} + \bar{\mathbf{E}}_{d,\rho}\bar{\mathbf{E}}_{d,\rho}^T & \mathbf{A}_\rho\mathbf{X}_{d,\rho}\mathbf{C}_\rho^T + \bar{\mathbf{E}}_{d,\rho}\bar{\mathbf{F}}_{d,\rho}^T \\ \mathbf{C}_\rho\mathbf{X}_{d,\rho}\mathbf{A}_\rho^T + \bar{\mathbf{F}}_{d,\rho}\bar{\mathbf{E}}_{d,\rho}^T & \mathbf{C}_\rho\mathbf{X}_{d,\rho}\mathbf{C}_\rho^T + \bar{\mathbf{F}}_{d,\rho}\bar{\mathbf{F}}_{d,\rho}^T \end{bmatrix} \begin{bmatrix} I \\ \mathbf{L}_{d,\rho} \end{bmatrix} = 0 \tag{16.20}$$

(ii) $\mathbf{W}_{\rho,opt}\mathbf{H}_{d,\rho} = I$, $\mathbf{H}_{d,\rho}\mathbf{H}_{d,\rho}^T = \mathbf{C}_\rho\mathbf{X}_{d,\rho}\mathbf{C}_\rho^T + \bar{\mathbf{F}}_{d,\rho}\bar{\mathbf{F}}_{d,\rho}^T$.

(iii) $\bar{\mathbf{G}}_{r_of,opt} = (\mathbf{A}_\rho - \mathbf{L}_{\rho,opt}\mathbf{C}_\rho, \bar{\mathbf{E}}_{f,\rho} - \mathbf{L}_{\rho,opt}\bar{\mathbf{F}}_{f,\rho}, \mathbf{W}_{\rho,opt}\mathbf{C}_\rho, \mathbf{W}_{\rho,opt}\bar{\mathbf{F}}_{f,\rho})$ is the ρ-step lifting of $\bar{G}_{rf,opt} = (A - L_{opt}C, \bar{E}_f, W_{opt}C, O)$.

Moreover,

$$\begin{aligned}(\mathbf{L}_{\rho,opt}, \mathbf{W}_{\rho,opt}) &= \arg\left(\max_{\mathbf{L}_\rho,\mathbf{W}_\rho} \frac{\|\bar{\mathbf{G}}_{r_of}\|_\infty}{\|\bar{\mathbf{G}}_{r_od}\|_\infty}\right) \\ &= \arg\left(\max_{\mathbf{L}_\rho,\mathbf{W}_\rho} \frac{\|\mathbf{\Gamma}_{r_of}\|}{\|\mathbf{\Gamma}_{r_od}\|}\right) = \arg\left(\max_{\mathbf{L}_\rho,\mathbf{W}_\rho} \mathbf{J}_{SSD,OBS,\infty/\infty,\rho}\right)\end{aligned}$$

and the optimal performance indexes achieved by residual generators (14.63) and (16.15) are the same, i.e.

$$\mathbf{J}_{SSD,OBS,\infty/\infty,\rho,opt} = J_{SSD,OBS,\infty/\infty,h,opt}$$

Proof: Let $\rho = 2$. (i)-(iii) can be easily verified by substituting

$$\begin{aligned}
\mathbf{X}_{d,\rho} &= X_d,\ \mathbf{L}_{o,\rho} = -(\mathbf{L}_{\rho,opt})^T,\ \mathbf{L}_{\rho,opt} = [\,(A - L_{opt}C)L_{opt}\ \ L_{opt}\,] \\
\mathbf{A}_\rho &= A^2, \bar{\mathbf{E}}_{d,\rho} = [\,A\bar{E}_d\ \bar{E}_d\,] \\
\mathbf{W}_{\rho,opt} &= \begin{bmatrix} W_{opt} & O \\ -W_{opt}CL_{opt} & W_{opt} \end{bmatrix}, \mathbf{H}_{d,\rho} = \begin{bmatrix} H_d & O \\ CL_{opt}H_d & H_d \end{bmatrix} \\
\mathbf{C}_\rho &= \begin{bmatrix} C \\ CA \end{bmatrix}, \bar{\mathbf{F}}_{d,\rho} = \begin{bmatrix} O & O \\ C\bar{E}_d & O \end{bmatrix}
\end{aligned}$$

into (i)-(iii) and taking into account (14.71)-(14.72). Assume that (i)-(iii) holds for $\rho = n$. Then, for $\rho = n+1$ and $\mathbf{L}^*_{n+1}$ given by (16.19), as

$$\begin{aligned}
\mathbf{L}_{n+1,opt} &= \left[\,(A - L_{opt}C)\mathbf{L}_{n,opt}\ \ L_{opt}\,\right] \\
\mathbf{A}_{n+1} &= A\mathbf{A}_n,\ \mathbf{C}_{n+1} = \begin{bmatrix} \mathbf{C}_n \\ C\mathbf{A}_{T,n} \end{bmatrix}
\end{aligned} \tag{16.21}$$

we get

$$\mathbf{A}_{n+1} - \mathbf{L}_{n+1,opt}\mathbf{C}_{n+1} = (A - L_{opt}C)\left(\mathbf{A}_n - \mathbf{L}_{n,opt}\mathbf{C}_n\right) = (A - L_{opt}C)^{n+1}$$

Substituting

$$\begin{aligned}
&\mathbf{X}_{d,n+1} = X_d,\ \mathbf{L}_{d,n+1} = -\left(\mathbf{L}_{n+1,opt}\right)^T, \\
&\bar{\mathbf{E}}_{d,n+1} = \left[\,A\bar{\mathbf{E}}_{d,n}\ \bar{E}_d\,\right],\ \bar{\mathbf{F}}_{d,n+1} = \begin{bmatrix} \bar{\mathbf{F}}_{d,n} & O \\ C\bar{\mathbf{E}}_{d,n} & O \end{bmatrix}
\end{aligned}$$

and (16.21) into the left side of DTARS (16.20) yields

$$\begin{aligned}
&\mathbf{A}_{n+1}\mathbf{X}_{d,n+1}\mathbf{A}_{n+1}^T - \mathbf{X}_{d,n+1} + \bar{\mathbf{E}}_{d,n+1}\bar{\mathbf{E}}_{d,n+1}^T \\
&+ \left(\mathbf{A}_{n+1}\mathbf{X}_{d,n+1}\mathbf{C}_{n+1}^T + \bar{\mathbf{E}}_{d,n+1}\bar{\mathbf{F}}_{d,n+1}^T\right)\mathbf{L}_{d,n+1} \\
&= A\left(\mathbf{A}_nX_d\mathbf{A}_n^T + \bar{\mathbf{E}}_{d,n}\bar{\mathbf{E}}_{d,n}^T - (\mathbf{A}_nX_d\mathbf{C}_n^T + \bar{\mathbf{E}}_{d,n}\bar{\mathbf{F}}_{d,n}^T)\mathbf{L}_{n,opt}^T\right) \\
&\times (A^T - C^TL_{opt}^T) - X_d + \bar{E}_d\bar{E}_d^T \\
&= AX_dA^T - AX_dC^TL^{*T} - X_d + \bar{E}_d\bar{E}_d^T = 0 \\
&\mathbf{C}_{n+1}\mathbf{X}_{d,n+1}\mathbf{A}_{n+1}^T + \bar{\mathbf{F}}_{d,n+1}\bar{\mathbf{E}}_{d,n+1}^T + \left(\mathbf{C}_{n+1}\mathbf{X}_{d,n+1}\mathbf{C}_{n+1}^T + \bar{\mathbf{F}}_{d,n+1}\bar{\mathbf{F}}_{d,n+1}^T\right)\mathbf{L}_{d,n+1} \\
&= \begin{bmatrix} \left(\mathbf{C}_nX_d\mathbf{A}_n^T + \bar{\mathbf{F}}_{d,n}\bar{\mathbf{E}}_{d,n}^T - \left(\mathbf{C}_nX_d\mathbf{C}_n^T + \bar{\mathbf{F}}_{d,n}\bar{\mathbf{F}}_{d,n}^T\right)\mathbf{L}_{n,opt}^T\right)(A^T - C^TL_{opt}^T) \\ C\left(\mathbf{A}_nX_d\mathbf{A}_n^T + \bar{\mathbf{E}}_{d,n}\bar{\mathbf{E}}_{d,n}^T - \left(\mathbf{A}_nX_d\mathbf{C}_n^T + \bar{\mathbf{E}}_{d,n}\bar{\mathbf{F}}_{d,n}^T\right)\mathbf{L}_{n,opt}^T\right)(A^T - C^TL_{opt}^T) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ CXA^T - CXC^TL_{opt}^T \end{bmatrix} = 0
\end{aligned}$$

Moreover, $\mathbf{A}_{n+1} - \mathbf{L}_{n+1,opt}\mathbf{C}_{n+1} = (A - L_{opt}C)^{n+1}$ is stable because $A - L_{opt}C$ is stable. Thus, $\mathbf{X}_{d,n+1} = X_d, \mathbf{L}_{d,n+1} = -\mathbf{L}_{n+1,opt}^T$ with $\mathbf{L}_{n+1,opt}$ given by (16.19) are the stabilizing solution of (16.20) when $\rho = n+1$. By induction, (i) holds for any $\rho \geq 2$.

Note that $\mathbf{L}_{n+1,opt}, \mathbf{W}_{n+1,opt}, \mathbf{H}_{n+1}, \bar{\mathbf{E}}_{f,n+1}, \bar{\mathbf{F}}_{f,n+1}$ can also be re-written as

$$\begin{aligned}
\mathbf{L}_{n+1,opt} &= \left[\,(A - L_{opt}C)^nL_{opt}\ \mathbf{L}_{n,opt}\,\right], \bar{\mathbf{E}}_{f,n+1} = [\,\mathbf{A}_n\bar{E}_f\ \bar{\mathbf{E}}_{f,n}\,] \\
\mathbf{W}_{n+1,opt} &= \begin{bmatrix} \mathbf{W}_{n,opt} & O \\ -W_{opt}C\mathbf{L}_{n,opt} & W_{opt} \end{bmatrix}, \mathbf{H}_{d,n+1} = \begin{bmatrix} \mathbf{H}_{d,n} & O \\ C\mathbf{L}_{n,opt}\mathbf{H}_{d,n} & H_d \end{bmatrix} \\
\bar{\mathbf{F}}_{f,n+1} &= \begin{bmatrix} O & O \\ \mathbf{C}_n\bar{E}_f & \bar{\mathbf{F}}_{f,n} \end{bmatrix}
\end{aligned} \tag{16.22}$$

It is easy to obtain $\mathbf{W}_{n+1,opt}\mathbf{H}_{d,n+1} = I$. From $\mathbf{H}_{d,n}\mathbf{H}_{d,n}^T = \mathbf{C}_nX_d\mathbf{C}_n^T + \bar{\mathbf{F}}_{d,n}\bar{\mathbf{F}}_{d,n}^T$, there is

$$\begin{aligned}
\mathbf{H}_{d,n}\mathbf{H}_{d,n}^T\mathbf{L}_{n,opt}^T &= \left(\mathbf{C}_n X_d \mathbf{C}_n^T + \bar{\mathbf{F}}_{d,n}\bar{\mathbf{F}}_{d,n}^T\right)\mathbf{L}_{n,opt}^T = \mathbf{C}_n X_d \mathbf{A}_n^T + \bar{\mathbf{F}}_{d,n}\bar{\mathbf{E}}_{d,n}^T \\
\mathbf{L}_{n,opt}\mathbf{H}_{d,n}\mathbf{H}_{d,n}^T\mathbf{L}_{n,opt}^T &= \mathbf{L}_{n,opt}\left(\mathbf{C}_n X_d \mathbf{A}_n^T + \bar{\mathbf{F}}_{d,n}\bar{\mathbf{E}}_{d,n}^T\right) \\
&= \mathbf{A}_n X_d \mathbf{A}_n^T + \bar{\mathbf{E}}_{d,n}\bar{\mathbf{E}}_{d,n}^T - X_d \\
\mathbf{H}_{d,n+1}\mathbf{H}_{d,n+1}^T &= \begin{bmatrix} \mathbf{H}_{d,n}\mathbf{H}_{d,n}^T & \mathbf{H}_{d,n}\mathbf{H}_{d,n}^T\mathbf{L}_{n,opt}^T C^T \\ C\mathbf{L}_{n,opt}\mathbf{H}_{d,n}\mathbf{H}_{d,n}^T & C\mathbf{L}_{n,opt}\mathbf{H}_{d,n}\mathbf{H}_{d,n}^T\mathbf{L}_{n,opt}^T C^T + H_d H_d^T \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{C}_n \\ C\mathbf{A}_n \end{bmatrix} X_d \begin{bmatrix} \mathbf{C}_n \\ C\mathbf{A}_n \end{bmatrix}^T + \begin{bmatrix} \bar{\mathbf{F}}_{d,n} & O \\ C\bar{\mathbf{E}}_{d,n} & O \end{bmatrix}\begin{bmatrix} \bar{\mathbf{F}}_{d,n} & O \\ C\bar{\mathbf{E}}_{d,n} & O \end{bmatrix}^T \\
&= \mathbf{C}_{n+1}\mathbf{X}_{d,\rho}\mathbf{C}_{n+1}^T + \bar{\mathbf{F}}_{d,n+1}\bar{\mathbf{F}}_{d,n+1}^T
\end{aligned}$$

As shown before, $\mathbf{A}_{n+1} - \mathbf{L}_{n+1,opt}\mathbf{C}_{n+1} = (A - L_{opt}C)^{n+1}$. Taking into account (16.21)-(16.22), we have

$$\begin{aligned}
\bar{\mathbf{E}}_{f,n+1} - \mathbf{L}_{n+1,opt}\bar{\mathbf{F}}_{f,n+1} &= \left[(\mathbf{A}_n - \mathbf{L}_{n,opt}\mathbf{C}_n)\bar{E}_f \;\; \bar{\mathbf{E}}_{f,n} - \mathbf{L}_{n,opt}\bar{\mathbf{F}}_{f,n} \right] \\
&= \left[(A - L_{opt}C)^n \bar{E}_f \;\; (A - L_{opt}C)^{n-1}\bar{E}_f \; \cdots \; \bar{E}_f \right] \\
\mathbf{W}_{n+1,opt}\mathbf{C}_{n+1} &= \begin{bmatrix} \mathbf{W}_{n,opt}\mathbf{C}_n \\ W_{opt}C(\mathbf{A}_n - \mathbf{L}_{n,opt}\mathbf{C}_n) \end{bmatrix} \\
&= \begin{bmatrix} W_{opt}C \\ \vdots \\ W_{opt}C(A - L_{opt}C)^{n-1} \\ W_{opt}C(A - L_{opt}C)^n \end{bmatrix} \\
\mathbf{W}_{n+1,opt}\bar{\mathbf{F}}_{f,n+1} &= \begin{bmatrix} \mathbf{W}_{n,opt}\bar{\mathbf{F}}_{f,n} & O \\ W_{opt}C\left(\bar{\mathbf{E}}_{f,n} - \mathbf{L}_{n,opt}\bar{\mathbf{F}}_{f,n}\right) & O \end{bmatrix} \\
&= \begin{bmatrix} O & \cdots & O \\ \vdots & \ddots & \vdots \\ W_{opt}C(A - L_{opt}C)^{n-1}\bar{E}_f & \cdots & O \end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\bar{\mathbf{G}}_{r_o f,opt} = \big(&\mathbf{A}_{n+1} - \mathbf{L}_{n+1,opt}\mathbf{C}_{n+1}, \bar{\mathbf{E}}_{f,n+1} - \mathbf{L}_{n+1,opt}\bar{\mathbf{F}}_{f,n+1}, \\
&\mathbf{W}_{n+1,opt}\mathbf{C}_{n+1}, \mathbf{W}_{n+1,opt}\bar{\mathbf{F}}_{f,n+1}\big)
\end{aligned}$$

is the $n+1$-step lifting of $\bar{G}_{rf,opt} = (A - L_{opt}C, \bar{E}_f, W_{opt}C, O)$. By induction, (ii)-(iii) hold for any $\rho \geq 2$.

