A Bi-Objective Optimization Model for Fare Structure Design in Public Transport

Philine Schiewe
Department of Mathematics and Systems Analysis, Aalto University, Finland

Anita Schöbel
Department of Mathematics, University of Kaiserslautern-Landau (RPTU), Germany
Fraunhofer Institute of Industrial Mathematics ITWM, Kaiserslautern, Germany

Reena Urban
Department of Mathematics, University of Kaiserslautern-Landau (RPTU), Germany

Abstract

Fare planning in public transport is important from the view of passengers as well as of operators. In this paper, we propose a bi-objective model that maximizes the revenue as well as the number of attracted passengers. The potential demand per origin-destination pair is divided into demand groups that have their own willingness how much to pay for using public transport, i.e., a demand group is only attracted as public transport passengers if the fare does not exceed their willingness to pay. We study the bi-objective problem for flat and distance tariffs and develop specialized algorithms to compute the Pareto front in quasilinear or cubic time, respectively. Through computational experiments on structured data sets we evaluate the running time of the developed algorithms in practice and analyze the number of non-dominated points and their respective efficient solutions.

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1 Introduction

Fare structures in public transport are an important design element that involves the interests of both (potential) passengers and operators alike. For passengers, fares are one among several criteria for mode and route choice. The affordability and the perceived fairness of fares significantly influence people's decisions to opt for public transport over other modes of transport, for example, their own car. When the fares exceed a certain price limit (willingness to pay), it is reasonable to assume a deterrent effect leading to a reduction in the attractiveness of public transport and, therefore, ridership. Conversely, for operators, fares directly impact the revenue. An increase of prices, for example, increases the income per sold ticket but might decrease the ridership and therefore the total number of sold tickets.

In this paper, we investigate the trade-off between revenue and number of passengers for different fare strategies. For each origin-destination (OD) pair, we consider multiple demand groups that differ in their willingness to pay. If the fare for an OD pair exceeds the willingness to pay of a demand group, this group does not use public transport. These demand groups could, for example, be captive and choice passengers, where the willingness to pay is dependent on whether an alternative mode like a car is available or not. Another categorization of demand groups could be based on age and income. We introduce a bi-objective model that optimizes fare structures to determine the Pareto front of revenue and number of passengers.

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A wide range of fare structures is implemented worldwide. In this paper, we introduce the revenue-passenger model for fare structures in general and then focus on flat and distance tariffs. In a flat tariff, all tickets have the same price. While this is very easy to understand, it also encourages using public transport for longer journeys. For shorter journeys a flat tariff might be perceived as unfair because passengers with a short journey pay the same price as passengers with a long journey. The opposite can be realized with a distance tariff, which accounts for the traveled distance. Here, the distance may either be the beeline distance between the start and the end station of the journey or the network distance of the respective path of the journey. In this paper, we consider affine distance tariffs, which are composed of a base amount and an additional price per kilometer. While it is easy to communicate, passengers need to know the exact distance of their journey to determine the fare. Slight deviations of the path may directly lead to a change in the fare. Other differential fare structures depend on a duration, time or quality component of the journey [12, 22, 9], but are not considered here.

Related Literature. In public transport, requirements and needs of several actors such as the passengers and the operators are involved, leading to multi-objective models [5]. However, multi-objective models are scarcely considered in fare planning, and the literature on fare planning so far focuses on single-objective models. The objective often is the minimization of the deviation from reference prices [15, 16, 1] or the maximization of either revenue or demand [10, 3, 26, 19, 17]. Moreover, studies analyze the impact of different fare structures such as flat, distance and zone tariffs on the route choice and travel time [18] and the revenue and number of passengers [13, 6]. We expand the literature by a bi-objective model with respect to revenue and number of passengers.

Contribution. First, we formulate a general model that can be applied for any fare strategy. Due to the characteristic of one of the objectives, the complete Pareto front can be determined with the ϵ-constraint method. Second, we study the specific problem for flat and distance tariffs. In both cases, we identify a finite candidate set, based on which we develop algorithms that compute the Pareto front in quasilinear or cubic time, respectively.

We also perform computational experiments on structured data sets and analyze the number of non-dominated points and their respective efficient solutions. The experiments emphasize the advantage in running time of the specialized algorithm for distance tariffs compared to the mixed-integer programming formulation derived from the general model.

2 Problem Formulation

Let a *public transport network (PTN)* (V, E) be given. The node set V represents a set of stops or stations and the edge set E represents the direct connections between them. For simplicity, we assume the PTN to be an undirected graph which is simple and connected. The PTN can be used to model railway, tram, or bus networks. In the following, we call the nodes of the PTN stations, also if bus networks with stops are under consideration.

By $D \subseteq \{(v_1, v_2) : v_1, v_2 \in V, v_1 \neq v_2\}$ we denote the set of origin-destination (OD) pairs. The potential passengers of an OD pair can be distinguished by their willingness to pay. This could for example reflect the degree of dependence on public transport or the income. For each OD pair $d \in D$, we denote by $G_d \in \mathbb{N}_{\geq 1}$ the number of demand groups, by $t_d^g \in \mathbb{N}_{\geq 1}$ the number of people belonging to group $g \in \{1, \dots, G_d\}$ with a *willingness to pay* of $w_d^g \in \mathbb{R}_{>0}$. In this model, a demand group uses public transport whenever the ticket price does not exceed its willingness to pay. Without loss of generality, we assume that $w_d^g > 0$ and $w_d^g \neq w_d^{g'}$ for all $g, g' \in \{1, \dots, G_d\}$, $g \neq g'$ and all OD pairs $d \in D$.

To simplify notation, we introduce the shorthand notation $[G] := \{1, \dots, G\}$ for $G \in \mathbb{N}_{\geq 1}$.

▶ **Definition 1** (Fare structure [25]). *Let a PTN be given, and let $\mathcal{W}$ be the set of all paths in the PTN. A* fare structure *is a function $\pi\colon \mathcal{W} \to \mathbb{R}_{\geq 0}$ that assigns a price to every path in the PTN.*

Usually, a fare strategy (e.g., a flat, distance or zone tariff) is desired instead of just determining a price for each OD pair. Such a desired fare strategy can be modeled by additional requirements, that can be formulated as constraints.

The objective is to maximize the revenue and the number of passengers simultaneously. While the revenue is the key objective of the operator, the number of passengers serves as an indicator of the success of the transition towards sustainable transport modes. This is particularly significant when public transport is used instead of private motorized transport modes such that the environmental impact of traveling is reduced.

Given a fare structure π, we denote the according price for OD pair $d \in D$ by π_d. The number of attracted passengers for OD pair $d \in D$ given π_d is then determined as

$$\mathsf{pass}(d \mid \pi_d) := \sum_{g \in [G_d]\colon \pi_d \leq w_d^g} t_d^g.$$

The total number of passengers with respect to fare structure π is

$$\mathsf{pass}(\pi) := \sum_{d \in D} \mathsf{pass}(d \mid \pi_d)$$

and the total revenue is

$$\mathsf{rev}(\pi) := \sum_{d \in D} \mathsf{pass}(d \mid \pi_d) \cdot \pi_d.$$

With this, we can now define the revenue-passenger model formally.

▶ **Definition 2** (The revenue-passenger model). *Given are*

- *a PTN (V, E) as an undirected graph,*
- *a set of OD pairs $D \subseteq \{(v_1, v_2) : v_1, v_2 \in V, v_1 \neq v_2\}$, $D \neq \emptyset$,*
- *the numbers of demand groups $G_d \in \mathbb{N}_{\geq 1}$ for each OD pair $d \in D$,*
- *the willingness to pay $w_d^g \in \mathbb{R}_{\geq 0}$ and the potential demand $t_d^g \in \mathbb{N}_{\geq 1}$ for each demand group $g \in [G_d]$ and each OD pair $d \in D$.*

The aim is to determine fare structures π that maximize the revenue $\mathsf{rev}(\pi)$ and the number of passengers $\mathsf{pass}(\pi)$, where a desired fare strategy might be required. The bi-objective problem hence is:

$$\begin{aligned} \max\ & \mathsf{rev}(\pi) \\ \max\ & \mathsf{pass}(\pi) \\ \text{s.t.}\ & \pi \text{ is of a desired fare strategy} \\ & \pi_d \geq 0 \quad \text{for all } d \in D. \end{aligned}$$

Each feasible fare structure π induces a two-dimensional vector of objective function values, i.e., we have a bi-objective problem. For multi-objective optimization, we refer to, e.g., [11]. As usual in multi-objective optimization, we are interested in finding the Pareto front and corresponding efficient solutions. Generally speaking, we aim to find those feasible fare structures that do not allow to improve one objective function without decreasing the other.

