Cheapest Paths in Public Transport: Properties and Algorithms

Anita Schöbel
Technische Universität Kaiserslautern, Germany
Fraunhofer-Institute for Industrial Mathematics ITWM, Kaiserslautern, Germany
schoebel@mathematik.uni-kl.de

Reena Urban
Technische Universität Kaiserslautern, Germany
urban@mathematik.uni-kl.de

Abstract

When determining the paths of the passengers in public transport, the travel time is usually the main criterion. However, also the ticket price a passenger has to pay is a relevant factor for choosing the path. The ticket price is also relevant for simulating the minimum income a public transport company can expect.

However, finding the correct price depends on the fare system used (e.g., distance tariff, zone tariff with different particularities, application of a short-distance tariff, etc.) and may be rather complicated even if the path is already fixed. An algorithm which finds a cheapest path in a very general case has been provided in [6], but its running time is exponential. In this paper, we model and analyze different fare systems, identify important properties they may have and provide polynomial algorithms for computing a cheapest path.

2012 ACM Subject Classification Applied computing → Transportation

Keywords and phrases Public Transport, Fare Systems, Modeling, Cheapest Paths

Digital Object Identifier 10.4230/OASIcs.ATMOS.2020.13

1 Introduction

Fare systems may be very diverse, containing a lot of different rules and regulations. Among them is the unit tariff in which all journeys cost the same, no matter how long they are, or kilometer-based distance tariffs which are used by most railway companies all over the world. Very popular in metropolitan regions are zone tariffs (used in many German regional transport networks and in many European cities, but also, e.g., in California). In most regions, these fare systems come with special regulations: Journeys with less than a given number of stops may get a special price, there might be network-wide tickets, or stations belonging to more than one zone. The underlying fare system is usually independent of the way tickets are bought: they can be provided as paper tickets from ticket machines or from online sales, by usage of smart cards in check-in-check-out systems, or by other mobile devices. Recently, some public transport companies offer the simple usage of a mobile device for charging the beeline tariff between the start coordinates and the end coordinates of the journey. Sometimes all these different fare systems are combined.

The question which we pursue in our paper is how to find the cheapest possibility to travel between two stations. This question is relevant for several reasons. First, the passengers would like to minimize their ticket prices as one among other criteria when planning their journeys. Second, a public transport company can only estimate its income if ticket prices are known for the demand. Simulating the journeys of the passengers together with their ticket prices for some given demand is common for dividing the income of traffic associations

20th Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2020).
Editors: Dennis Huisman and Christos D. Zaroliagis; Article No. 13; pp. 13:1–13:16
OpenAccess Series in Informatics
OASICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

between the single public transport providers. Third, for designing and improving fare systems, it is necessary to be able to compute (cheapest) ticket prices.

A model together with an algorithm for computing cheapest paths which are able to cover most of the possible regulations are developed in [6]. The idea is to define transition functions between tickets over partially ordered monoids. However, since so many particularities are covered, the algorithm needs exponential time. In this paper, we analyze properties of special fare systems, for example, a plain distance tariff and two variations of zone tariffs. This does not cover all particularities simultaneously, but allows to derive analytical properties and to design algorithms which are based on shortest path techniques and hence run in polynomial time for many common fare systems.

The properties we are going to investigate are the following:

No-stopover property: Can we be sure that passengers cannot save money by splitting a journey into two (or more) parts and buying separate tickets for each of these sub-journeys?

No-elongation property: Can we be sure that passengers cannot save money by buying a ticket for a longer journey although they only use a part of it?

In (real-world) fare systems, these two properties need not be satisfied. As will be shown, there is also no relation between them. The third property we investigate is the well-known subpath-optimality property from dynamic programming.

Subpath-optimality property: Is any subpath of a cheapest path again a cheapest path?

The first two properties are relevant from a real-world point of view, since they ensure that a fare system is consistent and does not trigger strange actions (e.g., buying a ticket for a longer path than needed) as a legal way of saving money. In Section 3 of [12] the authors say that a fare system without the no-stopover property would be "impractical and potentially confusing for the customer". Still, as we will see, this property is not always satisfied in real-world fare systems. The subpath-optimality property is relevant for the design of algorithms.

1.1 Related Literature

Literature on fare systems is scarce compared to papers on timetabling or scheduling in public transport. Early papers deal with the design of (fair) zone tariffs [9, 10, 3], a topic which is still ongoing using different types of objectives, e.g., the income of the public transport company [2, 7, 13]. Also the (backward) design of distance tariffs from zone tariffs has been studied [11]. The computation of cheapest paths has been considered for distance tariffs in a railway context in [12], while [4, 5] compute paths that visit the smallest number of tariff zones. Recently, [6] present the so-called ticket graph which models transitions between tickets via transition functions over partially ordered monoids and allows the design of an algorithm for finding cheapest paths in fare systems which do not have the subpath-optimality property. However, the running time of this approach need not be polynomial.

1.2 Our contribution

We present models for the following fare systems: unit tariff, distance tariff, beeline tariff, zone tariff, and zone tariff with metropolitan zone. For these fare systems we analyze the no-stopover property, the no-elongation property, and the subpath-optimality property. Furthermore, we develop polynomial algorithms for computing cheapest paths for all of these cases. This shows that for many practically used fare systems, cheapest paths can be computed in polynomial time although this is not the case for general fare systems [6].

2 Modeling Different Fare Systems

We consider different types of fare systems which we define in this section. We first specify what a fare system is. We are not aware of such a formal definition in the literature.

Let a *Public Transport Network (PTN)* be given. PTN $= (V, E)$ is a graph given by a set of stops or stations V and a set E of direct connections between them. For simplicity we assume the PTN to be an undirected graph which is simple and connected. The PTN can be used to model railway, tram, or bus networks. The price of a journey through a PTN depends not only on the start station and the end station of the journey, but also on the specific path that has been chosen. Note that this is sometimes included implicitly, e.g., in railway tickets which include origin and destination, but specify a geographical *travel corridor* which is allowed for the journey (given by intermediate stops between which the corridor can be looked up). Hence, we need the set of all paths of the PTN, which we denote by $\mathcal{W}$.

▶ **Definition 1.** *Let a PTN be given and let $\mathcal{W}$ be the set of all paths in the PTN. A* fare system *is a function $p : \mathcal{W} \to \mathbb{R}_{\geq 0}$ that assigns a price to every path in a PTN.*

Note that a fare system always involves a PTN.