Finally, according to Theorem 14.6 and (i)-(ii), $\mathbf{L}_{\rho,opt}$ and $\mathbf{W}_{\rho,opt}$ solves the optimization problem $\max\limits_{\mathbf{L}_\rho,\mathbf{W}_\rho} \frac{\|\bar{\mathbf{G}}_{r_o f}\|_\infty}{\|\bar{\mathbf{G}}_{r_o d}\|_\infty}$ and $\|\bar{\mathbf{G}}_{r_o d,opt}\|_\infty = 1$. As the lifting preserves the H_∞ norm, it follows from (iii) that

$$\|\bar{\mathbf{G}}_{r_o f,opt}\|_\infty = \|\bar{G}_{rf,opt}\|_\infty$$

Therefore, $\mathbf{J}_{SSD,OBS,\infty/\infty,\rho,opt} = J_{SSD,OBS,\infty/\infty,h,opt}$. □

According to Lemma 16.3, for SD system (14.1)-(14.3) with sampling period h, we can generate the residual signal either by (14.63) or by (16.15). Both schemes achieve the same optimal H_∞/H_∞ index and there is a one-to-one relationship between the optimal parameters of these two schemes. That means also using time-varying parameters $L(k), W(k)$ in residual generator (14.63) will not improve the optimal H_∞/H_∞ index.

16.3.2 Optimal H_∞/H_∞ index vs. sampling period

With the help of the above analysis, in this subsection we shall show that increasing sampling period h by an integer multiple will lead to a worse H_∞/H_∞ index $J_{SSD,OBS,\infty/\infty,h,opt}$, as stated in Theorem 16.3.

Theorem 16.3 Given SD system described by (14.1)-(14.3) and an arbitrary positive integer $\rho \geq 2$. Let $J_{SSD,OBS,\infty/\infty,h,opt}$ and $J_{SSD,OBS,\infty/\infty,\rho h,opt}$ denote, respectively, the optimal H_∞/H_∞ index achievable under sampling period h and ρh. Then, $J_{SSD,OBS,\infty/\infty,h,opt} \geq J_{SSD,OBS,\infty/\infty,\rho h,opt}$.

Proof: If the continuous-time process (14.1) is sampled with sampling period ρh, then the output samples of y are related to the control inputs u and the continuous-time signals f, d by

$$\begin{aligned} x(k+1) &= A_{\rho h}x(k) + B_{\rho h}\upsilon(k) + \int_{k\rho h}^{(k+1)\rho h} e^{A_c\tau}\left(E_{dc}d(\tau) + E_{fc}f(\tau)\right)d\tau \\ \psi(k) &= Cx(k) \end{aligned}$$

with $x(k) = x(k\rho h)$, $A_{\rho h} = \mathbf{A}_\rho = e^{\rho A_c h}$ and

$$B_{\rho h} = \int_0^{\rho h} e^{A_c\tau}B_c d\tau = \sum_{j=0}^{\rho-1} A^j B,\ B_{\rho h}\upsilon(k) = \mathbf{B}_\rho \begin{bmatrix} \upsilon(k) \\ \vdots \\ \upsilon(k) \end{bmatrix} \tag{16.23}$$

The residual signal $r_{\rho h}$ is obtained by a residual generator with constant free parameters $L_{\rho h}$ and $W_{\rho h}$

$$\begin{aligned} \breve{x}(k+1) &= A_{\rho h}\breve{x}(k) + B_{\rho h}\upsilon(k) + L_{\rho h}(\psi(k) - \breve{\psi}(k)) \\ \breve{\psi}(k) &= C\breve{x}(k) \\ r_{\rho h} &= W_{\rho h}(\psi(k) - \breve{\psi}(k)) \end{aligned} \tag{16.24}$$

Considering (16.23), we can bring (16.24) into the form of (16.15) with an additional structural constraint imposed on parameters $\mathbf{L}_\rho$ and $\mathbf{W}_\rho$ as follows

$$\mathbf{L}_\rho = \begin{bmatrix} L_{\rho h} & O \end{bmatrix}, \quad \mathbf{W}_\rho = \begin{bmatrix} W_{\rho h} & O \\ O & O \end{bmatrix} \tag{16.25}$$

If $\mathbf{L}_\rho$ and $\mathbf{W}_\rho$ satisfies (16.25), then

$$J_{SSD,OBS,\infty/\infty,\rho h}(L_{\rho h}, W_{\rho h}) = \mathbf{J}_{SSD,OBS,\infty/\infty,\rho}(\mathbf{L}_\rho, \mathbf{W}_\rho)$$

Hence,

$$\begin{aligned} &\max_{L_{\rho h}, W_{\rho h}} J_{SSD,OBS,\infty/\infty,\rho h} \\ &= \max_{\mathbf{L}_\rho, \mathbf{W}_\rho \text{ satisfying (16.25)}} \mathbf{J}_{SSD,OBS,\infty/\infty,\rho} \\ &\leq \max_{\mathbf{L}_\rho, \mathbf{W}_\rho} \mathbf{J}_{SSD,OBS,\infty/\infty,\rho} \end{aligned}$$

i.e.,

$$J_{SSD,OBS,\infty/\infty,\rho h,opt} \leq \mathbf{J}_{SSD,OBS,\infty/\infty,\rho,opt}$$

Recall that, according to Lemma 16.3,

$$\mathbf{J}_{SSD,OBS,\infty/\infty,\rho,opt} = J_{SSD,OBS,\infty/\infty,h,opt}, \forall\rho \geq 2$$

Thus, $J_{SSD,OBS,\infty/\infty,\rho h,opt} \leq J_{SSD,OBS,\infty/\infty,h,opt}$. □

Remark 16.1 We would like to point out that

$$J_{SSD,OBS,\infty/\infty,h,opt} \geq J_{SSD,OBS,\infty/\infty,\rho h,opt}$$

for any positive integer $\rho \geq 2$ doesn't mean

$$J_{SSD,OBS,\infty/\infty,h_1} \geq J_{SSD,OBS,\infty/\infty,h_2}$$

for any $h_1 < h_2$, as shown later in Example 16.3. That means, the optimal H_∞/H_∞ index $J_{SSD,OBS,\infty/\infty,h,opt}$ is not necessarily a monotonically decreasing function of sampling period h.

Remark 16.2 In [85], based on an example which is also studied here, it has been concluded that decreasing the sampling period may impair the optimal H_∞/H_∞ index and thus the H_∞/H_∞ index is not appropriate for comparison of different design techniques. As shown in Theorem 16.3 and the results achieved in our study on the same example (see the following example), it is indeed in general not the case.

Example 16.3 For the SD system given in Example 16.1, compare the optimal H_∞/H_∞ index.

To calculate $J_{SSD,OBS,\infty/\infty,h,opt}$, the optimization problem (14.65) is solved under different sampling periods h. The matrices $\bar{E}_{f,h}$ and $\bar{E}_{d,h}$ in (14.24) are calculated by the algorithm given in [20]. The H_∞-norm is calculated with the help of the Matlab command *sigma* evaluated over a fine grid of frequency points ($\Delta\omega = 0.00001$). Table 16.3 shows the optimal H_∞/H_∞ index $J_{SSD,OBS,\infty/\infty,h,opt}$ with respect to different sampling period h.

h	0.1	0.2	0.3	0.4	0.5
$J_{SSD,OBS,\infty/\infty,h,opt}$	0.5950	0.5940	0.5911	0.5857	0.5784
h	0.6	0.7	0.8	0.9	1.0
$J_{SSD,OBS,\infty/\infty,h,opt}$	0.5701	0.56180	0.5548	0.5504	0.5497
h	1.1	1.2	1.3	1.4	1.5
$J_{SSD,OBS,\infty/\infty,h,opt}$	0.5540	0.5613	0.5637	0.56179	0.5567
h	1.6	1.7	1.8	1.9	2.0
$J_{SSD,OBS,\infty/\infty,h,opt}$	0.5493	0.5404	0.5305	0.5195	0.5328

Table 16.3 The optimal H_∞/H_∞ index $J_{SSD,OBS,\infty/\infty,h,opt}$ with respect to different sampling periods h in Example 16.3

It can be seen that $J_{SSD,OBS,\infty/\infty,h_i,opt} \geq J_{SSD,OBS,\infty/\infty,\rho_i h_i,opt}, i = 1, \cdots, 10$, ρ_i is a positive integer, for $h_1 = 0.1s, 2 \leq \rho_1 \leq 20$, $h_2 = 0.2s, 2 \leq \rho_2 \leq 10$, $h_3 = 0.3s, 2 \leq \rho_3 \leq 6$, etc. Theorem 16.3 is thus verified. It is also shown by Table 16.3 that in this example $J_{SSD,OBS,\infty/\infty,h_1,opt} \geq J_{SSD,OBS,\infty/\infty,\rho_i h_2,opt}$ doesn't hold for some $h_1 < h_2$. For instance,

$$J_{SSD,OBS,\infty/\infty,h_1=1.1,opt} < J_{SSD,OBS,\infty/\infty,h_2=1.2,opt}$$

though $h_1 = 1.1 < h_2 = 1.2$. $J_{SSD,OBS,\infty/\infty,h,opt}$ here is related to $J^*_{\infty/\infty}(h)$ calculated in [85] by $J_{SSD,OBS,\infty/\infty,h,opt} = \frac{1}{J^*_{\infty/\infty}(h)}$. It is worth noticing that, when $h = 2.0$, the value of $J_{SSD,OBS,\infty/\infty,h,opt}$ is 0.5328 in Table 16.3, which is different from $\frac{1}{J^*_{\infty/\infty}(T)} = \frac{1}{1.0926} = 0.9152$ obtained in [85].

Example 16.4 Consider an SD system described by (14.1)-(14.3) with

$$A_c = \begin{bmatrix} -10 & -5 \\ 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$E_{dc} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, E_{fc} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{16.26}$$

The optimal H_∞/H_∞ index $J_{SSD,OBS,\infty/\infty,h,opt}$ is shown in Table 16.4.

h	0.1	0.2	0.3	0.4	0.5
$J_{SSD,OBS,\infty/\infty,h,opt}$	4.7785	4.3864	4.0836	3.8892	3.7655
h	0.6	0.7	0.8	0.9	1.0
$J_{SSD,OBS,\infty/\infty,h,opt}$	3.6837	3.6271	3.5860	3.5550	3.5310
h	1.1	1.2	1.3	1.4	1.5
$J_{SSD,OBS,\infty/\infty,h,opt}$	3.5119	3.4962	3.4833	3.4724	3.4632
h	1.6	1.7	1.8	1.9	2.0
$J_{SSD,OBS,\infty/\infty,h,opt}$	3.4552	3.4482	3.4422	3.4368	3.4321

Table 16.4 The optimal H_∞/H_∞ index $J_{SSD,OBS,\infty/\infty,h,opt}$ with respect to different sampling periods h in Example 16.4

It can be seen that

$$J_{SSD,OBS,\infty/\infty,h_i,opt} \geq J_{SSD,OBS,\infty/\infty,\rho_i h_i,opt}, i = 1, \cdots, 10$$

for $h_1 = 0.1s, 2 \leq \rho_1 \leq 20$, $h_2 = 0.2s, 2 \leq \rho_2 \leq 10$, $h_3 = 0.3s, 2 \leq \rho_3 \leq 6$, etc. It verifies Theorem 16.3.

16.4 Optimal H_-/H_∞ performance

The influence of sampling period on the optimal H_-/H_∞ index is rather different from that of the other performance indices.

According to (i)-(iii) of Lemma 16.3, $\mathbf{L}_{\rho,opt}$ and $\mathbf{W}_{\rho,opt}$ given by (16.19) also solves the optimization problem $\max_{\mathbf{L}_\rho,\mathbf{W}_\rho} \frac{\left\|\bar{\mathbf{G}}_{r_of}\right\|_-}{\left\|\bar{\mathbf{G}}_{r_od}\right\|_\infty}$, i.e.

$$\begin{aligned}(\mathbf{L}_{\rho,opt}, \mathbf{W}_{\rho,opt}) &= \arg\left(\max_{\mathbf{L}_\rho,\mathbf{W}_\rho} \frac{\left\|\bar{\mathbf{G}}_{r_of}\right\|_-}{\left\|\bar{\mathbf{G}}_{r_od}\right\|_\infty}\right) \\ &= \arg\left(\max_{\mathbf{L}_\rho,\mathbf{W}_\rho} \frac{\underline{\lambda}^{1/2}(\mathbf{\Gamma}_{r_of}\left(\mathbf{\Gamma}_{r_of}\right)^*)}{\left\|\mathbf{\Gamma}_{r_od}\right\|}\right) \\ &= \arg\left(\max_{\mathbf{L}_\rho,\mathbf{W}_\rho} \mathbf{J}_{SSD,OBS,-/\infty,\rho}\right) \end{aligned} \quad (16.27)$$

Because the lifting preserves the l_2-norm of the signals, it also preserves the H_- index of the corresponding system. Therefore, $\left\|\bar{\mathbf{G}}_{r_of,opt}\right\|_- = \left\|\bar{G}_{rf,opt}\right\|_-$. Taking into account $\left\|\bar{\mathbf{G}}_{r_od,opt}\right\|_\infty = 1$, there is

$$\mathbf{J}_{SSD,OBS,-/\infty,\rho,opt} = J_{SSD,OBS,-/\infty,h,opt} \quad (16.28)$$

However, it is worth noticing that W_ρ satisfying (16.25) is not of full column rank. **Given the matrices L_ρ and W_ρ satisfying (16.25),**

$$J_{SSD,OBS,-/\infty,\rho h}(L_{\rho h}, W_{\rho h}) = \mathbf{J}_{SSD,OBS,-/\infty,\rho}(\mathbf{L}_\rho, \mathbf{W}_\rho) \quad (16.29)$$

doesn't hold. Therefore, the reasoning in Theorem 16.3 can not be applied to study the optimal H_-/H_∞ index any more.

The following example documents the change of the optimal H_-/H_∞ index $J_{SSD,OBS,-/\infty,h,opt}$ with respect to the sampling period h.

Example 16.5 For the SD system given in Example 16.1, compare the optimal H_-/H_∞ index.

h	0.1	0.2	0.3	0.4	0.5
$J_{SSD,OBS,-/\infty,h,opt}$	0.1445	0.187495	0.187488	0.187498	0.1876
h	0.6	0.7	0.8	0.9	1.0
$J_{SSD,OBS,-/\infty,h,opt}$	0.1877	0.1880	0.1885	0.1893	0.1903
h	1.1	1.2	1.3	1.4	1.5
$J_{SSD,OBS,-/\infty,h,opt}$	0.1917	0.1935	0.1957	0.1983	0.2012
h	1.6	1.7	1.8	1.9	2.0
$J_{SSD,OBS,-/\infty,h,opt}$	0.2045	0.2082	0.2120	0.2163	0.2370

Table 16.5 The optimal H_-/H_∞ index $J_{SSD,OBS,-/\infty,h,opt}$ with respect to different sampling periods h in Example 16.5

It is interesting to observe that in this example,

$$J_{SSD,OBS,\infty/\infty,h_i,opt} \leq J_{SSD,OBS,\infty/\infty,\rho_i h_i,opt}, i = 1, \cdots, 10$$

for $h_1 = 0.1s, 2 \leq \rho_1 \leq 20$, $h_2 = 0.2s, 2 \leq \rho_2 \leq 10$, $h_3 = 0.3s, 2 \leq \rho_3 \leq 6$,etc. That means, increasing sampling period h by an integer multiple will even lead to a better H_-/H_∞ index $J_{SSD,OBS,-/\infty,h,opt}$.

