Visualizing and Unfolding Nets of 4-Polytopes

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Abstract

Over a decade ago, it was shown that every edge unfolding of the Platonic solids was without self-overlap, yielding a valid net. Recent work has extended this property to their higher-dimensional analogs: the 4-cube, 4-simplex, and 4-orthoplex. We present an interactive visualization that allows the user to unfold these polytopes by drawing on their dual 1-skeleton graph.

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Supplementary Material *Software (Web-Application)*: `https://sam.zhang.fyi/html/unfolding/`

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1 Unfolding Polytopes

The study of unfolding polyhedra was popularized by Albrecht Dürer in the early 16th century who first recorded examples of polyhedral *nets*, connected edge unfoldings of polyhedra that lay flat on the plane without overlap. Motivated by this, Shephard [8] conjectured that every convex polyhedron can be cut along certain edges to admit a net. This claim remains open.

Consider this question for higher-dimensional *polytopes*: The codimension-one faces of a polytope are *facets* and its codimension-two faces are *ridges*. The analog of an edge unfolding of polyhedron is the *ridge unfolding* of an n-dimensional polytope: the process of cutting the polytope along a collection of its ridges so that the resulting (connected) arrangement of its facets develops isometrically into an $\mathbb{R}^{n-1}$ hyperplane. Such an unfolding without overlap of its facets yields a valid *net*. Instead of trying to find one net for each convex polyhedron (as posed by Shephard), we consider a more aggressive property:

▶ **Definition 1.** *A polytope P is* all-net *if every ridge unfolding of P yields a valid net.*

A decade ago, Horiyama and Shoji [7] showed that the five Platonic solids are all-net. Recent work [4] has shown applications in protein science: polyhedral nets are used to find a balance between entropy loss and energy gain for the folding propensity of a given shape. The higher-dimensional analogs of the Platonic solids are the regular polytopes. Three classes of regular polytopes exist for all dimensions: the n-simplex, n-cube, and n-orthoplex (sometimes called the *cross-polytope* or the *cocube*). The following is from [2] and [3]:

▶ **Theorem 2.** *The 4-simplex, the 4-cube, and the 4-orthoplex are all-net.*

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▶ Remark 3. For $n > 4$, the n-simplex and n-cube are all-net, while the n-orthoplex fails. Three additional regular polytopes appear only in four-dimensions: the 24-cell, 120-cell, and 600-cell. Their all-net property remain unexplored.

A ridge unfolding of a convex 4-dimensional polytope is given by a series of cuts along its 2-dimensional ridges so that the polytope may be unwrapped and "laid flat" in $\mathbb{R}^3$. The goal of our visualization is to show the resulting net – the final placement of the unwrapped facets – rather than the unwrapping itself. Such an unfolding is specified by the combinatorics of the arrangement of its facets in the resulting net. In particular, a ridge unfolding of polytope P induces a *spanning tree* in the 1-skeleton of the dual of P: a tree whose nodes are the facets of the polytope and whose edges are the uncut ridges between the facets.

We now consider these associated graphs, the 1-skeleton of the duals of these polytopes: Since the 4-simplex is self-dual, its 1-skeleton is simply the *complete graph on 5 nodes* (corresponding to the 5 facets of the 4-simplex). The 4-cube is dual to the 4-orthoplex, whose 1-skeleton forms the 4-*Roberts graph*. The 8 nodes of this graph can arranged on a circle so that antipodal nodes represent opposite facets of the cube. Finally, the dual of the 4-orthoplex is the 4-cube, whose 1-skeleton forms the 4-*hypercube graph*. We chose a drawing of this graph where its 16 nodes are arranged on a circle.

The work of Buekenhout and Parker [1] has enumerated the spanning trees on these three graphs. Since unfoldings are in bijection with spanning trees, there are (up to symmetry), 3 distinct unfoldings of the 4-simplex, 261 distinct unfoldings of the 4-cube, and 110,912 distinct unfoldings of the 4-orthoplex. By Theorem 2 above, each of these unfoldings is a valid net. Our visualization software (`https://sam.zhang.fyi/html/unfolding/`) allows the user to interactively create all of these nets. The figures in this paper show three examples.

