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Research Article

Subjective Rationalizability in Hypergames

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A new solution concept for hypergames called subjective rationalizability is proposed. Hypergame theory is a game theoretical framework that deals with agents who may misperceive game structures and explicitly takes into account hierarchy of perceptions, that is, an agent's view about another agent's view and so on. An action of an agent is called subjectively rationalizable when the agent thinks it can be a best response to the other's choices, each of which the agent thinks each agent thinks is a best response to the other's choices, and so on. Then it is proved that subjective rationalizability is equivalent to the standard notion of rationalizability under a condition called inside common knowledge. The result makes the new solution concept a practical tool in hypergame analyses. Theoretically, it is characterized as such a concept that provides the precise implication, that is, predicted outcomes, of a given hypergame structure.

1. Introduction

Hypergame theory deals with misperceptions of agents (decision makers) in games by relaxing common knowledge often assumed in the standard game theory [1, 2]. It is the basic idea of hypergames that each agent is supposed to possess independently a subjective view about a game called her subjective game and make a decision based on it. The idea allows agents to hold different perceptions and thus enables us to capture realistic aspects of many interactive decision making situations given that misperceptions are everyday affairs in our life.

In game theory, Bayesian games are often referred to as the standard model to deal with incomplete information [3]. While a hypergame can technically be reformulated as a Bayesian game under specific conditions, the reformulation requires the agents to be aware of every possibility indeed relevant to the situation [4]. Therefore hypergames are unique in that they can directly deal with unawareness of agents.

Although hypergame theory has been developed in several ways, the framework the present study deals with explicitly takes into account hierarchy of perceptions, that is, an agent's view about another agent's view and so on [5–8]. It can describe not only situations in which an agent may perceive a game differently from the others but also situations in which she may notice that other agents may perceive the game differently, and moreover the other agents may also notice that the other agents may see different games and so on. Such a hierarchy of perceptions is formalized by using the concept of viewpoint. For example, agent i's view about agent j's view about agent k's view about a game is modeled as the subjective game of viewpoint kji. In this paper, we simply refer to this type of hypergame model as a hypergame. Then a hypergame is defined as the collection of subjective games for all the viewpoints.

In order to predict an agent's choice in one-shot hypergame, that is, a hypergame played only once, several solution concepts have been proposed [8]. They are typically based on the following idea. First, an analyzer fixes the level of hierarchy of perceptions and finds out an "equilibrium" (e.g., Nash equilibrium) in the subjective games of the lowest level of the hierarchy. Then it is supposed that best responses are taken sequentially at each level. For example, consider a two-level hypergame played by two agents, i and j. According to the idea, in order to analyze agent i's choice, we first need to know an equilibrium in the subjective game of viewpoint ji. If agent j's some action constitutes an equilibrium there, then, expecting agent j would take the action, agent i chooses a best response to it in viewpoint i's subjective game (see an example in Section 5).

There are at least two problems of the idea. First, it is not clear what assures that an agent takes an action that constitutes an equilibrium [9]. Rather, it has been shown that the precise implication of common knowledge of the game structure and rationality of the agents is rationalizability, a weaker concept than Nash equilibrium [10]. Second, there seems to be no substantial reason that we can fix the finite level of a hypergame. In the example above, agent j's choice in viewpoint ji's subjective game would depend on agent j's expectation about agent i's choice, which clearly depends on how agent j thinks how agent i perceives the game. Therefore it seems that we need to consider viewpoint iji's subjective game and so on.

In this paper, addressing these issues, we explore the possibility of extending the notion of rationalizability to hypergames so as to examine the precise prediction of an agent's choice in a hypergame. We consider an infinite hierarchy and suppose that agents act according to the following principle as an analogy of rationalizability: an agent takes a best response to actions of the others, each of which the agent thinks each agent thinks is a best response to actions of the others, and so on. If an agent's action satisfies the principle under a given hierarchy of perceptions, it is called subjectively rationalizable. Subjective rationalizability is defined not for agents but for viewpoints, so, for instance, agent i can think of agent j's choice as viewpoint ji's subjectively rationalizable action. Then agent i's choice can be predicted as viewpoint i's subjectively rationalizable action; thus we also call this the agent's subjectively rationalizable action. Hence, possible outcomes of a hypergame are given as any combinations of each agent's subjectively rationalizable action. Note that, throughout the paper, we distinguish clearly between "subjective rationalizability" and "rationalizability."

Subjective rationalizability, however, is apparently impractical because it requires us to calculate each agent's choice at each level in an infinite hierarchy of perceptions. We, however, prove that, under a condition called inside common knowledge [6], subjective rationalizability is equivalent to rationalizability and show that an agent's subjectively rationalizable action can be easily derived by applying the result. Here a particular viewpoint is said to have inside common knowledge when the viewpoint considers that the game structure is common knowledge. In general, something is called common knowledge if everyone knows it, everyone knows everyone knows it, and so on.

Following the introduction, we introduce the hypergame framework in Section 2 and subjective rationalizability in Section 3 and examine a relationship between subjective rationalizability and rationalizability in Section 4. Then, we discuss its implication with an example case in Section 5 and finally add a conclusion.

2. Hypergame Model

The theoretical basis of hypergames is normal form game, which is defined as follows.