▶ **Definition 3** (Efficient solution, non-dominated point and Pareto front, e.g., [11]). *Let an instance of the* revenue-passenger model *be given. A feasible solution π is called* efficient *and its objective value $(\mathsf{rev}(\pi), \mathsf{pass}(\pi))$ is called* non-dominated *if there does not exist another feasible solution π' with objective value $(\mathsf{rev}(\pi'), \mathsf{pass}(\pi'))$ such that $\mathsf{rev}(\pi') \geq \mathsf{rev}(\pi)$ and $\mathsf{pass}(\pi') \geq \mathsf{pass}(\pi)$ and at least one inequality holding strictly. The set of all non-dominated points is also called the* Pareto front*.*

Because the numbers of passengers t_d^g for all $d \in D$, $g \in \{1, \ldots, G_d\}$ are given as natural numbers, the objective function pass always attains integral values. Hence, the whole Pareto front can be computed systematically by applying the well-known ϵ-constraint method [11, 4]. This is done by restricting the number of passengers pass in the constraints while the revenue rev remains as the objective function. In this case, by increasing ϵ with a step width of 1, we do not miss any non-dominated point. Further, given a set of tuples of revenue and number of passengers, it is simple to *filter for the non-dominated points*: The points are first sorted in decreasing order by the number of passengers and as a second criterion by decreasing revenue. We then iterate over this sorted list and add a point to the Pareto front whenever the revenue is strictly higher than the highest value so far. This ensures that while the numbers of passengers is decreasing, the revenue is increasing, and we only keep non-dominated points.

3 Flat Tariffs

Flat tariffs are common in city centers and assign the same price to all paths. We start with a formal definition:

▶ **Definition 4** (Flat tariff, [25]). *Let a PTN be given, and let $\mathcal{W}$ be the set of all paths in the PTN. A fare structure π is a* flat tariff *w.r.t. a fixed price $f \in \mathbb{R}_{\geq 0}$ if $\pi(W) = f$ for all $W \in \mathcal{W}$.*

Let $S_{\text{will}} := \{w_d^g : d \in D, g \in [G_d]\}$ be the set of all willingness to pay values with $\max S_{\text{will}}$ the largest of these values and let

$$S := \left\{ (w, t) : w \in S_{\text{will}}, t = \sum_{d \in D} \sum_{g \in [G_d]:\, w_d^g = w} t_d^g \right\}$$

be the set of all tuples of willingness to pay and the respective demand with exactly this willingness to pay. Let $(w_1, t_1), \ldots, (w_{|S|}, t_{|S|})$ be a sorting of S such that $w_1 < \ldots < w_{|S|}$. In particular, we have $|S| \leq \sum_{d \in D} G_d$, with equality if and only if the willingness to pay is different for every demand group.

For a flat tariff π with fixed price $f \in \mathbb{R}_{\geq 0}$, the objective functions simplify to

$$\mathsf{rev}(\pi) = f \cdot \sum_{\substack{(w,t) \in S: \\ f \leq w}} t \quad \text{and} \quad \mathsf{pass}(\pi) = \sum_{\substack{(w,t) \in S: \\ f \leq w}} t.$$

Because a flat tariff π is uniquely determined by f, we write $\mathsf{rev}(f)$ and $\mathsf{pass}(f)$ instead of $\mathsf{rev}(\pi)$ and $\mathsf{pass}(\pi)$.

▶ **Definition 5** (F-RPM). *Given the input data as in Definition 2, the bi-objective revenue-passenger model for a flat tariff (*F-RPM*) is the following:*

$$\max \ \mathsf{rev}(f) = f \cdot \sum_{\substack{(w,t)\in S:\\ f\le w}} t$$

$$\max \ \mathsf{pass}(f) = \sum_{\substack{(w,t)\in S:\\ f\le w}} t$$

$$\text{s.t.} \quad f \in \mathbb{R}_{\geq 0}.$$

We now derive a finite candidate set for F-RPM.

▶ **Lemma 6.** *For all efficient solutions f to* F-RPM*, it holds that $f \in S_{\text{will}}$.*

Proof. Let $\bar{f}$ be an efficient solution, and assume that $\bar{f} \notin S_{\text{will}}$. First, we have that $\bar{f} < \max S_{\text{will}}$ because for $\bar{f} > \max S_{\text{will}}$ the objective function values are $(0, 0)$, which is not efficient since for $f := \max S_{\text{will}} = w_{|S|}$ the objective function values are $(\mathsf{rev}(f), \mathsf{pass}(f)) = (t_{|S|} \cdot w_{|S|}, t_{|S|})$, which dominates $(0, 0)$. Hence, $f' := \min\{w \in S_{\text{will}} : \bar{f} < w\}$ is well-defined and f' is the next higher price compared to $\bar{f}$ that is contained in S_{will}. Then $\bar{f} < f'$ and $\{(w,t) \in S : \bar{f} \leq w\} = \{(w,t) \in S : f' \leq w\}$ by definition of f' and because $\bar{f} \notin S_{\text{will}}$. This yields $\mathsf{pass}(\bar{f}) = \mathsf{pass}(f')$ and $\mathsf{rev}(\bar{f}) = \bar{f} \cdot \mathsf{pass}(\bar{f}) < f' \cdot \mathsf{pass}(\bar{f}) = \mathsf{rev}(f')$, which is a contradiction to $\bar{f}$ being efficient. ◀

Lemma 6 allows Algorithm 1 to compute the Pareto front in $\mathcal{O}(|S| \cdot \log(|S|))$. Note that $|S| \leq \sum_{d\in D} G_d$.

Algorithm 1 Solution method for F-RPM.

Input : Set S (as defined above) of F-RPM
Output : Set Γ of all non-dominated points

1 Sort S such that $w_1 < \ldots < w_{|S|}$.
2 Initialize $\overline{\mathsf{pass}} \leftarrow \sum_{s=1}^{|S|} t_s$; $\overline{\mathsf{rev}} \leftarrow w_1 \cdot \overline{\mathsf{pass}}$; $\Gamma \leftarrow \{(\overline{\mathsf{rev}}, \overline{\mathsf{pass}})\}$; $\mathsf{rev}^* \leftarrow \overline{\mathsf{rev}}$.
3 **for** $s = 2, \ldots, |S|$ **do**
4   Update $\overline{\mathsf{pass}} \leftarrow \overline{\mathsf{pass}} - t_{s-1}$.
5   Update $\overline{\mathsf{rev}} \leftarrow w_s \cdot \overline{\mathsf{pass}}$.
6   **if** $\overline{\mathsf{rev}} > \mathsf{rev}^*$ **then**
7   Update $\Gamma \leftarrow \Gamma \cup \{(\overline{\mathsf{rev}}, \overline{\mathsf{pass}})\}$.
8   Update $\mathsf{rev}^* \leftarrow \overline{\mathsf{rev}}$.
9 **return** Γ

▶ **Theorem 7.** *Algorithm 1 solves* F-RPM *in $\mathcal{O}(|S| \cdot \log(|S|))$.*

Proof. By Lemma 6 it suffices to consider the willingness to pay $w_s \in S_{\text{will}}$ as fixed prices of the flat tariff. Because w_1 is the unique optimum with respect to the objective function pass, $(\mathsf{rev}(w_1), \mathsf{pass}(w_1))$ is a non-dominated point and is added to Γ in line 2. In rev^* we store the maximum revenue that has occurred so far. Increasing the fixed price from w_{s-1} to w_s reduces the number of passengers by those that have a willingness to pay of w_{s-1}, which are t_{s-1} many. Hence, after the updates in lines 4 and 5, $\overline{\mathsf{rev}}$ and $\overline{\mathsf{pass}}$ are the revenue and the number of passengers for a flat tariff with fixed price w_s. Because the number of passengers is strictly decreased in every iteration, the tuple $(\overline{\mathsf{rev}}, \overline{\mathsf{pass}})$ is non-dominated whenever the revenue $\overline{\mathsf{rev}}$ is larger than any previous revenue, i.e., if $\overline{\mathsf{rev}} > \mathsf{rev}^*$. Therefore, in this case, the tuple is added to Γ and the maximum revenue rev^* is updated.