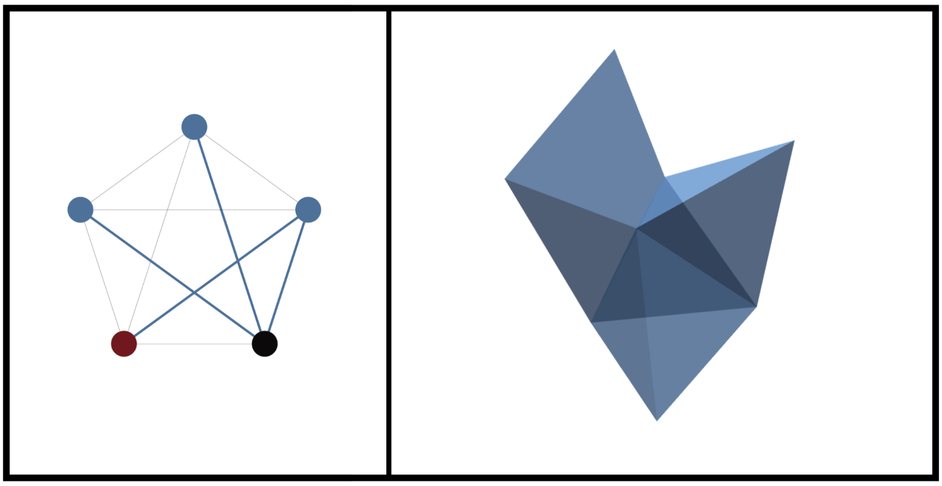

Figure 1 A user-drawn spanning tree and its corresponding unfolded 4-simplex net.

2 Unfolding Geometry

An unfolding is specified, step-by-step, by drawing a spanning tree. As it is being drawn, the corresponding net is formed by attaching new facets along the faces indicated by the tree.

In the case of the hypercube, the facets are cubes. The first cube is placed with its center (centroid) at the origin and its faces parallel to the coordinate planes. Each subsequent facet is attached to an exposed face f of one of the facets F in the existing structure as follows: the center P of F is translated one edge length in the direction perpendicular to F, to a new point Q (so that f bisects PQ). A new facet is then placed with Q as its center.

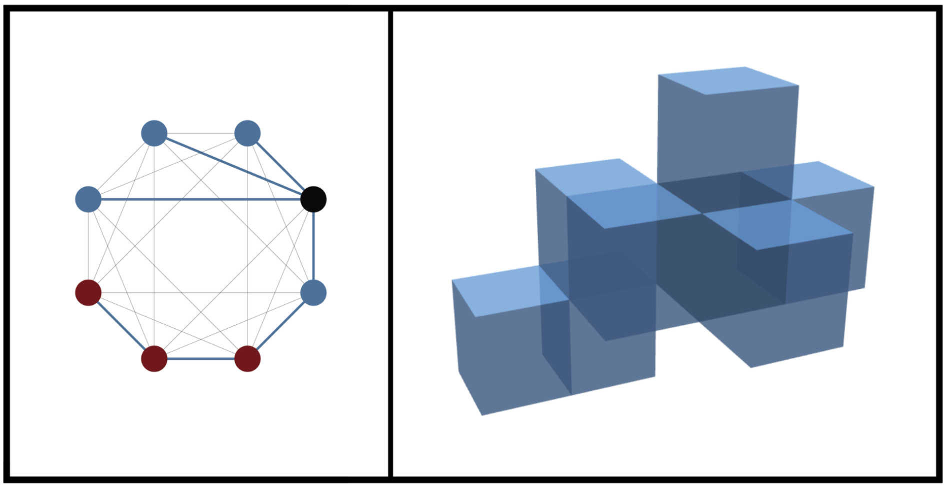

Figure 2 A user-drawn spanning tree and its corresponding unfolded 4-cube net.