Definition 1 (normal form games). $G = (I, A, u)$ is a *normal form game*, where

(i) I is the (nonempty) set of *agents*;

(ii) $A = \times_{i\in I} A_i$, where A_i is the (nonempty) set of agent i's *actions*; $a \in A$ is called an *outcome* (we do not deal with mixed extension of games in this paper);

(iii) $u = (u_i)_{i\in I}$, where $u_i : A \to \mathfrak{R}$ is agent i's *utility function*.

In a hypergame, each agent is assumed to have her own subjective view about the game she faces. The framework explicitly takes into account hierarchy of perceptions, that is, an agent's perception about another agent's perception and so on, by introducing the concept of viewpoints (in the literature, the concept of viewpoint may be described in the form of strings of agents [6, 8]; in particular, our definition of relevant viewpoints as well as the concept of their concatenations introduced in this section is mostly based on [6]). For example, viewpoint i means agent i's view, and viewpoint ji is agent j's view perceived by agent i. In general, viewpoint $i_1 i_2 \cdots i_n$ is interpreted as agent i_1's view perceived by agent $i_2 \cdots$ perceived by agent i_n. Each perception of each viewpoint is given as a normal form game and called the viewpoint's subjective game. An example of a hypergame is presented in Section 5.

Definition 2 (hypergames). $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ is a *hypergame*, where I is the finite set of agents involved in the situation and Σ is the set of *viewpoints* relevant to it. For any $\sigma \in \Sigma$, $G^\sigma = (I^\sigma, A^\sigma, u^\sigma)$ is a normal form game called viewpoint σ's *subjective game*, where

(i) I^σ is the (nonempty) set of agents perceived by viewpoint σ;

(ii) $A^\sigma = \times_{i\in I^\sigma} A_i^\sigma$, where A_i^σ is the (nonempty) set of agent i's actions perceived by viewpoint σ;

(iii) $u^\sigma = (u_i^\sigma)_{i\in I^\sigma}$, where $u_i^\sigma : A^\sigma \to \mathfrak{R}$ is agent i's utility function perceived by viewpoint σ.

Then Σ is defined as $\Sigma = I \cup \bigcup_{n=2}^{\infty}\{i_1 \cdots i_n \mid i_n \in I, i_{k-1} \in I^{i_k \cdots i_n} \setminus \{i_k\}$ for any $k(= 2, \dots, n)\}$.

Although the hypergame literature usually does not put any restrictions on the set of viewpoints, we exclude "unnatural" ones by specifying the set of viewpoints *relevant* to a hypergame and consider only these viewpoints. A viewpoint is said to be relevant when it is actually taken into account in some agent's decision making. We assume that, in a hypergame, any viewpoint σ, when formulating the decision situation, considers views of all the agents whom σ thinks are participating in the game and does not consider views of anybody else. For example, when agent i is in I, $i \in \Sigma$ by definition, and if another agent j is in I^i, viewpoint ji must be in Σ; otherwise, it is not included in Σ. Furthermore, we suppose that a viewpoint does not contain any successive agents. For example, since considering agent i's view perceived by agent i is redundant, we do not consider viewpoint ii; that is, $ii \notin \Sigma$; similarly, neither jii nor iij is included in Σ. In the subsequent discussion, when we refer to viewpoints, we only indicate viewpoints relevant in this sense.

We deal with concatenations of viewpoints. For example, by $\sigma'\sigma$ with $\sigma = i_1 \cdots i_n$ and $\sigma' = j_1 \cdots j_m$ (with $j_m \neq i_1$) we mean viewpoint $j_1 \cdots j_m i_1 \cdots i_n$. When $\sigma = i_1 \cdots i_n$ with $n \geq 2$, any viewpoint $i_m \cdots i_n$ with $n \geq m \geq 2$ is said to be *higher* than σ. On the other hand, any viewpoint $\tau\sigma$ with $\tau = j_1 \cdots j_l$ and $j_l \neq i_1$ is said to be *lower* than σ. For example, for viewpoint ji, viewpoint i is higher than ji while viewpoint kji is lower than it. Furthermore, for $\sigma = i_1 \cdots i_n$, let us denote i_1 by σ_1. We say σ_1 is the *lowest agent* in viewpoint σ and naturally assume $\sigma_1 \in I^\sigma$ for any $\sigma \in \Sigma$; that is, the lowest agent is always included in the agent set in any viewpoint's subjective game. For any $\sigma \in \Sigma$, let $\Sigma_\sigma = \{\sigma\} \cup \{\tau\sigma \mid \tau = j_1 \cdots j_l, j_l \neq \sigma_1$, and $\tau\sigma \in \Sigma\}$. Σ_σ is the union of σ itself and the set of viewpoints lower than σ.

3. Subjective Rationalizability

We define a new solution concept for hypergames, subjective rationalizability, as follows. (Note that, in Definition 3, a^*_σ is a particular action of σ_1, the lowest agent of viewpoint σ.)