For a path $W = (x_1, \ldots, x_n)$, we denote a subpath $(x_i, \ldots, x_j)$ with $1 \leq i < j \leq n$ by $[x_i, x_j]$. The price of a subpath is hence given as $p([x_i, x_j])$. The brackets $[\cdot]$ emphasize that $[x_i, x_j]$ describes a path and not only a pair of stations. To look at paths in the PTN and not at timetabled trips is a simplification because a ticket usually has a maximal duration how long it is valid. Stopovers may be allowed as long as this maximal duration is not exceeded. If a passenger combines two journeys, e.g., she first travels to the house of her uncle for a visit and then travels to university to get some important documents, she might be able to do this within the same ticket or she buys two separate tickets. In the latter case, we call her path a *compound path* and its price is the sum of the prices of the two tickets that she bought.

We now define the fare systems which we study in this paper. The simplest fare system is a *unit tariff* in which all trips cost the same.

▶ **Definition 2.** *Let a PTN be given and let $\mathcal{W}$ be the set of all paths in the PTN. A fare system p is a unit tariff w.r.t. $\bar{p} \geq 0$ if $p(W) = \bar{p}$ for all $W \in \mathcal{W}$.*

Unit tariff fare systems are often considered as unfair. They are used within (even big) cities. The contrary is that the price of a journey depends on the kilometers traveled. In a *distance tariff* the length of the journey is used, while in a *beeline tariff* the airline distance is the basis for the ticket price. To define these two fare systems, we use $l(W)$ to denote the length of a path $W = (x_1, \ldots, x_n)$ (in kilometers), and $l_2(W) = \|x_n - x_1\|$ as its beeline distance. In order to compute $l(W)$, we assume that each edge in the PTN has assigned its physical length, and to compute the beeline distance, we assume that the stations V of the PTN are embedded in the plane such that the Euclidean distance l_2 between every pair of stations can be computed.

▶ **Definition 3.** *Let a PTN be given and let $\mathcal{W}$ be the set of all paths in the PTN. A fare system p is a distance tariff w.r.t. $\bar{p}, f \geq 0$ if $p(W) = f + \bar{p} \cdot l(W)$ for all $W \in \mathcal{W}$.*

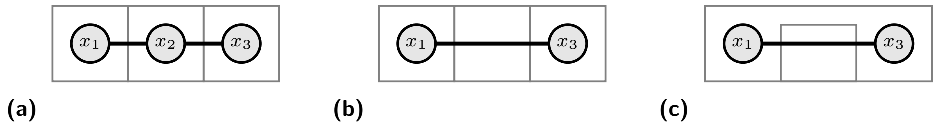


Figure 1 PTNs with zone partitions for Example 5.

▶ **Definition 4.** *Let a PTN be given and let $\mathcal{W}$ be the set of all paths in the PTN. A fare system p is a beeline tariff w.r.t. $\bar{p}, f \geq 0$ if $p(W) = f + \bar{p} \cdot l_2(W)$ for all $W \in \mathcal{W}$.*

Note that in case of $\bar{p} = 0$, both the distance tariff and the beeline tariff become unit tariffs. An important property of the beeline tariff is that it does not depend on the whole journey, but only on its start and end point. Most railway systems rely on distance tariffs (or modifications). Beeline tariffs are rather new and often used for mobile tickets on mobile phones or internet devices which track the journey of a passenger by using her GPS coordinates and determining the price based on the beeline distance after the journey is over.

Zone tariffs are somehow intermediate between unit tariffs and distance tariffs. The whole region is divided into *tariff zones* and the length of a journey is approximated by the number of zones it visits. The ticket price then depends only on the number of visited zones. This is considered as more fair than the unit tariff, but it is also more complicated. For modeling a zone tariff, we use the PTN. The geographical zones imply a partition $\mathcal{Z} = \{Z_1, \dots, Z_K\}$ of the set of stations V, i.e., $V = \bigcup_{i=1,\dots,K} Z_i$ and the Z_i are pairwise disjoint. For every edge $(x, y) \in E$ of the PTN, the value $b(x, y)$ of the *border function* denotes the number of zone borders crossed when traveling between x and y. If $b(x, y) = 0$, both stations x and y belong to the same zone, while the reverse direction need not hold, see Figure 1c in Example 5. We consider the border function as an additional edge weight which is given together with the PTN. From that, we can derive for a path $W = (x_1, \dots, x_n)$ the *zone function* $z(W) = z(x_1, \dots, x_n) := 1 + \sum_{i=1}^{n-1} b(x_i, x_{i+1})$, which determines the number of zones which are visited by the path W. We illustrate the way to count zones in the following example.

▶ **Example 5.** For the situations shown in Figure 1, we determine the value of the zone function for the x_1-x_3-path.

(a) We have $b(x_1, x_2) = 1$, $b(x_2, x_3) = 1$. For $W = (x_1, x_2, x_3)$ we get $z(W) = 1 + 1 + 1 = 3$.
(b) Here, $b(x_1, x_3) = 2$, hence for $W = (x_1, x_3)$ we have $z(W) = 1 + 2 = 3$.
(c) Although x_1 and x_3 belong to the same zone, the edge between them crosses another zone. Hence, $b(x_1, x_3) = 2$ and for $W = (x_1, x_3)$ we get $z(W) = 1 + 2 = 3$.

We now have the preliminaries to define a zone tariff.

▶ **Definition 6.** *Let a PTN be given and let $\mathcal{W}$ be the set of all paths in the PTN. A fare system p is a zone tariff w.r.t. the price function $P \colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ if $p(W) = P(z(W))$ for all $W \in \mathcal{W}$.*

The price function P assigns a ticket price to each number of visited zones. If it is constant, the zone tariff simplifies to a unit tariff.

Many zone tariffs include particularities. A common one is the definition of metropolitan zones in which a subset of zones $\mathcal{Z}_M \subseteq \mathcal{Z}$ is combined to a common zone $Z_M = \bigcup_{Z \in \mathcal{Z}_M} Z$, the *metropolitan zone*. For journeys which cross the metropolitan zone or start or end there, the zones are counted as in the basic zone tariff. For journeys within the metropolitan zone,

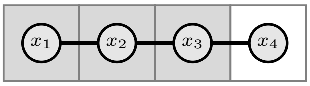


Figure 2 PTN with zones for Example 8.

a special price is fixed. A higher price might be charged if the metropolitan zone has a well-developed public transport network or is much larger than a usual zone. A lower price might be chosen in order to make public transport more attractive, e.g., in city regions to reduce the car traffic.