To find out the reason, recall that it is stated in Theorem 14.6 and 14.7 that

$$\begin{aligned} J_{SSD,OBS,-/\infty,h,opt} &= \left\|\bar{G}_{rf,opt}\right\|_- = \left\|W_{opt}C(zI - A + L_{opt}C)^{-1}\bar{E}_f\right\|_- \\ &= \frac{1}{\|W_{opt,-/\infty}C(zI - A + L_{opt,-/\infty}C)^{-1}\bar{E}_d\|_\infty} \end{aligned}$$

from which we obtain the following theorem.

Theorem 16.4 Given SD system described by (14.1)-(14.3) and an arbitrary positive integer $\rho \geq 2$. Let $J_{SSD,OBS,-/\infty,h,opt}$ and $J_{SSD,OBS,-/\infty,\rho h,opt}$ denote, respectively, the optimal H_-/H_∞ index achievable under sampling period h and ρh. Then, $J_{SSD,OBS,-/\infty,h,opt} \leq J_{SSD,OBS,-/\infty,\rho h,opt}$.

Proof: Similar to Theorem 16.3, it can be proven that $\frac{1}{J_{SSD,OBS,-/\infty,h,opt}} = \|W_{opt,-/\infty}C(zI - A + L_{opt,-/\infty}C)^{-1}\bar{E}_d\|_\infty$ decreases if the sampling period h increases by a multiple. Therefore, $J_{SSD,OBS,-/\infty,h,opt}$ increases with the increase of h. □

16.5 Extension to multirate sampled-data systems

In MSD systems, we make the simplifying assumption that

$$\begin{aligned} \psi_l(k^l) &= y_l(k^l\sigma_l h), \sigma_l \in N,\ l = 1, \cdots, m \\ u(t) &= v(k), kh \leq t < (k+1)h \end{aligned} \tag{16.30}$$

Theorem 16.5 Let $J_{SSD,PS,\infty/\infty,h,opt}$ and $J_{SSD,PS,\infty/\infty,\rho h,opt}$ denote, respectively, the optimal achievable ∞/∞ index of SD system (14.1)-(14.3) under sampling period h and ρh, and $J_{MSD,PS,\infty/\infty,(\sigma_1 h,\cdots,\sigma_m h),opt}$ that of MSD system described by (14.1) and (16.30). Assume that ρ is a common multiple of $\sigma_1, \cdots, \sigma_m$. Then,

$$\begin{aligned} J_{SSD,PS,\infty/\infty,h,opt} &\geq J_{MSD,PS,\infty/\infty,(\sigma_1 h,\cdots,\sigma_m h),opt} \\ &\geq J_{SSD,PS,\infty/\infty,\rho h,opt} \end{aligned} \tag{16.31}$$

Proof: The proof is similar to that of Theorem 16.1. The sampling period will change the pattern of N_ρ. □

Theorem 16.6 Let $J_{SSD,FRE,2/2,h,opt}$ and $J_{SSD,FRE,2/2,\rho h,opt}$ denote, respectively, the optimal achievable H_2/H_2 index of SD system (14.1)-(14.3) under sampling period h and ρh, and

$J_{MSD,FRE,2/2,(\sigma_1 h,\cdots,\sigma_m h),opt}$ that of MSD system described by (14.1) and (16.30). Assume that ρ is a common multiple of $\sigma_1, \cdots, \sigma_m$. Then,

$$J_{SSD,FRE,2/2,h,opt} \geq J_{MSD,FRE,2/2,(\sigma_1 h,\cdots,\sigma_m h),opt} \geq J_{SSD,FRE,2/2,\rho h,opt} \quad (16.32)$$

Proof: The proof follows from (16.11),

$$J_{MSD,FRE,2/2,(\sigma_1 h,\cdots,\sigma_m h),opt} = \lim_{s\to\infty} J_{MSD,PS,\infty/\infty,(\sigma_1 h,\cdots,\sigma_m h),opt}$$

and Theorem 16.2. □

Theorem 16.7 Let $J_{SSD,OBS,\infty/\infty,h,opt}$ and $J_{SSD,OBS,\infty/\infty,\rho h,opt}$ denote, respectively, the optimal achievable H_∞/H_∞ index of SD system (14.1)-(14.3) under sampling period h and ρh, and $J_{MSD,OBS,\infty/\infty,(\sigma_1 h,\cdots,\sigma_m h),opt}$ that of MSD system described by (14.1) and (16.30). Assume that ρ is a common multiple of $\sigma_1, \cdots, \sigma_m$. Then,

$$J_{SSD,OBS,\infty/\infty,h,opt} \geq J_{MSD,OBS,\infty/\infty,(\sigma_1 h,\cdots,\sigma_m h),opt} \geq J_{SSD,OBS,\infty/\infty,\rho h,opt} \quad (16.33)$$

Proof: The proof follows the same line as that of Theorem 16.3. The residual generator for the MSD system can be brought into the form of (16.15) with $\mathbf{L}_\rho$ having $n_z = \rho m - \sum_{j=1}^{m} \frac{\rho}{\sigma_j}$ columns of zeros and $\mathbf{W}_\rho$ having n_z rows and n_z columns of zeros. □

16.6 Concluding remarks

This chapter has studied the influence of sampling period on some widely accepted classes of optimal fault detection performance. The background of this study is the important role played by the sampling period in embedded networked control systems. The result achieved in this chapter can be applied to the optimal selection of the sampling period in embedded networked control systems by taking into account the communication and real-time computational aspect [173, 171, 185].

In the analysis, both the parity space approach and the observer based approach have been considered. It is shown that the optimal H_2/H_2, H_∞/H_∞ fault detection performance index will become worse, if the sampling period is increased by an integer multiple. However, the optimal H_-/H_∞ performance index shows different property. As a by-product, it is also shown (see Lemma 16.3) that in the single-rate SD systems a linear discrete-time observer-based residual generator with time-varying gain matrix and weighting matrix will not improve the optimal H_∞/H_∞ or the H_-/H_∞ fault detection performance index, compared with constant gain matrix and weighting matrix. The above results have also been extended to the multirate SD systems.

Part V

FD of Networked Control Systems

17

Modelling of NCS

Recently, networked control systems (NCS) receive more and more attention in the field of automatic control (see, for instance, [1, 2, 117, 140, 148] and the references therein). In NCS the information is exchanged among sensors, actuators, controllers and supervision stations through a digital network shared among multiple users. As an example, Fig. 17.1 shows a typical structure of the NCS, where the controller is located at a remote site, u_p represents the control inputs that activate the actuators, y_p the sensor readings, u and y the information obtained by the fault detection (FD) system about control inputs and sensor outputs. The NCS provides advantages such as less wiring, flexible structure and easy maintenance. Many different types of network have been promoted for different application situations, for instance, CAN, Ethernet, Profibus, WLAN, etc. Different from classical control systems, the dynamic behavior of the NCS is closely related to the characteristics of the network.

From the network side, the access of different nodes to a network of limited bandwidth is mainly coordinated through medium access control (MAC) protocol [108]. In general, the MAC protocols can be classified into schedule-based schemes, contention-based schemes and mixed schemes. In schedule-based MAC (e.g. TDMA, Time Division Multiple Access), a static or dynamic time schedule is used to allocate time slots among the nodes. It guarantees a deterministic network behavior but requires a synchronization mechanism and a careful design of the time schedule to reduce conservatism and make a full use of the network resource. Token based or polling based schemes can be regarded as quasi schedule-based schemes. In contention-based MAC (e.g. CSMA in Ethernet, CAN and WLAN), each node tries to access the network as soon as it wants to send a message. Therefore, the avoidance and resolution of collisions is the key part of the protocol, which can be realized, for instance, by setting priorities to the message (e.g. in CAN) or by specifying a random waiting time before re-transmission (e.g. in Ethernet). The mixed schemes (e.g. Flexray) divide each cycle time into schedule-based part and contention-based part.

The performance of the network in real-time applications is evaluated by the QoS (quality of service) parameters, such as network-induced delay, jitter, packet loss rate, packet error rate, synchronization and quantization error, etc. The QoS parameters depend not only on the network bandwidth, network type, MAC protocol, but also on the number of nodes, the amount of data flow, the distribution of data flow over the network as well as environmental disturbances.

The main purpose of NCS design is to achieve the best utilization of the network under limited resource (i.e. improved cost/efficiency) without sacrifice of system control and monitoring performance. To this aim, interdisciplinary study with integrated efforts from the communication and control society makes the NCS an exciting research area.

From the viewpoint of monitoring, there are different system setups that are of practical importance. In Fig. 17.2, the controller is a local controller and the information about the process (y_p or u_p, y_p) are sent over the network for the purpose of remote monitoring. The distributed NCS structure shown in Fig. 17.3 is often encountered in industrial automation systems. There is distributed intelligence, e.g. local PID controllers, in each subsystem. The local systems exchange information

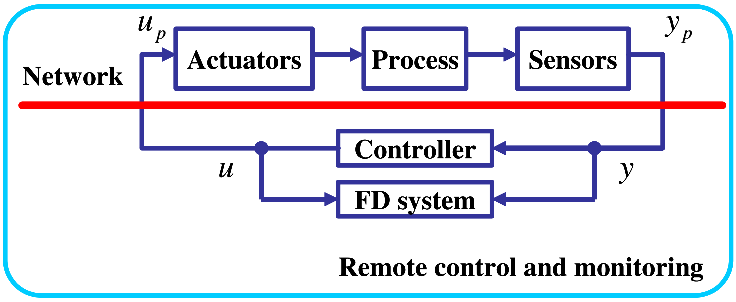

Fig. 17.1 A typical structure of NCS

through the network. A higher level control and monitoring unit can be implemented for the global supervision and coordination. The various system setups, network configurations (topology, protocol, coding and decoding algorithm) and different working modes of system components (sensors, actuators and controllers) further complicate the analysis and design of the NCS.

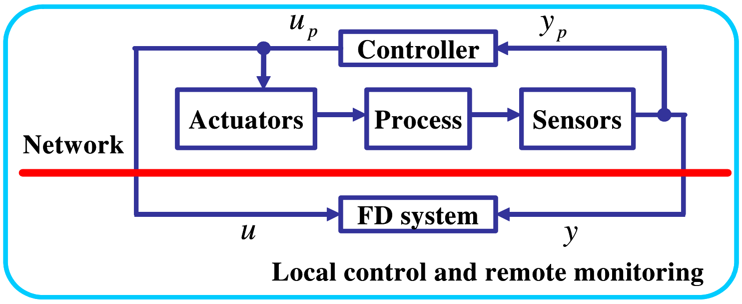

Fig. 17.2 Local control and remote monitoring

In case that multiple systems share one common network, the NCS can be classified into closed network and open network. Here the words closed or open don't mean closed-loop or open-loop. Instead, here *closed network* means that the scale of the network is modest, the number of nodes (users) is fixed and designable. Thus it is possible to optimize the network utilization and have a more active influence on the QoS of the network by designing system structure, data structure and information exchange strategy. In comparison, *open network* means that a lot of users have access to the network (for instance, in Internet) or there are strong unpredictable disturbances (for instance, in wireless network), in which the QoS of the network can be influenced to some extent but can not be completely determined by a small part of nodes (users).

The fault detection (FD) problem of the NCS has attracted much attention in the recent years, which aims at improving the safety and reliability [19, 60, 67, 121, 124] of the NCS. The development can be divided into three phases. In the first phase, the main problem under consideration is, for an NCS with given network configurations, how to design the FD system to reduce the impact of the QoS parameters on the fault detection performance. In the second phase, it is investigated how to reduce and optimize the network load at the application level, so that the nondeterminism in the network QoS is reduced and the FD system design can be simplified. Most recently, the co-design of the FD system, the controller and the network is discussed.

In this chapter, we shall give a description of the NCS and point out different network-induced factors that need consideration in the FD of NCS.

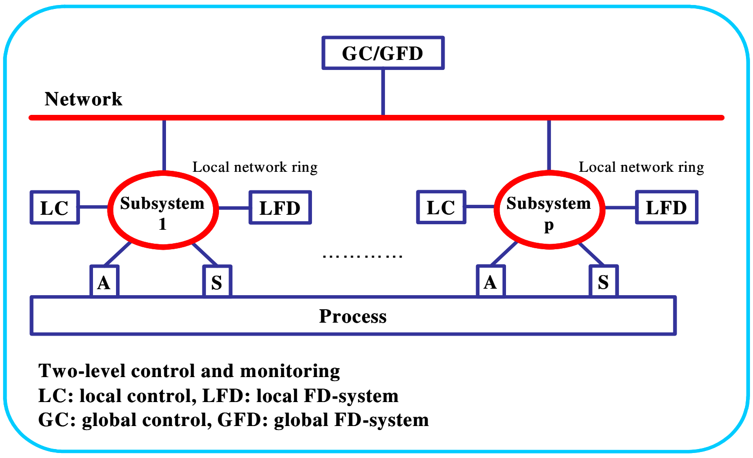

Fig. 17.3 Two-level control and monitoring

17.1 Process, sensors and actuators

To illustrate the basic ideas, we assume in the following that the process is linear. It is described either as a sampled-data (SD) system

$$\dot{x}(t) = A_c x(t) + B_c u_p(t) + E_{dc} d(t) + E_{fc} f(t) \tag{17.1}$$
$$y_p(t) = C x(t) \tag{17.2}$$
$$y_p(k) = \mathcal{S}(y_p(t)) = y_p(t_{y,k}) \tag{17.3}$$
$$u_p(t) = \mathcal{H}(u_p(k)), \quad t_{u,k} \le t < t_{u,k+1} \tag{17.4}$$

or as a discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu_p(k) + E_d d(k) + E_f f(k) \\ y_p(k) &= Cx(k) + Du_p(k) + F_d d(k) + F_f f(k) \end{aligned} \tag{17.5}$$

where $x \in \mathbf{R}^n$ denote the state vector, $u_p \in \mathbf{R}^{n_u}$ the control input vector, $y_p \in \mathbf{R}^m$ the measured output vector, $d \in \mathbf{R}^{n_d}$ the unknown disturbance vector and $f \in \mathbf{R}^{n_f}$ the fault vector, $A, B, E_d, E_f, A_c, B_c, E_{dc}, E_{fc}, C, D, F_d, F_f$ are known matrices of appropriate dimensions. In the SD system description, (17.3) and (17.4) represent the A/D converter and the D/A converter, respectively, $t_{y,k}$ denotes the time instants at which the outputs are sampled and $t_{u,k}$ the time instants at which the control inputs are updated.

If the process has a distributed structure and can be divided into p subsystems, then the vectors x, u_p, y_p are composed of the local vectors $x_i, u_{p,i}, y_{p,i}$, $i = 1, \cdots, p$, of the subsystems. For instance, (17.5) can be decomposed and re-written as

$$\begin{aligned} x_i(k+1) &= A_{ii} x_i(k) + \sum_{j=1, j\neq i}^{p} A_{ij} x_j(k) + B_i u_{p,i}(k) + E_{d,i} d(k) + E_{f,i} f(k) \\ y_{p,i}(k) &= C_i x_i(k) + D_i u_{p,i}(k) + F_{d,i} d(k) + F_{f,i} f(k) \end{aligned} \tag{17.6}$$

where A_{ij} $(i \neq j)$ represents the coupling between the subsystems and

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{bmatrix}, B = \begin{bmatrix} B_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & B_p \end{bmatrix}, C = \begin{bmatrix} C_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & C_p \end{bmatrix}$$

$$D = \begin{bmatrix} D_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & D_p \end{bmatrix}, E_d = \begin{bmatrix} E_{d,1} \\ \vdots \\ E_{d,p} \end{bmatrix}, E_f = \begin{bmatrix} E_{f,1} \\ \vdots \\ E_{f,p} \end{bmatrix}$$

$$F_d = \begin{bmatrix} F_{d,1} \\ \vdots \\ F_{d,p} \end{bmatrix}, F_f = \begin{bmatrix} F_{f,1} \\ \vdots \\ F_{f,p} \end{bmatrix}$$

The sensors and the actuators work in either clock-driven mode or event-driven mode. In the clock-driven mode, $t_{y,k}, t_{u,k}$ are pre-defined. In the event-driven mode, $t_{y,k}$ and $t_{u,k}$ are related to the status of an event.