Definition 3 (subjective rationalizability). Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hypergame. $a^*_\sigma \in A^\sigma_{\sigma_1}$ is called *subjectively rationalizable* for viewpoint σ if and only if there exists $(a^*_\tau)_{\tau\in\Sigma_\sigma}$ in $\times_{\tau\in\Sigma_\sigma} A^\tau_{\tau_1}$ which satisfies $\forall \tau \in \Sigma_\sigma$, $\forall a_\tau \in A^\tau_{\tau_1}$, $u^\tau_{\tau_1}(a^*_\tau, a^*_{-\tau}) \geq u^\tau_{\tau_1}(a_\tau, a^*_{-\tau})$, where $a^*_{-\tau} = (a^*_{i\tau})_{i\in I^\tau\setminus\{\tau_1\}}$. Then such $(a^*_\tau)_{\tau\in\Sigma_\sigma}$ is called a *best response hierarchy* of Σ_σ.

An action of the lowest agent in a viewpoint is called subjectively rationalizable for the viewpoint when it is a best response in the viewpoint's subjective game to some actions of the other agents, each of which is a best response in the subjective game of one-step lower viewpoint to some actions of the other agents, and so on. The concept of subjective rationalizability can be understood based on the following principle. The lowest agent in a viewpoint would take a best response to actions which she thinks the other agents would choose. When expecting the choices of the others, she considers that each of the other agents takes a best response to actions which she thinks the agent thinks the other agents would choose, and her inference goes on for further lower viewpoints. When agent i makes a decision in this way, her choice can be predicted as a subjectively rationalizable action of viewpoint i. Thus we may also say it is *agent i*'s subjectively rationalizable action.

For example, let us see Figure 1 and consider how agent i decides her choice according to the principle. In the figure, a circled letter indicates an agent and each arrow describes each agent's perception. Suppose agent i considers agents 1,. . ., m are participating in the game, and each of them is going to choose $a^*_{1i}, \ldots$, and a^*_{mi}, respectively because each of them takes a best response to such actions that agent i thinks each agent considers the other agents are going to choose. For example, agent i expects agent 1 to take a^*_{1i}, which is a best response in viewpoint $1i$'s subjective game to $(a^*_{21i}, \ldots, a^*_{n1i})$, actions of the other agents who are in the game in agent 1's subjective game perceived by agent i. Furthermore, agent i considers agent 1 thinks the others are going to choose

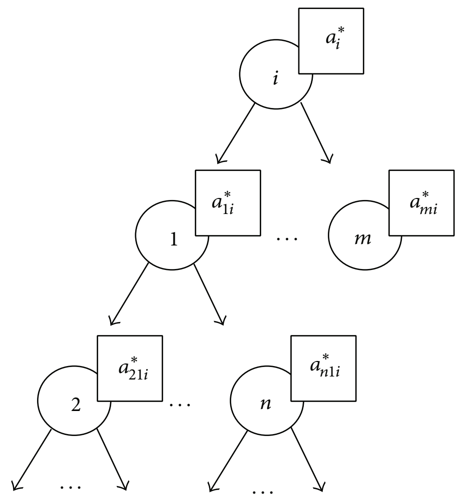

Figure 1: Subjective rationalizability.

$(a^*_{21i}, \ldots, a^*_{n1i})$ because agent i thinks agent 1 thinks each of the other agents takes a best response to such actions that agent i thinks agent 1 thinks each agent expects the other agents to choose, and so on. Then agent i takes a^*_i, a best response in her own subjective game to $(a^*_{1i}, \ldots, a^*_{mi})$: a^*_i is her subjectively rationalizable action. As a result, the actions depicted in the squares in the figure constitute a best response hierarchy: each action is a best response in the viewpoint's subjective game to the actions described in the one-step lower layer. Therefore, by definition, each action is subjectively rationalizable for each viewpoint.

An outcome likely to obtain as a result of each agent's decision making in this way is thus given as a combination of each agent's subjectively rationalizable action. Note that, given that a viewpoint's subjective game may contain some misperceptions, a subjectively rationalizable action may not be a best response to the others' actual choices from an objective (an analyzer's) point of view, and hence it is called "subjectively" rationalizable (see an example in Section 5).

We note that, in general, a viewpoint may not have any subjectively rationalizable actions. However, the next proposition assures the existence of a subjectively rationalizable action under a natural assumption on the hypergame structure.

Proposition 4. *In a hypergame $H = (I, (G^\sigma)_{\sigma\in\Sigma})$, for any $\sigma \in \Sigma$, if, for any $\tau \in \Sigma_\sigma$, $\forall i \in I^\tau \setminus \{\tau_1\}$, $A^{i\tau}_i \subseteq A^\tau_i$, then σ has at least one subjectively rationalizable action.*