Sorting S can be done in $\mathcal{O}(|S| \cdot \log(|S|))$ (see, e.g., [7]). The initialization of $\overline{\mathsf{pass}}$ in line 2 is executed in $\mathcal{O}(|S|)$, whereas all other initializations and updates are in $\mathcal{O}(1)$. Hence, the for-loop takes $\mathcal{O}(|S|)$ in total. Overall, we obtain a running time of $\mathcal{O}(|S| \cdot \log(|S|))$. ◀

4 Distance Tariffs

In a distance tariff, the fare is related to a distance $l(W)$ associated with the path $W \in \mathcal{W}$. Usually this is the beeline (Euclidean) distance between the start and the end station of the path or the network distance, i.e., the length of the path W in the PTN. We consider affine distance tariffs which means that the fares consist of a base amount and a price per kilometer. We start with a formal definition.

▶ **Definition 8** (Affine distance tariff, [25]). *Let a PTN be given, and let $\mathcal{W}$ be the set of all paths in the PTN. Let $l\colon \mathcal{W} \to \mathbb{R}_{\geq 0}$ be a distance function determining, e.g., the beeline or network distance. A fare structure π is an* affine distance tariff *w.r.t. a base amount $f \in \mathbb{R}_{\geq 0}$ and a price per kilometer $p \in \mathbb{R}_{\geq 0}$ if $\pi(W) = f + p \cdot l(W)$ for all $W \in \mathcal{W}$.*

For the optimization of affine distance tariffs, we consider that each OD pair $d \in D$ travels along a fixed path $W_d \in \mathcal{W}$. Hence, each OD pair is associated with a distance $l_d := l(W_d)$ based on the distance function l.

▶ **Definition 9** (D-RPM). *Given the input data as in Definition 2 and a distance $l_d \in \mathbb{R}_{\geq 0}$ associated with OD pair d for all $d \in D$, the bi-objective revenue-passenger model for a distance tariff (*D-RPM*) is the following:*

$$
\begin{aligned}
\max \quad & \mathsf{rev}(\pi) \\
\max \quad & \mathsf{pass}(\pi) \\
\text{s.t.} \quad & \pi_d = f + p \cdot l_d \quad \text{for all } d \in D \\
& f, p \in \mathbb{R}_{\geq 0}.
\end{aligned}
$$

Also here, we write $\mathsf{rev}(f, p)$ and $\mathsf{pass}(f, p)$ instead of $\mathsf{rev}(\pi)$ and $\mathsf{pass}(\pi)$ because an affine distance tariff is uniquely determined by f and p.

For the ϵ-constraint method, the following mixed-integer linear programming (MILP) formulation may be used:

$$
\begin{aligned}
\max_{f,p,\pi_d^g,x_d^g} \quad & \sum_{d \in D} \sum_{g \in [G_d]} t_d^g \cdot \pi_d^g \\
\text{s.t.} \quad & \epsilon \leq \sum_{d \in D} \sum_{g \in [G_d]} t_d^g \cdot x_d^g \\
& f + p \cdot l_d \leq w_d^g + M \cdot (1 - x_d^g) && \text{for all } d \in D, g \in [G_d] && (1) \\
& \pi_d^g \leq f + p \cdot l_d && \text{for all } d \in D, g \in [G_d] && (2) \\
& \pi_d^g \leq M \cdot x_d^g && \text{for all } d \in D, g \in [G_d] && (3) \\
& f, p, \pi_d^g \in \mathbb{R}_{\geq 0} && \text{for all } d \in D, g \in [G_d] \\
& x_d^g \in \{0, 1\} && \text{for all } d \in D, g \in [G_d].
\end{aligned}
$$

The variables f and p determine the base amount and the price per kilometer of the distance tariff. The binary variable x_d^g is 1 if and only if demand group g of OD pair d uses public transport. Finally, the variable π_d^g stores the price that is actually paid by the demand group g of OD pair d. Constraints (1) ensure that x_d^g is set correctly, meaning that it is only 1 if the price according to the distance tariff does not exceed the willingness to pay. Constraints (2) limit the price of a demand group to the price of the distance tariff and constraints (3) set the price paid by a demand group to 0 if it does not use public transport. Together constraints (2) and (3) set the price paid by a demand group to either 0 or the distance tariff price.

Before we show that M can be chosen based on natural bounds on f and p, we introduce some notation. Let

$$S_{\text{dem}}(f,p) := \{(d,g) : d \in D, g \in [G_d], w_d^g \geq f + p \cdot l_d\}$$

be the set of demand groups that are attracted in case of a distance tariff with base amount f and price per kilometer p, i.e., $\mathsf{pass}(f,p) = \sum_{(d,g)\in S_{\text{dem}}(f,p)} t_d^g$. For every efficient solution (f,p), it holds that $S_{\text{dem}}(f,p) \neq \emptyset$ because otherwise the objective function value is $(0,0)$ and is, analogously to the proof of Lemma 6, dominated by a solution $f' := w_{\bar{d}}^{\bar{g}}$ for some $\bar{d} \in D$, $\bar{g} \in [G_{\bar{d}}]$ and $p' := 0$, which attracts at least one demand group. Therefore, we can restrict $f \leq f^{\max} := \max\{w_d^g : d \in D, g \in [G_d]\}$ and $p \leq p^{\max} := \max\left\{\frac{w_d^g}{l_d} : d \in D, g \in [G_d]\right\}$. Let $l^{\max} := \max\{l_d : d \in D\}$. Setting $M := f^{\max} + p^{\max} \cdot l^{\max}$, we have for all $d \in D$, $g \in [G_d]$ that

$$\pi_d^g \overset{(2)}{\leq} f + p \cdot l_d \leq f^{\max} + p^{\max} \cdot l^{\max} = M,$$

i.e., M is sufficiently large for constraints (1) and (3).

This MILP has $\mathcal{O}(\sum_{d\in D} G_d)$ many variables and constraints and, as we will see later in the experiments, is hard to solve. We hence develop a polynomial-time method that exploits the specific problem structure.