17.2 Network-induced delay and jitter

From the FD viewpoint, the delays in the NCS can be classified into two different kinds of delays. One is the time taken by a sensor output packet to arrive the FD system, denoted by τ_y. The other is the time difference between u and u_p, denoted by τ_u. In Fig. 1, $u_p(t) = u(t - \tau_u)$. In Fig. 2, $u(t) = u_p(t - \tau_u)$. The delay is composed of transmission delays, processing delay (packet encapsulation, coding and decoding, queuing) and sometimes re-transmission delay (e.g. if TCP is used at the transport level). As mentioned before, the delay in NCS depends on many factors, such as the network type, MAC protocol, network load, etc. They can be constant (e.g. TDMA with static scheduling), time-varying, or stochastic (e.g. CSMA) [108, 148].

The variation of the network-induced delays is called jitter. It is the main reason why the delays caused by the network are sometimes difficult to be handled.

For the design of FD systems, it is important to

- check whether τ_y and τ_u can be measured (estimated) online,
- determine when the sensor outputs received by the FD system are sampled, and
- analyze how the inputs influence the sensor outputs received by the FD system.

Bearing these questions in mind, we shall analyze the network-induced delay and jitter in the NCS in this subsection.

The delays can be measured online, if the following two conditions are satisfied:

- the local clocks at sensors, actuators and controllers are well synchronized,
- a time-stamping mechanism is available.

The synchronization can be achieved, for instance, by using the IEEE 1588 protocol. In general, the smaller the synchronization error is, the higher will be the additional network load caused by the synchronization. A description of the synchronization error will be discussed later.

If it is possible to add a sequence number in the packets from clock-driven sensors or a time-stamp in the packets from event-driven sensors, then the sampling instants of the arriving sensor output packets can be easily determined. If the timing information is not available and the FD system read the sensor values from the buffer periodically, then the sampling instants of these sensor outputs can not be precisely determined.

If a time-stamp is attached to the packet of control input, the working mode of the actuator is known, and this timing information is transmitted by the actuator to the FD system, then the control input working on the process can be figured out. Otherwise, it is in general more difficult to determine the precise time instant when the control inputs from the controller have effect on the process.

Different from classical time delay systems, in NCS with the aid of network technique it is often possible to obtain some information about the delay, such as

- probability distribution $f(\tau)$,
- mean value $\mathbf{E}\tau$ and variance $\mathbf{Var}(\tau)$,
- upper and lower bound $\underline{\tau} \leq \tau \leq \bar{\tau}$.

Integration of such information into the analysis and design of fault detection systems will improve the FD performance.

17.3 Packet loss

A packet loss can be caused by collisions, buffer overrunning, or channel impairments (detected but unrepairable channel errors). From the viewpoint of real-time control and monitoring, a packet which arrives after a long time delay is sometimes also handled as a packet loss. To describe packet loss, a variable α is usually introduced to denote the status of packet arrival, $\alpha = 0$ if the packet is lost, $\alpha = 1$ if the packet arrives. For the design of FD schemes, at first it is important to have the following information:

- the availability of information on the state of α to the FD system
- the candidate signal used to replace the lost packet.

For the FD performance not only the probability or the frequency of the packet loss but also the distribution of the packet loss plays an important role. Typical assumptions on the available information about the packet loss in NCS are:

- The transition of α between 0 (packet loss) and 1 (packet arrival) obeys a Markov chain, which includes Bernoulli process as a special case [140]. The distribution of packet loss is described by the transition probability matrix
$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \tag{17.7}$$
with
$$\begin{aligned} p_{ij} &= \text{Prob}\{\alpha(k+1) = i \mid \alpha(k) = j\} \\ &= \text{Prob}\{\alpha(k+1) = i \mid \alpha(k) = j, \forall \alpha(k-1), \cdots, \alpha(0)\} \\ i, j &\in \{0, 1\}, \ \sum_i p_{ij} = 1 \end{aligned}$$
If the sensor outputs are sent by different packets, for instance, the outputs from each subsystem are sent separately and the arrival of different output packets at the same time instant happens independently, then the number of the states in the Markov chain will increase to 2^p.
- $\{\alpha(0), \alpha(1), \alpha(2), \cdots\}$ is a switching sequence with part information of the switching sequence such as maximal number of consecutive packet losses, minimal (or maximal) interval between neighboring packet losses, average frequency of packet losses over a time window of fixed length, etc.

If a packet loss happens, different schemes may be used to replace the lost information. Assume that a q-dimensional signal $\xi = \begin{bmatrix} \xi_1 & \cdots & \xi_q \end{bmatrix}^T$ is under consideration. If at time instant k the packet of ξ_j is lost and a zero-padding scheme is adopted, then $\xi_{sub,j}(k) = 0$ and the substitution signal $\xi_{sub} = \begin{bmatrix} \xi_{sub,1} & \cdots & \xi_{sub,q} \end{bmatrix}^T$ can be described by

$$\xi_{sub}(k) = \Lambda(k)\xi(k) \tag{17.8}$$

$$\Lambda(k) = \begin{bmatrix} \alpha_1(k) & 0 & \cdots & 0 \\ 0 & \alpha_2(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_q(k) \end{bmatrix}, \ \alpha_i(k) \in \{0,1\}, \quad i = 1, \cdots, q$$

Other schemes, such as "keep-the-old-value", "keep-a-constant-value" or "use-the-estimation", can be, respectively, expressed as

$$\begin{aligned} \xi_{sub}(k) &= \Lambda(k)\xi(k) + (I - \Lambda(k))\xi_{sub}(k-1) \\ \xi_{sub}(k) &= \Lambda(k)\xi(k) + (I - \Lambda(k))\xi_{const} \\ \xi_{sub}(k) &= \Lambda(k)\xi(k) + (I - \Lambda(k))\hat{\xi}(k) \end{aligned} \tag{17.9}$$

where ξ_{const} is a constant (e.g. default value), $\hat{\xi}$ is an estimation obtained from a model or an observer. Which compensation (interpolation) scheme is the most advantageous for the control and monitoring of the NCS, is still a topic of current research.

17.4 Quantization

In the NCS, quantizers are used to transform a continuous-valued signal to a finite number of bits. The approximation error caused by the quantization depends strongly on the type of the quantizer (uniform quantizer, non-uniform quantizer, adaptive quantizer, etc). Denote the signal before and after quantization, respectively, by ξ and ξ_{quan}. If a uniform quantizer is used, then the quantization error $\delta(k) = \xi(k) - \xi_{quan}(k)$ can be regarded as a bounded variable $\delta(k) \in [-b_{uni}, b_{uni}]$, where $2b_{uni}$ is the step size of the quantization. The basic idea of non-uniform quantizers is to choose non-equidistant step size by taking into account the distribution of the source signal ξ. A kind of widely accepted non-uniform quantizers is logarithmic quantizers [49]. In this case, the quantization is fine near zero and becomes coarse for large values. In general, for non-uniform quantizers the quantization error δ depends on the value of ξ and can be described by $\delta(k) = b_{non-uni}\xi(k)$, where $b_{non-uni}$ is related to the "zoom" factor of the non-uniform quantizer. Adaptive quantizers adjust the step size and the range of the quantization online according to the system forward or backward information.

17.5 Coding, decoding and packet error

The above mentioned quantization is indeed part of *source coding* aiming at a compression of the original source signal. Due to channel disturbance, the packet arriving the receiver may be different from the original packet send by the sender. To detect and correct such packet errors, redundancy will be further added in the packet before it is sent over the network, which is called *channel coding* [15, 126]. After decoding the received codeword, some of the packet errors can be detected and corrected. The part of undetectable packet errors is what needs to be considered in the FD algorithms. On the one side, sophisticated channel coding and decoding algorithms (e.g. turbo codes and low-density parity check codes) have been developed and could guarantee a very low packet error rate. On the other side, too much redundant information will increase network load and reduce useful data rate. The performance of the coding and decoding algorithm is evaluated by

- the code rate, which is decisive for the increased data amount and has influence on the delay, and
- the packet error rate (PER) and distribution, which can be calculated or estimated based on bit error rate (BER) and bit error distribution.

In the structure of the NCS as shown in Fig. 17.1 with distributed sensors and actuators, the coding and decoding problem is in the nature of

- distributed coding and central decoding problem at the side of sensors-FD system, and
- central coding and distributed decoding at the side of FD system-actuators.

17.6 Synchronization error

Clock synchronization is a well-known problem in distributed systems [90]. As mentioned in Section 17.2, the IEEE 1588 PTP (precise time protocol) can be applied to reduce the synchronization error to a certain level, whose performance also depends also on the network status. To study the influence of the synchronization error on the FD, let us have a look at distributed clock-driven sensors with a uniform sampling period T. Denote the time at the central (master) station as t, which is used as reference time for the synchronization, and the local time at each sensor as t_i, $i = 1, \cdots, m$. The synchronization error can be described by $t_i = t + \Delta_{syn,i}$, where $\Delta_{syn,i}$ is time-varying (slowly increasing during each synchronization cycle) but bounded. Therefore, the sensor output vector used for the FD is indeed

$$y(k) = \begin{bmatrix} \mathcal{S}_1(y_{p,1}(t_1)) \\ \vdots \\ \mathcal{S}_m(y_{p,m}(t_m)) \end{bmatrix} = \begin{bmatrix} y_{p,1}(kT + \Delta_{syn,1})) \\ \vdots \\ y_{p,m}(kT + \Delta_{syn,m})) \end{bmatrix} \tag{17.10}$$

17.7 Concluding remarks

In this chapter, we have given a brief description of network-induced delay, jitter, packet loss, quantization error, packet error and synchronization error. It is worthy of emphasis that, for a given network of finite bandwidth, the QoS parameters are closely related to each other, for instance:

- TCP and UDP are two often adopted protocols at the transport level. Compared with UDP, TCP will reduce the packet loss rate but at the price of retransmission, increased network load and delay.
- Synchronization error can be reduced by performing synchronization more frequently. However, this may also increase the data flow and the transmission delay.

To the authors' knowledge, due to the variety of the networks, an analytical expression of the relation among the QoS parameters that can be used to guide the global optimization of the NCS in terms of resource utilization and system performance is not yet available.

18

FD of NCS

The FD problem of the NCS consists in an early and reliable detection of the faults in the process components, sensors or actuators ($f \neq 0$) based on the information contained in the control input packets (u) and the sensor output packets (y). As the network is imperfect and will modify the system dynamics, the FD system should be robust to not only the disturbances but also the uncertainty caused by the network, while keeping to be sensitive to the faults. In this chapter, we shall focus on the problem of the *FD over the network* and outline the basic ideas and solution procedures of solving the FD problem in NCS with a given network of limited bandwidth. For the *FD of the network* the readers are referred to [5, 16].

The first step towards the FD system design is to derive the relation between the available information in the packets of sensor outputs and control inputs by taking into account the influence of the QoS parameters. It should be noticed that in the modelling of the NCS, it is important to consider different system setups. For the sake of clarity, in most investigations only one or two dominant factors will be considered and the interconnections between the QoS parameters are neglected.

Then, based on the derived NCS model, residual generator and residual evaluator can be designed. In some cases, the problem can be transformed into some standard FD problem formulations and solved with the help of the existing technique. In other cases, new residual generator and evaluation schemes need to be developed, especially due to the often stochastic nature of the QoS parameters. If not specified, the discussion in this section will be carried out in the framework of the system setup illustrated in Fig. 17.1.

To illustrate the basic ideas behind different handlings, we define a *benchmark scenario*, where the NCS components are described by (17.1)-(17.4), the sensors are clock-driven, the controller and the actuators are event-driven, $\tau_{y,k} + \tau_{proc,k} + \tau_{u,k} = \tau_k$, $\tau_k \leq T$, $\tau_{y,k}$ denotes the sensor-to-controller delay, $\tau_{proc,k}$ the processing time needed by the controller, $\tau_{u,k}$ the controller-to-actuator delay, τ_k is indeed the timing difference between the sensors and the actuators.

18.1 Handling of NCS as LPV systems

We consider the benchmark scenario. By discretizing the process model in (17.1)-(17.2) equidistantly at $t = kT$, the NCS can be modelled as

$$\begin{aligned} x(k+1) &= A_d x(k) + E_d d(k) + E_f f(k) + B_{d0}(\tau_k) u(k) + B_{d1}(\tau_k) u(k-1) \\ y(k) &= Cx(k) \end{aligned} \tag{18.1}$$

where

$$A_d = e^{A_c T},\ B_{d0}(\tau_k) = \int_0^{T-\tau_k} e^{A_c \zeta} B_c d\zeta,\ B_{d1}(\tau_k) = \int_{T-\tau_k}^{T} e^{A_c \zeta} B_c d\zeta$$

$$E_d d(k) = \int_{kT}^{(k+1)T} e^{A_c \zeta} E_{dc} d\zeta,\ \ E_f f(k) = \int_{kT}^{(k+1)T} e^{A_c \zeta} E_{fc} d\zeta$$

Suppose that a good synchronization is available, both the control input packets and the sensor output packets are time-stamped, and the time-stamps of the control input packets are transmitted by the actuator back to the FD system with negligible delay. In this situation, τ_k is available information and the above model has the form of a linear parameter varying (LPV) system [181]. Therefore, an LPV residual generator can be constructed as

$$\begin{aligned}\hat{x}(k+1) &= A_d\hat{x}(k) + B_{d0}(\tau_k)u(k) + B_{d1}(\tau_k)u(k-1) + L(y(k) - \hat{y}(k))\\ \hat{y}(k) &= C\hat{x}(k)\\ r(k) &= W(y(k) - \hat{y}(k))\end{aligned} \tag{18.2}$$

As the influence of the control inputs is completely compensated, the observer gain matrix L and the weighting matrix W as well as threshold selection can be determined using the standard technique. The effect of sampling or multirate sampling can be taken into account by using the approaches summarized in [172].

We take the modeling of *packet loss* as another example.

Assume that the NCS components are modeled by (17.5), the sensors and the actuators are distributed, the information of packet loss at the sensors' side, i.e. state of $\alpha_y(k) = \begin{bmatrix}\alpha_{y,1}(k) & \cdots & \alpha_{y,m}(k)\end{bmatrix}$ is known. We shall consider two cases.