Proof. Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hypergame and suppose, for a particular viewpoint $\sigma \in \Sigma$, for any $\tau \in \Sigma_\sigma$, $\forall i \in I^\tau \setminus \{\tau_1\}$, $A^{i\tau}_i \subseteq A^\tau_i$ (note that $A^{i\tau}_i$ is defined because $i \in I^{i\tau}$ is assumed in any case and nonempty according to Definition 2). This implies, for any $\tau \in \Sigma_\sigma$, for any $a_{-\tau} \in \times_{i\in I^\tau\setminus\{\tau_1\}} A^{i\tau}_i$, $a_{-\tau}$ is included in $\times_{i\in I^\tau\setminus\{\tau_1\}} A^\tau_i$; hence there exists $a'_\tau \in A^\tau_{\tau 1}$ such that for any $a_\tau \in A^\tau_{\tau 1}$, $u^\tau_{\tau_1}(a'_\tau, a_{-\tau}) \geq u^\tau_{\tau_1}(a_\tau, a_{-\tau})$, that is, τ_1's best response to $a_{-\tau}$ in G^τ. This implies the existence of a best response hierarchy of Σ_σ; that is, there exists $(a_\tau^*)_{\tau\in\Sigma_\sigma}$ such that $a_\tau^* \in A_{\tau_1}^\tau$ for all $\tau \in \Sigma_\sigma$ which satisfies $^\forall\tau \in \Sigma_\sigma$, $^\forall a_\tau \in A_{\tau_1}^\tau$, $u_{\tau_1}^\tau(a_\tau^*, a_{-\tau}^*) \geq u_{\tau_1}^\tau(a_\tau, a_{-\tau}^*)$, where $a_{-\tau}^* = (a_{i\tau}^*)_{i\in I^\tau\setminus\{\tau_1\}}$. Such $a_\sigma^* \in A_{\sigma_1}^\sigma$ in $(a_\tau^*)_{\tau\in\Sigma_\sigma}$ is σ's subjectively rationalizable action. □

The proposition describes the sufficient condition of the existence of subjectively rationalizable action for a particular viewpoint in a hypergame. It means that if an agent (she) thinks another agent (he) is aware that an action is available to him, then she never excludes the action from his action set in her own subjective game.

4. Equivalence of Subjective Rationalizability and Rationalizability under Inside Common Knowledge

The concept of subjective rationalizability is apparently impractical because calculation of a subjectively rationalizable action requires us to identify a best response hierarchy, which consists of infinite elements. In this section, we prove that, under a condition called inside common knowledge, a subjectively rationalizable action can be replaced by a rationalizable action.

In a normal form game, a rationalizable action is defined as such an action that survives iterative elimination of actions that cannot be a best response to any combinations of the others' actions [10].

Definition 5 (rationalizability). Let $G = (I, A, u)$ be a normal form game. $a_i^* \in A_i$ is called *rationalizable* for agent i if and only if $a_i^* \in \bigcap_{t=1}^\infty H_i(t)$, where $^\forall j \in I$, $H_j(t) = \{a_j' \in A_j \mid {}^\exists a_{-j} \in \times_{k\in I\setminus\{j\}} H_k(t-1)$, $^\forall a_j \in A_j$, $u_j(a_j', a_{-j}) \geq u_j(a_j, a_{-j})\}$ with $t = 1, 2, \ldots$, and $H_j(0) = A_j$. Let us denote the set of rationalizable actions of agent i in G by $R_i(G)$.

Then we introduce the concept of inside common knowledge (ICK) of viewpoints. In a hypergame, a viewpoint σ is said to have ICK when every viewpoint lower than σ has the same subjective game as that of σ [6] (Inohara's [6] original definition of the concept of inside common knowledge refers not to subjective games but to each agent's choice of action, which he calls her final strategy; both definitions are the same in that the concept deals with an agent's subjective belief of common knowledge of something. This kind of subjective common knowledge about the game structure is also discussed in [11]).

Definition 6 (inside common knowledge). Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hypergame. $\sigma \in \Sigma$ is said to have *inside common knowledge* if and only if $^\forall\tau \in \Sigma_\sigma$, $G^\tau = G^\sigma$.

ICK can be regarded as common knowledge perceived subjectively. For example, viewpoint ji's having ICK means that agent i thinks agent j considers the game is common knowledge among the agents. Since it is quite unlikely that, in reality, people consider an enormous number of different games in an infinite hierarchy of perception, it would be natural to assume ICK at some particular level in the hierarchy.

The next lemma assures that, under a viewpoint's ICK, a subjectively rationalizable action of the viewpoint coincides with a rationalizable action of the lowest agent of the viewpoint in its subjective game.

Lemma 7. *In a hypergame $H = (I, (G^\sigma)_{\sigma\in\Sigma})$, suppose $\sigma \in \Sigma$ has ICK. Then, $a_\sigma^* \in A_{\sigma_1}^\sigma$ is subjectively rationalizable for viewpoint σ if and only if $a_\sigma^* \in R_{\sigma_1}(G^\sigma)$.*

Proof. Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hypergame and suppose $\sigma \in \Sigma$ has ICK.

Proof of *if* part: suppose $a_\sigma^* \in A_{\sigma_1}^\sigma$ is rationalizable in G^σ, that is, by definition, $a_\sigma^* \in \bigcap_{t=1}^\infty H_{\sigma_1}(t)$, where $^\forall i \in I^\sigma$, $H_i(t) = \{a_i' \in A_i^\sigma \mid {}^\exists a_{-i} \in \times_{j\in I^\sigma\setminus\{i\}} H_j(t-1)$, $^\forall a_i \in A_i^\sigma$, $u_i^\sigma(a_i', a_{-i}) \geq u_i^\sigma(a_i, a_{-i})\}$ with $t = 1, 2, \ldots$, and $H_i(0) = A_i^\sigma$.