In order to describe the metropolitan zone, we could save which stations and which edges of the PTN belong to Z_M. Algorithmically, we define weights $z_M(e) \in \{0, 1\}$ for every edge $e \in E$ in the PTN by

$$z_M(e) := \begin{cases} 0 & \text{if } e \text{ is completely contained in } Z_M, \\ 1 & \text{otherwise.} \end{cases}$$

Together with the border function b, we save the values of $z_M(e)$ as information with the PTN. For a path W, we have $z_M(W) := \sum_{e \in E(W)} z_M(e)$. We say that a path W is *included in the metropolitan zone* Z_M if all of its stations and all of its edges are completely contained in Z_M, i.e., if the path never leaves the metropolitan zone. Formally, this means that $z_M(W) = 0$.

The formal definition of this fare system is:

▶ **Definition 7.** *Let a PTN be given and let $\mathcal{W}$ be the set of all paths in the PTN. A fare system p is a zone tariff with metropolitan zone Z_M, a price function $P\colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ and a price $P_M \in \mathbb{R}_{\geq 0}$ if we have for every path $W \in \mathcal{W}$ that*

$$p(W) = \begin{cases} P_M & \text{if } W \text{ is included in the metropolitan zone } Z_M, \text{ i.e., if } z_M(W) = 0, \\ P(z(W)) & \text{otherwise.} \end{cases}$$

▶ **Example 8.** As an example for metropolitan zones consider Figure 2. We have four zones and the zones highlighted in gray form a metropolitan zone. For the path $W_1 = (x_1, x_2, x_3)$ which is included in the metropolitan zone, the metropolitan price $p(W_1) = P_M$ is applied. On the other hand, the price for the path $W_2 = (x_1, x_2, x_3, x_4)$ is computed as in the basic zone tariff and is given by $p(W_2) = P(4)$.

Note that also zone tariffs with several metropolitan zones are possible (and can be defined as above). Paths traveling through a metropolitan zone may be also treated in other ways, e.g., the metropolitan zone always counts as two zones.

Each of the considered types of fare systems is uniquely defined by a PTN and some price information, e.g., the fix price f and the price per kilometer $\bar{p}$ for a distance tariff or the price function P for a zone tariff.

3 Properties of Fare Systems

Before analyzing the no-stopover property, the no-elongation property, and the subpath optimality property, we define them formally. For that, let a PTN $= (V, E)$ with a fare system p be given.

▶ **Definition 9.** *A path* $(x_1, \ldots, x_n) \in \mathcal{W}$ *satisfies the no-stopover property if*

$$p([x_1, x_n]) \leq p([x_1, x_i]) + p([x_i, x_n])$$

for all intermediate stops x_i *with* $i = 2, \ldots, n-1$. *A fare system satisfies the no-stopover property if it is satisfied for all paths in the PTN.*

The no-stopover property says that a compound path $[x_1, x_i] \circ [x_i, x_n]$ is never preferable to buying a ticket for the complete path $[x_1, x_n]$, i.e., making a stopover does never decrease the ticket price. If a single path satisfies the no-stopover property, it might nevertheless be beneficial to have multiple stopovers as the following path $W = (x_1, x_2, x_3, x_4)$ with four stations and the following ticket prices show: $p([x_1, x_4]) = 10$, $p(x_1, x_2) = p(x_2, x_3) = p(x_3, x_4) = 3$ and $p([x_1, x_3]) = p([x_2, x_4]) = 7$. However, in case that the no-stopover property holds for the whole fare system p, also multiple stopovers of a single path are not helpful, since for several stopovers at $x_{i_1}, \ldots, x_{i_k}$ we have that

$$\underbrace{\underbrace{p([x_1, x_{i_1}]) + p([x_{i_1}, x_{i_2}])}_{\geq p([x_1, x_{i_2}])} + p([x_{i_2}, x_{i_3}])}_{\geq ([x_1, x_{i_3}])} + \cdots + p([x_{i_k}, x_n]) \geq p([x_1, x_n]).$$

▶ **Definition 10.** *A path* $(x_1, \ldots, x_n) \in \mathcal{W}$ *fulfills the no-elongation property if it holds that* $p([x_1, x_{n-1}]) \leq p([x_1, x_n])$. *A fare system satisfies the no-elongation property if it is satisfied for all paths in the PTN.*

The no-elongation property says that buying a ticket for a longer path which includes the journey a passenger really wants to travel is never preferable to buying the ticket for the subpath, i.e., that $p([x_i, x_j]) \leq p(W)$ for $W = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$. This holds, since $p([x_i, x_j]) \leq p([x_i, x_{j+1}]) \leq \ldots \leq p([x_i, x_n])$ and $p([x_i, x_n]) \leq p([x_{i-1}, x_n]) \leq \ldots \leq p([x_1, x_n])$ by considering the reverse path $(x_n, \ldots, x_1)$.

In Section 4.3 we will see that the no-stopover property does not imply the no-elongation property, and Section 4.5 will show that the inverse implication does also not hold.

If the no-stopover and the no-elongation property both hold, then we do not need to consider compound paths when searching for the cheapest possibility to travel between two stations x and y and we do not need to look for paths between other pairs of stations that contain an x-y-path as subpath. In other words, if both, the no-elongation and the no-stopover property hold, there always exists a cheapest possibility to travel from x to y which can be realized by a (non-compound) path, i.e., by a path $[x, y]$ for which the passenger buys one single ticket with price $p([x, y])$. This will be used later on and simplifies the situation.

▶ **Definition 11.** *A cheapest path* $(x_1, \ldots, x_n) \in \mathcal{W}$ *satisfies the subpath-optimality property if every subpath* $[x_i, x_j]$, $2 \leq i \leq j \leq n-1$ *is again a cheapest path for its corresponding start and end station. A fare system satisfies the subpath-optimality property if it is satisfied for every cheapest path in the PTN.*

4 Results and Algorithms for Computing Cheapest Paths

4.1 Unit tariff

As a simple warm-up, let us start with the unit tariff. It has the property that every path in the PTN costs the same, so every path is a cheapest path.