In the first case, assume that the information of packet loss at the actuators' side (state of $\alpha_u(k) = \begin{bmatrix}\alpha_{u,1}(k) & \cdots & \alpha_{u,n_u}(k)\end{bmatrix}$) is delivered by the actuators to the FD system. Under this assumption, the control input $u_p(k)$ that works on the process can be precisely reconstructed from $u(k), \alpha_u(k)$, no matter which interpolation scheme is adopted by the actuator in the case of packet loss. Therefore, in this case we treat $u_p(k)$ as known and put the focus on the handling of $\alpha_y(k)$. Let $\bar{N}_y(k)$ be a matrix formed by the information of $\alpha_y(k)$,

$$\bar{N}_y(k) = \begin{bmatrix}\alpha_{y,1}(k) & O & \cdots & O\\ O & \alpha_{y,2}(k) & \ddots & \vdots\\ \vdots & \ddots & \ddots & O\\ O & \cdots & O & \alpha_{y,m}(k)\end{bmatrix}$$

and denote with $N_y(k)$ the matrix consisting only of the non-zero rows of $\bar{N}_y(k)$. Then, $y(k) = N_y(k)y_p(k)$, where $y(k)$ is the output packets received by the FD system at time k. As a result, the NCS can be modelled by

$$\begin{aligned}x(k+1) &= Ax(k) + Bu_p(k) + E_d d(k) + E_f f(k)\\ y(k) &= N_y(k)\left(Cx(k) + Du_p(k) + F_d d(k) + F_f f(k)\right)\end{aligned} \tag{18.3}$$

Applying the basic idea of the parity space approach, a residual generator can be constructed as

$$r(k) = V_s(k)\left(\begin{bmatrix}y(k-s)\\ \vdots\\ y(k)\end{bmatrix} - H_{u,s}(k)\begin{bmatrix}u_p(k-s)\\ \vdots\\ u_p(k)\end{bmatrix}\right) \tag{18.4}$$

where $V_s(k)$ is the time-varying parity matrix satisfying $V_s(k)H_{o,s}(k) = 0$, $H_{o,s}(k)$ and $H_{u,s}(k)$ are time-varying matrices depending on $N_y(k)$ as

$$\begin{aligned}H_{o,s}(k) &= \begin{bmatrix}N_y(k-s)C\\ N_y(k-s+1)CA\\ \vdots\\ N_y(k)CA^s\end{bmatrix}\\ H_{u,s}(k) &= \begin{bmatrix}N_y(k-s)D & O & \cdots & O\\ N_y(k-s+1)CB & N_y(k-s+1)D & \ddots & \vdots\\ \vdots & & \ddots & O\\ N_y(k)CA^{s-1}B & \cdots & & N_y(k)D\end{bmatrix}\end{aligned} \tag{18.5}$$

The residual dynamics is

$$r(k) = V_s(k)\left(H_{d,s}(k)\begin{bmatrix} d(k-s) \\ \vdots \\ d(k) \end{bmatrix} + H_{f,s}(k)\begin{bmatrix} f(k-s) \\ \vdots \\ f(k) \end{bmatrix}\right)$$

$$H_{d,s}(k) = \begin{bmatrix} N_y(k-s)F_d & O & \cdots & O \\ N_y(k-s+1)CE_d & N_y(k-s+1)F_d & \ddots & \vdots \\ \vdots & & \ddots & O \\ N_y(k)CA^{s-1}E_d & \cdots & & N_y(k)F_d \end{bmatrix}$$

$$H_{f,s}(k) = \begin{bmatrix} N_y(k-s)F_f & O & \cdots & O \\ N_y(k-s+1)CE_f & N_y(k-s+1)F_f & \ddots & \vdots \\ \vdots & & \ddots & O \\ N_y(k)CA^{s-1}E_f & \cdots & & N_y(k)F_f \end{bmatrix}$$

As $N_y(k)$ is a time-varying matrix, the optimal time-varying parity matrix $V_s(k)$ is determined at each time instant by solving one of the following optimization problems

$$\max_{V_s(k),V_s(k)H_{o,s}(k)=0} J_{\infty/\infty} = \max_{V_s(k),V_s(k)H_{o,s}(k)=0} \frac{\bar{\sigma}\left(V_s(k)H_{f,s}(k)H_{f,s}^T(k)V_s^T(k)\right)}{\bar{\sigma}\left(V_s(k)H_{d,s}(k)H_{d,s}^T(k)V_s^T(k)\right)} \tag{18.6}$$

$$\max_{V_s(k),V_s(k)H_{o,s}(k)=0} J_{-/\infty} = \max_{V_s(k),V_s(k)H_{o,s}(k)=0} \frac{\underline{\sigma}\left(V_s(k)H_{f,s}(k)H_{f,s}^T(k)V_s^T(k)\right)}{\bar{\sigma}\left(V_s(k)H_{d,s}(k)H_{d,s}^T(k)V_s^T(k)\right)} \tag{18.7}$$

$$\max_{V_s(k),V_s(k)H_{o,s}(k)=0} J_{i/\infty} = \max_{V_s(k),V_s(k)H_{o,s}(k)=0} \frac{\sigma_i\left(V_s(k)H_{f,s}(k)H_{f,s}^T(k)V_s^T(k)\right)}{\bar{\sigma}\left(V_s(k)H_{d,s}(k)H_{d,s}^T(k)V_s^T(k)\right)} \tag{18.8}$$

In the second case, assume that the information of packet loss at the actuators' side (state of $\alpha_u(k) = \begin{bmatrix} \alpha_{u,1}(k) \cdots \alpha_{u,n_u}(k) \end{bmatrix}$) is not available to the FD system. Under this assumption, the control input $u_p(k)$ that works on the process is only partly known. Introduce a matrix $N_u(k)$ to represent the information of $\alpha_u(k)$,

$$N_u(k) = \begin{bmatrix} \alpha_{u,1}(k) & O & \cdots & O \\ O & \alpha_{u,2}(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & \alpha_{u,n_u}(k) \end{bmatrix}$$

Because $u_p(k) = N_u(k)u(k)$, the NCS is now modelled by

$$\begin{aligned} x(k+1) &= Ax(k) + BN_u(k)u(k) + E_d d(k) + E_f f(k) \\ y(k) &= N_y(k)\left(Cx(k) + DN_u(k)u(k) + F_d d(k) + F_f f(k)\right) \end{aligned} \tag{18.9}$$

where $u(k)$ is the control input packets delivered by the controller (co-located with the FD system) to the actuator. A parity relation based residual generator can be constructed as

$$r(k) = V_s(k)\left(\begin{bmatrix} y(k-s) \\ \vdots \\ y(k) \end{bmatrix} - H_{u,s}(k)\begin{bmatrix} u(k-s) \\ \vdots \\ u(k) \end{bmatrix}\right)$$

$$V_s(k)H_{o,s}(k) = 0$$

The residual dynamics is

$$r(k) = V_s(k)\left(H_{f,s}(k)\begin{bmatrix} f(k-s) \\ \vdots \\ f(k)\end{bmatrix} + H_{d,s}(k)\begin{bmatrix} d(k-s) \\ \vdots \\ d(k)\end{bmatrix} + H_{u,s}(k)\begin{bmatrix} \Delta u(k-s) \\ \vdots \\ \Delta u(k)\end{bmatrix}\right)$$

where $\Delta u(k-s) = u_p(k-s) - u(k-s)$. Due to the lack of information on $N_u(k)$, the control inputs will influence the residual $r(k)$. This effect can be taken into account in the selection of $V_s(k)$ by decoupling, optimization or adaptive threshold.

18.2 Handling of NCS as uncertain systems

In this subsection, we shall consider three different example cases, where the network induced factors can be modeled as model uncertainty.

For the benchmark scenario, assume that the delays are not online measurable. The NCS model (18.1) can be re-written as

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k-1) + g(\tau_k, u(k), u(k-1)) + E_d d(k) + E_f f(k) \\ y(k) &= Cx(k) \end{aligned} \tag{18.10}$$

where

$$B_d = \int_0^T e^{A_c\zeta} B_c d\zeta, \;\; g(\tau_k, u(k), u(k-1)) = B_{d0}(\tau_k)(u(k) - u(k-1))$$

As the delay is unknown, the influence of the control inputs on the residual signal can not be precisely described. This effect is captured by the additional term $g(\tau_k, u(k), u(k-1))$. To extract more information, $B_{d0}(\tau_k)$ can be re-written as [41]

$$B_{d0}(\tau_k) = B_{d0}(\tau_o) + \Delta B$$

The *nominal* part $B_{d0}(\tau_o) = \int_0^{T-\tau_o} e^{A_c\zeta} B_c d\zeta$ can be determined based on the available deterministic or statistical information of the delays, for instance, $\tau_o = \frac{\underline{\tau}+\bar{\tau}}{2}$ (median) or $\tau_o = \mathbf{E}\tau$ (expectation). The *model uncertainty* ΔB is caused by the deviation of τ_k from the nominal value τ_o and structured as $\Delta B = e^{A_c(T-\tau_o)} \int_0^{\tau_o - \tau_k} e^{A_c\zeta} B_c d\zeta$. Finally, based on the model

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k-1) + E_d d(k) + E_f f(k) \\ &\quad + B_{d0}(\tau_o)(u(k) - u(k-1)) + \Delta B(u(k) - u(k-1)) \\ y(k) &= Cx(k) \end{aligned} \tag{18.11}$$

an observer based residual generator can be designed to suppress the influence of ΔB and the disturbances on the residual signal. The threshold for the decision can be adapted to the control update. If there is further information about the delay distribution, it can be integrated in the threshold selection using, for instance, probabilistic robustness technique [39]. From the above discussion it is not difficult to see that the (unknown) jitter is what makes the FD in the NCS more difficult.

The second typical example that can be well modeled as uncertainty is the synchronization error. Consider the benchmark scenario with a synchronization mechanism. Due to the synchronization error (see the discussion in Section 17.6), the model (18.1) is extended to

$$\begin{aligned} x(k+1) &= A_d x(k) + E_d d(k) + E_f f(k) \\ &\quad + (B_{d0}(\tau_k) + \Delta B_{d0})u(k) + (B_{d1}(\tau_k) + \Delta B_{d1})u(k-1)) \\ y(k) &= (C + \Delta C)x(k) + \Delta D u(k) + \Delta E_d d(k) + \Delta E_f f(k) \end{aligned} \tag{18.12}$$

with $\Delta B_{d0}, \Delta B_{d1}, \Delta C, \Delta D, \Delta E_d, \Delta E_f$ are related to the synchronization error $\Delta_{syn,i}, i = 1, \cdots, m$.

In the third example, we shall look at the influence of quantization errors caused by non-uniform quantizers. Assume that the process is described by (17.1)-(17.4) and the NCS is configured as shown in Fig. 17.1. Let

$$\Delta_{y,q} = \begin{bmatrix} \Delta_{y_1,q} & O & \cdots & O \\ O & \Delta_{y_2,q} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & \Delta_{y_m,q} \end{bmatrix}, \quad \Delta_{u,q} = \begin{bmatrix} \Delta_{u_1,q} & O & \cdots & O \\ O & \Delta_{u_2,q} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & \Delta_{u_{n_u},q} \end{bmatrix}$$

where $\Delta_{y_j,q}$, $j = 1, \cdots, m$, and $\Delta_{u_i,q}$, $i = 1, \cdots, n_u$, denote the step size of the quantizers in the j-th output and the i-th input, respectively. Substituting

$$y_q(k) = y_p(k) + \delta_y(k), \quad \delta_y(k) = \Delta_{y,q} y_p(k)$$
$$u_p(k) = u(k) + \delta_u(k), \quad \delta_u(k) = \Delta_{u,q} u(k)$$

into process model (17.5), the outputs received by the FD system are indeed

$$x(k+1) = Ax(k) + B(I + \Delta_{u,q})u(k) + E_d d(k) + E_f f(k)$$
$$y_q(k) = (I + \Delta_{y,q})\left(Cx(k) + D(I + \Delta_{u,q})u(k) + F_d d(k) + F_f f(k)\right)$$

18.3 Handling of NCS as systems with unknown inputs

For the benchmark scenario, Ye et al. propose to approximate the additional term $g(\tau_k, u(k), u(k-1))$ in (18.10) caused by the network-induced delay by [160, 162]

$$g(\tau_k, u(k), u(k-1)) \approx E_\tau(k) d_\tau(k) \tag{18.13}$$

where $d_\tau(k)$ is an unknown term composed by the unknown time-varying delay, and $E_\tau(k)$ is a time-varying matrix depending on the control input. This can be done, for instance, by Taylor expansion or Pade approximation [157]. Based on the NCS model,

$$x(k+1) = A_d x(k) + B_d u(k-1) + E_d d(k) + E_\tau(k) d_\tau(k) + E_f f(k)$$
$$y(k) = Cx(k)$$

a time-varying parity relation based residual generator and residual evaluation scheme can be designed. Compared with the modeling in Subsection 18.2, the influence of the delay is handled as an unknown input, whose influence, together with the unknown disturbances d, should be decoupled or optimally suppressed. To reduce the dimension of the unknown inputs caused by the delay $d_\tau(k)$, a PCA (principle component analysis) based method is proposed by [162]. Furthermore, the assumption on smaller than one sampling period delays can be relaxed [156].

Now we take another example from [196]. For the sake of consistency, we describe the scenario considered there in a slightly different way. Different from the benchmark scenario, assume that both the sensors and the actuators are clock driven. The sensor output packets are time-stamped. However, the actuator doesn't feed back the arrival time of the control input packets to the FD system. The delay of control input packets is stochastic, can be longer than one sampling period but bounded. If the actuator doesn't receive any new control input signal during $[kT, (k+1)T)$, then it will keep the old value. In this case, the NCS can be modelled as

$$x(k+1) = A_d x(k) + B_d u(k-i) + E_d d(k) + E_f f(k)$$
$$y(k) = Cx(k) \tag{18.14}$$

where $i \in \{1, 2, \cdots, \bar{\rho}\}$, $\bar{\rho} \in \mathbf{N}$, represents the uncertainty caused by the network. The control inputs that work on the process during $[kT, (k+1)T)$ are indeed the i-th step delay of the control inputs delivered by the controller. However, as $u(k-i)$ is unknown, its expectation is used for the purpose of residual generation as follows [196]

$$\begin{aligned}\hat{x}(k+1) &= A_d\hat{x}(k) + B_d\mathbf{E}(u(k-i)) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) \\ r &= Q(z)(y - \hat{y})\end{aligned}$$

As a result, the residual dynamics is governed by

$$\begin{aligned}e(k+1) &= (A_d - LC)e(k) + B_d\left(u(k-i) - \mathbf{E}(u(k-i))\right) + E_d d(k) + E_f f(k) \\ r &= Q(z)Ce(z)\end{aligned}$$

Note that the control input u may influence the residual signal and needs to be taken into account in the design of L and $Q(z)$. The residual generator is designed to be robust to d, u and sensitive to f.

The influence of quantization errors caused by uniform quantizers can be modeled as bounded unknown inputs as well [106]. If both $u(k)$ and $y_p(k)$ are sent over the network, as shown in Fig. 17.1, then

$$\begin{aligned}x(k+1) &= Ax(k) + B(u(k) + \delta_u(k)) + E_d d(k) + E_f f(k) \\ y_q(k) &= Cx(k) + D(u(k) + \delta_u(k)) + F_d d(k) + F_f f(k) - \delta_y(k)\end{aligned}$$

with $\delta_u(k)$ and $\delta_y(k)$ being unknown but bounded signals.

18.4 Handling of NCS as hybrid systems

The NCS with packet loss can be conveniently modelled as hybrid systems. Consider the packet loss scenario described in Section 18.1. Assume that the state of both $\alpha_y(k)$ and $\alpha_u(k)$ are known.

