An $(\aleph_0, k+2)$-Theorem for k-Transversals

Chaya Keller
Ariel University, Israel

Micha A. Perles
Einstein Institute of Mathematics, Hebrew University, Jerusalem, Israel

Abstract

A family $\mathcal{F}$ of sets satisfies the (p,q)-property if among every p members of $\mathcal{F}$, some q can be pierced by a single point. The celebrated (p,q)-theorem of Alon and Kleitman asserts that for any $p \geqslant q \geqslant d+1$, any family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^d$ that satisfies the (p,q)-property can be pierced by a finite number $c(p,q,d)$ of points. A similar theorem with respect to piercing by $(d-1)$-dimensional flats, called $(d-1)$-transversals, was obtained by Alon and Kalai.

In this paper we prove the following result, which can be viewed as an $(\aleph_0, k+2)$-theorem with respect to k-transversals: Let $\mathcal{F}$ be an infinite family of sets in $\mathbb{R}^d$ such that each $A \in \mathcal{F}$ contains a ball of radius r and is contained in a ball of radius R, and let $0 \leqslant k < d$. If among every $\aleph_0$ elements of $\mathcal{F}$, some $k+2$ can be pierced by a k-dimensional flat, then $\mathcal{F}$ can be pierced by a finite number of k-dimensional flats.

This is the first (p,q)-theorem in which the assumption is weakened to an $(\infty, \cdot)$ assumption. Our proofs combine geometric and topological tools.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases convexity, (p,q)-theorem, k-transversal, infinite (p,q)-theorem

Digital Object Identifier 10.4230/LIPIcs.SoCG.2022.50

Funding *Chaya Keller*: Research partially supported by the Israel Science Foundation (grant no. 1065/20).

Acknowledgements The authors are grateful to Andreas Holmsen for valuable suggestions and information.

1 Introduction

1.1 Background

Helly's theorem and the (p,q)-theorem. The classical Helly's theorem [19] asserts that if $\mathcal{F}$ is a family of compact convex sets in $\mathbb{R}^d$ and every $d+1$ (or fewer) members of $\mathcal{F}$ have a non-empty intersection, then the whole family $\mathcal{F}$ has a non-empty intersection.

For a pair of positive integers $p \geqslant q$, a family $\mathcal{F}$ of sets in $\mathbb{R}^d$ is said to satisfy the (p,q)-property if $|\mathcal{F}| \geq p$, none of the sets in $\mathcal{F}$ is empty, and among every p sets of $\mathcal{F}$, some q have a non-empty intersection, or equivalently, can be pierced by a single point. A set $P \subset \mathbb{R}^d$ is called a *transversal* for $\mathcal{F}$ if it has a non-empty intersection with every member of $\mathcal{F}$, or equivalently, if every member of $\mathcal{F}$ is pierced by an element of P. In this language, Helly's theorem states that any family of compact convex sets in $\mathbb{R}^d$ that satisfies the $(d+1,d+1)$-property, has a singleton transversal.

One of the best-known generalizations of Helly's theorem is the (p,q)-theorem of Alon and Kleitman (1992), which resolved a 35-year old conjecture of Hadwiger and Debrunner [18].

▶ **Theorem 1** (the (p,q)-theorem [3]). *For any triple of positive integers $p \geqslant q \geqslant d+1$ there exists $c = c(p,q,d)$ such that if $\mathcal{F}$ is a family of compact convex sets in $\mathbb{R}^d$ that satisfies the (p,q)-property, then there exists a transversal for $\mathcal{F}$ of size at most c.*

38th International Symposium on Computational Geometry (SoCG 2022).
Editors: Xavier Goaoc and Michael Kerber; Article No. 50; pp. 50:1–50:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In the 30 years since the publication of the (p,q)-theorem, numerous variations, generalizations and applications of it were obtained (see, e.g., the surveys [13, 20]). We outline below three variations to which our results are closely related.

(p,q)-theorems for k-transversals. The question whether Helly's theorem can be generalized to *k-transversals* – namely, to piercing by k-dimensional flats (i.e., k-dimensional affine subspaces of $\mathbb{R}^d$) – goes back to Vincensini [32], and was studied extensively. Santaló [31] observed that there is no Helly-type theorem for general families of convex sets, even with respect to 1-transversals in the plane. Subsequently, numerous works showed that Helly-type theorems for 1-transversals and for $(d-1)$-transversals in $\mathbb{R}^d$ can be obtained under additional assumptions on the sets of the family (see [20] and the references therein). A few of these results were generalized to k-transversals for all $1 \leqslant k \leqslant d-1$ (see [5, 6]).

Concerning (p,q)-theorems, the situation is cardinally different. In [1], Alon and Kalai obtained a (p,q)-theorem for *hyperplane transversals* (that is, for $(d-1)$-transversals in $\mathbb{R}^d$). The formulation of the theorem involves a natural generalization of the (p,q)-property:

For a family $\mathcal{G}$ of objects (e.g., the family of all hyperplanes in $\mathbb{R}^d$), a family $\mathcal{F}$ is said to satisfy the (p,q)-property with respect to $\mathcal{G}$ if among every p members of $\mathcal{F}$, some q can be pierced by an element of $\mathcal{G}$. A set $P \subset \mathcal{G}$ is called a *transversal* for $\mathcal{F}$ with respect to $\mathcal{G}$ if every member of $\mathcal{F}$ is pierced by an element of P.

▶ **Theorem 2** ([1]). *For any triple of positive integers $p \geqslant q \geqslant d+1$ there exists $c = c(p,q,d)$ such that if $\mathcal{F}$ is a family of compact convex sets in $\mathbb{R}^d$ that satisfies the (p,q)-property with respect to piercing by hyperplanes, then there exists a hyperplane transversal for $\mathcal{F}$ of size at most c.*

As an open problem at the end of their paper, Alon and Kalai [1] asked whether a similar result can be obtained for k-transversals, for $1 \leqslant k \leqslant d-2$. The question was answered on the negative by Alon, Kalai, Matoušek and Meshulam [2], who showed by an explicit example that no such (p,q)-theorem exists for line transversals in $\mathbb{R}^3$.

(p,q)-theorems without convexity. Numerous works obtained variants of the (p,q)-theorem in which the convexity assumption on the sets is replaced by a different (usually, topological) assumption. Most of the results in this direction base upon a result of Alon et al. [2], who showed that a (p,q)-theorem can be obtained whenever a *fractional Helly theorem* can be obtained, even without a convexity assumption on the elements of $\mathcal{F}$. In particular, the authors of [2] obtained a (p,q)-theorem for finite families of sets which are a *good cover*, meaning that the intersection of every sub-family is either empty or contractible. Matoušek [24] showed that bounded VC-dimension implies a (p,q)-theorem, and Pinchasi [29] proved a (p,q)-theorem for geometric hypergraphs whose ground set has a small union complexity. Recently, several more general (p,q)-theorems were obtained for families with a bounded Radon number, by Moran and Yehudayoff [26], Holmsen and Lee [21], and Patáková [28].

(p,q)-theorems for infinite set families. While most of the works on (p,q)-theorems concentrated on finite families of sets, several papers studied (p,q)-theorems for infinite set families.

It is well-known that Helly's theorem for infinite families holds under the weaker assumption that all sets are convex and closed, and at least one of them is bounded. In 1990, Erdős asked whether a (p,q)-theorem can be obtained in this weaker setting as well. Specifically, his conjecture – which was first published in [8] – was that a $(4,3)$-theorem holds for infinite families of convex closed sets in the plane in which at least one of the sets is bounded.

