Subdivision Methods for Sum-Of-Distances Problems: Fermat-Weber Point, n-Ellipses and the Min-Sum Cluster Voronoi Diagram

Ioannis Mantas
Faculty of Informatics, Università della Svizzera italiana (USI), Lugano, Switzerland

Evanthia Papadopoulou
Faculty of Informatics, Università della Svizzera italiana (USI), Lugano, Switzerland

Martin Suderland
Faculty of Informatics, Università della Svizzera italiana (USI), Lugano, Switzerland

Chee Yap
Courant Institute, New York University (NYU), NY, USA

Abstract

Given a set P of n points, the *sum of distances function* of a point x is $d_P(x) := \sum_{p \in P} ||x - p||$. Using a *subdivision approach* with *soft predicates* we implement and visualize approximate solutions for three different problems involving the sum of distances function in $\mathbb{R}^2$. Namely, (1) finding the *Fermat-Weber point*, (2) constructing *n-ellipses* of a given set of points, and (3) constructing the *nearest Voronoi diagram under the sum of distances function*, given a set of point clusters as sites.

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Supplementary Material *Audiovisual (Video)*: https://youtu.be/wgG8uqLIizo

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1 Introduction

Let P denote a set of n points in $\mathbb{R}^2$. The *sum of distances*, or *Fermat distance*, function of a point $x \in \mathbb{R}^2$ to a set P is $d_P(x) := \sum_{p \in P} ||x - p||$, where $|| \cdot ||$ denotes the Euclidean distance. We are considering the following problems involving the Fermat distance function.

- The **Fermat(-Weber) point** of a set of points P is a point in $\mathbb{R}^2$ that minimizes the Fermat distance, i.e., $p_P^* := min_{x \in \mathbb{R}^2} d_P(x)$. The *Fermat radius* is the distance realizing the Fermat point, i.e., $d_P^* := d_P(p_P^*)$. See Figure 1 (left) for an illustration.
- An ***n*-ellipse** of a set of n points P of *radius* r, is the level set of the Fermat distance function $d_P^{-1}(r) := \{x \in \mathbb{R}^2 \mid d_P(x) = r\}$. An n-ellipse is non-empty only if $r \geqslant d_P^*$. See Figure 1 (middle) for an illustration.
- The **min-sum Voronoi diagram** of a family $\mathcal{S}$ of point sets, called *clusters*, is the subdivision of $\mathbb{R}^2$ into maximal regions, such that the region of a cluster $P \in \mathcal{S}$ is the locus of points closer to P than to any other cluster in $\mathcal{S}$, i.e., $\text{vreg}(P) := \{x \in \mathbb{R}^2 \mid d_P(x) < d_Q(x) \; \forall \, Q \in \mathcal{S} \backslash \{P\}\}$. See Figure 1 (right) for an illustration.

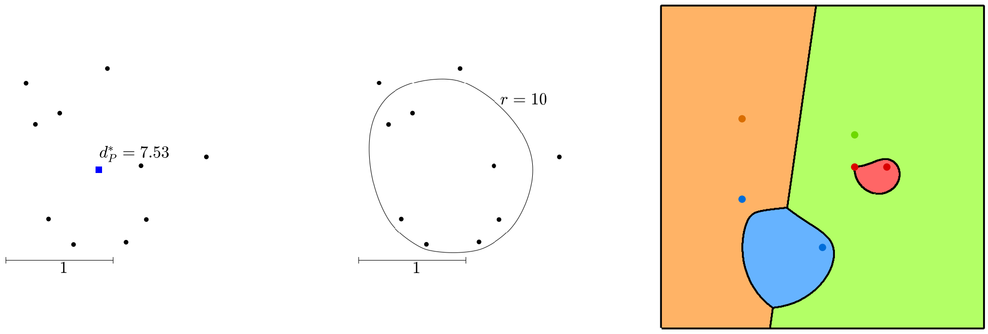

Figure 1 Illustration of the problems considered. (left) The Fermat point (■) of 10 points. (middle) An n-ellipse of 10 points of radius 10. (right) The min-sum Voronoi diagram of 4 clusters.

