Weak Coloring Numbers of Intersection Graphs

Zdeněk Dvořák
Charles University, Prague, Czech Republic

Jakub Pekárek
Charles University, Prague, Czech Republic

Torsten Ueckerdt
Karlsruhe Institute of Technology, Germany

Yelena Yuditsky
Université Libre de Bruxelles, Brussels, Belgium

Abstract

Weak and strong coloring numbers are generalizations of the degeneracy of a graph, where for a positive integer k, we seek a vertex ordering such that every vertex can (weakly respectively strongly) reach in k steps only few vertices that precede it in the ordering. Both notions capture the sparsity of a graph or a graph class, and have interesting applications in structural and algorithmic graph theory. Recently, Dvořák, McCarty, and Norin observed a natural volume-based upper bound for the strong coloring numbers of intersection graphs of well-behaved objects in $\mathbb{R}^d$, such as homothets of a compact convex object, or comparable axis-aligned boxes.

In this paper, we prove upper and lower bounds for the k-th weak coloring numbers of these classes of intersection graphs. As a consequence, we describe a natural graph class whose strong coloring numbers are polynomial in k, but the weak coloring numbers are exponential. We also observe a surprising difference in terms of the dependence of the weak coloring numbers on the dimension between touching graphs of balls (single-exponential) and hypercubes (double-exponential).

2012 ACM Subject Classification Mathematics of computing → Combinatoric problems; Mathematics of computing → Graph coloring

Keywords and phrases geometric intersection graphs, weak and strong coloring numbers

Digital Object Identifier 10.4230/LIPIcs.SoCG.2022.39

Funding *Zdeněk Dvořák*: Supported by the ERC-CZ project LL2005 (Algorithms and complexity within and beyond bounded expansion) of the Ministry of Education of Czech Republic.
Jakub Pekárek: Supported by the ERC-CZ project LL2005 (Algorithms and complexity within and beyond bounded expansion) of the Ministry of Education of Czech Republic.

Acknowledgements This research was carried out at the workshop on Generalized Coloring Numbers organized by Michał Pilipczuk and Piotr Micek in February 2021. We would like to thank the organizers and all participants for creating a friendly and productive environment. Special thanks go to Stefan Felsner for fruitful discussions.

1 Introduction

It is well known that if every subgraph of a graph G has average degree at most d, then G is *d-degenerate*, that is, there exists a linear ordering of the vertices of G such that each vertex has at most d neighbors that precede it in the ordering. Conversely, every subgraph of a d-degenerate graph has average degree at most $2d$. This fact is often used in design of algorithms for sparse graphs, where a result is obtained by processing the vertices one by one in the degeneracy ordering.

For algorithmic problems that involve interactions over larger distances, a stronger notion of sparsity is needed. Such a notion of *bounded expansion* was developed by Nešetřil and Ossona de Mendez [12] and can be formulated in terms of the dependence of the density of

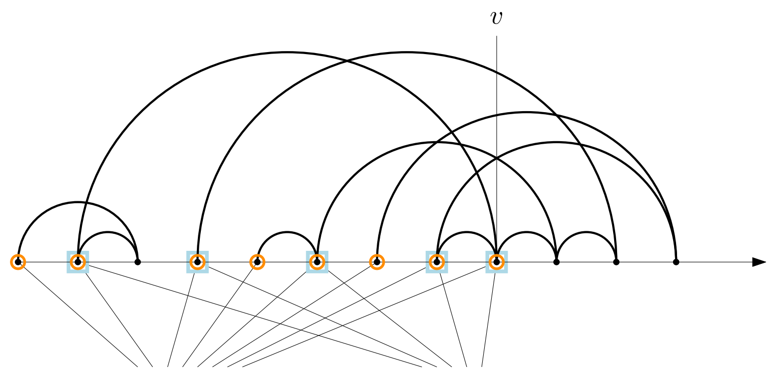

Figure 1 A vertex ordering $\prec$ of a graph G, and the sets $\mathrm{wreach}_{G,\prec,k}(v)$ and $\mathrm{sreach}_{G,\prec,k}(v)$ for a vertex v and $k = 3$.

minors or topological minors that appear in the considered graphs on the depths of these minors (we do not give a precise definition since it is somewhat technical and we do not need it in this paper). As was shown by Zhu [15], there is also an equivalent degeneracy-like characterization of bounded expansion, in terms of generalized coloring numbers, that is weak and strong coloring numbers defined below. The generalized coloring numbers were previously introduced by Kierstead and Yang [10] in the context of marking and coloring games on graphs.