Following Erdős and Grünbaum, who refuted Erdős' conjecture and replaced it by a weaker conjecture of his own, several papers studied versions of the (p, q)-theorem for infinite families (see [25, 27]). These papers aimed at replacing the compactness assumption (which can be removed completely for finite families) by a weaker assumption.

1.2 Our contributions

In this paper we study variants of the (p, q)-theorem for infinite families $\mathcal{F}$ of sets in $\mathbb{R}^d$. Our basic question is whether the assumption of the theorem can be replaced by the following weaker infinitary assumption, which we naturally call *an* $(\aleph_0, q)$*-property*: Among every $\aleph_0$ elements of $\mathcal{F}$, there exist some q that can be pierced by a single point (or more generally, by an element of $\mathcal{G}$). We show that despite the apparently weaker condition, (p, q)-theorems can be obtained in several settings of interest.

An $(\aleph_0, 2)$-theorem for closed balls in $\mathbb{R}^d$. Our first result concerns the classical setting of point transversals and considers families of closed balls in $\mathbb{R}^d$. For such families, Danzer [9] obtained in 1956 a $(2, 2)$-theorem in the plane, answering a question of Gallai. Grünbaum [17] obtained a $(2, 2)$-theorem in $\mathbb{R}^d$, Kim et al. [22] obtained a $(p, 2)$ theorem in the plane for all $p \geqslant 2$, and finally, Dumitrescu and Jiang [11] obtained a $(p, 2)$-theorem in $\mathbb{R}^d$ for all $p \geqslant 2$. We show that an $(\aleph_0, 2)$-theorem holds as well.

▶ **Theorem 3.** *Let $\mathcal{F}$ be an infinite family of closed balls in $\mathbb{R}^d$. If among every $\aleph_0$ elements of $\mathcal{F}$, some two intersect, then $\mathcal{F}$ can be pierced by a finite number of points.*

We note that unlike the standard (p, q)-theorems, there does not exist a universal constant $c = c(d)$ such that every family of closed balls in $\mathbb{R}^d$ can be pierced by at most c points. Indeed, for any $m \in \mathbb{N}$, if the family consists of $\aleph_0$ copies of m pairwise disjoint balls then it satisfies the $(\aleph_0, 2)$-property (and actually, even the much stronger $(\aleph_0, \aleph_0)$-property), yet it clearly cannot be pierced by less than m points.

An $(\aleph_0, k + 2)$-theorem for "fat" sets in $\mathbb{R}^d$, with respect to k-transversals. Our main result concerns (p, q)-theorems with respect to k-transversals. In this setting, the construction presented in [2, Sec. 9] suggests that no $(\aleph_0, k + 2)$-theorem with respect to k-transversals can be obtained for general families of convex sets in $\mathbb{R}^d$ where $k < d - 1$, since even the stronger $(d + 1, d + 1)$-property does not imply a bounded-sized k-transversal. However, we show that if the convexity assumption is replaced by an assumption that the elements of the family are "fat",[1] then an $(\aleph_0, k + 2)$-theorem can be obtained.

▶ **Definition 4.** *Let $0 < r \leqslant R$. A family $\mathcal{F}$ of sets in $\mathbb{R}^d$ is called* (r, R)-fat *if any $A \in \mathcal{F}$ contains a ball of radius r and is contained in a ball of radius R.*

▶ **Theorem 5.** *Let $0 < r \leqslant R$, $0 \leqslant k \leqslant d - 1$, and let $\mathcal{F}$ be an infinite (r, R)-fat family of sets in $\mathbb{R}^d$. If among every $\aleph_0$ elements of $\mathcal{F}$, some $k + 2$ can be pierced by a k-flat, then $\mathcal{F}$ can be pierced by a finite number of k-flats.*

[1] We note that a "fatness" assumption was considered in the context of (p, q)-theorems for families of convex sets in the plane, by Gao and Zerbib [16].

Theorem 5 allows significantly weakening the (p,q)-property assumption of "classical" (p,q)-theorems into an (∞,q)-property assumption, it applies to k-transversals for all $0 \leqslant k \leqslant d-1$ (while the (p,q)-theorem for k-transversals holds only for $k = 0, d-1$), and it does not require the sets in the family to be convex.

On the other hand, it requires a significant additional assumption – namely, that the elements of the family are "fat". We show by an explicit construction that this assumption is essential.

▶ **Proposition 6.** *There exists an infinite family $\mathcal{F}$ of open discs in the plane that satisfies the $(3,3)$-property (and so, also the $(\aleph_0, 3)$-property) with respect to 1-transversals (i.e., piercing by lines), but cannot be pierced by a finite number of lines.*

Note that such a strong example could not be obtained for families of closed discs in the plane, since by Theorem 2, a family of compact convex sets in the plane that satisfies the $(3,3)$-property with respect to piercing by lines, admits a bounded-sized line transversal.

An infinite Ramsey-type theorem. In [23], Larman et al. observed that every $(p,2)$-theorem can be used to obtain a Ramsey-type theorem. Using a similar argument (presented in Sec. 7), Theorem 5 can be used to obtain the following Ramsey-type result.

▶ **Corollary 7.** *Let $0 < r \leqslant R$, $0 \leqslant k \leqslant d-1$, and let $\mathcal{F}$ be an infinite (r,R)-fat family of sets in $\mathbb{R}^d$. Denote $\alpha = |\mathcal{F}|$. Then one of the following holds:*

- *There exists $S \subset \mathcal{F}$ with $|S| = \aleph_0$ such that no $k{+}2$ elements of S can be pierced by a k-flat.*
- *There exists $S' \subset \mathcal{F}$ with $|S'| = \alpha$, such that every $k+2$ elements of S' can be pierced by a k-flat.*

For $\alpha > \aleph_0$ and $k \geqslant 1$, the assertion of Corollary 7 is significantly stronger than the best possible "generic" Ramsey theorem that can be obtained in the same setting. Indeed, the corresponding Ramsey-type theorem concerns (blue, red)-colorings of all r-element subsets of a set with cardinality α, for $r \geqslant 3$. In this setting, Erdős and Rado [14, Thm. 28] showed that in general, one cannot guarantee even the existence of either a set of $r+1$ elements all of whose r-tuples are blue or a set of cardinality α all of whose r-tuples are red. Corollary 7 provides either an "all-blue" set with cardinality $\aleph_0$ or an "all-red" set with cardinality α (of course, for the specific coloring in which a $(k+2)$-tuple is colored blue if it can be pierced by a k-flat). This provides yet another example of the phenomenon that graphs and hypergraphs arising in geometry satisfy much stronger forms of Ramsey's theorem than arbitrary graphs and hypergraphs. This phenomenon was demonstrated in several works in the finite setting (see [4, 7, 15, 23]), and our result provides an infinitary example.

Organization of the paper. In Section 2 we present some definitions, notations, and basic observations. In Section 3 we prove a lemma which shall be used in the proof of Theorem 5. Then, in Section 4 we prove Theorem 5. The construction of Proposition 6 is presented in Section 5, and the proof of Theorem 3 is given in Section 6. A more detailed comparison of Corollary 7 with generic Ramsey results is presented in Section 7. We conclude the paper with an open problem in Section 8.

2 Definitions, Notations, and Basic Observations

2.1 Definitions and notations

We use the following classical definitions.