▶ **Theorem 12.** *Let p be a unit tariff w.r.t. $\bar{p}$. Then p satisfies the no-stopover property, the no-elongation property and the subpath-optimality property.*

Proof. Consider a path $W = (x_1, \ldots, x_n)$. For any $i \in \{2, \ldots, n-1\}$, the two corresponding subpaths $[x_1, x_i]$ and $[x_i, x_n]$ satisfy $p([x_1, x_i]) + p([x_i, x_n]) = \bar{p} + \bar{p} \geq \bar{p} = p(W)$, so a stopover at x_i does not decrease the ticket price. Also, no subpath $[x_i, x_j]$ can be more expensive than the original path W, hence the no-elongation property is satisfied. Since every path is a cheapest path, also the subpath-optimality property is satisfied. ◀

Finding a cheapest path hence reduces to finding an arbitrary path between two stations which can be done, e.g., by breadth-first search (in which case we would end up with a path with a minimum number of edges). We receive:

▶ **Corollary 13.** *For the unit tariff, a cheapest path can be found in polynomial time.*

4.2 Distance tariff

For the distance tariff, we have for any pair of paths $W_1, W_2 \in \mathcal{W}$ that $l(W_1) \leq l(W_2)$ is equivalent to $p(W_1) \leq p(W_2)$ by definition. Hence, a shortest path is always a cheapest path and vice versa. Consequently, we can use any shortest path algorithm (with corresponding speed-up techniques) for finding a cheapest path.

▶ **Lemma 14.** *For a distance tariff, a cheapest path can be found in polynomial time.*

Also, all three properties are satisfied for distance tariff fare systems.

▶ **Theorem 15.** *Let p be a distance tariff w.r.t. $f, \bar{p}$. Then p satisfies the no-stopover property, the no-elongation property and the subpath-optimality property.*

Proof. For the no-stopover property consider a path $W = (x_1, \ldots, x_n)$ with a possible stopover at x_i, $i \in \{2, \ldots, n-1\}$. Since $l(W) = l([x_1, x_i]) + l([x_i, x_n])$, we know that

$$\begin{aligned} p([x_1, x_i]) + p([x_i, x_n]) &= f + \bar{p} \cdot l([x_1, x_i]) + f + \bar{p} \cdot l([x_i, x_n]) \\ &= f + \underbrace{f + \bar{p} \cdot l([x_1, x_n])}_{=p([x_1, x_n])} \geq p([x_1, x_n]), \end{aligned}$$

hence the no-stopover property holds. For the no-elongation property note that $l([x_1, x_{n-1}]) \leq l([x_1, x_n])$ and hence $p([x_1, x_{n-1}]) \leq p([x_1, x_n])$ is satisfied. The subpath-optimality property is satisfied, since it holds for classical shortest paths. ◀

4.3 Beeline tariff

For the beeline tariff, we use the Euclidean (airline) distance to determine the price of a ticket. This means that the ticket price is only dependent on the location of the start and end station, but not on the specific path chosen to travel between them. Consequently, *all* paths between two stations x and y are cheapest paths and hence can be found in polynomial time, e.g., by breadth-first search.

▶ **Lemma 16.** *For a beeline tariff, a cheapest path can be found in polynomial time.*

One might assume that a beeline tariff satisfies all three of our properties, but this is only the case for the no-stopover and the subpath-optimality property.

▶ **Theorem 17.** *Let p be a beeline tariff w.r.t. $f, \bar{p}$. Then p satisfies the no-stopover property and the subpath-optimality property.*

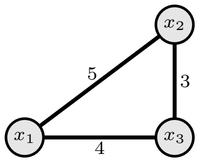


Figure 3 PTN in which the no-elongation property is not satisfied for the beeline tariff.

Proof. The no-stopover property holds, since the Euclidean distance satisfies the triangle inequality, and hence $p([x_1, x_n]) \leq p([x_1, x_i]) + p([x_i, x_n])$ for all $x_1, x_i, x_n \in \mathbb{R}^2$ independent of the specific path $W = (x_1, \ldots, x_n)$. Since all paths are cheapest paths, the subpath-optimality property trivially holds. ◀

However, the no-elongation property is *not* satisfied for the beeline tariff in general as the following small example demonstrates. This example shows that for every $f \geq 0$ and $\bar{p} > 0$, there is a PTN such that the induced beeline tariff does not satisfy the no-elongation property, even if going back to the start station is not allowed.

▶ **Example 18.** Consider any beeline tariff regarding the PTN depicted in Figure 3. The path $W_1 = (x_1, x_2)$ costs $f + 5 \cdot \bar{p}$, which is more than the costs $f + 4 \cdot \bar{p}$ of the elongated path $W_2 = (x_1, x_2, x_3)$.

In our example, passengers would save money by buying a ticket for the path W_2, but leaving the bus already at station x_2. This is avoided in practice, since passengers are tracked by their mobile devices and hence need to checkout at a station which is really visited.

We remark that instead of the Euclidean distance also other metrics can be used, see [15].

4.4 Zone tariff

For zone tariffs, the analysis is a bit more involved as for the fare systems discussed so far. The first observation is that for zone tariffs, the zone prices $P(k)$ determine if the no-stopover property holds.

▶ **Theorem 19.** *Let a price function $P \colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ be given. All zone tariffs w.r.t. P satisfy the no-stopover property if and only if $P(k) \leq P(i) + P(k - i + 1)$ for all $k \geq 3$, $i \in \{2, \ldots, \lfloor \frac{k+1}{2} \rfloor\}$, $i, k \in \mathbb{N}$.*

In particular, if all zone tariffs w.r.t. P satisfy the no-stopover property, then the increase of the price function is bounded by $P(k) \leq (k - 1)P(2)$ for $k \geq 2$.

The proof of the theorem is given in the appendix.