Given a linear ordering $\prec$ of the vertices of a graph G and an integer $k \geq 0$, a vertex u is *weakly k-reachable* from a vertex v if $u \preceq v$ and there exists a path in G from v to u of length at most k with all internal vertices greater than u in $\prec$, and *strongly k-reachable* if there exists such a path with all internal vertices greater than v in $\prec$; see Figure 1 for an illustration. Let $\mathrm{wreach}_{G,\prec,k}(v)$ and $\mathrm{sreach}_{G,\prec,k}(v)$ denote the sets of vertices that are weakly and strongly k-reachable from v, respectively. We define *weak and strong coloring numbers* for a given ordering $\prec$ as

$$\mathrm{wcol}_{\prec,k}(G) = \max_{v\in V(G)} |\,\mathrm{wreach}_{G,\prec,k}(v)|$$
$$\mathrm{scol}_{\prec,k}(G) = \max_{v\in V(G)} |\,\mathrm{sreach}_{G,\prec,k}(v)|$$

The weak and strong coloring numbers of a graph are then obtained by minimizing over all linear orderings of $V(G)$.

$$\mathrm{wcol}_{k}(G) = \min_{\prec} \mathrm{wcol}_{\prec,k}(G)$$
$$\mathrm{scol}_{k}(G) = \min_{\prec} \mathrm{scol}_{\prec,k}(G)$$

Note that for $k = 1$, both $\mathrm{wreach}_{G,\prec,1}(v) \setminus \{v\}$ and $\mathrm{sreach}_{G,\prec,1}(v) \setminus \{v\}$ consist of the neighbors of v that precede it in the ordering $\prec$, and thus $\mathrm{scol}_1(G) = \mathrm{wcol}_1(G)$ coincide with the *coloring number* of the graph G, equal to the degeneracy of G plus one.

1.1 Properties and applications of generalized coloring numbers

The following basic claims can be found for example in [12]. One can easily check that both $\mathrm{wcol}_k(G)$ and $\mathrm{scol}_k(G)$ are non-decreasing in k and that $\mathrm{scol}_k(G) \leq \mathrm{wcol}_k(G) \leq (\mathrm{scol}_k(G))^k$ for any positive integer k. Moreover, for every $k \geq |V(G)|$, $\mathrm{scol}_k(G)$ is equal to the treewidth of G and $\mathrm{wcol}_k(G)$ is equal to the treedepth of G. A greedy coloring algorithm applied along the corresponding vertex ordering shows that the chromatic number of G is at most $\mathrm{scol}_1(G) = \mathrm{wcol}_1(G)$, the acyclic chromatic number of G is at most $\mathrm{scol}_2(G)$, and the star chromatic number of G is at most $\mathrm{wcol}_2(G)$.

Algorithmic applications of the generalized coloring numbers include for example:

- Generating sparse neighborhood covers used in decision algorithms for problems expressible in the first-order logic [8].
- Constant-factor approximation for distance versions of domination number and independence number [2], with further applications in fixed-parameter algorithms and kernelization [5].
- Practical algorithm for counting the number of appearances of fixed subgraphs [13].

As we mentioned before, Zhu [15] proved that generalized coloring numbers are bounded exactly for graph classes with bounded expansion (which include planar graphs and more generally all proper classes closed under taking minors or topological minors, graphs with bounded maximum degree, graphs that can be drawn in the plane with a bounded number of crossings per edge, intersection graphs of balls with bounded clique number, and many others). More precisely, for any class $\mathcal{G}$ with bounded expansion, there exist functions $f^s_{\mathcal{G}}$ and $f^w_{\mathcal{G}}$ such that for every graph $G \in \mathcal{G}$ and every positive integer k, we have $\mathrm{scol}_k(G) \leq f^s_{\mathcal{G}}(k)$ and $\mathrm{wcol}_k(G) \leq f^w_{\mathcal{G}}(k)$. However, the general bounds arising from Zhu's result are rather weak, and since the time complexity of the aforementioned algorithms depends on the generalized coloring numbers, we are interested in more precise bounds for specific graph classes.