- For $0 \leqslant k \leqslant d-1$, a k-flat in $\mathbb{R}^d$ is a k-dimensional affine subspace of $\mathbb{R}^d$ (namely, a translation of a k-dimensional linear subspace of $\mathbb{R}^d$). In particular, a 0-flat is a point, an 1-flat is a line, and a $(d-1)$-flat is a hyperplane.
- The *direction* of a k-flat ($k > 0$) in $\mathbb{R}^d$ is defined as follows. First, the k-flat is translated such that it will pass through the origin. Then, its direction is defined as the great $(k-1)$-sphere in which the k-flat intersects the sphere $\mathcal{S}^{d-1}$. (This definition follows [6].)
- A k_1-flat and a k_2-flat are called *parallel* if the direction of one of them is contained in the direction of the other. (Equivalently, this means that if both are translated so that they will pass through the origin, then one translation will be included in the other. Note that this relation is not transitive, and that two flats of the same dimension are parallel, if and only if one of them is a translation of the other.)
- For $\epsilon > 0$, an (open) ϵ-neighborhood of a point $x \in \mathcal{S}^{d-1}$ on the sphere is $B^\circ(x,\epsilon) \cap \mathcal{S}^{d-1}$, where $B^\circ(x,\epsilon)$ is the open ball with radius ϵ centered at x.
- A family $\mathcal{F} = \{B_\alpha\}_\alpha$ of sets in $\mathbb{R}^d$, is *independent w.r.t. k-flats* if no k-flat $\pi \subset \mathbb{R}^d$ intersects $k+2$ B_α's or more.

In the proofs of the theorems in the sequel, we mostly consider families $\mathcal{F}$ of closed unit balls in $\mathbb{R}^d$, $d \geqslant 1$, no two of them are equal. We always assume w.l.o.g. that $\mathcal{F}$ does not contain a ball centered at the origin, since all such balls are pierced by a single point, and hence by a single k-flat. We use the following definitions and notations:

- For $B = B(x,1) \in \mathcal{F}$, the *direction* of B is the point $\hat{x} = x/||x||_2$. Of course, $\hat{x} \in \mathcal{S}^{d-1}$.
- For any $\hat{x} \in \mathcal{S}^{d-1}$ (which is not necessarily a direction of a ball in $\mathcal{F}$) and any $\epsilon > 0$, the (open) ϵ-neighborhood of $\hat{x}$ in $\mathcal{F}$ is

  $$\mathcal{F}_{\hat{x},\epsilon} = \{B(y,1) \in \mathcal{F} : \hat{y} \in B^\circ(\hat{x},\epsilon) \cap \mathcal{S}^{d-1}\},$$

  that is, the set of all elements of $\mathcal{F}$ whose directions are in an ϵ-neighborhood of $\hat{x}$.
- For convenience, we often focus on the point $\hat{x} = (0,0,\ldots,0,1) \in \mathcal{S}^{d-1}$, on the line $\ell = \{t\hat{x} : t \in \mathbb{R}\}$, and on projections onto the hyperplane orthogonal to ℓ (i.e., projections onto the first $d-1$ coordinates.)
  For each $B = B(x,1) \in \mathcal{F}$, we denote by $B' \subset \mathbb{R}^{d-1}$ and $x' \in \mathbb{R}^{d-1}$ the projections of B and x, repsectively. The d'th coordinate of x, omitted in the projection, is denoted by $x(d)$.

2.2 Basic claims and observations

We use the two following simple claims.

▷ Claim 8. Let $\hat{x} = (0,0,\ldots,0,1)$, and let $\{B(x_n,1)\}_{n=1,2,\ldots}$ be a sequence of pairwise disjoint unit balls in $\mathbb{R}^d$ such that $\lim_{n\to\infty} \hat{x}_n = \hat{x}$. Then $\lim_{n\to\infty} x_n(d) = \infty$.

Proof. Let $0 < M \in \mathbb{R}, \epsilon > 0$. There exists $n_1 \in \mathbb{N}$ such that for any $n > n_1$, $\hat{x}_n$ is in the ϵ-neighborhood of $\hat{x}$. The set

$$\mathcal{F}_{\hat{x},\epsilon} \cap \{B(x_n,1) : n > n_1, x_n(d) < M\}$$

is contained in a finite area (which is a function of ϵ, d and M). By the disjointness of the balls in $\mathcal{F}$, there exists $n_2 > n_1$ such that for any $n > n_2$, $x_n(d) \geqslant M$. ◁

▷ Claim 9. Let $\mathcal{F} \subset \mathbb{R}^d$ be a family of balls of radius 1, and let G be a family of balls of radius $r > 0$, with the same centers. Then for any $0 \leqslant k \leqslant d-1$, $\mathcal{F}$ can be pierced by a finite set of k-flats if and only if G can be pierced by a finite set of k-flats.

Proof. Assume w.l.o.g. that $r > 1$. If $\mathcal{F}$ can be pierced by finitely many k-flats, then the same clearly holds for G as well, as the elements of $\mathcal{F}$ are contained in corresponding elements of G.

Assume that G can be pierced by finitely many k-flats, and take a finite family H of k-flats that pierces it. Replace each k-flat π in H by a sufficiently dense net of k-flats parallel to it, whose distance from π is at most $2r$. It is clear that the resulting finite family of k-flats pierces $\mathcal{F}$. ◁

3 A Technical Lemma

In the proof of Theorem 5, we shall need the following lemma.

▶ **Lemma 10.** *Let $\mathcal{F}$ be a family of closed unit balls in $\mathbb{R}^d$, let $0 \leqslant k \leqslant d-1$, and let $\hat{x} = (0, 0, \ldots, 0, 1)$. Assume that for any $\epsilon > 0$, the set $\mathcal{F}_{\hat{x},\epsilon}$ cannot be pierced by a finite collection of k-flats.*

Then there exists a sequence of balls, $\{B(x_n, 1)\}_{n=1,2,3,\ldots} \subset \mathcal{F}$ such that $\lim_{n\to\infty} \hat{x}_n = \hat{x}$ and the sequence cannot be pierced by a finite family of k-flats.

We derive Lemma 10 from the following proposition.

▶ **Proposition 11.** *Let $\mathcal{F}$ be a family of closed unit balls in $\mathbb{R}^d$, let $0 \leqslant k \leqslant d-1$ and $m \in \mathbb{N}$. If any finite subfamily of $\mathcal{F}$ can be pierced by at most m k-flats, then $\mathcal{F}$ can be pierced by at most m k-flats.*

We first derive Lemma 10 from Proposition 11, and then present the proof of the proposition.

Proof of Lemma 10, assuming Proposition 11. Let $\mathcal{F}, \hat{x}$ be as in the statement of the lemma, and assume that for any $\epsilon > 0$, the set $\mathcal{F}_{\hat{x},\epsilon}$ cannot be pierced by a finite collection of k-flats.