▶ **Example 20.** We provide some examples:

- If the price function P is decreasing for $k \geq 2$, i.e., if $P(k + 1) \leq P(k)$ for all $k \geq 2$, then every zone tariff w.r.t. P satisfies the no-stopover property. This is true since $P(i) + P(k - i + 1) \geq 2P(k) \geq P(k)$ for $2 \leq i \leq k$.
  However, this is an unrealistic price function, since longer trips are cheaper than shorter trips, which is considered as unfair.
- If the price function P is affine and increasing, i.e., if $P(k) = f + k \cdot \bar{p}$ with $\bar{p} \geq 0$, then every zone tariff w.r.t. P satisfies the no-stopover property. This can be verified by computing
  $P(i) + P(k - i + 1) = f + \bar{p} \cdot i + f + \bar{p} \cdot (k - i + 1) = 2f + \bar{p} \cdot (k + 1) \geq f + \bar{p} \cdot k = P(k)$,
  and is a realistic choice of prices for a zone tariff.

- For general increasing price functions, the no-stopover property need not be satisfied. An example is a zone tariff in which a path passes through three consecutive zones and in which the zone prices are $P(1) = 1, P(2) = 2$ and $P(3) = 5$.

For the no-elongation property, there is the following criterion.

▶ **Theorem 21.** *Let a price function P be given. All zone tariffs w.r.t. P satisfy the no-elongation property if and only if P is increasing.*

Proof. Let p be a zone tariff w.r.t. an increasing price function P. Note that for a path $W = (x_1, \ldots, x_n) \in \mathcal{W}$, we have that $z([x_1, \ldots, x_{n-1}]) \leq z(W)$. Since P is increasing, we obtain that $p([x_1, x_{n-1}]) = P(z([x_1, x_{n-1}])) \leq P(z(W)) = p(W)$.

If P is not increasing, there is some $k \in \mathbb{N}_{\geq 2}$ such that $P(k) < P(k-1)$. We construct a zone tariff in which the no-elongation property is not satisfied: Consider a zone tariff w.r.t. P in which there is a path $(x_1, \ldots, x_k)$ with $z([x_1, x_k]) = k$ and $z([x_1, x_{k-1}]) = k - 1$. Then we have $p([x_1, x_k]) = P(k) < P(k-1) = p([x_1, x_{k-1}])$. ◀

We now turn our attention to cheapest paths. The first observation interestingly shows that for price functions which are non-increasing there need not even exist a cheapest path.

▶ **Lemma 22.** *Let P be a price function for which there is some $n \in \mathbb{N}_{\geq 1}$ such that for all $k \geq n$ there is some $k' > k$ with $P(k') < P(k)$, and let p be a zone tariff w.r.t P. Then a cheapest path need not exist for p.*

The proof constructs an instance, i.e., a PTN $= (V, E)$ and two stations $x, y \in V$ such that no cheapest x-y-path exists in the induced zone tariff, and it can be found in the appendix. This happens, for example, when the prices are strictly decreasing. Cheapest paths always exist if the prices become constant for more than n zones. Still, there might be cheapest paths with large detours compared to a shortest path. All these situations are avoided if the price function is increasing.

▶ **Lemma 23.** *If p is a zone tariff with an increasing price function P, there exist cheapest paths.*

Proof. Since P is increasing, longer paths can never be better, hence we only have to consider simple paths. Since there is a finite number of simple paths between a given pair of stations, a cheapest path must exist. ◀

Note that even in the case of an increasing price function, a cheapest path need not be unique and there might even be two cheapest paths visiting different numbers of zones (if the price function becomes constant).

▶ **Theorem 24.** *Let the zone tariff p satisfy the no-stopover property, and let $x, y \in V$.*

1. *If the price function P is increasing, then any x-y-path W which visits a minimum number of zones $z(W)$ is a cheapest path.*
2. *If the price function P is strictly increasing, then an x-y path W is a cheapest path between x and y if and only if it visits a minimum number of zones.*

Proof. First note that compound and elongated paths need not be considered, since the no-stopover and the no-elongation property (due to Theorem 21) are satisfied. The first part is clear due to the monotonicity of P. For the second part, we have $P(k) < P(k + 1)$ for all $k \geq 1$. Hence, an x-y-path is cheapest if and only if it visits a minimum number of zones. ◀

(a) Illustration of Example 25. **(b)** Illustration of Example 27.

Figure 4 PTNs with zones for Examples 25 and 27.

Note that the assumption of the no-stopover property is necessary in Theorem 24. This is illustrated in the following example. Here we consider a price function P which is increasing and a zone tariff p w.r.t P which does not satisfy the no-stopover property. We construct a path which visits a minimum number of zones, but which is not a cheapest path.

▶ **Example 25.** Consider the PTN depicted in Figure 4a and an increasing price function P with $P(1) = 1$, $P(2) = 2$, $P(3) = 10$. The path $W_1 = (x_1, x_5)$ which visits three zones costs $p(W_1) = 10$, whereas the compound path $W_2 = (x_1, x_2) \circ (x_2, x_3) \circ (x_3, x_4) \circ (x_4, x_5)$ which visits more zones costs only $p(W_2) = 4 \cdot 2 = 8$.

We finally turn our attention to the subpath-optimality property.

▶ **Theorem 26.** *Let p be a zone tariff with a strictly increasing price function P. If p satisfies the no-stopover property, then the subpath-optimality property is satisfied.*

The proof is in the appendix. Note that the subpath-optimality property is not satisfied in general without the assumption of *strict* monotonicity as the following example shows.

▶ **Example 27.** Consider the zone tariff induced by the price function P given by $P(1) = 1$, $P(2) = 2$ and $P(k) = 3$ for $k \geq 3$ together with the PTN shown in Figure 4b. The path $W = (x_1, x_2, x_3, x_4, x_5)$ is a cheapest x_1-x_5-path. However, the subpath (x_1, x_2, x_3, x_4) of W with costs 3 is not a cheapest x_1-x_4-path because the path (x_1, x_4) with costs 2 is cheaper.

In order to construct an algorithm for computing cheapest paths in a zone tariff, we assume that the price function is increasing and that the no-stopover property holds. We can then use Theorem 24 and look for a path which visits the minimum number of zones. This can be done by applying a shortest path algorithm to the PTN with edge weights given by the border function $b(e)$.

Algorithm 1 Zone tariff: finding a cheapest path.