1.2 Bounds on generalized coloring numbers

Quite a bit is known about the maximum possible values of generalized coloring numbers of many natural graph classes, as summarized in the following table:

Class	scol_k	wcol_k
treewidth $\leq t$	$t+1$ [7]	$\binom{k+t}{t}$ [7]
outerplanar	3	$\Theta(k \log k)$ [9]
planar	$\Theta(k)$ [14]	$\Omega(k^2 \log k)$ [9] $O(k^3)$ [14]
genus g	$O(gk)$ [14]	$O(gk + k^3)$ [14]
no K_t minor	$O(t^2 k)$ [14]	$\Omega(k^{t-2})$ [7] $O(k^{t-1})$ [14]
no K_t topological minor	$\Omega((t-3)^{k/4})$ [6, attributed to Norin] $t^{O(k)}$ [7]	

Moreover, Dvořák et al. [3] observed that in many classes of intersection graphs of geometric objects in $\mathbb{R}^d$, a non-increasing ordering of the objects according to their volume easily implies that their strong coloring number is at most $O(k^d)$. The starting point of this paper is the investigation of the same ordering from the perspective of the weak coloring numbers.

1.3 Strong coloring numbers of intersection graphs

Let S be a finite set of subsets of $\mathbb{R}^d$, which we call *objects*. The *intersection graph* of S is the graph G with $V(G) = S$ and with $uv \in E(G)$ if and only if $u \cap v \neq \emptyset$. For an integer $t \geq 1$, we say that the set S is *t-thin* if every point of $\mathbb{R}^d$ is contained in the interior of at most t objects from S; in the case $t = 1$, we say S is a *touching representation* of G. For example, a famous result of Koebe [11] states that a graph is planar if and only if it has a touching representation by balls in $\mathbb{R}^2$. Another example can be found in [4], where it is shown that the graphs in any proper minor-closed class have touching representation by *comparable* axis-aligned boxes in bounded dimension. That is, by a set S of axis-aligned boxes which has the additional property that for every $u, v \in S$, a translation of u is a subset of v or vice versa. As observed in [3], there is a very natural way of bounding the strong coloring numbers for thin intersection graphs of certain classes of objects by ordering the vertices in a non-increasing order according to the size of the objects that represent the vertices. Note that, by the definitions of the coloring numbers, if it is possible to show an upper bound on the strong coloring number in this ordering (or any ordering) then it implies an upper bound on the strong coloring number of the intersection graph. In particular, this approach works in the case the objects in S are

- scaled and translated copies of the same centrally symmetric compact convex object (this includes intersection graphs of balls and of axis-aligned hypercubes); or
- *b-ball-like* for some real number $b \geq 1$, i.e., every $v \in S$ is a compact convex set satisfying $\mathrm{vol}(v) \geq \mathrm{vol}(B(\mathrm{diam}(v)/2))/b$, where $B(a)$ is the ball in $\mathbb{R}^d$ of radius a, $\mathrm{diam}(v)$ is the maximum distance between any two points of v, and $\mathrm{vol}(v)$ is the volume of v; or
- comparable axis-aligned boxes.

As we are going to build on this argument, let us give a sketch of it. A linear ordering $\prec$ of a finite set of compact objects S is *sizewise* if for all $u, v \in S$ such that $u \prec v$, we have $\mathrm{diam}(u) \geq \mathrm{diam}(v)$. Roughly, the idea behind the proof of the next lemma is that in a sizewise ordering, the number of objects it is possible to strongly k-reach from a given object v, is bounded by the maximum order of a t-thin system of objects of larger size which can be placed in a scaled instance of v.

▶ **Lemma 1.** *Let d and t be positive integers. Let S be a t-thin finite set of compact convex objects in $\mathbb{R}^d$ and let G be the intersection graph of S. Let $\prec$ be a sizewise linear ordering of S. For each integer $k \geq 1$,*

(a) *if S consists of scaled and translated copies of the same centrally symmetric object, or if S is a set of comparable axis-aligned boxes, then $\mathrm{scol}_{\prec,k}(G) \leq t(2k+1)^d$, and*

(b) *if S consists of b-ball-like objects for a real number $b \geq 1$, then $\mathrm{scol}_{\prec,k}(G) \leq bt(2k+2)^d$.*

Proof. Consider a vertex $v \in V(G)$; we need to provide an upper bound on $|\,\mathrm{sreach}_{G,\prec,k}(v)|$. For any $m \geq 0$, in case (a) let $B_m(v)$ be the object obtained by scaling v by the factor of $2m+1$, with the center p of v being the fixed point; i.e., $B_m(v) = \{p + (2m+1)(q-p) \colon q \in v\}$. In case (b), let $B_m(v)$ be a ball of radius $(m+1)\,\mathrm{diam}(v)$ centered at an arbitrarily chosen point of v.