We construct the sequence of balls $\{B(x_n, 1)\}_{n=1,2,3,\ldots} \subset \mathcal{F}$ as follows. We take a sequence $\{\epsilon_m\}_{m=1,2,3,\ldots}$, where $\epsilon_m = 1/m$. For each $m \in \mathbb{N}$, we find in $\mathcal{F}_{\hat{x},\epsilon_m}$ a finite family G_m of balls that cannot be pierced by m k-flats (this is possible by Proposition 11). We define the sequence $\{B(x_n, 1)\}_{n=1,2,3,\ldots}$ as $\bigcup_{m\in\mathbb{N}} G_m$. Namely, we arbitrarily order the balls in each G_m and add them to the sequence, allowing repetitions, starting with $m = 1$, proceeding to $m = 2$, etc.. We have $\lim_{n\to\infty} \hat{x}_n = \hat{x}$, since for any $\epsilon > 0$, only a finite number of $B(x_n, 1)$'s do not belong to $\mathcal{F}_{\hat{x},\epsilon}$. Furthermore, $\{B(x_n, 1)\}_{n=1,2,3,\ldots}$, cannot be pierced by m k-flats (for any $m \in \mathbb{N}$) since it contains the family G_m that cannot be pierced by m k-flats by its construction. Hence, $\{B(x_n, 1)\}_{n=1,2,3,\ldots}$ cannot be pierced by a finite number of k-flats, as asserted. ◀

Proof of Proposition 11. Any k-flat $\pi \subset \mathbb{R}^d$ can be represented as

$$\pi = \{c + \lambda_1 v_1 + \ldots + \lambda_k v_k : \lambda_i \in \mathbb{R}\},$$

where c is the point on π closest to the origin, and $\{v_1, \ldots, v_k\}$ is an orthonormal basis of the vector subspace $\pi - \pi = \{x - y : x, y \in \pi\}$ which is parallel to π. (This actually means that the vector $\vec{0c}$ is orthogonal to each v_i.)

Assign to each k-flat π all the corresponding $(k+1)$-tuples of the type $\{c, v_1, \ldots, v_k\}$. Note that while c is uniquely determined by π, the orthogonal basis is not. We obtain a representation of all k-flats in $\mathbb{R}^d$ as $(k+1)$-tuples of d-vectors

$$\mathcal{A} = \{(c, v_1, \ldots, v_k) : c, v_i \in \mathbb{R}^d \wedge \forall i \neq j, v_i \perp v_j \wedge v_i \perp c \wedge ||v_i|| = 1\} \subset \mathbb{R}^{d(k+1)}.$$

By the conditions of the proposition, we can assume w.l.o.g. that there exists a finite sub-family $\mathcal{F}_0 \subset \mathcal{F}$ that cannot be pierced by $m-1$ k-flats. Let

$$\mathcal{A}^m = \{(c^1, v_1^1, \ldots, v_k^1, \ldots, c^m, v_1^m, \ldots, v_k^m) : \forall 1 \leqslant j \leqslant m, (c^j, v_1^j, \ldots, v_k^j) \in \mathcal{A}\} \subset \mathbb{R}^{d(k+1)m}$$

represent m-tuples of k-flats in $\mathbb{R}^d$.

Note that $\mathcal{A}^m$ is not compact (as a subset of $\mathbb{R}^{d(k+1)m}$), since $||c^j||$ may be arbitrarily large. However, for any fixed closed unit ball $B = B(x_0, 1) \subset \mathbb{R}^d$, the subset $\Pi_B \subset \mathcal{A}^m$ that represents all m-tuples of k-flats intersecting $B \cup \mathcal{F}_0$, is a compact subset of $\mathcal{A}^m$. Indeed, all the m coordinates c^j, satisfy $||c^j|| \leqslant \max\{||x_0|| + 1, \max_{B' \in \mathcal{F}_0} dist(B', 0) + 2\}$.

Consider the family $\{\Pi_B\}_{B \in \mathcal{F}}$ where Π_B represents all m-tuples of k-flats that pierce $B \cup \mathcal{F}_0$. (For each such m-tuple of k-flats, we take all possible $(d(k+1)m)$-tuples that represent it.) Each Π_B is compact, and by the assumption, any finite sub-family $\{\Pi_{B_i}\}_{i=1}^n$ has non-empty intersection (that contains the representation of some m-tuple of k-flats that together intersect $\mathcal{F}_0, B_1, \ldots, B_n$). Therefore, by the finite intersection property of compact sets, the whole family $\{\Pi_B\}_{B \in \mathcal{F}}$ has non-empty intersection. Any element in this non-empty intersection represents an m-tuple of k-flats that pierce together all the balls in $\mathcal{F}$. ◀

▶ Remark 12. Proposition 11 holds not only when $\mathcal{F}$ is a family of unit balls, but actually for any family $\mathcal{F}$ of non-empty compact sets in $\mathbb{R}^d$.

4 Proof of the Main Theorem

We restate Theorem 5, in a formulation that will be more convenient for the proof:

▶ **Theorem 5** (restated). *Let $R, r > 0$ and let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ such that each $S \in \mathcal{F}$ contains a ball of radius r and is contained in a ball of radius R. Let $0 \leqslant k \leqslant d-1$. Then one of the two following conditions must hold:*

- *$\mathcal{F}$ can be pierced by a finite number of k-flats.*
- *$\mathcal{F}$ contains an infinite sequence of sets that are independent w.r.t. k-flats (i.e., no k-flat pierces $k+2$ of them).*

First, we observe that it is sufficient to prove Theorem 5 for families of closed unit balls in $\mathbb{R}^d$.

▷ Claim 13 (Reduction to closed unit balls). If the assertion of Theorem 5 holds for all families of closed unit balls in $\mathbb{R}^d$, then it holds in the full generality stated in the theorem.

Proof. Let $\mathcal{F}$ be a family as in the assumption. Construct a family $\mathcal{F}_1$ by taking, for each $S \in \mathcal{F}$, a closed ball of radius r contained in S. (Note that we can make sure that the centers of these balls are distinct, possibly at the price of reducing their radii to $r/2$.) Then, construct another family $\mathcal{F}_2$ by taking, for each $S \in \mathcal{F}$, a closed ball of radius $2R$ that contains S, with the same center as the corresponding ball in S_1.

Apply Theorem 5 to $\mathcal{F}_2$. If it contains an infinite sequence of balls that are independent w.r.t. k-flats, then so does $\mathcal{F}$, since for each element of the sequence, we can take the element of $\mathcal{F}$ that corresponds to it, and the resulting sequence of elements of $\mathcal{F}$ will clearly be independent as well.

Otherwise, $\mathcal{F}_2$ can be pierced by a finite number of k-flats. Hence, by Claim 9, $\mathcal{F}_1$ can be pierced by a finite number of k-flats as well. This implies that $\mathcal{F}$ can be pierced by a finite number of k-flats, since any element of $\mathcal{F}$ contains an element of $\mathcal{F}_1$. Therefore, the assertion of the theorem holds for $\mathcal{F}$. ◁

By Claim 13, it is sufficient to prove Theorem 5 for families of closed unit balls in $\mathbb{R}^d$. A second reduction, before procceeding to the proof, is passing to *pairwise disjoint* unit balls.

▷ Claim 14 (Reduction to pairwise disjoint balls). If the assertion of Theorem 5 holds for all families of pairwise disjoint closed unit balls in $\mathbb{R}^d$, then it holds in the full generality stated in the theorem.

Proof. By Claim 13, it is sufficient to prove that if Theorem 5 holds for all families of pairwise disjoint closed unit balls in $\mathbb{R}^d$, then it holds for any family of closed unit balls.

Indeed, assume correctness for all families of pairwise disjoint closed unit balls in $\mathbb{R}^d$, and let $\mathcal{F}$ be a family of arbitrary closed unit balls in $\mathbb{R}^d$. First, we pass to a subfamily $\bar{\mathcal{F}} \subset \mathcal{F}$ of pairwise disjoint balls, which is maximal under inclusion:

Consider the family $\mathcal{G}$ of all subsets of $\mathcal{F}$ in which all balls are pairwise disjoint. View $\mathcal{G}$ as a poset with respect to inclusion. As each chain in $\mathcal{G}$ has a maximal element (which is the union of its elements), by Zorn's lemma $\mathcal{G}$ has a maximal element. This maximal element $\bar{\mathcal{F}} \subset \mathcal{F}$ is a set of pairwise disjoint balls, which is maximal under inclusion, among all the pairwise disjoint subfamilies.