By extending the results in [186], a residual generator is constructed as

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu_p(k) + L(\alpha_y(k))(y_{sub}(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) + Du_p(k) \\ r &= W(\alpha_y(k))(y_{sub}(k) - \hat{y}(k))\end{aligned}$$

As $\alpha_y(k)$ is available information, the observer parameters L and W can be adapted according to the state of $\alpha_y(k)$. Since no new information is available for generating residual signals if no packet arrives, the "use-the-estimation" scheme is suitable for the purpose of FD. As mentioned in Subsection 17.3, the candidate signal $y_{sub}(k)$ can be written as

$$y_{sub}(k) = \Lambda_y(k)y_p(k) + (I - \Lambda_y(k))\hat{y}(k)$$

where $\Lambda_y(k)$ is a diagonal matrix formed by the information of $\alpha_y(k)$.

$$\Lambda_y(k) = \begin{bmatrix}\alpha_{y,1}(k) & O & \cdots & O \\ O & \alpha_{y,2}(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & \alpha_{y,m}(k)\end{bmatrix} = \Lambda_y(\alpha_y(k))$$

Note that

$$y_{sub}(k) - \hat{y}(k) = \Lambda_y(k)(y_p(k) - \hat{y}(k))$$

The resulting residual dynamics is described by a hybrid system

$$
\begin{aligned}
e(k+1) &= A_{r,i}e(k) + E_{dr,i}d(k) + E_{fr,i}f(k) \\
r(k) &= W_i(C_{r,i}e(k) + F_{dr,i}d(k) + F_{fr,i}f(k))
\end{aligned}
$$

where

$$
\begin{aligned}
A_{r,i} &= A - L_i\Lambda_{y,i}C,\ E_{dr,i} = E_d - L_i\Lambda_{y,i}F_d,\ E_{fr,i} = E_f - L_i\Lambda_{y,i}F_f \\
C_{r,i} &= \Lambda_{y,i}C,\ F_{dr,i} = \Lambda_{y,i}F_d,\ F_{fr,i} = \Lambda_{y,i}F_f \\
\Lambda_{y,i} &= \Lambda_y(\alpha_y(k) = i) \\
L_i &= L(\alpha_y(k) = i),\ W_i = W(\alpha_y(k) = i),\ i = 0, 1, \cdots, 2^m - 1
\end{aligned}
$$

Depending on the assumptions about packet losses, design parameters L_i and W_i can be synthesized with the help of the theory of Markov jump systems or switched systems. In the case of sensor output packet loss the state of the Markov chain or the switching sequence is known.

If, however, the FD system doesn't have the information of packet loss at the actuators' side, then $\alpha_u(k)$ is unknown and the influence of the control inputs on the residual can not be totally eliminated. Let $u_{sub}(k)$ represent the real control inputs working on the process. The dynamics of the NCS is governed by

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu_{sub}(k) + E_d d(k) + E_f f(k) \\
y_p(k) &= Cx(k) + Du_{sub}(k) + F_d d(k) + F_f f(k)
\end{aligned}
$$

Implement the residual generator as

$$
\begin{aligned}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(\alpha_y(k))(y_{sub}(k) - \hat{y}(k)) \\
\hat{y}(k) &= C\hat{x}(k) + Du(k) \\
r(k) &= W(\alpha_y(k))(y_{sub}(k) - \hat{y}(k))
\end{aligned}
$$

The residual dynamics is in this case governed by

$$
\begin{aligned}
e(k+1) &= A_{r,i}e(k) + E_{dr,i}d(k) + E_{fr,i}f(k) + B_{r,i}\Delta_u(k) \\
r(k) &= W_i(C_{r,i}e(k) + F_{dr,i}d(k) + F_{fr,i}f(k) + D_{r,i}\Delta_u(k))
\end{aligned}
$$

where $\Delta_u(k) = u_{sub}(k) - u(k)$ represents the influence of missed control input packets, $B_{r,i} = B - L_i\Lambda_{y,i}D$, $D_{r,i} = \Lambda_{y,i}D$.

Recently, the influence of packet errors on the FD system has been studied by [107]. That means, the FD system receives a sensor output packet, whose data field is composed of a number of bits. If the channel is rather noisy, the value of the output obtained by decoding the received codeword could be different from the real value of the output. By applying decoding algorithms, a part of bit errors can be detected and corrected. The focus is thus to handle the part of bit errors that can not be detected or can not be corrected by the decoding algorithms. Therefore, the residual dynamics can be modeled as hybrid systems with the state of the Markov chain or the switching sequence unknown or only partly known.

In the recent years, increasingly the packet loss and the delays are formulated unifiedly by one Markov chain [135, 73].

18.5 Residual evaluation in NCS

In the last sections, it is shown that the FD problem of the NCS can be transformed into that of LPV systems, uncertain systems with additive or multiplicative unknown inputs, Markov systems, switched systems, etc, depending on the nature and the available information of the network-induced factors (see Fig. 18.1).

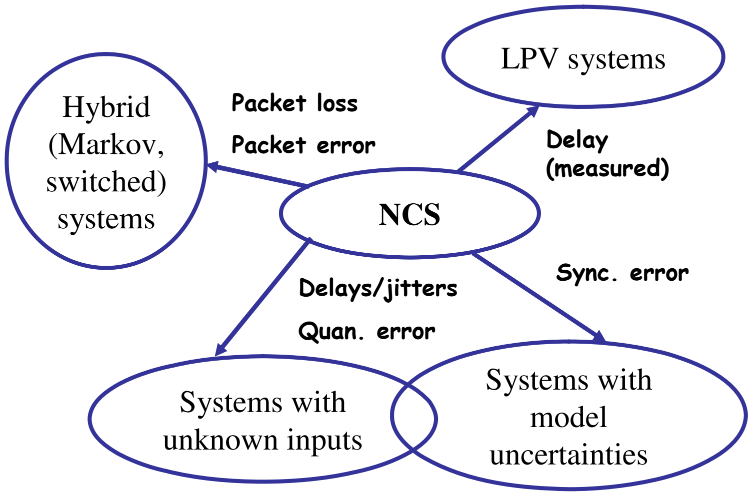

Fig. 18.1 Modelling of NCS with a given network

In this section, we shall consider how to evaluate the residual signal in the NCS. In principle, the residual can be evaluated as discussed in Chapter 3. For instance, we can use adaptive threshold to take into account the influence of the control inputs on the residual. Besides of these basic principles, we would like to call attention that the residual evaluation should be handled carefully if the NCS is formulated as stochastic systems. For instance, the $\mathbf{H}_\infty$-norm of a Markov system is defined by

$$\|G\|_\infty = \frac{\mathbf{E}\left(\|r\|_2\right)}{\|d\|_2} \Leftrightarrow \mathbf{E}\left(\|r\|_2\right) \leq \|G\|_\infty \|d\|_2$$

Denote with β_d the upper bound of $\|d\|_2$. Different from deterministic systems, setting the threshold as $J_{th} = \|G\|_\infty \beta_d$ could result in a high false alarm rate, because $\|G\|_\infty \beta_d$ is the upper bound of $\mathbf{E}\left(\|r\|_2\right)$ but may be much lower than that of $\|r\|_2$.

In some cases, it is possible to figure out the probability distribution of the residual signal in the fault-free case, which can be used to calculate the false alarm rate for a given threshold or to set a risk-dependent threshold [155].

In more complex situations, the recently proposed residual evaluation scheme in [38] can be applied to estimate the bounds of both the expectation $\mathbf{E}\left(\|r\|_{2,\eta}\right)$ and the variance $\boldsymbol{\sigma}\left(\|r\|_{2,\eta}\right)$ of

$$\|r\|_{2,\eta} = \sqrt{\sum_{j=0}^{\eta} r^T(k-j)r(k-j)}$$

where η denotes the length of time window used for the residual evaluation. Based on it, the threshold can be set using the "n-$\boldsymbol{\sigma}$" principle.

To make full use of the stochastic information, advanced residual generation and evaluation schemes are worthy of further investigation [39, 189].

18.6 Concluding remarks

In this chapter, the residual generation and evaluation approaches in the NCS are introduced. As mentioned in Section 17.7, different QoS parameters are closely correlated. Therefore, the questions such as "tolerate longer delay or drop out more packets" are of practical interest. However, as it is difficult to describe the interconnections analytically, it is not an easy task to consider this aspect in the design. A combined modeling of the delay and the packet loss by a common Markov chain adopted by [73, 135] is one step towards this direction. In open NCS with frequent QoS parameter changes, adaptive FD schemes or fuzzy T-S model based design can be developed.

19

Integrated design of communication and FD strategy

From the last chapter we can see that it is sometimes difficult to handle stochastic variations in the QoS parameters, which becomes a serious problem if the network is overloaded [22]. To reduce such variations, the ideal way is to improve the network capacity. Alternatively, reducing the flow rate over the network would also decrease the uncertainty and simplify the handling.

At the network level, the traffic over the network can be reduced and smoothed in different ways, for instance, by [65]:

- applying the classical mechanisms to optimize a network, for instance, to control the priority levels of the messages, to control the scheduling policies in the communication buffers, to change the parameters of protocols or to smooth the traffic,
- finding a good distribution of network components on the network, e.g. connect the system components with intensive information exchange in a sub-network, which can be carried out based on the graph theory, splitting algorithms, spectral algorithms or genetic algorithms.

At the application level, the flow rate over the network mainly depends on quantization density, sampling rate and the number of signals to be transmitted, as the amount of useful data exchanged over the network can be schematically described by

$$N = \sum_{i=1}^{m} f_i q_i$$

where m is the number of signals to be transmitted over the network, f_i the sampling rate of the i-th signal, and q_i the number of bits used to represent the i-th signal, $i = 1, 2, \cdots, m$. A reduction of the flow rate can thus be achieved, as shown in Fig. 19.1, by

- a coarser quantization,
- a lower sampling rate,
- partial access of the sensors and the actuators to the network at each time instant,
- reduction of information exchange between the subsystems, or
- transmission of multiple data in one packet to reduce header overhead.

The distribution of data flows over the network plays an important role as well. As mentioned before, the most part of the delay, especially the large variance of delays in closed NCS with contention based MAC protocol is caused by the waiting time resulting from the collision. For instance, in Ethernet based NCS, the collision happens when two nodes try to access the network at the same time. After a collision is detected, both sending nodes will wait and try to send the messages again after a back-off time. Each node selects the back-off time independently and randomly among a given set of numbers. Therefore, the distribution of network flows should also be regarded as a design parameter and taken into account in network dimensioning and FD system design.

Aiming at achieving an integrated design of communication and FD strategy, in this chapter we shall discuss several possibilities of reducing network load, analyze the influence on the FD performance and give the corresponding FD scheme.

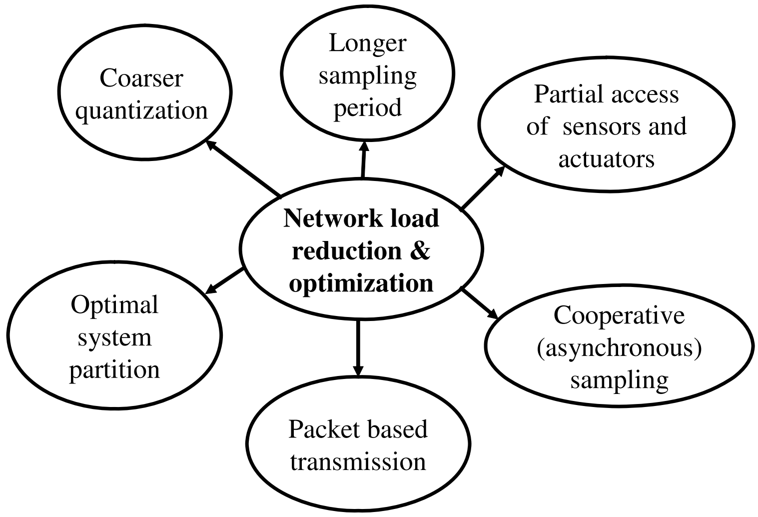

Fig. 19.1 Network flow reduction and optimization at the application level

19.1 Selection of sampling mechanism

Sampling mechanism, including sampling period and timing of sampling instants, is an important parameter in the NCS. It has direct influence on the rate and distribution of the data flows, the FD performance as well as the real-time computational efforts.

19.1.1 Sampling period

Based on the sampled-data system description (17.1)-(17.4) with clock-driven and synchronized sensors and actuators, it has been shown in Chapter 16 that increasing sampling period will reduce the flow rate but also decrease the FD performance $J_{SSD,PS,\infty/\infty,opt}$, the optimal FD performance achievable in the parity space approach, and $J_{SSD,OBS,\infty/\infty,opt}$, the optimal H_∞/H_∞ performance achievable with the observer-based design. The sampling period should thus be selected to get a suitable compromise between the flow rate and the FD performance.

19.1.2 Timing of sampling instants

For closed and synchronized NCS of modest scale, a cooperative asynchronous sampling and transmission scheme is suggested in [173]. The basic idea is to regulate the distribution of the communication over the network so that collisions can be considerably reduced. Depending on the available computing power, a time-varying or time-invariant FD system can be designed.

To illustrate the basic idea, we consider an NCS with a number of m distributed sensors that sample the corresponding outputs with a uniform sampling period h. The conventional sampling mechanism is to sample all m outputs simultaneously at $t = kh$, $k = 0, 1, 2, \cdots$. If these sampled values are sent through the network to the central unit, i.e. following the *simultaneous sampling and simultaneous sending* scheme, then collisions may happen and cause unnecessary delays. In order to avoid bursty traffic and collisions, each sensor will access the network one after the other. One way to achieve this is to sample the outputs still simultaneously at $t = kh$, but, after the i-th $(i = 1, \cdots, m)$ output is sampled, the sampled value will be held up in a local buffer and sent after a time gap of δ_i, $0 < \delta_1 < \delta_2 < \cdots < \delta_m \leq h$. We call this scheme as *simultaneous sampling and sequential sending.* In this case, the standard parity space approach or observer-based approach can be directly applied to generate the residual signal at time kh.

Alternatively, we could also sample the i-th plant output at time instants $t_{\tilde{k}} = kh + \delta_i,\ k = 0, 1, 2, \cdots,\ \tilde{k} = km + i,\ i = 1, \cdots, m$, and sent over the network to the central unit immediately. This scheme is named as *asynchronous sampling and sequential sending*. The time interval between two successive sending instants $\delta_{i+1} - \delta_i$ can be equidistant or selected according to the transmitted packet length and the processing speed of the central unit. Let C_i denote the i-th row of the matrix C. Depending on the computing power of the central unit, the central FD system may be designed in two different ways.

In the case of *sufficient computing power*, a periodic residual generator as follows can be implemented to calculate the residual signal immediately after the arrival of each sensor value

$$\begin{aligned} \hat{x}(\tilde{k}+1) &= A_{\tilde{k}}\hat{x}(\tilde{k}) + B_{\tilde{k}}u(\tilde{k}) + L_{\tilde{k}}\left(y_i(\tilde{k}) - \hat{y}_i(\tilde{k})\right) \\ \hat{y}_i(\tilde{k}) &= C_i\hat{x}(\tilde{k}) \\ r(\tilde{k}) &= W_{\tilde{k}}\left(y_i(\tilde{k}) - \hat{y}_i(\tilde{k})\right) \end{aligned} \tag{19.1}$$

where $L_{\tilde{k}}$ and $W_{\tilde{k}}$ are periodic matrices,

$$A_{\tilde{k}} = e^{A_c(t_{\tilde{k}+1} - t_{\tilde{k}})},\ B_{\tilde{k}} = \int_0^{t_{\tilde{k}+1} - t_{\tilde{k}}} e^{A_c\tau}B_c d\tau$$

In the case of *limited computing power*, the residual signal can be calculated with an interval of h. In this case, the system dynamics can be described by a lifted LTI model [179] and, as a result, the residual generator can be designed as

$$\begin{aligned} \hat{x}(k+1) &= \bar{A}\hat{x}(k) + \bar{B}u(k) + L\left(\bar{y}(k) - \tilde{y}(k)\right) \\ \tilde{y}(k) &= \bar{C}\hat{x}(k) + \bar{D}u(k) \\ r(k) &= W\left(\bar{y}(k) - \tilde{y}(k)\right) \end{aligned} \tag{19.2}$$

where

$$\hat{x}(k) = x(kh), u(k) = u(kh),\ \bar{y}(k) = \begin{bmatrix} y_1(kh+\delta_1) \\ \vdots \\ y_m(kh+\delta_m) \end{bmatrix}$$

$$\bar{A} = e^{A_c h},\ \bar{B} = \int_0^h e^{A_c\tau}B_c d\tau$$

$$\bar{C} = \begin{bmatrix} C_1 e^{A_c\delta_1} \\ \vdots \\ C_m e^{A_c\delta_m} \end{bmatrix},\ \bar{D} = \begin{bmatrix} C_1\int_0^{\delta_1} e^{A_c\tau}B_c d\tau \\ \vdots \\ C_m\int_0^{\delta_m} e^{A_c\tau}B_c d\tau \end{bmatrix}$$

Though the above scheme is originally proposed for the implementation at the application level [173], it can be used in combination with the TDMA protocol at the network level. We call such a scheme as *cooperative and adaptive sampling*.