For $\tau \in \Sigma_\sigma$ and $i \in I^\tau \setminus \{\tau_1\}$, let $S_{i\tau} = \{a_i \in A_i^\tau \mid {}^\exists(a_{\tau_1}', a_{-\tau_1,i}) \in S_\tau \times (\times_{j\in I^\tau\setminus\{\tau_1,i\}} A_j^\tau)$, $^\forall a_{\tau_1} \in A_{\tau_1}^\tau$, $u_{\tau_1}^\tau(a_{\tau_1}', a_i, a_{-\tau_1,i}) \geq u_{\tau_1}^\tau(a_{\tau_1}, a_i, a_{-\tau_1,i})\}$ with $S_\sigma = \{a_\sigma^*\}$. $S_{i\tau}$ is interpreted as the set of agent i's actions that can constitute action profile of all the agents except for τ_1 to which some action of τ_1 in S_τ is a best response in G^τ. Due to σ's ICK, for any $\tau \in \Sigma_\sigma$ and $i \in I^\tau\setminus\{\tau_1\}$, $S_{i\tau} = \{a_i \in A_i^\sigma \mid {}^\exists(a_{\tau_1}', a_{-\tau_1,i}) \in S_\tau \times (\times_{j\in I^\sigma\setminus\{\tau_1,i\}} A_j^\sigma)$, $^\forall a_{\tau_1} \in A_{\tau_1}^\sigma$, $u_{\tau_1}^\sigma(a_{\tau_1}', a_i, a_{-\tau_1,i}) \geq u_{\tau_1}^\sigma(a_{\tau_1}, a_i, a_{-\tau_1,i})\}$.

Suppose $S_{i\tau} = \phi$ for some $\tau \in \Sigma_\sigma$ and $i \in I^\tau \setminus \{\tau_1\}$. This means that any actions of τ_1 in S_τ cannot be a best response to any action profiles of the others in G^σ; that is, $H_{\tau_1}(1) \cap S_\tau = \phi$. If $\tau = \sigma$, then $H_{\sigma_1}(1) \cap S_\sigma = \phi \Rightarrow a_\sigma^* \notin H_{\sigma_1}(1) \Rightarrow a_\sigma^* \notin R_{\sigma_1}(G^\sigma)$. If $\tau = i_1 \cdots i_n\sigma$ with $n \geq 1$, then $H_{i_1}(1) \cap S_{i_1\cdots i_n\sigma} = \phi \Rightarrow H_{i_2}(2) \cap S_{i_2\cdots i_n\sigma} = \phi \Rightarrow \cdots \Rightarrow H_{i_n}(n) \cap S_{i_n\sigma} = \phi \Rightarrow H_{\sigma_1}(1) \cap S_\sigma = \phi \Rightarrow a_\sigma^* \notin R_{\sigma_1}(G^\sigma)$. Thus, in any case, $a_\sigma^* \notin R_{\sigma_1}(G^\sigma)$, but this contradicts $a_\sigma^* \in R_{\sigma_1}(G^\sigma)$. Hence we have $S_{i\tau} \neq \phi$ for any $\tau \in \Sigma_\sigma$ and $i \in I^\tau \setminus \{\tau_1\}$.

Therefore, there exists $(a_\tau)_{\tau\in\Sigma_\sigma}$ such that $a_\tau \in S_\tau$ for all $\tau \in \Sigma_\sigma$. Then, in $(a_\tau)_{\tau\in\Sigma_\sigma}$, $^\forall\tau \in \Sigma_\sigma$, $^\forall\overline{a}_\tau \in A_{\tau_1}^\tau$, and $u_{\tau_1}^\tau(a_\tau, a_{-\tau}) \geq u_{\tau_1}^\tau(\overline{a}_\tau, a_{-\tau})$, where $a_{-\tau} = (a_{i\tau})_{i\in I^\tau\setminus\{\tau_1\}}$. That is, this $(a_\tau)_{\tau\in\Sigma_\sigma}$ is a best response hierarchy of Σ_σ. Since $a_\sigma = a_\sigma^*$, a_σ^* is subjectively rationalizable.

Proof of *only if* part: suppose $a_\sigma^* \in A_{\sigma_1}^\sigma$ is a subjectively rationalizable action of viewpoint σ; that is, by definition, there exists a best response hierarchy of σ. Due to σ's ICK, this implies that there exists $(a_\tau^*)_{\tau\in\Sigma_\sigma}$ such that $a_\tau^* \in A_{\tau_1}^\sigma$ for all $\tau \in \Sigma_\sigma$ which satisfies $^\forall\tau \in \Sigma_\sigma$, $^\forall a_\tau \in A_{\tau_1}^\sigma$, and $u_{\tau_1}^\sigma(a_\tau^*, a_{-\tau}^*) \geq u_{\tau_1}^\sigma(a_\tau, a_{-\tau}^*)$, where $a_{-\tau}^* = (a_{i\tau}^*)_{i\in I^\sigma\setminus\{\tau_1\}}$.