By assuming correctness of Theorem 5 for families of pairwise disjoint closed unit balls, either $\bar{\mathcal{F}}$ contains an infinite sequence $\{\bar{\mathcal{F}}_n\}_{n\in\mathbb{N}}$ of balls that are independent w.r.t. k-flats, or $\bar{\mathcal{F}}$ can be pierced by a finite number of k-flats.

In the first case, $\{\bar{\mathcal{F}}_n\}_{n\in\mathbb{N}} \subset \bar{\mathcal{F}} \subset \mathcal{F}$ satisfies the second assertion of Theorem 5. In the second case, by the maximality of $\bar{\mathcal{F}}$, any ball in $\mathcal{F}$ intersects some ball in $\bar{\mathcal{F}}$. Therefore, by replacing each k-flat in the finite piercing set of $\bar{\mathcal{F}}$, by a sufficiently dense net of k-flats surrounding it and parallel to it, we obtain a finite piercing set of k-flats for $\mathcal{F}$, that satisfies the first assertion of Theorem 5. ◁

The proof of Theorem 5 is by induction, passing from $(k-1, d-1)$ to (k, d). The induction basis is the case $k = 0$ of Theorem 5, reduced to a family of closed unit balls, by Claim 13. (The reduction to disjoint balls is not needed here.) We observe:

▶ **Observation 15.** *Let $\mathcal{F}$ be a family of (not neccessarily disjoint) closed unit balls in $\mathbb{R}^d$. Then one of the two following conditions must hold:*

- *$\mathcal{F}$ can be pierced by a finite number of points.*
- *$\mathcal{F}$ contains an infinite sequence of pairwise disjoint balls.*

Proof. Consider the set $A = \{x \in \mathbb{R}^d | B(x, 1) \in \mathcal{F}\}$ of all centers of balls in $\mathcal{F}$. If A is bounded in some $B(0, R) \subset \mathbb{R}^d$, then clearly a finite set of points pierces all elements of $\mathcal{F}$. Otherwise, A is unbounded, hence $\mathcal{F}$ contains an infinite sequence of pairwise disjoint balls, that can be obtained inductively. ◀

For $d = 1$, the assertion of Theorem 5, after applying the reduction of Claim 13, is exactly Observation 15. For $d \geqslant 2$, we shall prove the following version, which is sufficient due to the reductions of Claims 13 and 14:

▶ **Theorem 16.** *Let $d \geqslant 2$ and $0 \leqslant k \leqslant d-1$. Let $\mathcal{F}$ be a family of pairwise disjoint closed unit balls in $\mathbb{R}^d$, and assume w.l.o.g. that $\mathcal{F}$ does not contain a ball centered at the origin. Then:*

1. *If for any $\hat{x} \in \mathcal{S}^{d-1}$, there exists $\epsilon(\hat{x}) = \epsilon > 0$ such that $\mathcal{F}_{\hat{x},\epsilon}$ can be pierced by finitely many of k-flats, then $\mathcal{F}$ can be pierced by finitely many of k-flats.*
2. *If the condition of (1) does not hold, then $\mathcal{F}$ contains an infinite sequence of balls that are independent w.r.t. k-flats (i.e., no k-flat pierces $k+2$ of them).*

Proof. First, we give the proof of the first assertion. Assume that for any $\hat{x} \in \mathcal{S}^{d-1}$, there exists $\epsilon = \epsilon(\hat{x}) > 0$ such that $\mathcal{F}_{\hat{x},\epsilon}$ can be pierced by a finite number of k-flats. Pick such an $\epsilon(\hat{x})$ for each $\hat{x} \in \mathcal{S}^{d-1}$, and obtain an open covering of $\mathcal{S}^{d-1}$ by open balls $B(\hat{x}, \epsilon(\hat{x}))$, for all $\hat{x} \in \mathcal{S}^{d-1}$.

By the compactness of the sphere, we can find a finite sub-cover, generated by balls around $\hat{x}_1, \ldots, \hat{x}_n$. As each $\mathcal{F}_{\hat{x}_i,\epsilon(\hat{x}_i)}$ can be pierced by a finite number of k-flats, we can pierce all elements of $\mathcal{F}$ by a finite collection of k-flats (which is the union of the k-flats that pierce $\mathcal{F}_{\hat{x}_i,\epsilon(\hat{x}_i)}$, for $i = 1, \ldots, n$).

Now we move to the second assertion. Assume that for some $\hat{x} \in \mathcal{S}^{d-1}$ and for any $\epsilon > 0$, the family $\mathcal{F}_{\hat{x},\epsilon}$ cannot be pierced by a finite number of k-flats. We assume w.l.o.g. that $\hat{x} = (0, 0, \ldots, 1)$. We shall construct a sequence of elements of $\mathcal{F}$ that is independent w.r.t. k-flats. The construction goes by induction, which reduces from k-flats in $\mathbb{R}^d$ to $(k-1)$-flats in $\mathbb{R}^{d-1}$.

Induction basis: $k = 0$. This case, which concerns piercing by points, follows by the argument of Observation 15.

Induction step: From $(k-1, d-1)$ to (k, d). Assume that we proved the assertion for families in $\mathbb{R}^{d-1}$, with respect to piercing by $(k-1)$-flats, and consider a family $\mathcal{F} \subset \mathbb{R}^d$ of pairwise disjoint closed unit balls.

First, we use Lemma 10 to find a sequence $G = \{B(x_n, 1)\}_{n=1,2,\ldots}$ of elements of $\mathcal{F}$ such that $\lim_{n\to\infty} \hat{x}_n = \hat{x}$ and the sequence cannot be pierced by a finite number of k-flats. From now on, we restrict ourselves to this sequence.

We project each $B(x_n, 1) \in G$ onto its first $d-1$ coordinates. Let the resulting set be G', and similarly to the proof of Claim 14, let $G'' \subset G'$ be a subset of pairwise disjoint balls, maximal under inclusion in G'. By the induction hypothesis, either G'' (and therefore G') can be pierced by a finite number of $(k-1)$-flats in $\mathbb{R}^{d-1}$, or else it contains a sequence of $(d-1)$-dimensional balls that are independent w.r.t. $(k-1)$-flats.

The first option cannot happen, as otherwise, one could pierce G with a finite number of k-flats (which are the pre-images of the $(k-1)$-flats-transversal in $\mathbb{R}^{d-1}$ under the projection), contrary to the choice of G. Hence, there exists a sub-sequence $\bar{G} = \{B(x_{n_l}, 1)\}_{l=1,2,\ldots} \subset G$ of balls whose projections are independent w.r.t. $(k-1)$-flats in $\mathbb{R}^{d-1}$. Note that as $\lim_{n\to\infty} \hat{x}_n = \hat{x}$, we have $\lim_{l\to\infty} \hat{x}_{n_l} = \hat{x}$. From now on, we restrict ourselves to this sequence and construct inductively a subsequence of it that will be independent w.r.t. k-flats in $\mathbb{R}^d$.

We construct the subsequence $\{B(x^n, 1)\}_{n=1}^{\infty}$ inductively. ($\{x^n\}_{n=1}^{\infty}$ is a subsequence of $\{x_{n_l}\}_{l=1}^{\infty}$.)