▶ **Lemma 10.** *For every efficient solution (f,p), at least one willingness to pay is met exactly, i.e., there is at least one willingness to pay w_d^g for some OD pair $d \in D$ and demand group $g \in [G_d]$ such that $w_d^g = f + p \cdot l_d$.*

Proof. Let $(\bar{f},\bar{p})$ be an efficient solution. Assume that no willingness to pay is met exactly. Then there must be a willingness to pay w_d^g for some $d \in D$, $g \in [G_d]$ such that $w_d^g > \bar{f} + \bar{p} \cdot l_d$. Otherwise $S_{\text{dem}}(\bar{f},\bar{p}) = \emptyset$, which is not possible for an efficient solution. Hence, we can set $\delta := \min\{w_d^g - \bar{f} - \bar{p} \cdot l_d > 0 : d \in D, g \in [G_d]\}$. Increasing $\bar{f}$ to $f' := \bar{f} + \delta$, we have

$$\{(d,g) : d \in D, g \in [G_d], \bar{f} + \bar{p} \cdot l_d \leq w_d^g\} = \{(d,g) : d \in D, g \in [G_d], f' + \bar{p} \cdot l_d \leq w_d^g\}$$

and hence $\mathsf{pass}(\bar{f},\bar{p}) = \mathsf{pass}(f',\bar{p})$ and $\mathsf{rev}(\bar{f},\bar{p}) < \mathsf{rev}(\bar{f},\bar{p}) + \delta \cdot \mathsf{pass}(\bar{f},\bar{p}) = \mathsf{rev}(f',\bar{p})$ which is a contradiction to $(\bar{f},\bar{p})$ being an efficient solution. ◀

Lemma 10 shows that we can interpret our problem as least absolute deviation (LAD) regression problem (see, e.g., [2, 23, 8, 24]): Given a set of points with weights, find a line that minimizes the sum of vertical distances to the given points. In our case, the existing points are (l_d, w_d^g) with weights t_d^g for every OD pair $d \in D$ and every demand group $g \in [G_d]$. We want to fit these points by a line with intercept f and slope p, i.e., $x \mapsto f + p \cdot x$. For the evaluation of the fit we distinguish two cases:

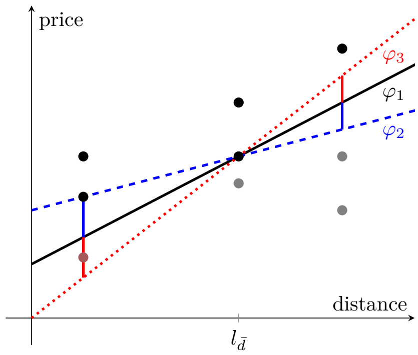


Figure 1 Example of three OD pairs with distances 1, 4 and 6 and with three groups each. They are marked based on their distance and willingness to pay. Line φ_1 (black, solid) meets exactly one willingness to pay exactly. The point $(l_{\bar{d}}, w_{\bar{d}}^{\bar{g}})$ is the fixed point. The line can be rotated clockwise to line φ_2 (blue, dashed), or it can be rotated counterclockwise to line φ_3 (red, dotted). For φ_2 the willingness to pay of two groups is met exactly. The black groups are always attracted, the gray are attracted in none of the scenarios, and the gray-red group is only attracted in case of φ_3 (red, dotted). Vertical lines show the price difference for each OD pair in the different scenarios.

- If (l_d, w_d^g) is on or above the line, we add t_d^g to $\mathsf{pass}(f,p)$. For $\mathsf{rev}(f,p)$, we could have achieved the full amount of the willingness to pay w_d^g, but we realize only the point on the line, i.e., $f + p \cdot l_d$. The vertical distance $w_d^g - (f + p \cdot l_d)$ between the point (l_d, w_d^g) and the line is what we lose and hence what we want to minimize.
- If (l_d, w_d^g) is below the line, the OD pair is lost and hence does not contribute to any of the two objective functions.

Lemma 10 then says that any optimal line passes through at least one of the points. For unrestricted LAD lines it is furthermore known that there always exists an optimal line that passes through two of the points. In our case, the parameters of the line are restricted to be positive, i.e., $f \geq 0$ and $p \geq 0$. Taking this restriction into account leads to the statement of Theorem 11.

▶ **Theorem 11.** *For every non-dominated point, there is an efficient solution (f,p) such that one of the following holds: The willingness to pay of*
- *two groups is met exactly, i.e., there are $d_i \in D$, $g_i \in [G_{d_i}]$ for $i \in \{1,2\}$ with $d_1 \neq d_2$ and $w_{d_i}^{g_i} = f + p \cdot l_{d_i}$,*
- *one group is met exactly and $p = 0$, i.e., there is some $d \in D$, $g \in [G_d]$ with $w_d^g = f$,*
- *one group is met exactly and $f = 0$, i.e., there is some $d \in D$, $g \in [G_d]$ with $w_d^g = p \cdot l_d$.*

Proof. Let (f_1, p_1) be an efficient solution, in particular $f_1, p_1 \in \mathbb{R}_{\geq 0}$. By Lemma 10, there is at least one willingness to pay that is met exactly. We consider the case that only exactly one willingness to pay $\bar{w} := w_{\bar{d}}^{\bar{g}}$ for some $\bar{d} \in D$, $\bar{g} \in [G_{\bar{d}}]$ is met exactly and that neither $p_1 = 0$ nor $f_1 = 0$. We fix $(\bar{w}, l_{\bar{d}})$ and rotate the line $f_1 + p_1 \cdot x$ in the following ways as illustrated in Figure 1:

We choose (f_2, p_2) as the optimal solution to

$$\begin{aligned} \min_{f,p}\ & p \\ \text{s.t. } & \bar{w} = f + p \cdot l_{\bar{d}} \\ & S_{\text{dem}}(f_1, p_1) \subseteq S_{\text{dem}}(f,p) \\ & p_1 \geq p \geq 0 \\ & f \geq 0. \end{aligned}$$

Hence, (f_2, p_2) still meets $\bar{w}$ for OD pair $\bar{d}$. The line $f_2 + p_2 \cdot x$ is less steep than $f_1 + \bar{p}_1 \cdot x$, with a non-negative slope for which all demand groups that are attracted by (f_1, p_1) are also attracted by (f_2, p_2).

Analogously, we choose (f_3, p_3) as the optimal solution to

$$\begin{aligned} \max_{f,p}\ & p \\ \text{s.t.}\ \ & \bar{w} = f + p \cdot l_{\bar{d}} \\ & S_{\text{dem}}(f_1, p_1) \subseteq S_{\text{dem}}(f, p) \\ & p \geq p_1 \\ & f \geq 0. \end{aligned}$$

Note that (f_2, p_2) and (f_3, p_3) are of the form as in the claim. Because of the assumption that only one willingness to pay is met exactly and $p_1 > 0$ and $f_1 > 0$, we have that $p_2 < p_1 < p_3$ and $f_3 < f_1 < f_2$. For $i \in \{1, 2, 3\}$, we define $\varphi_i \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ x \mapsto f_i + p_i \cdot x$.

By construction, we ensure that the numbers of attracted passengers are not decreased when changing from (f_1, p_1) to (f_2, p_2) or to (f_3, p_3). We show that for at least one of these options also the revenue does not decrease. Note that the revenue related to OD pairs with distance $l_{\bar{d}}$ does not change because the price is kept fixed. Hence, we divide the attracted demand groups based on their respective distances l_d compared to $l_{\bar{d}}$ as follows:

$$S_{\text{dem},L} := \{(d, g) \in S_{\text{dem}}(f_1, p_1) : l_d < l_{\bar{d}}\} \quad \text{and} \quad S_{\text{dem},R} := \{(d, g) \in S_{\text{dem}}(f_1, p_1) : l_d > l_{\bar{d}}\}.$$

For all $d \in D$, we set $\delta_d := |l_{\bar{d}} - l_d|$. From that, we obtain that $l_d = l_{\bar{d}} - \delta_d$ for $(d, g) \in S_{\text{dem},L}$ and $l_d = l_{\bar{d}} + \delta_d$ for $(d, g) \in S_{\text{dem},R}$, which yields for all $i \in \{1, 2, 3\}$ and all $d \in D$ that

$$\varphi_i(l_d) = f_i + p_i \cdot (l_{\bar{d}} \mp \delta_d) = f_i + p_i \cdot l_{\bar{d}} \mp p_i \cdot \delta_d = \varphi_i(l_{\bar{d}}) \mp p_i \cdot \delta_d.$$