For each $u \in \mathrm{sreach}_{G,\prec,k}(v)$, observe that $u \cap B_{k-1}(v) \neq \emptyset$, as u is joined to v through a path with at most $k-1$ internal vertices, each represented by an object smaller or equal to v in size. In case (a), observe that there exists a translation u' of v such that $u' \subseteq u$ and $u' \cap B_{k-1}(v) \neq \emptyset$. In case (b), let u' be a scaled translation of u such that $u' \subseteq u$, $u' \cap B_{k-1}(v) \neq \emptyset$, and $\mathrm{diam}(u') = \mathrm{diam}(v)$. Note that in the former case we have $\mathrm{vol}(u') = \mathrm{vol}(v) = (2k+1)^{-d}\,\mathrm{vol}(B_k(v))$, and in the latter case we have

$$\begin{aligned}\mathrm{vol}(u') &= \frac{\mathrm{diam}^d(v)}{\mathrm{diam}^d(u)}\,\mathrm{vol}(u) \geq \frac{\mathrm{diam}^d(v)}{b\,\mathrm{diam}^d(u)}\,\mathrm{vol}(B(\mathrm{diam}(u)/2)) \\ &= b^{-1}\,\mathrm{vol}(B(\mathrm{diam}(v)/2)) = b^{-1}(2k+2)^{-d}\,\mathrm{vol}(B_k(v)).\end{aligned}$$

In either case, observe that $u' \subseteq B_k(v)$, and since S is t-thin, we have

$$\sum_{u \in \mathrm{sreach}_{G,\prec,k}(v)} \mathrm{vol}(u') \le t\,\mathrm{vol}(B_k(v)).$$

Therefore, $|\,\mathrm{sreach}_{G,\prec,k}(v)| \le t(2k+1)^d$ in case (a) and $|\,\mathrm{sreach}_{G,\prec,k}(v)| \le bt(2k+2)^d$ in case (b). ◀

That is, the strong coloring numbers of these graph classes are polynomial in k, with a uniform ordering of vertices that works for all values of k. For weak coloring numbers, a general upper bound is as follows.

▶ **Observation 2.** *For any graph G, a linear ordering $\prec$ of its vertices, and an integer $k \ge 1$,*

$$\mathrm{wcol}_{\prec,k}(G) \le \sum_{i=1}^{k} \mathrm{scol}_{\prec,i}(G)\,\mathrm{wcol}_{\prec,k-i}(G).$$

In particular, if there exists $c > 1$ such that $\mathrm{scol}_{\prec,k}(G) \le c^k$ for every $k \ge 1$, then $\mathrm{wcol}_{\prec,k}(G) \le (2c)^k$ for every $k \ge 1$.

For graphs from the classes described in Lemma 1, we obtain an exponential bound on the weak coloring numbers, more precisely $\mathrm{wcol}_k(G) \le \left(2t3^d\right)^k$ in case (a) and $\mathrm{wcol}_k(G) \le \left(2bt4^d\right)^k$ in case (b).

2 Our results

Joret and Wood (see [6]) conjectured that every class of graphs with polynomial strong coloring numbers also has polynomial weak coloring numbers (more precisely, this claim is implied by their conjecture regarding weak coloring numbers of graphs of polynomial expansion). This turns out not to be the case; Grohe et al. [7] showed that the class of graphs obtained by subdividing all edges of each graph the number of times equal to its treewidth has superpolynomial weak coloring numbers, while their strong coloring numbers are linear. However, one could still expect this conjecture to hold for "natural" graph classes, and thus we ask whether the weak coloring numbers are polynomial for the graph classes described in Lemma 1. On the positive side, we obtain the following result.

▶ **Theorem 3.** *Let d and t be positive integers. Let S be a t-thin finite set of compact convex objects in $\mathbb{R}^d$ and let G be the intersection graph of S. Let $\prec$ be a sizewise linear ordering of S. For each integer $k \ge 1$:*

(a) *If S consists of scaled and translated copies of the same centrally symmetric object, then*

$$\mathrm{wcol}_{\prec,k}(G) \le t\max(1, \lceil \log_2 k\rceil)(4k-1)^d \binom{k + t5^d + 2}{t5^d + 2}.$$

(b) *If S consists of b-ball-like objects for a real number $b \ge 1$, then*

$$\mathrm{wcol}_{\prec,k}(G) \le tb\max(1, \lceil \log_2 k\rceil)(4k)^d \binom{k + tb6^d + 2}{tb6^d + 2}.$$