The first $k+1$ elements can be chosen arbitrarily. Assume that we already chose the balls $B(x^1, 1), \ldots, B(x^m, 1)$, for $m \geqslant k+1$. To choose $B(x^{m+1}, 1)$, we first look at each $(k+1)$-tuple of balls $(B(x^{i_1}, 1), \ldots, B(x^{i_{k+1}}, 1))$ separately. By assumption, the corresponding projections on the first $d-1$ coordinates cannot be pierced by a $(k-1)$-flat in $\mathbb{R}^{d-1}$. This implies that no k-flat that is parallel to the line $\ell = \{t\hat{x} : t \in \mathbb{R}\}$ can pierce all the $k+1$ balls $B(x^{i_1}, 1), \ldots, B(x^{i_{k+1}}, 1)$.

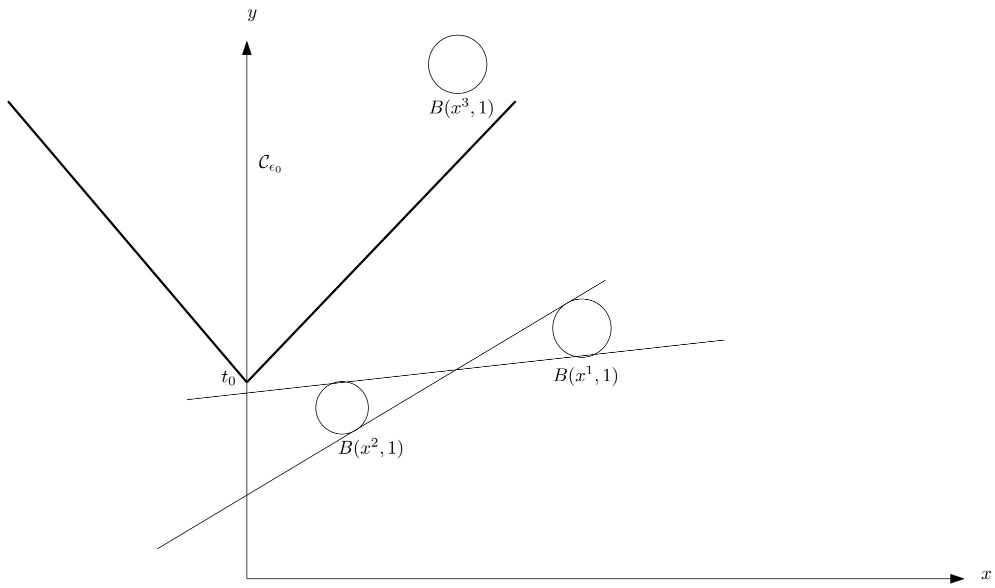

Figure 1 An illustration for the proof of Theorem 16 for $d = 2, k = 1$.

Consider the family U of all k-flats that pierce $(B(x^{i_1}, 1), \ldots, B(x^{i_{k+1}}, 1))$. As none of them is parallel to ℓ, neither of their directions[2] contains the point $(0, 0, \ldots, 0, 1) = \hat{x} \in \mathcal{S}^{d-1}$. By compactness of the elements of $\mathcal{F}$, this implies that there exists $\epsilon_0 > 0$, such that all these directions are disjoint with the ϵ_0-neighborhood of $\hat{x}$ on $\mathcal{S}^{d-1}$.

Now, let $\mathcal{C}_{\epsilon_0}$ be the unbounded cone whose vertex is the origin and whose intersection with $\mathcal{S}^{d-1}$ is the boundary of $\epsilon_0/2$-neighborhood of $\hat{x}$ on $\mathcal{S}^{d-1}$. (Informally, this is a cone of small aperture around the positive direction of the d'th axis.) We claim that there exists $t_0 \in \mathbb{R}$ such that for any $t > t_0$, the translation $(0, 0, \ldots, 0, t) + \mathcal{C}_{\epsilon_0}$ is disjoint from all k-flats in U (see Figure 1).

To see this, for each k-flat $L \in U$ we define a function $f_L : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ by $f_L(t) = \min\{||x'|| : x \in L \wedge x(d) = t\}$ (for all the relevant notations, see the end of Section 2.1). It is clear that f_L attains a minimum, and that since L is not parallel to ℓ, this minimum is attained in a single point, $t = \text{argmin}(f_L) \in \mathbb{R}$. Now, we define a function $g : U \to \mathbb{R}$ by $g(L) = \text{argmin}(f_L)$. By compactness of the elements of $\mathcal{F}$, this function attains a maximum, t_0. As the direction of any $L \in U$ is disjoint with the ϵ_0-neighborhood of $\hat{x}$ on $\mathcal{S}^{d-1}$, it follows that $L \cap ((0, 0, \ldots, 0, t) + \mathcal{C}_{\epsilon_0}) = \emptyset$, for all $t > t_0$.

We are now ready to choose the ball $B(x^{m+1}, 1)$. We go over all $(k+1)$-tuples of balls $(B(x^{i_1}, 1), \ldots, B(x^{i_{k+1}}, 1))$ with $1 \leqslant i_1 < i_2 < \ldots < i_{k+1} \leqslant m$. For each of them, we find (ϵ_0, t_0) such that for any $t > t_0$, any k-flat that pierces $(B(x^{i_1}, 1), \ldots, B(x^{i_{k+1}}, 1))$ is disjoint with the cone $(0, 0, \ldots, 0, t) + \mathcal{C}_{\epsilon_0}$, where $\mathcal{C}_{\epsilon_0}$ is as defined above.

Let ϵ_1 be the minimum of the ϵ_0 values, and let t_1 be the maximum of the t_0 values. It is clear that if we make sure that $B(x^{m+1}, 1)$ is entirely included in the cone $(0, 0, \ldots, 0, t_1 + 1) + \mathcal{C}_{\epsilon_1}$, then no k-flat will pierce both $B(x^{m+1}, 1)$ and a $(k+1)$-tuple $(B(x^{i_1}, 1), \ldots, B(x^{i_{k+1}}, 1))$. We can indeed choose $B(x^{m+1}, 1)$ in this way, by Claim 8. This completes the proof. ◀

[2] See Section 2.1 for the needed definitions.

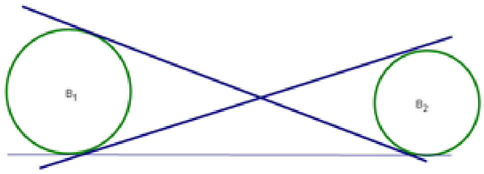

Figure 2 An illustration for Section 5.

5 Proof of Proposition 6

In this section we prove Proposition 6. Namely, we construct an infinite family of open discs in the plane that satisfies the $(3, 3)$-property with respect to line transversals, but cannot be pierced by a finite number of lines.

Proof of Proposition 6. Let $\mathcal{F} = \{\mathcal{F}_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$, where $\mathcal{F}_n = B(n, 1/n)$ is an open disc centered at $(n, \frac{1}{n})$ with radius $\frac{1}{n}$. The family $\mathcal{F}$ does not admit a finite line transversal, since the x-axis meets no element of $\mathcal{F}$, any line that is parallel to the x-axis meets finitely many elements of $\mathcal{F}$, and any line that forms a positive angle with the x-axis, intersects a finite subfamily of $\mathcal{F}$.

On the other hand, any $\mathcal{F}' \subset \mathcal{F}$ which is independent w.r.t. lines, satisfies $|\mathcal{F}'| \leqslant 2$. Indeed, consider the two leftmost discs $B_1, B_2 \in \mathcal{F}'$. The right wedge that the two common inner tangents of B_1 and B_2 form, contains all elements of $\mathcal{F}$ that are to the right of B_1 and B_2 (see Figure 2). Therefore, any element of $\mathcal{F}$ that lies to the right of B_1 and B_2 is pierced by a line that passes through B_1 and B_2, and hence cannot be contained in $\mathcal{F}'$. ◀

We note that no similar example could be constructed with closed balls, since by the Danzer-Grünbaum-Klee theorem [10], such a family would be pierced by a single line.