Contribution. In this work we present algorithms on how to find approximate solutions to the three aforementioned problems within a starting *box* (axis-aligned rectangle), using a *subdivision approach* augmented with *soft predicates*. This box is recursively split in a quadtree fashion. Deciding whether a box should be split or not, is done with respect to some *tests*, which we perform on this box. We typically derive the tests from *predicates*, evaluated with *interval arithmetic*. In the rest of the paper, we briefly describe how our algorithms work in each of the three problems, accompanied by illustrations from our visualization tool. All algorithms directly generalize for weighted input points P.

2 Problem 1: Finding the Fermat point

Finding the Fermat point (or Fermat-Weber point [25]) is an old geometric problem dating back to P. Fermat (1607–1665), which has attracted the attention of researchers of the last centuries. Unless P is a collinear point set of even size, the Fermat point is unique. Unfortunately, the coordinates of p_P^* are roots of polynomials of degree exponential in n, more precisely up to 2^n, see [5, 19]. For this reason there has been a profound interest in approximating the Fermat point; see indicatively [4, 8, 9, 10, 13, 21, 12].

Algorithm overview. Our algorithm returns a point $\widetilde{p_F}$ which is an ε approximation to the Fermat point, in the sense that $||\widetilde{p_F} - p_P^*|| \leqslant \varepsilon$; see our paper [15] for details including improvements using Newton's method. An illustration of the algorithm execution on two instances is shown in Figures 2 and 3. The algorithm starts with an initial box B_0 containing P, which guarantees that $p_P^* \in B_0$. During the subdivision, we keep and split boxes B that might contain p_P^* (**green boxes** in Figures 2 and 3). Boxes that are guaranteed not to contain p_P^* are discarded (**red boxes**); this is determined using an *exclusion test*. The algorithm stops when the set of remaining boxes (green) fit into a bounding box of radius ε; this *stopping test* guarantees that the center of the bounding box is within ε distance to p_P^*.

3 Problem 2: Constructing n-ellipses

Constructing n-ellipses is also a very old geometric problem dating back to E. von Tschirnhaus (1651–1708) [24]. When $n = 1$, the curve d_P^{-1} is a circle, and when $n = 2$, it is the classic ellipse. An n-ellipse is a convex piecewise smooth curve, with singularities occurring at

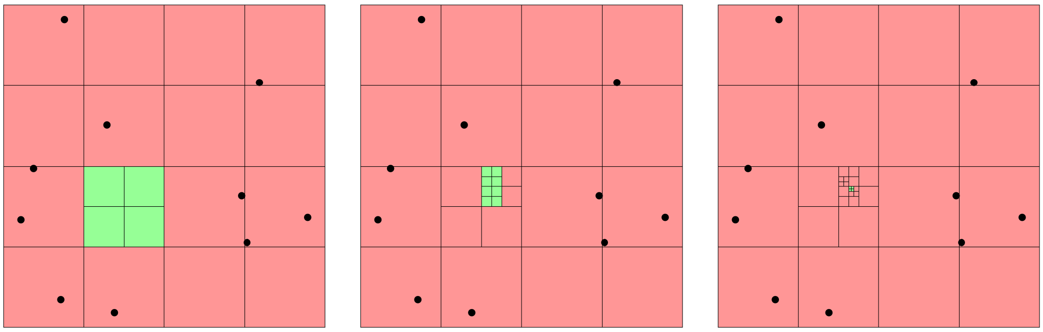

Figure 2 Different steps during the execution of the Fermat point algorithm ("easy" instance).

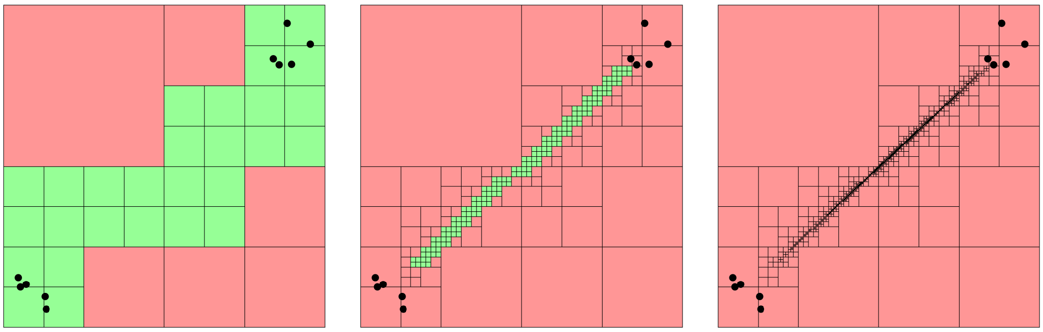

Figure 3 Different steps during the execution of the Fermat point algorithm ("difficult" instance).

points of P [18, 23]. Further, analogously to the Fermat point, the polynomial equations defining the n-ellipses have algebraic degree exponential in n [19], hence there is an interest in designing approximation algorithms to construct n-ellipses.