19.2 Partial information transmission based on communication sequences

In order to reduce the network load and thus avoid the uncertainty caused by transmission delays and packet losses, a so-called periodic communication sequence (PCS) introduced by [78, 166] can be employed for the allocation of network resource. Instead of transmitting each control input and measured output signal at each sampling time, only a part of sensors and actuators have access to the network.

19.2.1 Description of communication sequence

Assume that at any time only maximal ω_y sensors and ω_u actuators will be allowed to access the network, $1 \leq \omega_y \leq m$, $1 \leq \omega_u \leq n_u$. To describe such a scheme, time varying matrices $M(k)$ and $N(k)$ are introduced. The output vector used for FD is

$$y(k) = N(k)y_p(k)$$

where $N(k) \in \mathbf{R}^{\omega_y \times m}$ is a time-varying matrix formed by selecting ω_y rows of the identity matrix. The control input vector u_p working on the process depends on the interpolation scheme at the actuators' side (see the discussion in Subsection 17.3), which is represented by a time-varying diagonal matrix $M(k) \in \mathbf{R}^{n_u \times n_u}$ with ω_u ones and $n_u - \omega_u$ zeros on the diagonal. Often periodic transmissions are preferred from the viewpoint of simplified design and implementation efforts, which lead to periodically time-varying matrices $M(k), N(k)$.

By substituting the relation between y and y_p into (17.5), it is seen that the NCS with periodic partial transmission can be described as a periodic system [169]

$$\begin{aligned} x(k+1) &= Ax(k) + Bu_p(k) + E_d d(k) + E_f f(k) \\ y(k) &= N(k)(Cx(k) + Du_p(k) + F_d d(k) + F_f f(k)) \end{aligned}$$

19.2.2 Design of FD system

It is reasonable to assume that the transmission schemes $M(k), N(k)$ are known to the FD system. Thus, u_p can be calculated from $M(k), u(k)$. In the framework of the parity space approach, a periodic parity relation based residual generator can be built as

$$r(k) = V_s(k) \left(\begin{bmatrix} y(k-s) \\ \vdots \\ y(k) \end{bmatrix} - H_{u,s}(k) \begin{bmatrix} u_p(k-s) \\ \vdots \\ u_p(k) \end{bmatrix} \right)$$

where $V_s(k)$ is the periodic parity matrix satisfying $V_s(k)H_{o,s}(k) = 0$, $H_{o,s}(k)$ and $H_{u,s}(k)$ are periodic matrices depending on $N(k)$ as

$$H_{o,s}(k) = \begin{bmatrix} N(k-s)C \\ N(k-s+1)CA \\ \vdots \\ N(k)CA^s \end{bmatrix}$$

$$H_{u,s}(k) = \begin{bmatrix} N(k-s)D & O & \cdots & O \\ N(k-s+1)CBM(k-s) & N(k-s+1)D & \ddots & \vdots \\ \vdots & & \ddots & O \\ N(k)CA^{s-1}BM(k-s) & \cdots & & N(k)D \end{bmatrix}$$

Based on the residual dynamics

$$r(k) = V_s(k)\left(H_{d,s}(k)\begin{bmatrix} d(k-s) \\ \vdots \\ d(k) \end{bmatrix} + H_{f,s}(k)\begin{bmatrix} f(k-s) \\ \vdots \\ f(k) \end{bmatrix}\right)$$

$$H_{d,k} = \begin{bmatrix} N(k-s)F_d & O & \cdots & O \\ N(k-s+1)CE_d & N(k-s+1)F_d & \ddots & \vdots \\ \vdots & & \ddots & O \\ N(k)CA^{s-1}E_d & \cdots & & N(k)F_d \end{bmatrix}$$

$$H_{f,k} = \begin{bmatrix} N(k-s)F_f & O & \cdots & O \\ N(k-s+1)CE_f & N(k-s+1)F_f & \ddots & \vdots \\ \vdots & & \ddots & O \\ N(k)CA^{s-1}E_f & \cdots & & N(k)F_f \end{bmatrix}$$

The FD system can be designed by applying the FD approaches for periodic systems introduced in Chapter 10. Therefore, in the following we shall concentrate on analyzing the influence of partial transmission on the FD performance.

19.2.3 Influence on full decoupling

For this purpose, notice that

$$H_{o,s}(k) = \bar{N}(k)H_{o,s},\ H_{d,s}(k) = \bar{N}(k)H_{d,s},\ H_{f,s}(k) = \bar{N}(k)H_{f,s} \tag{19.3}$$

where $H_{o,s}, H_{d,s}, H_{f,s}$ are the coefficient matrices in the parity relation of the original system (17.5) (see (2.4)), $\bar{N}(k)$ is a periodic matrix of full row rank and is uniquely decided by the communication sequence,

$$\bar{N}(k) = \begin{bmatrix} N(k-s) & & O \\ & \ddots & \\ O & & N(k) \end{bmatrix}$$

The subsequent discussion will be carried out in the framework of the parity space approach.

At first suppose that a full decoupling of the residual signal r from the unknown disturbances d is achievable in the original system (17.5). Denote the basis matrix of the left null space of $\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix}$ by N_{decoup}, i.e.

$$N_{decoup}\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix} = 0$$

If the NCS under the given periodic communication sequence is still decouplable, then there should be a nonzero matrix $N_{decoup}(k)$ such that

$$N_{decoup}(k)\begin{bmatrix} H_{o,s}(k) & H_{d,s}(k) \end{bmatrix} = 0,\ \forall k \tag{19.4}$$

Substituting (19.3) into (19.4) yields

$$N_{decoup}(k)\bar{N}(k)\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix} = 0 \tag{19.5}$$

It shows that $N_{decoup}(k)\bar{N}(k)$ must also lie in the left null space of $\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix}$. Since

$$\dim N_{decoup} = (s+1)m - rank\left(\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix}\right)$$

and

$$\begin{aligned} &\dim N_{decoup}(k) \\ &= (s+1)\omega_m - rank\left(\bar{N}(k)\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix}\right) \\ &\le (s+1)\omega_m - \left((s+1)\omega_m + rank\left(\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix}\right) - m(s+1)\right) \\ &= (s+1)m - rank\left(\begin{bmatrix} H_{o,s} & H_{d,s} \end{bmatrix}\right) \end{aligned}$$

it can be seen that

$$\dim N_{decoup}(k) \le \dim N_{decoup}$$

Due to PCS, the dimension of the decoupling space will possibly decrease. Even if the original system is decouplable, the realizability of full decoupling may be lost due to reduced information in an NCS with periodic communication sequence.

19.2.4 Influence on optimal FD performance

In the next, we shall consider the optimal FD performance. Aiming at improving the robustness and sensitivity of the FD system, in the case of perfect communication the following optimization problem

$$\max_{v_s} J_{PS} = \max_{v_s} \frac{v_s H_{f,s} H_{f,s}^T v_s^T}{v_s H_{d,s} H_{d,s}^T v_s^T} \quad \text{subject to} \quad v_s H_{o,s} = 0 \tag{19.6}$$

is solved to get the optimal parity vector. Now we compare it with the optimization problem (19.7) of the NCS

$$\max_{v_s(k)} J_{PS,k} = \max_{v_s(k)} \frac{v_s(k) H_{f,s}(k) H_{f,s}^T(k) v_s^T(k)}{v_s(k) H_{d,s}(k) H_{d,s}^T(k) v_s^T(k)} \quad \text{subject to} \quad v_s(k) H_{o,s}(k) = 0 \tag{19.7}$$

Denote the basis matrix of the left null space of $H_{o,s}$ and $H_{o,s}(k)$, respectively, by N_{parity} and $N_{parity}(k)$. Optimization problems (19.7) and (19.6) are, respectively, equivalent to the unconstrained optimization problems

$$\max_{\substack{v_s(k) \\ v_s(k) H_{o,s}(k)=0}} J_{PS,k} = \max_{p_s(k)} \frac{p_s(k) N_{parity}(k) H_{f,s}(k) H_{f,s}^T(k) N_{parity}^T(k) p_s^T(k)}{p_s(k) N_{parity}(k) H_{d,s}(k) H_{d,s}^T(k) N_{parity}^T(k) p_s^T(k)} \tag{19.8}$$

$$\max_{\substack{v_s \\ v_s H_{o,s}=0}} J_{PS} = \max_{p_s} \frac{p_s N_{parity} H_f H_f^T N_{parity}^T p_s^T}{p_s N_{parity} H_d H_d^T N_{parity}^T p_s^T} \tag{19.9}$$

where p_s and $p_s(k)$ are freely selectable vectors of compatible dimensions. From

$$N_{parity}(k) H_{o,s}(k) = N_{parity}(k) \bar{N}(k) H_{o,s} = 0 \tag{19.10}$$

it can be seen that $N_{parity}(k)\bar{N}(k)$ is a linear combination of the basis matrix N_{parity} and there exists a matrix Q, such that

$$N_{parity}(k)\bar{N}(k) = Q N_{parity} \tag{19.11}$$

Because $\bar{N}(k), N_{parity}(k)$ and N_{parity} are of full row rank, it can be seen

$$rank(N_{parity}(k)\bar{N}(k)) = rank(N_{parity}(k)) = rank(Q N_{parity}) = rank(Q) \tag{19.12}$$

i.e., matrix Q is also of full row rank and it transforms the parity space spanned by N_{parity} onto a lower dimensional subspace $N_{parity}(k)\bar{N}(k)$. It follows from (19.3) and (19.11) that

$$\begin{aligned} J_{PS,k} &= \frac{p_s(k) N_{parity}(k) \bar{N}(k) H_{f,s} H_{f,s}^T \bar{N}^T(k) N_{parity}^T(k) p_s^T(k)}{p_s(k) N_{parity}(k) \bar{N}(k) H_{d,s} H_{d,s}^T \bar{N}^T(k) N_{parity}^T(k) p_s^T(k)} \\ &= \frac{p_s(k) Q N_{parity} H_{f,s} H_{f,s}^T N_{parity}^T Q^T p_s^T(k)}{p_s(k) Q N_{parity} H_{d,s} H_{d,s}^T N_{parity}^T Q^T p_s^T(k)} \end{aligned}$$

As a result, optimization problem (19.8) is equivalent to

$$\max_{\substack{v_s(k)\\ v_s(k)H_{o,s}(k)=0}} J_{PS,k} = \max_{\bar{p}_s(k)} \frac{\bar{p}_s(k)N_{parity}H_f H_f^T N_{parity}^T \bar{p}_s^T(k)}{\bar{p}_s(k)N_{parity}H_d H_d^T N_{parity}^T \bar{p}_s^T(k)}$$

where $\bar{p}_s(k) = p_s(k)Q$. That means the space of feasible solutions $\bar{p}_s(k)$ is dependent on Q. Compared with optimization problem (19.9), where p_s can be freely selected, we see that

$$J_{PS,k,opt} \le J_{PS,opt}$$

The optimal performance index $J_{PS,k,opt}$ of optimization problem (19.8) could reach $J_{PS,opt}$ if and only $v_{s,opt}$ lies in the space spanned by $N_{parity}(k)\bar{N}(k)$. From the above analysis we see that, compared with perfect communication, the optimal performance achievable in the NCS with a given PCS may decrease [169]. This is the price paid for the reduced network load.

19.2.5 Selection of communication sequence

If the communication capacity of the network is given, the optimal selection of the PCS aiming at the best achievable FD performance can be formulated as: Given ω_y and ω_u, find the value of period θ and a periodic matrix $N(k) = N(k+\theta)$ with a structure specified in Section 19.2.1 to maximize the value of $J_{PS,k,opt}$, i.e.,

$$\min_{\theta,\ N(k)} \left\{ \max_{1\le k\le\theta} J_{PS,k,opt} \right\} \tag{19.13}$$

Optimization problem (19.13) can be solved through an exhaustive search by evaluating $J_{PS,k,opt}$ for all possible $N(k)$, if θ, m and n_u are small. A generalized eigenvalue-eigenvector problem needs to be solved to get the optimal performance index $J_{PS,k,opt}$ of (19.8) for each possibility of $N(k)$.

19.3 Transmission of multiple data in one packet

It is well-known that a standard Ethernet packet contains a data field with length varying between 46 Bytes and 1500 Byte, which is much longer than a single sensor data obtained by a 8 bit or 12 bit A/D converters. Therefore, it is possible to reduce the bandwidth used for header transmission by encapsulating multiple sensor data in one packet.

To analyze the influence of such a scheme on FD system design, we assume that the control input packets and the sensor output packets arrive the FD system without delay or packet loss. Let h denote the sampling period and $\rho \in \mathbf{N}$ denote the number of data encapsulated in one packet. Note that the dynamics of SD system (17.1)-(17.4) during $[k\rho h, (k+1)\rho h)$ can be described by

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k) + \int_{k\rho h}^{(k+1)\rho h} e^{A_c\tau}\left(E_{dc}d(\tau) + E_{fc}f(\tau)\right)d\tau \\ \bar{y}(k) &= \bar{C}\bar{x}(k) + \bar{D}\bar{u}(k) \end{aligned} \tag{19.14}$$

where $\bar{x}(k) = x(k\rho h)$, $\bar{u}(k)$ and $\bar{y}(k)$ are the lifting of $u(k)$ and $y(k)$ defined by

$$\bar{u}(k) = \begin{bmatrix} u(k\rho) \\ u(k\rho+1) \\ \vdots \\ u(k\rho+\rho-1) \end{bmatrix}, \bar{y}(k) = \begin{bmatrix} y(k\rho) \\ y(k\rho+1) \\ \vdots \\ y(k\rho+\rho-1) \end{bmatrix}$$
$$\bar{A} = A^{\rho}, \ \bar{B} = \begin{bmatrix} A^{\rho-1}B \ A^{\rho-2}B \cdots \ B \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\rho-1} \end{bmatrix}, \ \bar{D} = \begin{bmatrix} O & O & \cdots & O \\ CB & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ CA^{\rho-2}B & \cdots & CB & O \end{bmatrix}$$
$$A = e^{A_c h}, B = \int_0^h e^{A_c t} B_c dt$$

If at time instant $(k+1)\rho h$ the FD system receives the packets containing the information of the output samples $y((k\rho+j)h)$ and input samples $u((k\rho+j)h)$, $j = 0, \cdots, \rho-1$, then a residual signal r can be generated by the following residual generator

$$\begin{aligned} \tilde{x}(k+1) &= \bar{A}\tilde{x}(k) + \bar{B}\bar{u}(k) + \bar{L}(\bar{y}(k) - \tilde{y}(k)) \\ \tilde{y}(k) &= \bar{C}\tilde{x}(k) + \bar{D}\bar{u}(k) \\ r(k) &= \bar{W}(\bar{y}(k) - \tilde{y}(k)) \end{aligned} \tag{19.15}$$

where $\bar{L}$ and $\bar{W}$ are constant free parameters. Recall that it is shown in Chapter 16 that the residual signal r generated by (19.15) is a lifting of the residual generated by the observer-based residual generator with sampling period h as follows

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) \\ r(k) &= W(y(k) - \hat{y}(k)) \end{aligned} \tag{19.16}$$

if and only if $\tilde{x}(0) = \hat{x}(0)$ and

$$\bar{L} = \begin{bmatrix} (A-LC)^{\rho-1}L \cdots \ (A-LC)L \ L \end{bmatrix}$$
$$\bar{W} = \begin{bmatrix} W & O & \cdots & O \\ -WCL & W & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ -WC(A-LC)^{\rho-2}L & \cdots & -WCL & W \end{bmatrix} \tag{19.17}$$

We would like to point out that the weighting matrix $\bar{W}$ plays an important role in avoiding the loss of FD performance. The additional detection delay caused by packet-based data transmission is ρh in the worst case. But (19.15) can achieve the FD performance that is achieved by (19.16).