Then for the difference in revenues, we have

$$\begin{aligned} \mathsf{rev}(f_2, p_2) - \mathsf{rev}(f_1, p_1) \geq & \sum_{(d,g) \in S_{\text{dem},L}} t_d^g \left(\varphi_2(l_{\bar{d}}) - p_2\delta_d\right) + \sum_{(d,g) \in S_{\text{dem},R}} t_d^g \left(\varphi_2(l_{\bar{d}}) + p_2\delta_d\right) \\ & - \sum_{(d,g) \in S_{\text{dem},L}} t_d^g \left(\varphi_1(l_{\bar{d}}) - p_1\delta_d\right) - \sum_{(d,g) \in S_{\text{dem},R}} t_d^g \left(\varphi_1(l_{\bar{d}}) + p_1\delta_d\right) \\ = & \sum_{(d,g) \in S_{\text{dem},L}} t_d^g \cdot \delta_d \cdot (p_1 - p_2) + \sum_{(d,g) \in S_{\text{dem},R}} t_d^g \cdot \delta_d \cdot (p_2 - p_1) \\ = & \underbrace{(p_1 - p_2)}_{>0} \underbrace{\left(\sum_{(d,g) \in S_{\text{dem},L}} t_d^g \cdot \delta_d - \sum_{(d,g) \in S_{\text{dem},R}} t_d^g \cdot \delta_d \right)}_{:=\Delta} \end{aligned}$$

and analogously

$$\begin{aligned} \mathsf{rev}(f_3, p_3) - \mathsf{rev}(f_1, p_1) \geq & \sum_{(d,g) \in S_{\text{dem},L}} t_d^g \left(\varphi_3(l_{\bar{d}}) - p_3\delta_d\right) + \sum_{(d,g) \in S_{\text{dem},R}} t_d^g \left(\varphi_3(l_{\bar{d}}) + p_3\delta_d\right) \\ & - \sum_{(d,g) \in S_{\text{dem},L}} t_d^g \left(\varphi_1(l_{\bar{d}}) - p_1\delta_d\right) - \sum_{(d,g) \in S_{\text{dem},R}} t_d^g \left(\varphi_1(l_{\bar{d}}) + p_1\delta_d\right) \\ = & \sum_{(d,g) \in S_{\text{dem},L}} t_d^g \cdot \delta_d \cdot (p_1 - p_3) + \sum_{(d,g) \in S_{\text{dem},R}} t_d^g \cdot \delta_d \cdot (p_3 - p_1) \\ = & \underbrace{(p_3 - p_1)}_{>0} \underbrace{\left(\sum_{(d,g) \in S_{\text{dem},R}} t_d^g \cdot \delta_d - \sum_{(d,g) \in S_{\text{dem},L}} t_d^g \cdot \delta_d \right)}_{=-\Delta} \end{aligned}$$

Because $\Delta \geq 0$ or $-\Delta \geq 0$, at least one difference is non-negative. If the absolute revenue difference $|\Delta|$ is strictly larger than zero, then (f_1, p_1) is dominated by the respective other distance tariff, which is a contradiction. If it is equal to zero, but the number of passengers has increased, then (f_1, p_1) is again dominated, which is a contradiction. And if it is equal to zero and the number of passengers is the same, then all solutions belong to the same non-dominated point and we can choose the one that satisfies one of the claimed criteria.

Hence, for every non-dominated point, there is an efficient solution satisfying one of the criteria in the claim. ◀

▶ **Corollary 12.** *The problem is tractable, i.e., the number of non-dominated points is polynomial in the input, namely in* $\mathcal{O}((\sum_{d \in D} G_d)^2)$.

Proof. The claim follows from Theorem 11 because there are at most $\sum_{d \in D} G_d$ non-dominated points for efficient solutions (f, p) with $f = 0$ or $p = 0$, and at most $(\sum_{d \in D} G_d)^2$ non-dominated points that meet the willingness to pay of two demand groups exactly. ◀

Algorithm 2 Solution method for D-RPM.

Input : Instance of D-RPM
Output : Set Γ of all non-dominated points

Initialize $\Gamma_1 \leftarrow \emptyset$; $\Gamma_2 \leftarrow \emptyset$; $\Gamma_3 \leftarrow \emptyset$.
// Determine points with a solution with $p = 0$.
Apply Algorithm 1 and let Γ_1 be its result.
// Determine points with a solution with $f = 0$.
Set $f \leftarrow 0$.
for $d \in D$ **do**
  for $g \in [G_d]$ **do**
    Set $p \leftarrow \frac{w_d^g}{l_d}$.
    Update $\Gamma_2 \leftarrow \Gamma_2 \cup \{(\mathsf{rev}(f, p), \mathsf{pass}(f, p))\}$.
// Determine points with a solution that meets the willingness to pay of two groups exactly.
for $d, d' \in D$ with $l_d < l_{d'}$ **do**
  for $g \in [G_d]$, $g' \in [G_{d'}]$ **do**
    Set $p \leftarrow \frac{w_{d'}^{g'} - w_d^g}{l_{d'} - l_d}$.
    Set $f \leftarrow w_d^g - p \cdot l_d$.
    if $f > 0$ and $p > 0$ **then**
      Update $\Gamma_3 \leftarrow \Gamma_3 \cup \{(\mathsf{rev}(f, p), \mathsf{pass}(f, p))\}$.
// Filter for non-dominated points.
Filter $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ for non-dominated solution as described in Section 2. Let Γ be the filtered result.
return Γ

▶ **Theorem 13.** *Algorithm 2 computes the set of all non-dominated points of* D-RPM *in* $\mathcal{O}((\sum_{d \in D} G_d)^3)$.

Proof. Theorem 11 gives a characterization of efficient solutions from which all non-dominated solutions can be determined. In line 2 of Algorithm 2, a superset of all non-dominated solutions with an efficient solution (f, p) with $p = 0$ is determined, in lines 3 to 7 a superset of all those with $f = 0$ and in lines 8 to 13 of all those that meet the willingness to pay of at least two groups exactly are computed. Combinations of demand groups with the same distance are omitted because this would yield an infeasible vertical line. Therefore, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ contains all non-dominated points. In line 14, all dominated solutions are removed and, hence, Γ is the set of all non-dominated points.

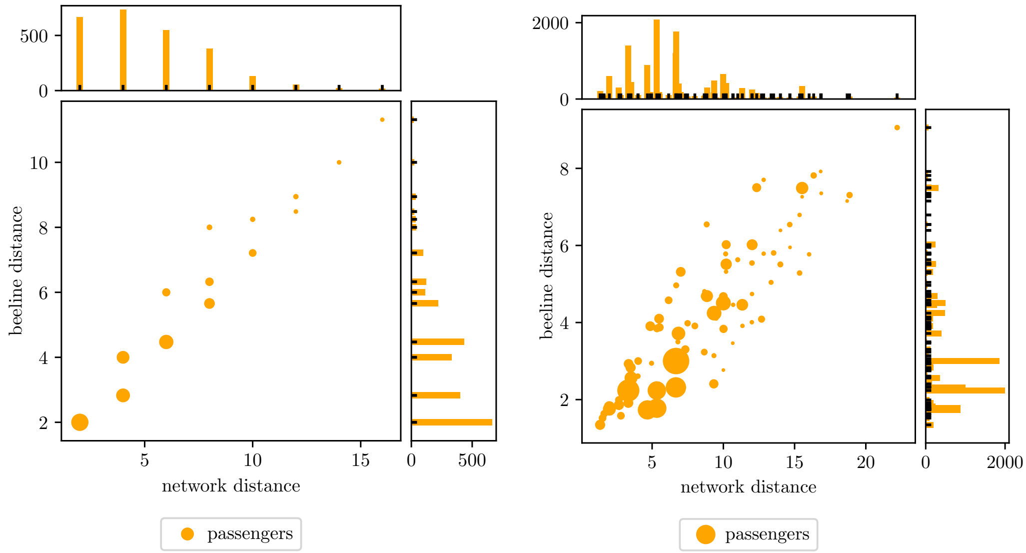
The computations in lines 2 to 7 are in $\mathcal{O}((\sum_{d \in D} G_d)^2)$. In lines 8 to 13, we iterate over the combinations of two demand groups and again iterate over the demand groups for determining the revenue and the number of passengers in line 13. This is done in $\mathcal{O}((\sum_{d \in D} G_d)^3)$. Filtering $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ for non-dominated solutions in line 14 is done in $\mathcal{O}((\sum_{d \in D} G_d)^2 \cdot \log(\sum_{d \in D} G_d))$. Hence, in total, the algorithm is in $\mathcal{O}((\sum_{d \in D} G_d)^3)$. ◀

▶ Remark 14. Note that the running time is significantly influenced by the number of OD pairs with the same distance because the for-loop in line 8 of Algorithm 2 is only performed for OD pairs d' with a larger distance than that of OD pair d, but in particular not for those with the same distance. Hence, the loops over the demand groups and the computation of the objective function value are omitted for OD pair combinations with the same distance.