Algorithm overview. Our algorithm returns a curve E which is *isotopic* to d_P^{-1} and the Hausdorff distance between the two curves is at most ε; refer to our paper [15] for details. An illustration of different steps of the algorithm is shown in Figure 4.

In a nutshell, the algorithm can be considered as an *"online" PV-construction* [16, 22]. The PV-construction yields isotopic approximations to a target curve, assuming that this curve is regular. The n-ellipse, though, is not regular when it passes through P [23]. During the subdivision, we keep and split boxes B until the PV-construction is possible in each of them; these boxes either definitely contain a piece of d_P^{-1} (**green boxes** in Figure 4) or might do so (**orange boxes**). Boxes guaranteed not to contain a piece of d_P^{-1} are discarded (**red boxes**). To ensure that E is an ε-approximation to d_P^{-1}, we split the boxes in which we draw edges until they have size ε. Boxes near P, which are additionally close to the n-ellipse (**gray boxes**), require special treatment. For each such group of gray boxes we connect the two incoming sides of the n-ellipse by just a single edge, if the group fits into a small bounding box of size ε.

Elliptic contour plotting. The described algorithm can also be used to produce isotopic ε-approximate *elliptic contour plots*, which are roughly equally spaced. By adapting the algorithm, we can simultaneously construct multiple ellipses of different radii within the same box subdivision (each ellipse corresponding to a contour line). See Figure 5 for an examples.

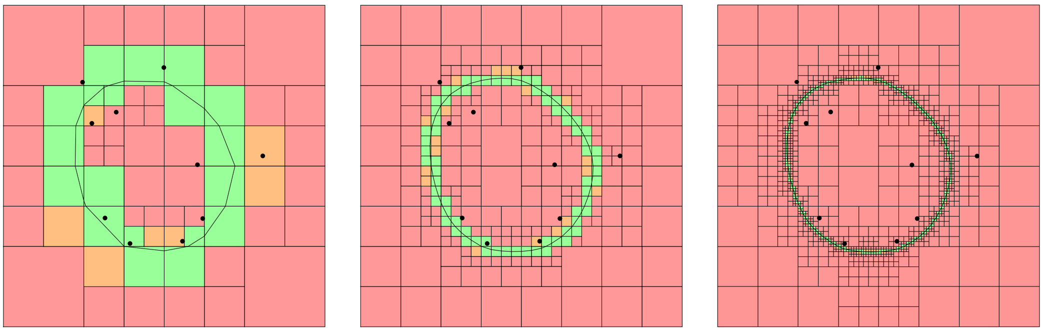

Figure 4 Different steps during the execution of the n-ellipses algorithm.

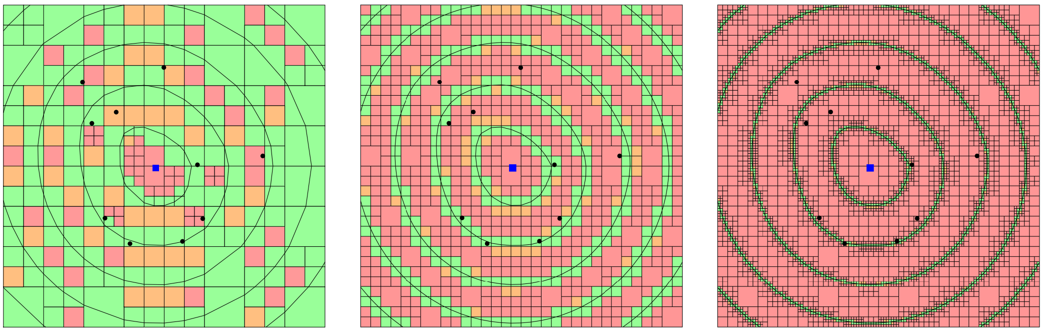

Figure 5 Different steps during the execution of the elliptic contour plotting algorithm.