19.4 Optimal partition of subsystems

In distributed systems, as described by (17.6), the partition of subsystems and the information exchange strategy among subsystems will influence the structural property of the NCS and the FD performance. To analyze fault detectability and isolability, an approach based on the graph theory is proposed by [136]. It is apparent that loose coupling between subsystems will reduce the requirement on communication. On the other side, [58] have shown that, based on consensus filters, suitable overlapping partition of the subsystems can be used to improve the FD of faults affecting the shared variables. An optimal scheme for partition of subsystems is worthy of further discussion.

19.5 Local encoder and transmission of local residual signals

For a distributed system described by (17.6), if the information of $u_{p,i}(k)$ and $y_{p,i}(k)$ of the local systems are available to the central control and monitoring system (CCMS), then an observer-based residual generator can be constructed as

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu_p(k) + L(y_p(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) + Du_p(k) \\ r(k) &= W(y_p(k) - \hat{y}(k))\end{aligned} \tag{19.18}$$

Since the CCMS can send the state estimation vector $\hat{x}(k)$ to all subsystems. Then a local encoder can be built as

$$r_{S_i}(k) = y_{p,i}(k) - C_i\hat{x}_i(k) - D_iu_{p,i}(k) \tag{19.19}$$

where $r_{S_i}(k), i = 1, \cdots, p$, is the local residual signal. That means, instead of the measurements of the local outputs $y_{p,i}(k)$, the local residual signal $r_{S_i}(k)$ will be sent from the subsystems to the CCMS. Then the central observer (19.18) can be implemented as

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu_p(k) + L\begin{bmatrix} r_{S_1}(k) \\ r_{S_2}(k) \\ \vdots \\ r_{S_p}(k)\end{bmatrix} \\ r(k) &= W\begin{bmatrix} r_{S_1}(k) \\ r_{S_2}(k) \\ \vdots \\ r_{S_p}(k)\end{bmatrix}\end{aligned} \tag{19.20}$$

As it is often the case that

$$\sup \|r_{S_i}(k)\|_{peak} \ll \sup \|y_{p,i}(k)\|_{peak} \tag{19.21}$$

the number of the bits needed for transmitting $r_{S_i}(k)$ should be less than that needed for transmitting $y_{p,i}(k)$ at the same quantization error.

Note that the observer (19.20) still needs the values of the local control inputs $u_{p,i}(k), i = 1, \cdots, p$, through the network from the subsystems. Fortunately, if an observer-based state feedback controller in the form of

$$u_{p,i}(k) = -K_i\hat{x}(k) + W_iw(k) \tag{19.22}$$

is used as the local controller, then this part of communication can be avoided by computing (19.22) simultaneously at the CCMS, as shown in Fig. 19.2.

The above scheme is characterized by a low flow rate over the network and the main computation of (19.20) is carried out by the state observer of the CCMS. On the other side, we notice that, when the global network fails, the local FD will stop running, as the information of the state estimation is no longer available. Therefore, to be tolerant to the faults of the global network, the local FD unit must be able to perform its work autonomously to a certain degree, which motivates the study in the next section.

19.6 Distributed realization of observers

To improve the tolerance to the network faults, in each subsystem a residual generator in the form of

$$\begin{aligned}\hat{x}_{S_i}(k+1) &= A_{ii}\hat{x}_{S_i}(k) + B_iu_{p,i}(k) + L_{S_i}(y_{p,i}(k) - \hat{y}_{S_i}(k)) \\ \hat{y}_{S_i}(k) &= C_i\hat{x}_{S_i}(k) + D_iu_{p,i}(k) \\ r_{S_i}(k) &= y_{p,i}(k) - \hat{y}_{S_i}(k)\end{aligned} \tag{19.23}$$

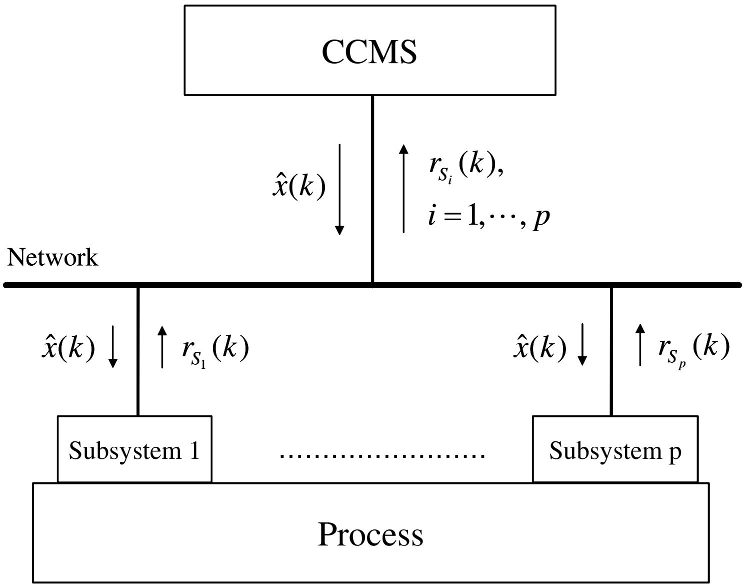

Fig. 19.2 Information exchange between the CCMS and the subsystems in the local encoder scheme

will be built, where L_{S_i} is the gain matrix of the local observer based residual generator. The local residual generator is only based on the local information $y_{p,i}(k), u_{p,i}(k)$ and doesn't need any information from the CCMS or the other subsystems. Even if the global network fails, the local residual generator should still guarantee the detection of local faults. The dynamics of the i-th local residual signal is governed by

$$\begin{aligned} e_{S_i}(k+1) &= (A_{ii} - L_{S_i}C_i)e_{S_i}(k) + \sum_{j=1, j\neq i}^{p} A_{ij}x_j(k) \\ &\quad + (E_{d,i} - L_{S_i}F_{d,i})d(k) + (E_{f,i} - L_{S_i}F_{f,i})f(k) \\ r_{S_i}(k) &= C_i e_{S_i}(k) + F_{d,i}d(k) + F_{f,i}f(k) \end{aligned} \tag{19.24}$$

where $e_{S_i}(k) = x_i(k) - \hat{x}_{S_i}(k)$. The detection of the local faults follows from a residual evaluation with the residual evaluation function

$$r_{S_i,ev}(k) = \sqrt{r_{S_i}^T(k) r_{S_i}(k)} \tag{19.25}$$

and the decision logic

$$r_{S_i,ev}(k) \begin{cases} \leq J_{th,S_i} \Rightarrow \text{ The } i\text{-th subsystem is fault-free} \\ > J_{th,S_i} \Rightarrow \text{ The } i\text{-th subsystem is faulty} \end{cases} \tag{19.26}$$

where J_{th,S_i} is the local threshold defined by

$$J_{th,S_i} = \sup_{d, x_j, j\neq i, f=0} r_{S_i,ev}(k) \tag{19.27}$$

In order to reduce the miss detection rate, J_{th,S_i} can be minimized by selecting the local observer gain matrix L_{S_i} with the help of the LMI technique.

Now we shall discuss in this case how to design the observer-based FD system in the CCMS. The basic idea is still to try to make use of the local residual signals $r_{S_i}(k)$, $i = 1, \cdots, p$, generated by (19.23), instead of transmitting $y_{p,i}(k)$ directly, to reduce the amount of data transmission by the same quantization error, as in general (19.21) holds.

The following theorem provides such a possibility [37].

Theorem 19.1 Given the system model (17.6). Let a central observer based residual generator be constructed as

$$\hat{x}_C(k+1) = A\hat{x}_C(k) + Bu_p(k) + L\begin{bmatrix} r_1(k) \\ r_2(k) \\ \vdots \\ r_p(k) \end{bmatrix} \tag{19.28}$$

$$\hat{x}_C(k) = \begin{bmatrix} \hat{x}_{C,1}(k) \\ \hat{x}_{C,2}(k) \\ \vdots \\ \hat{x}_{C,p}(k) \end{bmatrix},\ u_p(k) = \begin{bmatrix} u_{p,1}(k) \\ u_{p,2}(k) \\ \vdots \\ u_{p,p}(k) \end{bmatrix},\ L = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_p \end{bmatrix} \tag{19.29}$$

$$\lambda_i(k+1) = A_{ii}\lambda_i(k) - \sum_{\substack{j=1 \\ j\neq i}}^{p} A_{ij}\hat{x}_{C,j}(k) + L_{S_i}r_{S_i}(k) - L_i\begin{bmatrix} r_1(k) \\ r_2(k) \\ \vdots \\ r_p(k) \end{bmatrix} \tag{19.30}$$

$$\begin{bmatrix} r_1(k) \\ r_2(k) \\ \vdots \\ r_p(k) \end{bmatrix} = \begin{bmatrix} r_{S_1}(k) \\ r_{S_2}(k) \\ \vdots \\ r_{S_p}(k) \end{bmatrix} + \begin{bmatrix} C_1\lambda_1(k) \\ C_2\lambda_2(k) \\ \vdots \\ C_p\lambda_p(k) \end{bmatrix} \tag{19.31}$$

$$r_C(k) = W\begin{bmatrix} r_1(k) \\ r_2(k) \\ \vdots \\ r_p(k) \end{bmatrix},\ \hat{x}_C(0) = \begin{bmatrix} \hat{x}_{S_1}(0) \\ \hat{x}_{S_2}(0) \\ \vdots \\ \hat{x}_{S_m}(0) \end{bmatrix},\ \lambda_i(0) = 0 \tag{19.32}$$

where $r_C(k)$ is the residual signal, L is the gain matrix of the central observer, $r_{S_i}(k)$, $i = 1, \cdots, p$, is the local residual signal sent by the i-th local subsystem to the CCMS, $\lambda_i(k)$, $i = 1, \cdots, p$, is some correction term. Then $\hat{x}_C(k)$ and $r_C(k)$ are the same as $\hat{x}(k)$ and $r(k)$ generated by (19.18), as long as $\hat{x}_C(0) = \hat{x}(0)$.

Proof: Let $\xi_i(k) = \hat{x}_{C,i}(k) - (\hat{x}_{S_i}(k) - \lambda_i(k))$. Then from (19.28), (19.30) and (19.23) we get

$$\xi_i(k+1) = A_{ii}\xi_i(k)$$

As $\xi_i(0) = \hat{x}_{C,i}(0) - (\hat{x}_{S_i}(0) - \lambda_i(0)) = 0$, there is $\xi_i(k) = 0$, $\forall k$, i.e. $\hat{x}_{C,i}(k) = \hat{x}_{S_i}(k) - \lambda_i(k)$, and the feedback signal $r_i(k)$, $i = 1, \cdots, p$, in the central observer state equation can be re-written as

$$\begin{aligned} r_i(k) &= y_{p,i}(k) - C_i\hat{x}_{S_i}(k) - D_iu_{p,i}(k) + C_i\lambda_i(k) \\ &= y_{p,i}(k) - C_i\hat{x}_{C,i}(k) - D_iu_{p,i}(k) \end{aligned} \tag{19.33}$$

Substituting (19.33) into (19.28) and (19.32) gives

$$\begin{aligned} \hat{x}_C(k+1) &= A\hat{x}_C(k) + Bu_p(k) + L\left(y_p(k) - C\hat{x}_C(k) - Du_p(k)\right) \\ r_C(k) &= W\left(y_p(k) - C\hat{x}_C(k) - Du_p(k)\right) \end{aligned} \tag{19.34}$$

Comparing (19.34) with (19.18) shows that $\hat{x}_C(k) = \hat{x}(k)$ and $r_C(k) = r(k)$, if and only if $\hat{x}_C(0) = \hat{x}(0)$. □

From (19.34) in the proof it can be seen that the parameters L, W in the central residual generator described by (19.28)-(19.31) can be selected with the known standard decoupling or optimization techniques.

The communication and FD scheme introduced in this section can be summarized, as shown in Fig. 19.3 as:

- In each subsystem there is a local FD system consisting of (19.23), (19.25) and (19.26).
- Each local subsystem transmits the local residual signal $r_{S_i}(k)$, $i = 1, \cdots, p$, to the central monitoring system.
- There is a central residual generator constructed as (19.28)-(19.31), which makes use of the information from all subsystems for fault detection.

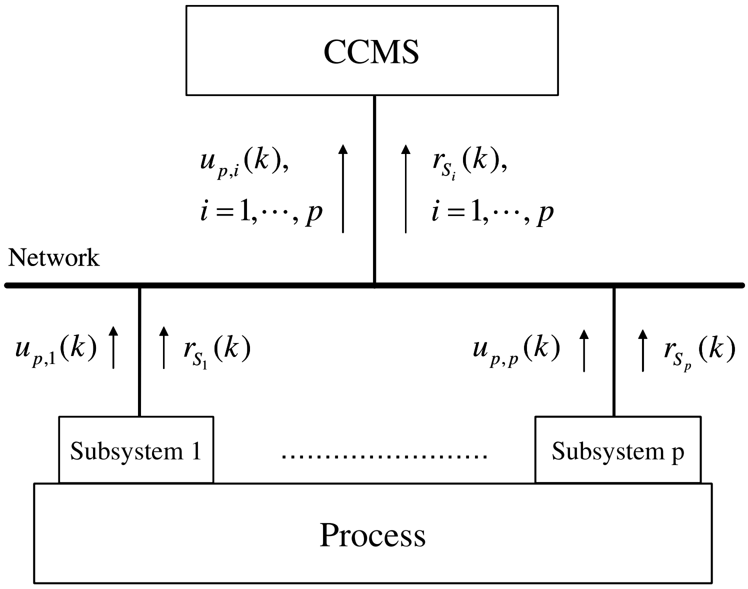

Fig. 19.3 Information exchange between the CCMS and the subsystems in the scheme of distributed realization of observers

19.7 Conclusion

In this chapter, different possibilities of reducing the network load at the application level and increasing the effective utilization of the limited bandwidth have been discussed. The structural change caused by different information exchange strategies has been pointed out. For the analysis and design, time-varying system theory has been applied. It is also possible to combine different approaches. From the system engineering viewpoint, the next step should be to allocate the limited resource to the most needed place by a systemwide planning and optimization.

One of the key problems in NCS design is to cope with the compromise between the utilization of the network and the real-time behaviour. The current discussion on the future-oriented bus systems in the automobile industry indicates the time-triggered technology for transmission of safety-critical data to meet the real-time requirements. With the advanced sensing and actuating technology, periodic and time-varying system theory can help the analysis and design of the controllers and the FD systems. In this framework, it is also possible to integrate the communication and the computing in embedded NCS.

It is interesting to notice the difference between the FD problem and the control problem of the NCS. For the FD system design, residual evaluation needs more attention and corresponding tools can be developed, especially if the QoS parameters under consideration are stochastic. An in-depth study of open problems will surely not only improve the reliability of the NCS but also contribute to the theoretical development of the related areas.

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