5 Computational Experiments

The revenue-passenger model introduced in this paper is tested on artificial instances based on the data sets `grid` and `mandl` from the open source software library LinTim [20, 21]. The PTNs provided for each of the data sets can be used to compute network and beeline distances between any pair of stations. The distributions of the demand with respect to the network and beeline distances is shown in Figure 2. Data set `mandl` consists of 172 OD pairs that have 72 different network distances and 84 beeline distances. While data set `grid` even has 567 OD pairs, these belong only to 8 network distances and 14 beeline distances. An overview of the parameters for generating the artificial instances is given in Table 1: The demand data provided in LinTim is split into $G \in \{1, 3, 5\}$ demand groups to create the input demand data of the revenue-passenger model in four different ways (EQUAL, RANDOM, INCREASING, DECREASING). The willingness to pay for each group is generated using a flat tariff (w-FLAT) or an affine distance tariff where the distance is derived from the network (w-NETWORK) or the Euclidean distance (w-BEELINE). The parameters f and p are chosen from three options for affine distance tariffs and one option for flat tariffs. In total, this yields 252 instances per data set. The instances are solved for the revenue-passenger models F-RPM and D-RPM, determining FLAT, NETWORK distance and BEELINE distance tariffs. The solution methods are implemented in Python and the experiments are run on a machine with an Intel(R) Core(TM) i5-1335U and 32 GB of RAM.

Running Time. The running times of Algorithm 1 and Algorithm 2 are depicted in Figure 3. According to Theorem 7 and Theorem 13 the running time of Algorithm 1 is quasilinear in the total number of demand groups while the running time of Algorithm 2 is cubic. This can be observed in the running times: F-RPM can be solved in 0.08 seconds for all `grid` instances and in 0.14 seconds for all `mandl` instances, while the running time of D-RPM increases to up to 13 seconds for `grid` and to 46 seconds for `mandl`, respectively.



(a) Demand data for data set `grid`.

(b) Demand data for data set `mandl`.

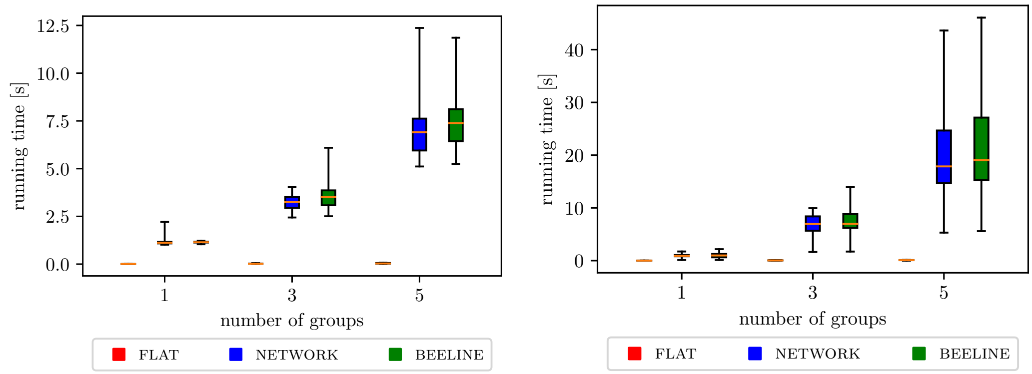
Figure 2 Demand data with respect to the different PTNs. The size of a point reflects on the demand. Above and on the right hand side of the plots, the demand with the same network or beeline distance, respectively, is aggregated.

Table 1 Parameters for generating artificial instances.

Parameter	Value	Explanation
demand groups	$G \in \{1, 3, 5\}$	number of groups $G_d = G$ for all OD pairs $d \in D$
demand split	EQUAL RANDOM INCREASING DECREASING	$\forall d \in D, \forall g \in [G] : t_d^g = \left\lceil \frac{t_d}{G} \right\rceil$ $\forall d \in D, \forall g \in [G] : t_d^g \in \{1, \dots, t_d\}$ random with $\sum_{g=1}^{G} t_d^g = t_d$ $\forall d \in D, \forall g \in [G-1] : t_d^g = \left\lceil \frac{t_d}{2^g} \right\rceil$ and $t_d^G = \left\lfloor \frac{t_d}{2^{G-1}} \right\rfloor$ $\forall d \in D, \forall g \in [G-1] : t_d^g = \left\lceil \frac{t_d}{2^{G+1-g}} \right\rceil$ and $t_d^1 = \left\lfloor \frac{t_d}{2^{G-1}} \right\rfloor$
willingness to pay	w-FLAT w-NETWORK w-BEELINE	tariff used to generate willingness to pay
tariff parameter	A B C	$\forall g \in [G] : f_g = g, p_g = 0.2$ $\forall g \in [G] : f_g = g, p_g = 0.6 - 0.1g$ $\forall g \in [G] : f_g = 1, p_g = 0.1g$

Figure 2 shows that the input data of `grid` is very structured and that only a few different distances occur, especially for the network distance. As suggested in Remark 14, this reflects on the running times, which is smaller for `grid` than for `mandl`, even though `grid` has roughly three times as many OD pairs as `mandl`.

These running times are orders of magnitude smaller compared to the running times of the MILP-based approach using Gurobi 10.01 [14] for solving the MILP of D-RPM. Figure 4 and Table 2 show that Algorithm 2 for D-RPM, that exploits the structure of distance tariffs, is much faster than the MILP-based approach. For NETWORK D-RPM, in 68% of the instances of `grid` with 5 demand groups, it was in some iteration not possible to even



(a) Data set `grid`. **(b)** Data set `mandl`.

Figure 3 Running times in seconds for computing the complete Pareto fronts with Algorithm 1 for F-RPM and with Algorithm 2 for NETWORK and BEELINE D-RPM.

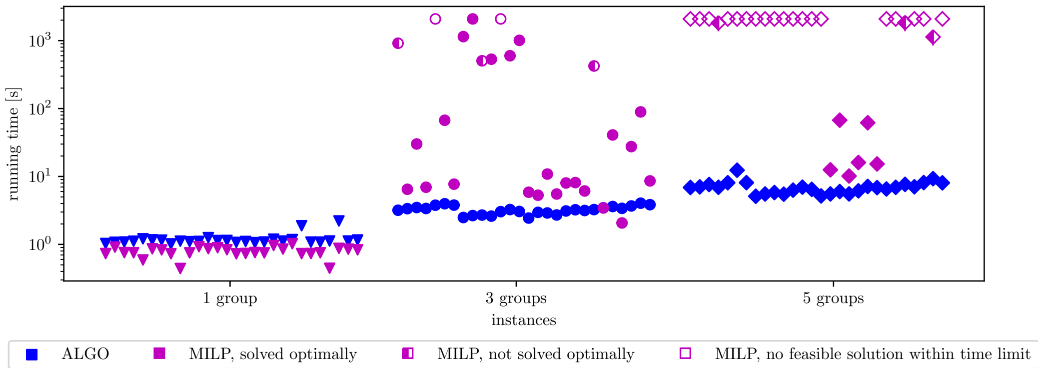


Figure 4 Running times in seconds of NETWORK D-RPM with Algorithm 2 (ALGO) and with the MILP-based method (MILP) on data set `grid` with a logarithmic scale. Each marker represents the running time for computing the Pareto front of a single instance. The time limit for solving each MILP within the ϵ-constraint method is set to 300 seconds. If a MILP could not be solved to optimality within this time limit but a feasible solution was found, then we continue with this feasible solution and label the instance as "MILP, not solved optimally". If no feasible solution is found, the procedure terminates and we label the instance as "MILP, no feasible solution within time limit" and depict it with the maximum running time in this figure.