Moreover, there exists k_0 (depending only on d) such that if S consists of balls, then for every $k \ge k_0$,

$$\mathrm{wcol}_{\prec,k}(G) \le t\max(1, \lceil \log_2 k\rceil)(4k-1)^d \binom{k + 2t + 2}{2t + 2}.$$

Asymptotically, the bounds in (a) and (b) in the above theorem are doubly exponential in the dimension d and singly exponential in t (and b), and for fixed d and t, they depend on k polynomially. Note that the bounds are for the full weak coloring numbers (minimized over all orderings), not just with respect to the sizewise ordering. Theorem 3 is qualitatively tight in several surprising aspects, summarized in the following result.

▶ **Theorem 4.** *For every positive integer k:*

(i) *There exists a touching graph F_k of comparable axis-aligned boxes in $\mathbb{R}^3$ such that* $\mathrm{wcol}_{2k}(F_k) \geq 2^{k+1} - 1$.

(ii) *For every t, there exists a t-thin set of axis-aligned squares in $\mathbb{R}^2$ whose intersection graph $H_{k,t}$ satisfies* $\mathrm{wcol}_{2k}(H_{k,t}) \geq \binom{k+t}{t}$.

(iii) *For every $d \geq 1$, the graph $H_{k,2^d-1}$ can also be represented as a touching graph of axis-aligned hypercubes in $\mathbb{R}^{d+2}$.*

That is:

(i) The class of touching graphs of comparable axis-aligned boxes in $\mathbb{R}^3$ has polynomial strong coloring numbers by Lemma 1, but exponential weak coloring numbers by Theorem 4(i). This provides a rather natural counterexample to the conjecture of Joret and Wood.
Let us remark that touching graphs of rectangles in $\mathbb{R}^2$ are obtained from planar graphs by adding crossing edges into faces of size four (when four of the boxes share corners), and such graphs have polynomial weak coloring numbers (this follows e.g. from their product structure [1]). Hence, the dimension three in the previous claim cannot be decreased.

(ii) Lemma 1 shows that the strong coloring numbers depend linearly on the thinness t of the representation, while the bounds on the weak coloring numbers in Theorem 3 contain t in the exponent. As shown in Theorem 4(ii), in dimension at least two this cannot be avoided (if we want a bound which is not exponential in k) and Theorem 3 cannot be strengthened so that only the multiplicative constant would depend on t.
Let us also remark that t-thin intersection graphs of intervals in $\mathbb{R}$ are interval graphs of clique number at most $2t$. As was pointed to us by Gwenaël Joret, any interval graph of clique number ω satisfies $\mathrm{wcol}_k(G) \leq \binom{\omega+1}{2}(k+1)$, as shown by an ordering obtained by placing first the vertices of a maximal system of pairwise disjoint cliques of size ω and then recursively processing the remainder of the graph which has clique number smaller than ω. Hence, the dimension two in the previous claim cannot be decreased.

(iii) In the case (a) of Theorem 3, and in particular for the touching graphs of axis-aligned hypercubes, the exponent must be exponential in the dimension, in a contrast to the case of touching graphs of balls.

3 Upper bounds

In order to prove Theorem 3 for all the classes at once, let us formulate an abstract graph property $P(f, a, e)$ on which the proof is based. For a graph G, a function $r\colon V(G) \to \mathbb{R}^+$ and $u, v \in V(G)$, let us define $\lambda_r(u, v)$ as the minimum of $\sum_{x \in V(Q) \setminus \{u,v\}} r(x)$ over all paths Q from u to v in G. For a function $f\colon \mathbb{Z}_0^+ \to \mathbb{Z}^+$ and positive integers a and e, we say that (G, r) has *the property* $P(f, a, e)$ if

(i) for each $v \in V(G)$ and integers $s \geq 1$ and $p \geq 0$, there are at most $f(p)$ vertices $u \in V(G)$ such that $r(u) \geq sr(v)$ and $\lambda_r(u, v) \leq psr(v)$, and

(ii) for each $v \in V(G)$ and each positive integer s, every sequence $u_1, u_2, \ldots$ of distinct vertices of G such that $\lambda_r(u_i, v) \leq sr(v)$ and $r(u_i) \geq a^i sr(v)$ for each i has length at most e.