We define $H_i(t)$ for any $i \in I^\sigma$ with $t = 1, 2, \ldots$ in the same way as shown in the proof of if part above. Let us prove $a_\sigma^* \in H_{\sigma_1}(n)$ with some integer $n \geq 1$. For any $i_1 \cdots i_n\sigma \in \Sigma_\sigma$, $a_{i_1\cdots i_n\sigma}^* \in H_{i_1}(0)$ because $H_{i_1}(0) = A_{i_1}^\sigma$. Then, $^\forall i_2 \cdots i_n\sigma \in \Sigma_\sigma$, $a_{i_2\cdots i_n\sigma}^* \in H_{i_2}(1) \Rightarrow {}^\forall i_3 \cdots i_n\sigma \in \Sigma_\sigma$, $a_{i_3\cdots i_n\sigma}^* \in H_{i_3}(2) \Rightarrow \cdots \Rightarrow {}^\forall i_n\sigma \in \Sigma_\sigma$, and $a_{i_n\sigma}^* \in H_{i_n}(n-1) \Rightarrow a_\sigma^* \in H_{\sigma_1}(n)$.

Since this holds for any integer $n \geq 1$, $a_\sigma^* \in H_{\sigma_1}(t)$ for any integer $t \geq 1$; thus we have $a_\sigma^* \in \bigcap_{t=1}^{\infty} H_{\sigma_1}(t)$. Hence $a_\sigma^* \in R_{\sigma_1}(G^\sigma)$. □

Then the next proposition states the following things. Suppose there exists a best response hierarchy of viewpoint σ. If some viewpoint τ which is lower than σ or σ itself has ICK, then, in the hierarchy, the actions for all the viewpoints lower than τ (including τ itself) can be replaced by a rationalizable action of the lowest agent of τ in τ's subjective game, when calculating subjectively rationalizable actions. That is, the definition of subjective rationalizability can also be described as follows under the condition of ICK.

Proposition 8. *Let $\sigma \in \Sigma$ in $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ and suppose $\upsilon \in \Sigma_\sigma$ has ICK. $a_\sigma^* \in A_{\sigma_1}^\sigma$ is subjectively rationalizable for viewpoint σ if and only if there exists $((a_\tau^*)_{\tau\in\Sigma_\sigma\setminus\Sigma_\upsilon}, a_\upsilon^*)$ such that $a_\tau^* \in A_{\tau_1}^\tau$ for all $\tau \in \Sigma_\sigma \setminus \Sigma_\upsilon$ and $a_\upsilon^* \in A_{\upsilon_1}^\upsilon$ which satisfies the following:*

(i) $^\forall\tau \in \Sigma_\sigma \setminus \Sigma_\upsilon$, $^\forall a_\tau \in A_{\tau_1}^\tau$, *and* $u_{\tau_1}^\tau(a_\tau^*, a_{-\tau}^*) \geq u_{\tau_1}^\tau(a_\tau, a_{-\tau}^*)$, *where* $a_{-\tau}^* = (a_{i\tau}^*)_{i\in I^\tau\setminus\{\tau_1\}}$;

(ii) $a_\upsilon^* \in R_{\upsilon_1}(G^\upsilon)$.

Proof. Let $H = (I, (G^\sigma)_{\sigma\in\Sigma})$ be a hypergame and $\sigma \in \Sigma$. Suppose $\upsilon \in \Sigma_\sigma$ has ICK.

Proof of *if* part: suppose there exists $((a_\tau^*)_{\tau\in\Sigma_\sigma\setminus\Sigma_\upsilon}, a_\upsilon^*)$ such that $a_\tau^* \in A_{\tau_1}^\tau$ for all $\tau \in \Sigma_\sigma \setminus \Sigma_\upsilon$ and $a_\upsilon^* \in A_{\upsilon_1}^\upsilon$ which satisfies both of the two conditions described in Proposition 8. Then, due to Lemma 7, a_υ is υ's subjectively rationalizable action in G^υ; that is, there exists $(a_\omega^*)_{\omega\in\Sigma_\upsilon}$ such that $a_\omega^* \in A_{\omega_1}^\omega$ for all $\omega \in \Sigma_\upsilon$, and $^\forall\omega \in \Sigma_\upsilon$, $^\forall a_\omega \in A_{\omega_1}^\omega$, and $u_{\omega_1}^\omega(a_\omega^*, a_{-\omega}^*) \geq u_{\omega_1}^\omega(a_\omega, a_{-\omega}^*)$, where $a_{-\omega}^* = (a_{i\omega}^*)_{i\in I^\omega\setminus\{\omega_1\}}$. Therefore $((a_\tau^*)_{\tau\in\Sigma_\sigma\setminus\Sigma_\upsilon}, (a_\omega^*)_{\omega\in\Sigma_\upsilon})$ is a best response hierarchy of σ, and hence a_σ^* in $((a_\tau^*)_{\tau\in\Sigma_\sigma\setminus\Sigma_\upsilon}, a_\upsilon^*)$ is σ's subjectively rationalizable action.

Proof of *only if* part: suppose $a_\sigma^* \in A_{\sigma_1}^\sigma$ is σ's subjectively rationalizable action, and let $(a_\tau^*)_{\tau\in\Sigma_\sigma}$ be a best response hierarchy of σ. Then, for $\upsilon \in \Sigma_\sigma$, $(a_\omega^*)_{\omega\in\Sigma_\upsilon}$ in it is a best response hierarchy of viewpoint υ; thus a_υ^* is υ's subjectively rationalizable action. Since υ has ICK, $a_\upsilon^* \in R_{\upsilon_1}(G^\upsilon)$ due to Lemma 7. Hence $((a_\tau^*)_{\tau\in\Sigma_\sigma\setminus\Sigma_\upsilon}, a_\upsilon^*)$ satisfies both of the two conditions in Proposition 8. □

When we want to know viewpoint σ's subjectively rationalizable action, instead of identifying a best response hierarchy with infinite elements as described in Definition 3, we just need to have rationalizable actions in subjective games of viewpoints having ICK and identify a "reduced" best response hierarchy based on them as described in the proposition.