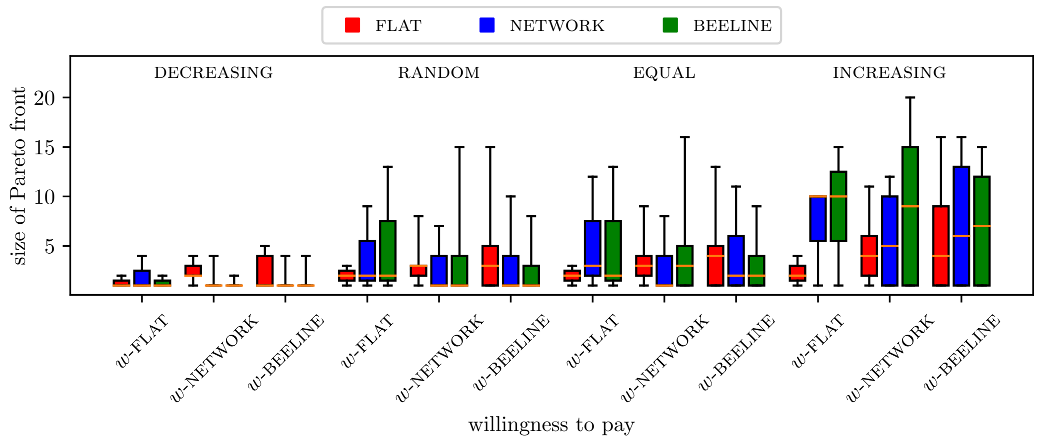
Table 2 Mean, minimum and maximum running times in seconds for solving NETWORK D-RPM on the `grid` instances with Algorithm 2 (ALGO) and with the MILP-based method (MILP). Only the instances that were solved optimally are considered.

	running time ALGO			running time MILP		
groups	**mean**	**min**	**max**	**mean**	**min**	**max**
1	1.19	1.02	2.21	0.78	0.44	1.02
3	3.23	2.44	4.04	248.36	2.07	2086.07
5	6.95	5.11	12.37	30.53	10.15	67.35
all	3.79	1.02	12.37	103.81	0.44	2086.07

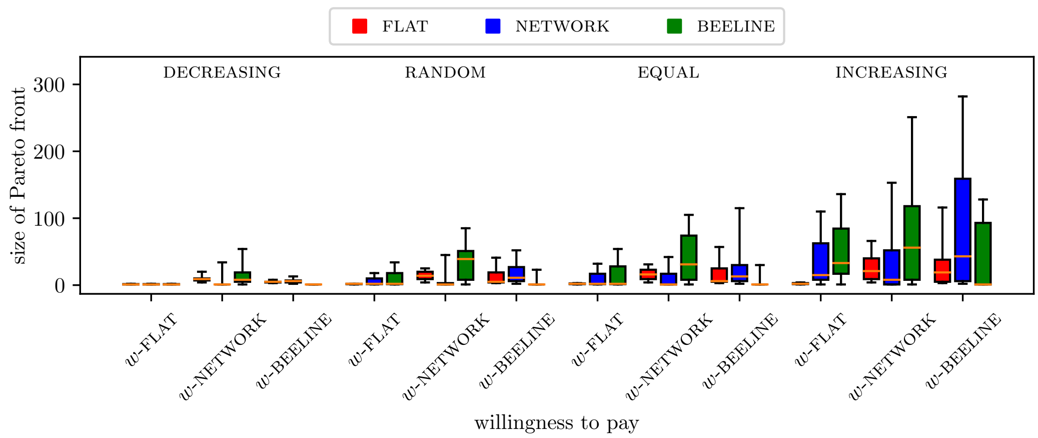
determine a feasible solution within the time limit. With 3 demand groups this still happened to 7% of the instances. In both cases, 11% terminated with a feasible, but not necessarily optimal solution. Just in case of only 1 demand group, the MILP-based approach performs slightly better than Algorithm 2 with a mean running time of 0.78 seconds compared to 1.19 seconds. Note that strengthening the formulation of the MILP could improve the running time of the MILP-based approach.

Size of the Pareto Front. Figure 5 shows the number of points on the Pareto front for the different options for the demand splits and for the generation of the willingness to pay. We can observe two main effects:

First, a DECREASING demand split leads to a small size of the Pareto front. This is because the price increase cannot compensate for loosing large demand groups with a low willingness to pay. We see the reverse effect for INCREASING which leads to the most points on the Pareto front because loosing only small demand groups with a low willingness to pay is compensated in the revenue by the increased price.



(a) Data set `grid`.



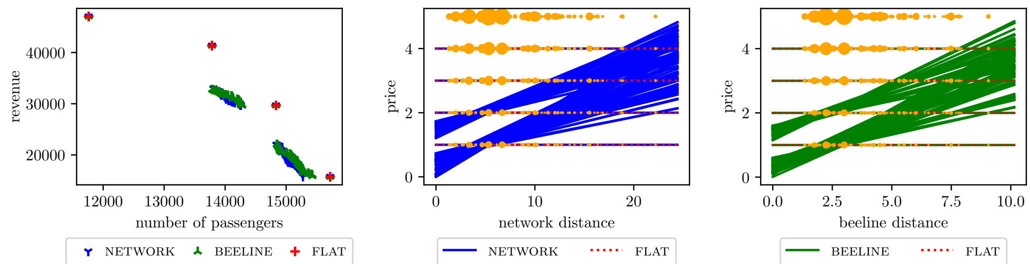
(b) Data set `mandl`.

Figure 5 Size of the Pareto front dependent on the demand split and the tariff used to generate the willingness to pay.

Second, if the fare strategy of the input tariff (for generating the willingness to pay) and the output tariff coincide, the size of the Pareto front is smaller. In this case, an output tariff might be chosen exactly as the willingness to pay of one demand group level, which is in general not possible if they differ.

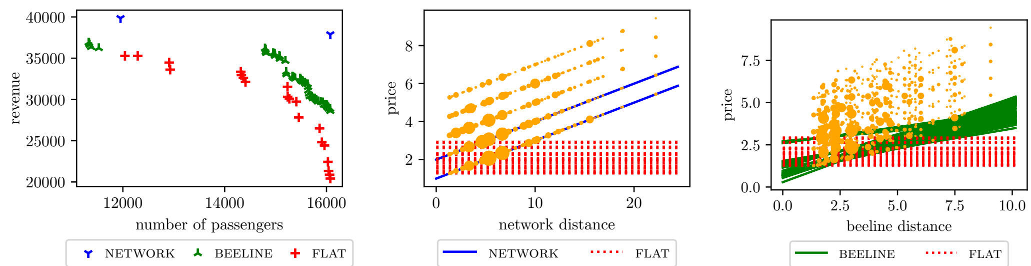
Structure of the Pareto Front, Efficient Tariffs and Input Data. Figure 6 to Figure 9 show the Pareto fronts in (a) and corresponding efficient solutions in (b) and (c) for selected parameter settings for the `mandl` instances. Additionally, (b) and (c) show the demand as points (l_d, w_d^g) weighted with the number of potential passengers t_d^g. The figures for `grid` can be found in Appendix A. In these cases, we can see well that coinciding input and output tariffs lead to a small sized Pareto front that even dominates many of the points of the other tariff types. For w-BEELINE, the Pareto front of BEELINE D-RPM dominates the Pareto front of NETWORK D-RPM, and vice versa for w-NETWORK. Particularly in Figure 6, it is visible that the distinct points on the Pareto front belong to solutions that are a FLAT tariff. Only in this setting with the willingness to pay being generated by w-FLAT, we obtain a Pareto front for F-RPM that is not dominated by both, the Pareto fronts of BEELINE and NETWORK D-RPM. This is however not surprising because a flat tariff is a special case of a distance tariff.

Moreover, in many cases, the efficient tariffs are located on the lower levels of the demand groups, meaning that it is not beneficial to increase the price to the highest willingness to pay. For example, Figure 6 constitutes an exception, where it is an efficient solution to choose a flat tariff with a fixed price equal to the second highest willingness to pay. However, we also see here that the highest willingness to pay does not lead to an efficient solution.