4 Problem 3: Constructing the min-sum Voronoi diagram

The min-sum Voronoi diagram of a set of point clusters is the nearest *cluster Voronoi diagram* under the Fermat distance function; refer to Figure 6 for some instances. This diagram has not been studied before, except a special case for input clusters of size 2 [6]. Various other cluster Voronoi diagrams have been considered such as the (min-max) *Hausdorff Voronoi diagram* [2, 11, 20], and the (max-min) *farthest color Voronoi diagram* [1, 14, 17].

Each cluster may have a different size, in fact, the diagram can be seen as a weighted Voronoi diagram of point sites [3], where the weight of each point is determined by the cluster size. Only the clusters of the smallest size may have unbounded faces, see Figure 6(left). Further, given two clusters their bisector is smooth everywhere unless it passes through a cluster point, see Figure 6 (left).

The diagram has $\Omega(n + m^2)$ worst-case complexity, where m is the number of clusters and n is the total number of points. (1) Choose two clusters of $n/2$ points on a circle, such that the points are equally spaced and alternate between the clusters, see Figure 6 (middle). The diagram then consists of n cones emanating from the origin. (2) Choose $m = n/2$ many clusters of size 2, such that the line segments formed by connecting the 2 points of each cluster form a grid structure, see Figure 6 (right). The diagram splits into $\Omega(m^2)$ many faces.

Algorithm overview. Our algorithm returns a plane graph which is an approximation of the min-sum Voronoi diagram of $\mathcal{S}$ with ε Hausdorff distance. It is based on a variant of the algorithm presented in [7]; refer therein for details. In brief, the edges are drawn based on the PV-construction, and in order to get an ε-approximation, prior to drawing the edges, the boxes are split until they are of size ε. Refer to Figure 7 for an illustration of the algorithm.

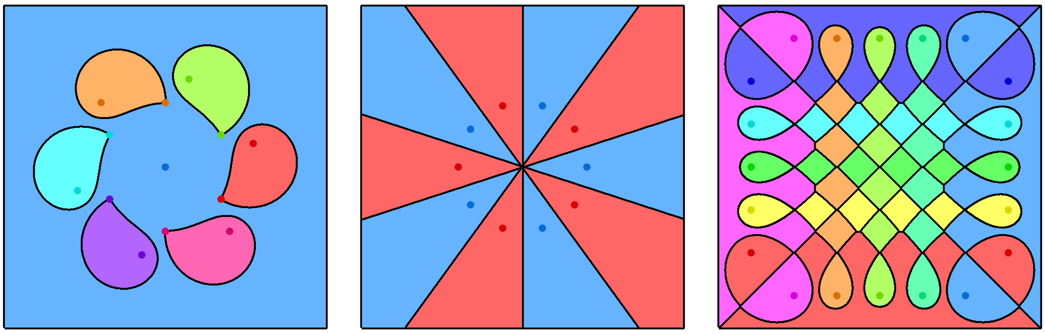

Figure 6 Three instances of a min-sum Voronoi diagram.

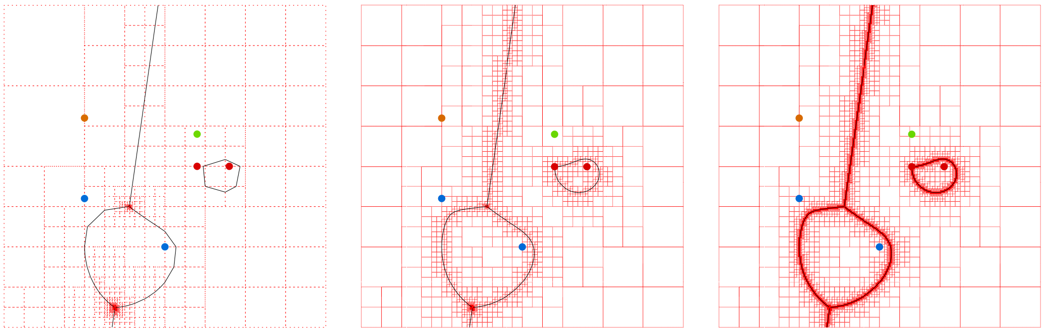

Figure 7 Different steps during the execution of the algorithm for min-sum Voronoi diagram.

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