To see an example, reconsider the example of Figure 1. Suppose now all the viewpoints one-step lower than viewpoint i, namely, viewpoint $1i$,..., mi, have ICK; that is, agent i considers every other agent thinks the game is common knowledge though each agent may perceive different games (Figure 2). Then Proposition 8 implies that if a_{1i}^* is agent 1's rationalizable action in viewpoint $1i$'s subjective game,..., and a_{mi}^* is agent m's rationalizable action in viewpoint mi's subjective game, then agent i's best response in viewpoint i's subjective game to $(a_{1i}^*, \ldots, a_{mi}^*)$, that is, a_i^*, is a subjectively rationalizable action for agent i. In this way, we can calculate an agent's subjectively rationalizable action much more simply by applying the proposition.

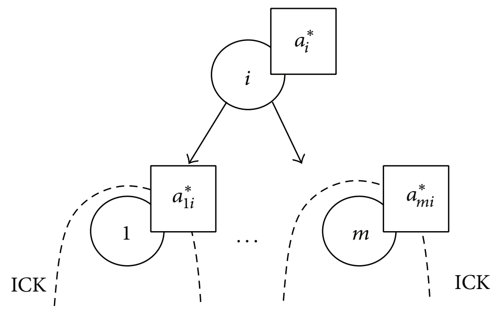

Figure 2: Subjective rationalizability under ICK.

5. Example and Discussions

We show an example of the concepts and findings presented so far. In particular, our focus is to illustrate how subjective rationalizability can give us new insights compared to existing solution concepts, as well as how the results obtained in the previous section can be applied when analyzing a hypergame.

Let us consider a two-agent hypergame with $I = \{i, j\}$. i's view is given as viewpoint i's subjective game, G^i, as shown in Table 1. i's decision depends on i's expectation about j's choice, which depends on i's view about how j perceives the game. Suppose that i thinks j thinks the game they play is not Table 1 but Table 2, which is viewpoint ji's subjective game, G^{ji}. It is slightly different with G^i in terms of i's utility. This means that i thinks j misperceives i's utility function. Furthermore, we assume that i thinks j thinks the game G^{ji} is common knowledge, namely, viewpoint ji's ICK. This means, while iji, $jiji$, $\ldots \in \Sigma_{ji}$, G^{iji}, G^{jiji}, $\ldots$ are the same as G^{ji}.

If, following the conventional way of analysis, we regard the situation as a two-level hypergame and apply Nash-type solution concept, the analysis would go as follows: G^{ji} has the unique Nash equilibrium, (c, z). Then, expecting j to choose z, i takes the best response to it in G^i, namely, c.

In G^{ji}, however, j's other actions, x and y, appear rather attractive for j. Indeed not only z but also these two actions are rationalizable for j. Due to Lemma 7, given that viewpoint ji has ICK, all of these three actions are subjectively rationalizable for viewpoint ji. Then, Proposition 8 implies that i's action that can be a best response to any of these actions in G^i is subjectively rationalizable for viewpoint i, namely b and c. Therefore, if i makes decision according to the idea of subjective rationalizability, i may choose not only c but also b.

Hence, in general, the prediction power of subjective rationalizability is weaker than conventional solution concepts. This is the case with rationalizability whose prediction power is weaker than Nash equilibrium. Rather, it often tells

Table 1: Viewpoint i's subjective game.

$i \setminus j$	x	y	z
a	2, 2	2, 3	0, 0
b	3, 3	3, 2	0, 0
c	0, 0	0, 0	1, 1

Table 2: Viewpoint ji's subjective game.

$i \setminus j$	x	y	z
a	3, 2	2, 3	0, 0
b	2, 3	3, 2	0, 0
c	0, 0	0, 0	1, 1

Table 3: Viewpoint j's subjective game.

$i \setminus j$	x	y
a	3, 2	2, 3
b	2, 2	3, 3

us nothing about prediction of a game because any actions can be rationalizable in some games, for example, battle of sexes and chicken games. Rationalizability, however, is insightful in that it provides the precise implication of common knowledge of the game structure as well as rationality of agents. Likewise, we characterize subjective rationalizability as such a tool that tells us the precise implication of a given hypergame structure. We also note that we would obtain the same result if we regard the situation above as a two-level hypergame and simply apply not Nash equilibrium but rationalizability; however our findings ensure that the agent's choice predicted by subjective rationalizability is theoretically consistent with this idea.

Possible outcomes of a hypergame are given as any combinations of each agent's subjectively rationalizable action. Therefore, in a realized outcome, an agent may notice her choice is actually not rationalizable; that is, it may not be a best response to the opponent's choice. For example, in the case above, suppose agent j actually perceives the game shown in Table 3 and G^j: j is unaware of i's action c as well as j's own action z. If viewpoint j has ICK, since y is the only rationalizable action in G^j, it is the only subjectively rationalizable action for agent j. Among viewpoint i's two subjectively rationalizable actions, b is a best response to y, but c is not. Hence it is called "subjectively" rationalizable.