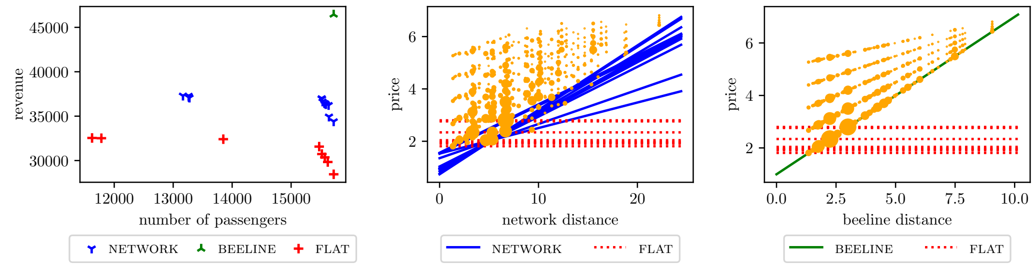
(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 6 Instance of `mandl` with 5 demand groups and parameters INCREASING/w-FLAT/A.



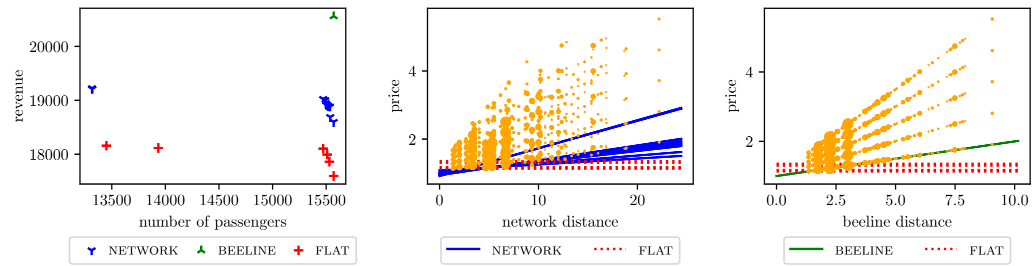
(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 7 Instance of `mandl` with 5 demand groups and parameters RANDOM/w-NETWORK/A.



(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 8 Instance of `mandl` with 5 demand groups and parameters DECREASING/w-BEELINE/B.



(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 9 Instance of `mandl` with 5 demand groups and parameters EQUAL/w-BEELINE/C.

6 Discussion and Outlook

In this paper, we have introduced a bi-objective model for fare planning maximizing the revenue and the number of passengers with different demand groups per OD pair. Specialized algorithms for flat and distance tariffs showed a significant reduction in running time in computational experiments on artificial data sets from the software library LinTim.

Another common fare strategy are zones tariffs. For counting zone tariffs, for which the price depends on the number of traversed zones, the MILP-based method can be performed by applying the MILP proposed by [19] and adding the constraint restricting the number of passengers. However, because this MILP has a high running time in practice, it cannot be expected to compute the whole Pareto front, even for small instances. Future work could encompass the design of a specialized algorithm for zone tariffs.

Computational experiments show that it is worth to look into the revenue-passenger model for the specific fare strategies. Exploiting the structure of tariffs, leads to methods that allow for the computation of the complete Pareto front. This yields a wide range of information for public transport operators to choose a tariff that serves their financial requirements as well as promotes public transport with the aim to attract and increase the demand.

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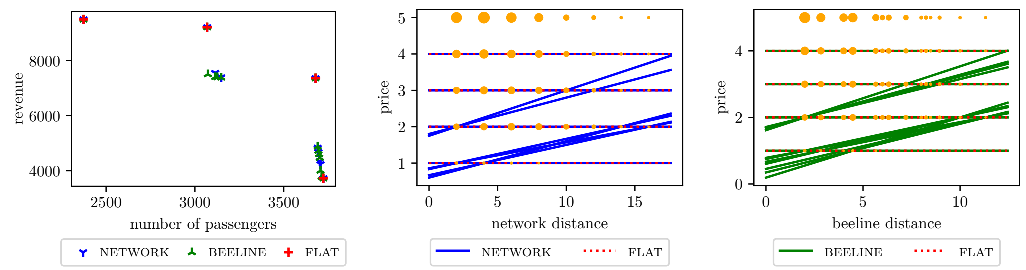
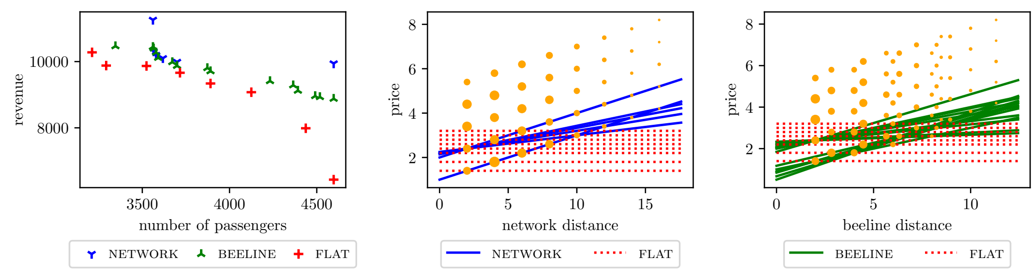
A Pareto Fronts and Efficient Solutions for Selected `grid` Instances

Figure 10 to Figure 13 show the Pareto fronts in (a) and corresponding efficient solutions in (b) and (c) for selected parameter settings for the `grid` instances. Additionally, (b) and (c) show the demand as points (l_d, w_d^g) weighted with the number of potential passengers t_d^g.



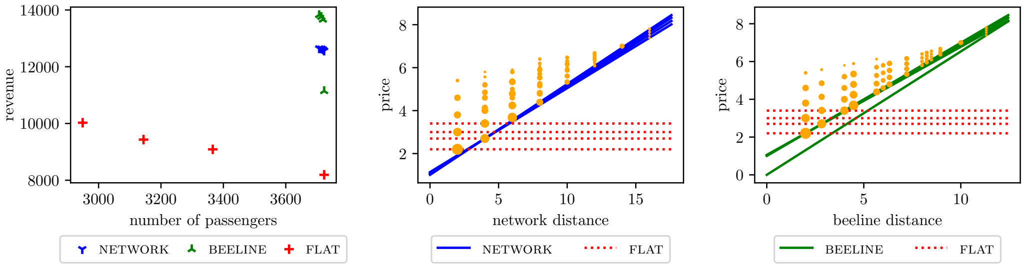
(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 10 Instance of `grid` with 5 demand groups and parameters INCREASING/w-FLAT/A.



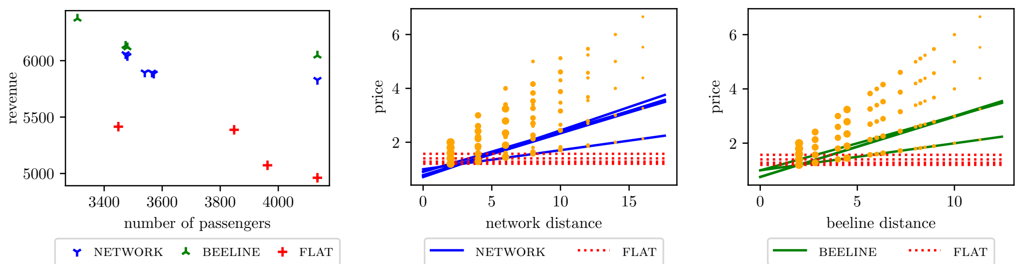
(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 11 Instance of `grid` with 5 demand groups and parameters RANDOM/w-NETWORK/A.



(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 12 Instance of `grid` with 5 demand groups and parameters DECREASING/w-BEELINE/B.



(a) Pareto fronts. **(b)** NETWORK and FLAT tariff. **(c)** BEELINE and FLAT tariff.

Figure 13 Instance of `grid` with 5 demand groups and parameters EQUAL/w-BEELINE/C.