6 An 0-flat Transversal With no Restriction on the Radii

In this section we prove Theorem 3, which is a much stronger version of Observation 15. This stronger version holds with no restriction on the radii. Let us restate the theorem in a formulation which is more convenient for us:

▶ **Theorem 3** (restated). *Let $\mathcal{F}$ be a family of closed balls in $\mathbb{R}^d$ (with no restriction on the radii). Then one of the two following conditions must hold:*

- *$\mathcal{F}$ can be pierced by a finitely many points.*
- *$\mathcal{F}$ contains an infinite sequence of pairwise disjoint balls.*

Before proceeding into the proof, we prove a reduction to the case where all elements of $\mathcal{F}$ are contained in a closed bounded ball $B(0, R) \subset \mathbb{R}^d$.

▷ Claim 17. Let $R > 0$ and assume we proved Theorem 3 where any ball in $\mathcal{F}$ is contained in $B(0, R)$. Then Theorem 3 holds.

Proof. Define the distance of a closed ball $B \subset \mathbb{R}^d$ from the origin, $dist(B, 0)$, as the Euclidian distance between the origin and the point $x \in B$ which is closest to the origin.

If the set $\{dist(B, 0) : B \in \mathcal{F}\}$ is unbounded in $\mathbb{R}^d$, then one can inductively construct an infinite sequence of pairwise disjoint balls in $\mathcal{F}$, whose distance from the origin tends to infinity.

From now on we assume that there exists some $0 < R \in \mathbb{R}$ such that for any $B \in \mathcal{F}$, $dist(B, 0) \leqslant R - 2$. Replace each $B \in \mathcal{F}$ whose radius $r(B) > 1$, by some closed smaller ball $B' \subset B$ with $r(B') = 1$, such that $dist(0, B) = dist(0, B')$. Let $\mathcal{F}'$ be the obtained family. Any ball in $\mathcal{F}'$ is contained in $B(0, R)$.

By the assumption of our claim, either $\mathcal{F}'$ can be pierced by finitely many points, or $\mathcal{F}'$ contains an infinite sequence $\mathcal{F}'' \subset \mathcal{F}'$ of pairwise disjoint balls. In the first case, the finite piercing set of $\mathcal{F}'$ pierces $\mathcal{F}$ as well.

In the second case, remove from $\mathcal{F}''$ all balls with radius 1. There are only finitely many such balls, since $\mathcal{F}'' \subset \mathcal{F}' \subset B(0, R)$, and the elements of $\mathcal{F}''$ are pairwise disjoint. After removing from $\mathcal{F}''$ all balls with radius 1, we are left with an infinite subfamily of balls each of which belongs to $\mathcal{F}$ (since the transition from $\mathcal{F}$ to $\mathcal{F}'$ involved only the radius-1 balls of $\mathcal{F}'$), which are pairwise disjoint. ◁

Proof of Theorem 3. By Claim 17 we can assume that there exists $R > 0$ such that each ball in $\mathcal{F}$ is contained in $B(0, R)$. We can assume w.l.o.g. that $\mathcal{F}$ contains no ball of radius 0. Indeed, if $\mathcal{F}$ contains finitely many such balls, we can remove them without changing the assertion. Otherwise, $\mathcal{F}$ contains an infinite sequence of radius-0 balls, and then we are done again.

Each $x \in B(0, R)$ is of exactly one of the two following types:

Type (a): For each $\delta > 0$, there exists some $B \in \mathcal{F}$, $B \cap B^\circ(x, \delta) \neq \emptyset$, with $r(B) < \delta$ and $B \cap \{x\} = \emptyset$.

Type (b): There exists $0 < \delta = \delta(x)$ such that for any $B \in \mathcal{F}$ with $B \cap B^\circ(x, \delta) \neq \emptyset$, the following holds: Either $r(B) \geqslant \delta$ or $B \cap \{x\} \neq \emptyset$.

If $B(0, R)$ contains some point x of type (a), then there exists an infinite sequence of pairwise disjoint balls in $\mathcal{F}$ (that tends to $\{x\}$). Indeed, start with $\delta_0 = 1$ and pick some $B_0 \in \mathcal{F}, b_0 \cap B^\circ(x, \delta_0) \neq \emptyset$, with $r(B_0) < \delta_0$ and $B_0 \cap \{x\} = \emptyset$. Since B_0 is closed, it has a positive distance ϵ from x. Let $\delta_1 = \frac{\epsilon}{10}$ and pick some $B_1 \in \mathcal{F}$, $B_1 \cap B^\circ(x, \delta_1) \neq \emptyset$, $r(B_1) < \delta_1$ and $B_1 \cap \{x\} = \emptyset$. Continue in the same manner to construct an infinite sequence of pairwise disjoint balls in $\mathcal{F}$.

The remainig case is where each $x \in B(0, R)$ is of type (b). Then for each $x \in B(0, R)$ there exists $0 < \delta = \delta(x)$ such that any $B \in \mathcal{F}$ that that intersects $B^\circ(x, \delta)$ can be pierced by finitely many points, say, by $f(x)$ points. (Note that the exact value of $f(x)$ depends on the choice of $\delta = \delta(x)$.) By the finite intersection property of compact sets in $\mathbb{R}^d$, the open cover $\bigcup_{x \in B(0,R)} B^\circ(x, \delta(x))$ of $B(0, R)$ has a finite sub-cover $B(0, R) \subset \bigcup_{i=1}^{k} B^\circ(x_i, \delta(x_i))$. Since all the balls in $\mathcal{F}$ that intersect $B^\circ(x_i, \delta(x_i))$ can be pierced by $f(x_i)$ points, it follows that all the elements of $\mathcal{F}$ can be pierced by at most $\Sigma_{i=1}^{k} f(x_i)$ points. ◀

7 Comparison of Corollary 7 with Generic Infinite Ramsey-type Theorems

We begin with a restatement of Corollary 7.

▶ **Corollary 7** (restated). *Let $0 < r \leqslant R$, $0 \leqslant k \leqslant d-1$, and let $\mathcal{F}$ be an infinite (r, R)-fat family of sets in $\mathbb{R}^d$. Denote $\alpha = |\mathcal{F}|$. Then one of the following holds:*

- *There exists $S \subset \mathcal{F}$ with $|S| = \aleph_0$ s.t. no $k+2$ elements of S can be pierced by a k-flat.*
- *There exists $S' \subset \mathcal{F}$ with $|S'| = \alpha$, s.t. every $k+2$ elements of S' can be pierced by a k-flat.*

Proof. If the first condition does not hold, then $\mathcal{F}$ satisfies the $(\aleph_0, k+2)$ property, and hence by Theorem 5, $\mathcal{F}$ can be pierced by a finite number of k-flats $L_1, L_2, \ldots, L_n$. Denote $\mathcal{F}_i = \{A \in \mathcal{F} : A \cap L_i \neq \emptyset\}$. At least one of the families $\mathcal{F}_i$ is of cardinality α, and every $k+2$ elements of it can be pierced by a k-flat. ◀

For $\alpha = \aleph_0$, Corollary 7 is not interesting, as it follows directly from the infinite Ramsey theorem [30]. For $\alpha > \aleph_0$ and $k = 0$ (i.e., piercing by points), Corollary 7 is already significantly stronger than the conclusion of the "diagonal" Ramsey's theorem, which guarantees only a countable monochromatic subset. However, it is still uninteresting since it follows from the Erdős-Dushnik-Miller theorem [12], which asserts that for any infinite α, any (blue, red)-coloring of a graph on α vertices contains either a monochromatic blue set of cardinality $\aleph_0$ or a monochromatic red set of cardinality α.