6. Conclusion

The contribution of the present study is mainly twofold. First, we have proposed a new solution concept in hypergames called subjective rationalizability. It is based on an idea that every agent takes a best response to expected choices of the others, every agent thinks every agent takes a best response to expected choices of the others, and so on. It is characterized as such a tool that provides the precise implication of a given hypergame structure. Second, we have proved the equivalence of subjective rationalizability and the standard notion of rationalizability under ICK. The result is useful to avoid the problem that it is practically impossible to calculate subjectively rationalizable actions in the way described by the original definition and thus makes the concept applicable to hypergame analysis.

Hypergame analysis has been applied to conflict analysis traditionally [2, 12, 13] and recently to a wider range of areas such as information security [14], service science [15] and systems thinking [16]. As far as the author knows, all of them apply solution concepts based on the conventional ideas mentioned in the introduction. Subjective rationalizability provides new insights that are theoretically consistent with the idea of rationalizability in game theory; thus it can contribute to improving analytical power of hypergames and making the analyses more convincing.

Subjective rationalizability is a solution concept for one-shot hypergames. Therefore, if an analyzer's interest is in a situation in which a hypergame is played multiple times and agents may update their views in the course of decisions, we need an additional framework to deal with it. For example, in the example shown in Section 5, if agent i takes a subjectively rationalizable action, c, then the other agent j first notices such an action is available to i; thus j would revise the subjective view before playing the next period game so as to include it in i's action set. Such a learning process of an agent in repetitive hypergames has been studied only in a few references [17, 18]. We consider the notion of subjective rationalizability can be one plausible candidate for a solution concept of each period game in such a repetitive hypergame situation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] P. Bennett, "Toward a theory of hypergames," *Omega*, vol. 5, no. 6, pp. 749–751, 1977.

[2] P. G. Bennett and M. R. Dando, "Complex strategic analysis: a hypergame study of the fall of France," *Operational Research Society Journal*, vol. 30, no. 1, pp. 23–32, 1979.

[3] J. C. Harsanyi, "Games with incomplete information played by "Bayesian" players. I. The basic model," *Management Science*, vol. 14, pp. 159–182, 1967.

[4] Y. Sasaki and K. Kijima, "Hypergames and Bayesian games: a theoretical comparison of the models of games with incomplete information," *Journal of Systems Science and Complexity*, vol. 25, no. 4, pp. 720–735, 2012.

[5] P. G. Bennett, "Hypergames: developing a model of conflict," *Futures*, vol. 12, no. 6, pp. 489–507, 1980.

[6] T. Inohara, "Interperceptional equilibrium as a generalization of Nash equilibrium in games with interperception," *IEEE Transactions on Systems, Man, and Cybernetics A Systems and Humans.*, vol. 30, no. 6, pp. 625–638, 2000.

[7] M. Wang, K. W. Hipel, and N. M. Fraser, "Modeling misperceptions in games," *Behavioral Science*, vol. 33, no. 3, pp. 207–223, 1988.

[8] M. Wang, K. W. Hipel, and N. M. Fraser, "Solution concepts in hypergames," *Applied Mathematics and Computation*, vol. 34, no. 3, pp. 147–171, 1989.

[9] H. Gintis, *The Bounds of Reason: Game Theory and the Unification of the Behavioral Sciences*, Princeton University Press, New Jersey, NJ, USA, 2009.

[10] D. G. Pearce, "Rationalizable strategic behavior and the problem of perfection," *Econometrica*, vol. 52, no. 4, pp. 1029–1050, 1984.

[11] M. Kaneko, "Epistemic logics and their game theoretic applications: introduction," *Economic Theory*, vol. 19, no. 1, pp. 7–62, 2002.

[12] T. Inohara, K. W. Hipel, and S. Walker, "Conflict analysis approaches for investigating attitudes and misperceptions in the War of 1812," *Journal of Systems Science and Systems Engineering*, vol. 16, no. 2, pp. 181–201, 2007.

[13] M. C. Shupe, W. M. Wright, K. W. Hipel, and N. M. Fraser, "Nationalization of the Suez canal: a hypergame analysist," *Journal of Conflict Resolution*, vol. 24, no. 3, pp. 477–493, 1980.

[14] Y. Imamverdiyev, "A hypergame model for information security," in *Proceedings of the 6th International Information Security and Cryptology Conference*, Ankara, Turkey, 2013.

[15] S. Novani and K. Kijima, "Value co-creation by customer-to-customer communication: social media and face-to-face for case of airline service selection," *Journal of Service Science and Management*, vol. 5, no. 1, pp. 101–109, 2013.

[16] Y. Sasaki, R. P. Hämäläinen, and E. Saarinen, "Modeling systems of holding back as hypergames and their connections with systems intelligence," *Systems Research and Behavioral Science*, 2014.

[17] B. Gharesifard and J. Cortes, "Evolution of players' misperceptions in hypergames under perfect observations," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1627–1640, 2012.

[18] S. Takahashi, N. Hinago, T. Inohara, and B. Nakano, "Evolutionary approach to three-person hypergame situation," in *Proceedings of the IEEE International Conference on Systems, Man, and Cybernetics (SMC '99)*, vol. 4, pp. 254–259, October 1999.