Input : PTN (V, E), two stations $x, y \in V$
Output : x-y-path W

Compute a shortest x-y-path W in the PTN by applying a shortest path algorithm using the border function $b(e)$ as edge weight for $e \in E$.
return W

▶ **Corollary 28.** *For a zone tariff with an increasing price function and which satisfies the no-stopover property, a cheapest path can be found in polynomial time by Algorithm 1.*

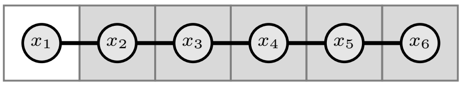


Figure 5 PTN with zones for Example 29.

4.5 Zone tariff with metropolitan zone

We now add a metropolitan zone to a basic zone tariff. In order to simplify our analysis, we make the following assumptions:

- the price function $P \colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ is increasing,
- the underlying basic zone tariff satisfies the no-stopover property.

We first provide an example that the no-stopover property need not be satisfied for zone tariffs with metropolitan zones.

▶ **Example 29.** Consider the PTN depicted in Figure 5. The zones highlighted in gray form a metropolitan zone. Let $P \colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0},\ k \mapsto k$ be a linear price function and $P_M := P(2) = 2$. We want to travel from x_1 to x_6. The path $(x_1, x_2, x_3, x_4, x_5, x_6)$ costs $p([x_1, x_6]) = P(6) = 6$. For this zone tariff with metropolitan zone, it is advantageous to exit and reenter at x_2, i.e., to use the compound path $(x_1, x_2) \circ (x_2, x_3, x_4, x_5, x_6)$ and benefit from the metropolitan zone, since the price is given by $p([x_1, x_2]) + p([x_2, x_6]) = P(2) + P_M = 2 + 2 = 4$.

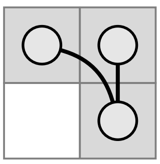
Note that the situation described above occurs in real-world, e.g., in the Verkehrsverbund Rhein-Neckar, see [15]. In order to analyze in which cases the no-stopover property nevertheless holds, we need to define the *maximum metropolitan zone distance* $D_{\max}$ which finds for any pair of stations x, y, both in the metropolitan zone Z_M, an x-y-path included in Z_M visiting a minimum number of zones, and then takes the maximum over all these values:

$$D_{\max} := \max_{x,y \in Z_M} \min_{x\text{-}y\text{-paths } W \text{ included in } Z_M} z(W).$$

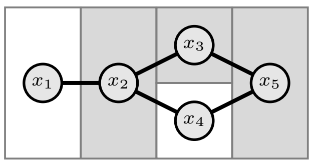
$D_{\max}$ depends on the PTN (including the metropolitan zone) and is always finite. We remark that $D_{\max}$ can be larger than the number of zones belonging to the metropolitan zone, see Figure 6a, where three zones belong to a metropolitan zone, but $D_{\max} = 4$. Further, we assume that every passenger who travels within the metropolitan zone Z_M uses a path with a minimum number of zones. This yields that for every path W included in Z_M we have that $z(W) \leq D_{\max}$. With this notation we state the following result.

▶ **Theorem 30.** *Let a price function P, a metropolitan price P_M, and an integer $d \in \mathbb{N}_{\geq 1}$ be given. All zone tariffs with metropolitan zone w.r.t. P and P_M on a PTN with $D_{\max} = d$ satisfy the no-stopover property if and only if $P(d + k) \leq P_M + P(k + 1)$ for all $k \in \mathbb{N}_{\geq 1}$.*

The proof of this theorem is provided in the appendix.



(a) $D_{\max}$ is larger than the number of zones in $\mathcal{Z}_M$.



(b) PTN with zones for Example 33.

Figure 6 Two PTNs with zones, both including a metropolitan zone.

▶ **Example 31.** For the linear price function $P(k) = k$ and $P_M := P(2) = 2$, we know from Theorem 30 that the no-stopover property is satisfied for all zone tariffs with metropolitan zone on a PTN with $D_{\max} = d$ if and only if $P(d+k) \leq P_M + P(k+1)$ for all $k \geq 1$. Plugging in our price function P, we receive that $d + k \leq 2 + (k+1)$. This is equivalent to $d \leq 3$.

The no-elongation property is satisfied even with metropolitan zone if $P_M \leq P(2)$.

▶ **Theorem 32.** *Let a price function P and price P_M be given. Then all zone tariffs with metropolitan zone w.r.t. P and P_M satisfy the no-elongation property if and only if $P_M \leq P(2)$.*

Proof. Let p be a zone tariff with metropolitan zone Z_M w.r.t. P and P_M, and let $W = (x_1, \ldots, x_n) \in \mathcal{W}$. We distinguish three cases.

- If W is included in Z_M, then $p([x_1, x_{n-1}]) = P_M = p(W)$.
- If $[x_1, x_{n-1}]$ is included in Z_M, but W is not, then W visits at least two zones and it holds that $p([x_1, x_{n-1}]) = P_M \leq P(2) \leq p(W)$ by assumption. On the other hand, if $P_M > P(2)$ and W visits exactly two zones, we obtain that $p([x_1, x_{n-1}]) = P_M > P(2) = p(W)$ and the no-elongation property does not hold.
- If $[x_1, x_{n-1}]$ is not included in Z_M, then the prices of W and its subpath $[x_1, x_{n-1}]$ are computed as in the basic zone tariff. Hence, we have $p([x_1, x_{n-1}]) \leq p(W)$ by monotonicity of P and Theorem 21. ◀

Finally, also the subpath-optimality property need not be satisfied.

▶ **Example 33.** Consider the PTN shown in Figure 6b. The zones highlighted in gray form a metropolitan zone. Let $P\colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$, $k \mapsto k$ be a linear price function and $P_M := P(2) = 2$. In the induced zone tariff with metropolitan zone, the no-stopover property is satisfied because $D_{\max} = 3$, see Example 31. A cheapest x_1-x_5-path is given by $W_1 = (x_1, x_2, x_4, x_5)$, since this paths costs $p(W_1) = P(4) = 4$ and the alternative path $W_2 = (x_1, x_2, x_3, x_5)$ costs $p(W_2) = P(4) = 4$ as well. The subpath (x_2, x_4, x_5) of W_1 with costs $P(3) = 3$ is not a cheapest x_2-x_5-path, though. A cheaper path is given by (x_2, x_3, x_5) with costs $P_M = 2$.

Note that the example above also shows that a path which visits a minimum number of zones is not necessarily a cheapest path if a path within a metropolitan zone exists. Although the subpath-optimality property does not hold, we can make use of the following lemma to find a cheapest path.