In the case of the simplex and the orthoplex, the facets are tetrahedra. Unlike the cube, a tetrahedron cannot be conveniently embedded in $\mathbb{R}^3$, making calculations there difficult. It can be much more elegantly placed in $\mathbb{R}^4$, with its vertices at $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. Each subsequent facet is then attached to an exposed face f of a facet F by reflection across f. This reflection will fix all of f, hence all the vertices of F except for one, say P. Thanks to the $\mathbb{R}^4$ embedding, its reflection, Q, can be calculated by a simple matrix multiplication. A new facet is then constructed whose vertices are those of f, along with Q. In this construction, the unfolded net will lie in the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$. Once vertex coordinates have been calculated, it is necessary to rotate the shape into standard 3-dimensional space $x_4 = 0$ before displaying the final result.

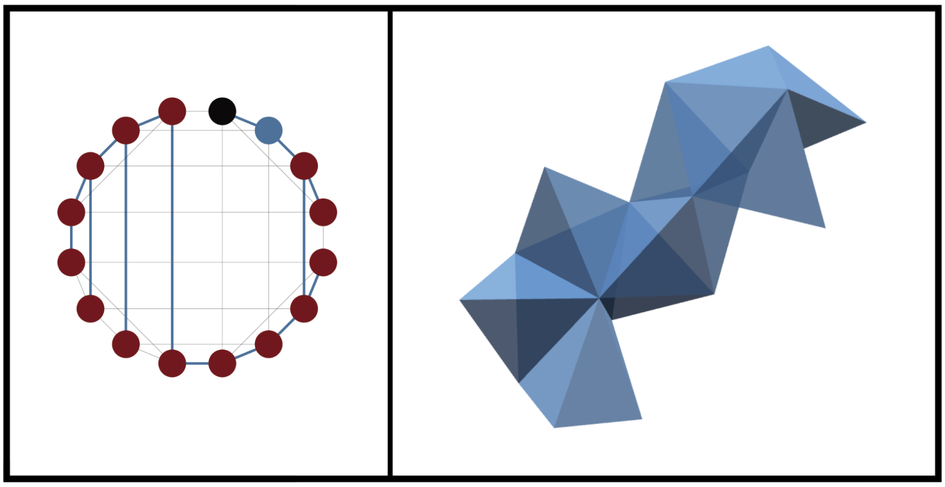

Figure 3 A user-drawn spanning tree and its corresponding unfolded 4-orthoplex net.

3 Implementation

Our visualization is an interactive, open source browser application implemented using HTML5 and JavaScript, and the application and source code can be accessed at `https://sam.zhang.fyi/html/unfolding/`. The user can select whether to unfold the 4-cube, the 4-simplex, or the 4-orthoplex. The user performs the unfolding by drawing a spanning tree on the graph of the 1-skeleton of the dual polytope. The graph of the 1-skeleton is represented using the JavaScript library JSXGraph [6], and the unfolded faces in $\mathbb{R}^3$ are drawn in WebGL using ThreeJS [5]. We used built-in features of ThreeJS to allow the user to scroll to zoom and click and drag to rotate the unfolded object.

The architecture of the application reuses common components for unfolding the cube, simplex, and orthoplex. In particular, we implement our own spanning tree data structure, which together with the underlying structure of the graph of the 1-skeleton of the dual polytope allows us to determine the set of valid moves. We maintain an undo stack of a single move, so that the visualization displays the outcome of a move when a valid node on the 1-skeleton is moused over, and saves the move if and only if a click is registered on the node before the mouse is moved off of the node. Otherwise, the move is undone when the mouse leaves the node.

There are a variety of choices for embedding the 1-skeleton of the dual polytope onto the plane, though we can pick elegant choices that position all of the nodes around the circumference of a circle. For the simplex, we have a standard visualization of a clique, and for the cube, we draw the 1-skeleton (a Roberts graph) as a clique with the opposite edges removed. For the orthoplex, we embed its 1-skeleton in a way that all of the edges either form part of the "circumference" of the graph or are parallel to the plane's vertical and horizontal axes.

We emphasize the current node by highlighting it as black on both the 1-skeleton as well as in the unfolding. Unvisited nodes are colored blue (if accessible) and red (if inaccessible), while visited nodes are shaded dark blue and dark red, appropriately. We arbitrarily fix a node as the starting one. Due to the ability of the user to pan the camera around the unfolding, all unfoldings up to rotation, but not reflection, are identified in the visualization. We introduced a minor amount of transparency into the unfolding so that the user can more clearly see the structure of the overall object.

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