The interesting case is $k \geqslant 1$ – i.e., piercing by k-flats with $k \geqslant 1$, which is the hard case in Theorem 5. Here, the corresponding Ramsey-type theorem concerns (blue, red)-colorings of all r-element subsets of a set with cardinality α, for $r \geqslant 3$. In this setting, Erdős and Rado [14, Thm. 28] showed that in general, one cannot guarantee even the existence of either a set of $r + 1$ elements all of whose r-tuples are blue or a set of cardinality α all of whose r-tuples are red. Corollary 7 provides either an "all-blue" set with cardinality $\aleph_0$ or an "all-red" set with cardinality α (of course, for the specific coloring in which a $(k + 2)$-tuple is colored blue if it can be pierced by a k-flat).

Therefore, in its "main" setting of $k \geqslant 1$, Theorem 5 provides an infinite Ramsey theorem which is significantly stronger than the best possible "generic" Ramsey theorems. Moreover, the assertion of Corollary 7 cannot be strengthened to obtain a first possibility with $|S| > \aleph_0$, since once no $k + 2$ elements of S can be pierced by a k-flat, all elements of S must be pairwise disjoints; hence $|S| \leqslant \aleph_0$.

8 Open Problem

A natural open problem which arises in light of Theorem 3 and Proposition 6 is, whether an $(\aleph_0, k + 2)$-theorem (like Theorem 5) can be obtained for families of closed balls, without the "fatness" assumption. For $1 \leqslant k < d - 1$, such a theorem cannot be obtained for general families of compact convex sets, as shown by the construction of Alon et al. [2]. However, it still might hold for families of balls.

References

1 N. Alon and G. Kalai. Bounding the piercing number. *Discrete Comput. Geom.*, 13:245–256, 1995.

2 N. Alon, G. Kalai, J. Matoušek, and R. Meshulam. Transversal numbers for hypergraphs arising in geometry. *Adv. Appl. Math.*, 29:79–101, 2002.

3 N. Alon and D. J. Kleitman. Piercing convex sets and the Hadwiger-Debrunner (p,q)-problem. *Advances in Mathematics*, 96(1):103–112, 1992.

4 N. Alon, J. Pach, R. Pinchasi, R. Radoičić, and M. Sharir. Crossing patterns of semi-algebraic sets. *J. Combin. Theory, Ser. A*, 111:310–326, 2005.

5 J. L. Arocha, J. Bracho, and L. Montejano. Flat transversals to flats and convex sets of a fixed dimension. *Advances in Mathematics*, 213(2):902–918, 2007.

6 B. Aronov, J. E. Goodman, and R. Pollack. A Helly-type theorem for higher-dimensional transversals. *Comput. Geom.*, 21:177–183, 2002.

7 I. Bárány and G. Kalai. Helly-type problems, 2021. `arXiv:2108.08804`.

8 V. Boltyanski and A. Soifer. *Geometric études in combinatorial mathematics*. Center for Excellence in Mathematical Education, Colorado Springs, CO, 1991.

9 L. Danzer. Zur lösung des gallaischen problems über kreisscheiben in der euklidischen ebene. *Studia Sci. Math. Hungar.*, 21:111–134, 1986.

10 L. Danzer, B. Grünbaum, and V. Klee. Helly's theorem and its relatives. In V. Klee, editor, *Convexity, Proceedings of Symposium in Pure Mathematics*, volume 7, pages 100–181. American Mathematical Society, Providence, RI, 1963.

11 A. Dumitrescu and M. Jiang. Piercing translates and homothets of a convex body. *Algorithmica*, 61(1):94–115, 2011.
12 B. Dushnik and E. W. Miller. Partially ordered sets. *American Journal of Mathematics*, 63(3):600–610, 1941.
13 J. Eckhoff. A survey of the Hadwiger-Debrunner (p, q)-problem. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry*, volume 25 of *Algorithms and Combinatorics*, pages 347–377. Springer Berlin Heidelberg, 2003.
14 P. Erdős and R. Rado. A partition calculus in set theory. *Bulletin of the American Mathematical Society*, 62(5):427–489, 1956.
15 J. Fox, J. Pach, and C. D. Tóth. Intersection patterns of curves. *J. London Math. Society*, 83:389–406, 2011.
16 S. Gao and S. Zerbib. The (2,2) and (4,3) properties in families of fat sets in the plane. *SIAM Journal of Discrete Math.*, 33(3):1326–1337, 2019.
17 B. Grünbaum. On intersections of similar sets. *Portugaliae Mathematica*, 18:155–164, 1959.
18 H. Hadwiger and H. Debrunner. Über eine variante zum Hellyschen satz. *Archiv der Mathematik*, 8(4):309–313, 1957.
19 E. Helly. Uber mengen konvexer körper mit gemeinschaftlichen punkte. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 32:175–176, 1923.
20 A. Holmsen and R. Wenger. Helly-type theorems and geometric transversals. In J. O'Rourke J. E. Goodman and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry, 3rd Edition*, pages 91–123. CRC Press LLC, Boca Raton, FL, 2017.
21 A. F. Holmsen and D. Lee. Radon numbers and the fractional Helly theorem. *Isr. J. Math.*, 241:433–447, 2021.
22 S. J. Kim, K. Nakprasit, M.J. Pelsmajer, and J. Skokan. Transversal numbers of translates of a convex body. *Discrete Math.*, 306:2166–2173, 2006.
23 D. Larman, J. Matoušek, J. Pach, and J. Töröcsik. A ramsey-type result for planar convex sets. *Bulletin of London Math. Soc.*, 26:132–136, 1994.
24 J. Matoušek. Bounded VC-dimension implies a fractional Helly theorem. *Discrete Comput. Geom.*, 31(2):251–255, 2004.
25 A. Montejano, L. Montejano, E. Roldán-Pensado, and P. Soberón. About an Erdős–Grünbaum conjecture concerning piercing of non-bounded convex sets. *Discrete Comput. Geom.*, 53(4):941–950, 2015.
26 S. Moran and A. Yehudayoff. On weak ϵ-nets and the Radon number. *Discret. Comput. Geom.*, 64(4):1125–1140, 2020.
27 T. Müller. A counterexample to a conjecture of Grünbaum on piercing convex sets in the plane. *Discrete Math.*, 313(24):2868–2871, 2013.
28 Z. Patáková. Bounding Radon number via Betti numbers. In *36th International Symposium on Computational Geometry, SoCG 2020*, volume 164 of *LIPIcs*, pages 61:1–61:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.
29 R. Pinchasi. A note on smaller fractional Helly numbers. *Discrete Comput. Geom.*, 54:663–668, 2015.
30 F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc. (Ser. 2)*, 30(1):264–286, 1930.
31 L. Santaló. Un teorema sobre conjuntos de paralelepipedos de aristas paralelas. *Publ. Inst. Mat. Univ. Nac. Litoral*, 2:49–60, 1940.
32 P. Vincensini. Figures convexes et variétés linéaires de l'espace euclidien à n dimensions. *Bull Sci. Math.*, 59:163–174, 1935.