▶ **Lemma 34.** *Let p be a zone tariff with metropolitan zone Z_M, price function P and price $P_M \leq P(2)$. Assume that p satisfies the no-stopover property and let $x, y \in V$. If there exists an x-y-path which is included in Z_M, then this is a cheapest path. Otherwise, an x-y path which visits a minimum number of zones is a cheapest path.*

Proof. Consider the first case in which there is an x-y-path W which is included in Z_M. This path costs P_M. Any path that leaves the metropolitan zone costs at least $P(2) \geq P_M$, hence W is a cheapest path. If there is no path included in Z_M, the price of a path is computed by the basic zone tariff and Theorem 24 can be applied. ◀

We conclude that we can compute a cheapest path in polynomial time in this case by Algorithm 2. It is correct due to Lemma 34, hence we get the following result.

▶ **Corollary 35.** *Let p be a zone tariff with metropolitan zone Z_M, price function P and price $P_M \leq P(2)$ which satisfies the no-stopover property. Then a cheapest path can be found in polynomial time.*

Algorithm 2 ZM2: finding a cheapest path.

Input : PTN (V, E), two stations $x, y \in V$
Output : x-y-path W

Compute a shortest x-y-path in the PTN by applying a shortest path algorithm using $z_M(e)$ as edge weight for all $e \in E$.
if $z_M(W) = 0$ **then**
  return W
else
  Apply Algorithm 1 for finding a cheapest path regarding the basic zone tariff.
end

5 Conclusion

In this paper, we have provided models for many common fare systems, studied their properties and provided polynomial algorithms, all of them based on shortest paths. As a further step, we plan to investigate speed-up techniques for shortest paths (e.g., in [16, 1]) in order to make the computation of cheapest paths more efficient and to evaluate these experimentally. Here it is particularly interesting to use the embedding of the PTN in the plane, bidirectional search and the structure of the zones (for zone tariffs). The next step is to include the ticket price as one criterion besides the travel time when determining the routes for the passengers. This can be done efficiently if ticket prices can be computed by common shortest path algorithms in the same network as the travel time, but with adapted edge weights, as in [8]. Also, planning fare systems under different criteria (such as fairness, income, low transition costs) is an interesting topic for further research.

Currently, we work on results for combining fare systems and for adding further particularities such as a short-distance tariff, see [15]. We finally plan to include the ticket prices in route choice models and integrate them into planning lines and timetables along the lines of [14], but with an underlying realistic passengers' behavior.

References

1 H. Bast, D. Delling, A.V. Goldberg, M. Müller–Hannemann, T. Pajor, P. Sanders, D. Wagner, and R.F. Werneck. Route planning in transportation networks. In L. Kliemann and P. Sanders, editors, *Algorithm Engineering: Selected Results and Surveys*, volume 9220 of *LNCS State of the Art*, pages 19–80. 2016.

2 R. Borndörfer, M. Karbstein, and M.E. Pfetsch. Models for fare planning in public transport. *Discrete Applied Mathematics*, 160(18):2591–2605, 2012.

3 R. Borndörfer, M. Neumann, and M.E. Pfetsch. Models for fare planning in public transport. Technical report, ZIB, 2005.

4 D. Delling, , T. Pajor, and R.F. Werneck. Round-based public transit routing. *Transportation Science*, 49(3):591–604, 2015.

5 D. Delling, J. Dibbelt, and T. Pajor. Fast and exact public transit routing with restricted Pareto sets. In *Proceedings of the Twenty-First Workshop on Algorithm Engineering and Experiments (ALENEX)*, pages 54–65, 2019.

6 R. Euler and R. Borndörfer. A graph- and monoid-based framework for price-sensitive routing in local public transportation networks. In V. Cacchiani and A. Marchetti-Spaccamela, editors, *19th Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2019)*, volume 75 of *OpenAccess Series in Informatics (OASIcs)*, pages

12:1–12:15, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi:10.4230/OASIcs.ATMOS.2019.6.

7 A. Galligari, M. Maischberger, and F. Schoen. Local search heuristics for the zone planning problem. *Optimization Letters*, 11:195–207, 2017.

8 T. Gunkel, M. Müller-Hannemann, and M. Schnee. 16. Improved search for night train connections. In Christian Liebchen, Ravindra K. Ahuja, and Juan A. Mesa, editors, *7th Workshop on Algorithmic Approaches for Transportation Modeling, Optimization, and Systems (ATMOS'07)*, volume 7 of *OpenAccess Series in Informatics (OASIcs)*, Dagstuhl, Germany, 2007. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi:10.4230/OASIcs.ATMOS.2007.1178.

9 H.W. Hamacher and A. Schöbel. On fair zone design in public transportation. In J.R. Daduna, I. Branco, and J.M.P. Paixao, editors, *Computer-Aided Transit Scheduling*, number 430 in Lecture Notes in Economics and Mathematical Systems, pages 8–22. Springer, Berlin, Heidelberg, 1995.

10 H.W. Hamacher and A. Schöbel. Design of zone tariff systems in public transportation. *OR*, 52(6):897–908, 2004.

11 S. Maadi and J.-D. Schmöcker. Theoretical evaluation on the effects of changes from a zonal to a distance-based fare structure. In *Proceedings of CASPT'18, Brisbane*. 2018.

12 M. Müller-Hannemann and M. Schnee. Paying less for train connections with MOTIS. In Leo G. Kroon and Rolf H. Möhring, editors, *5th Workshop on Algorithmic Methods and Models for Optimization of Railways (ATMOS'05)*, volume 2 of *OpenAccess Series in Informatics (OASIcs)*, Dagstuhl, Germany, 2006. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi:10.4230/OASIcs.ATMOS.2005.657.

13 B. Otto and N. Boysen. Zone-based tariff design in public transportation networks. *Networks*, 69(4):349–366, 2017.

14 P. Schiewe and A. Schöbel. Periodic timetabling with integrated routing: Towards applicable approaches. *Transportation Science*, 2020. Online first.

15 R. Urban. Analysis and computation of cheapest paths in public transport. Master's thesis, Technische Universität Kaiserslautern, 2020.

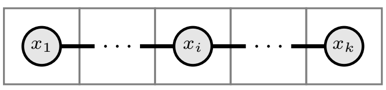
16 D. Wagner and T. Willhalm. Speed-up techniques for shortest-path computations. In *STACS 2007*, 2007.

Appendix

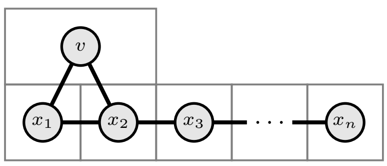
Proof of Theorem 19

Proof. If the inequality does not hold for some k and i, then consider the zone tariff w.r.t. P for the PTN depicted in Figure 7a. The path $(x_1, \dots, x_k)$ costs $P(k)$, whereas the compound path $[x_1, x_i] \circ [x_i, x_k]$ costs $P(i) + P(k - i + 1)$, which is cheaper by assumption. Hence, the no-stopover property is not satisfied for this zone tariff.

Now, we suppose that the inequality holds. Let p be any zone system w.r.t. P. For a path $W \in \mathcal{W}$, we define $k := z(W)$ and let $W_1 \circ W_2$ be a corresponding compound path.



(a) Illustration of Theorem 19.



(b) Illustration of Lemma 22.

Figure 7 PTNs with zones for Theorem 19 and Lemma 22.

Without loss of generality, we suppose that $z(W_1) \leq z(W_2)$, the other case is analogous. It holds $z(W_1) + z(W_2) = k + 1$. Note that for $k = 1$ the only possible decomposition is given by $z(W_1) = z(W_2) = 1$, and for $k = 2$ it is $z(W_1) = 1$ and $z(W_2) = 2$. The corresponding inequalities are $P(1) \leq 2P(1)$ and $P(2) \leq P(1) + P(2)$, which are clearly satisfied. Furthermore, for $z(W_1) = 1$ the inequality $P(k) \leq P(1) + P(k)$ is fulfilled for all $k \in \mathbb{N}_{\geq 1}$. Hence, let $k \in \mathbb{N}_{\geq 3}$ and $i := z(W_1) \in \{2, \ldots, \lfloor \frac{k+1}{2} \rfloor\}$. Then $z(W_1) + z(W_2) = k + 1$ is equivalent to $z(W_2) = k - z(W_1) + 1 = k - i + 1$. By assumption we have

$$p(W) = P(k) \leq P(i) + P(k - i + 1) = p(W_1) + p(W_2).$$

Thus, the no-stopover property is satisfied for all zone tariffs w.r.t. P if the inequalities hold.

For the second part of the theorem, let the no-stopover property be satisfied for all zone tariffs w.r.t. P, i.e., due to the first part of this proof we know that $P(k) \leq P(k+1-i) + P(i)$ for all $k \geq 3$, $i \in \{2, \ldots, \lfloor \frac{k+1}{2} \rfloor$, $i, k \in \mathbb{N}$. We prove the claim by induction over k and consider the condition for $i = 2$. For $k = 2$, the inequality is clearly fulfilled. For $k \geq 3$, we have

$$P(k) \leq P(2) + P(k - 1) \leq P(2) + (k - 2)P(2) = (k - 1)P(2).$$

This proves the claim. ◀

Proof of Lemma 22

Proof. Consider the situation shown in Figure 7b in which we want to travel from $x = x_1$ to $y = x_n$. The path with a minimum number of zones visits n zones. Note that for all $k > n$, we can construct an x_1-x_n-path with length k as follows: If $k - n$ is even, we choose the path $(x_1, \ldots, x_n)$ and additionally commute sufficiently often, i.e., $\frac{k-n}{2}$ times between two neighboring nodes. If $k - n$ is odd, we choose the upper path $(x_1, v, x_2, \ldots, x_n)$ and additionally commute $\frac{k-n-1}{2}$ times between two neighboring nodes. Hence, the set of all possible costs for x_1-x_n-paths is given by $\{P(k) : k \geq n\}$. This set does not have a minimum: Assume that $P(k)$ is the minimum for some $k \geq n$. By assumption there is some $k' > k$ such that $P(k') < P(k)$, a contradiction to $P(k)$ being the minimum. Thus, in this zone tariff, there is no cheapest x_1-x_n-path. ◀

Proof of Theorem 26

Proof. Since the no-stopover property and the no-elongation property are satisfied, we do not need to consider compound paths when searching for the cheapest possibility to travel between two stations. Now assume that a cheapest path W has a subpath W_1 which is not a cheapest path. Then there exists a cheaper path W_2 between the same stations as W_1. Due to the strict monotonicity, W_2 visits fewer zones than W_1. Replacing W_1 by W_2 in W yields a new path W' which visits fewer zones than W and is hence cheaper, a contradiction. ◀

Proof of Theorem 30

Proof. First, assume there is some k such that $P(d + k) > P_M + P(k + 1)$. Consider the PTN depicted in Figure 8, where $D_{\max} = d$. In the induced zone tariff with metropolitan zone, the path $(x_1, \ldots, x_{d+k})$, which costs $P(d + k)$, is more expensive than the compound path $[x_1, x_d] \circ [x_d, x_{d+k}]$, which costs $P_M + P(k + 1)$.

Conversely, we suppose that the inequalities are satisfied. Let a PTN with $D_{\max} = d$ be given. We show that the induced zone tariff with metropolitan zone satisfies the no-stopover

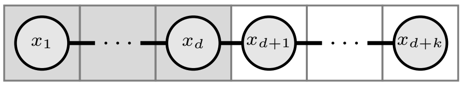


Figure 8 PTN with zones for Theorem 30.

property. By our assumptions, the no-stopover property is fulfilled for the basic zone tariff. Furthermore, it is satisfied for paths included in Z_M, since $P_M < 2P_M$. Hence, we will now consider paths $W \in \mathcal{W}$ which are not included in Z_M, but allow to apply the metropolitan price for a subpath by making a stopover. Such a path must start or end in Z_M. Let W consist of the subpaths W_1 and W_2 where W_1 is included in Z_M without loss of generality. We have $z(W_1) \leq D_{\max} = d$. Hence, it holds

$$\begin{aligned} p(W) = P(z(W)) = P(z(W_1) + z(W_2) - 1) &\overset{P \text{ incr.}}{\leq} P(d + z(W_2) - 1) \\ \leq P_M + P(z(W_2)) = p(W_1) + p(W_2) \end{aligned}$$

and the no-stopover property is satisfied. ◀