Let us remark that $P(f, a, e)$ implies $P(f, a', e)$ for every $a' \geq a$, and (i) implies (ii) with $a = 1$ and $e = f(1)$. The following lemma is proved similarly to Lemma 1. In the lemma, for the role of the function r, we use diam. Intuitively, part (i) says that the number of objects with large diam that can be reached with a path with bounded diam from some object v is bounded. Part (ii) says that the number of objects with an increasing diam that can reach an object v with a path of bounded diam is also bounded.

▶ **Lemma 5.** *Let d and t be positive integers. Let S be a t-thin finite set of compact convex objects in $\mathbb{R}^d$ and let G be the intersection graph of S. For $v \in V(G)$, let $r(v) = \operatorname{diam}(v)$.*

(a) *If S consists of scaled and translated copies of the same centrally symmetric object, then (G, r) has the property $P(p \mapsto t(2p+3)^d, 1, t5^d)$.*
(b) *If S consists of b-ball-like objects for $b \geq 1$, then (G, r) has the property $P(p \mapsto tb(2p+4)^d, 1, tb6^d)$.*
(c) *If S consists of balls, then there exists a such that (G, r) has the property $P(p \mapsto t(2p+3)^d, a, 2t)$.*

Proof. Consider a vertex $v \in V(G)$ and integers $s \geq 1$ and $p \geq 0$. For any $m \geq 0$, in cases (a) and (c) let $B_m(v)$ be the object obtained by scaling v by the factor of $2m+1$, with the center of v being the fixed point. In case (b), let $B_m(v)$ be a ball of radius $(m+1)\operatorname{diam}(v)$ centered at an arbitrarily chosen point of v. Let U be the set of vertices $u \in V(G)$ such that $r(u) \geq sr(v)$ and $\lambda_r(u, v) \leq psr(v)$. Observe that for any $u \in U$, we have $u \cap B_{ps}(v) \neq \emptyset$. Let u' be a scaled translation of u such that $u' \subseteq u$, $u' \cap B_{ps}(v) \neq \emptyset$, and $\operatorname{diam}(u') = s\operatorname{diam}(v)$. For each $m \geq 0$, in cases (a) and (c), we have

$$\operatorname{vol}(u') = s^d \operatorname{vol}(v) = \left(\tfrac{s}{2m+1}\right)^d \operatorname{vol}(B_m(v)),$$

and in case (b) we have

$$\operatorname{vol}(u') \geq b^{-1} s^d \operatorname{vol}(B(\operatorname{diam}(v)/2)) = b^{-1}\left(\tfrac{s}{2m+2}\right)^d \operatorname{vol}(B_m(v)).$$

In either case, we have $u' \subseteq B_{(p+1)s}(v)$, and since S is t-thin, it follows that

$$|U| \leq t\left(\tfrac{2(p+1)s+1}{s}\right)^d \leq t(2p+3)^d$$

in cases (a) and (c), and

$$|U| \leq tb\left(\tfrac{2(p+1)s+2}{s}\right)^d \leq tb(2p+4)^d$$

in case (b). Hence, the part (i) of the property $P(f, a, e)$ is verified, and by the observations made before the lemma, this finishes the proof for the cases (a) and (b).

Let us now consider the part (ii) in case (c). Let Q be a half-space whose boundary hyperplane touches $B_s(v)$ and is otherwise disjoint from $B_s(v)$. There exists l such that $\operatorname{vol}(Q \cap B_{ls}(v)) \geq \left(\frac{1}{2} - \frac{1}{6t}\right)\operatorname{vol}(B_{ls}(v))$; let us fix smallest such l. For $a \geq 1$, let C_a be a ball touching $B_s(v)$ of radius $as\operatorname{rad}(v)$. I.e. $C_a \subseteq Q$. Note that

$$\lim_{a\to\infty} \frac{\operatorname{vol}(C_a \cap B_{ls}(v))}{\operatorname{vol}(B_{ls}(v))} = \frac{\operatorname{vol}(Q \cap B_{ls}(v))}{\operatorname{vol}(B_{ls}(v))},$$

and thus there exists a such that $\operatorname{vol}(C_a \cap B_{ls}(v)) \geq \left(\frac{1}{2} - \frac{1}{5t}\right)\operatorname{vol}(B_{ls}(v))$; let us fix smallest such a.

Consider a sequence $u_1, u_2, \ldots, u_n$ of distinct vertices of G such that $\lambda_r(u_i, v) \le sr(v)$ and $r(u_i) \ge a^i sr(v)$ for each i. In particular, note that $\mathrm{rad}(u_i) \ge \mathrm{rad}(C_a)$ for each i. From the observation made in the first paragraph of the proof, we have $u_i \cap B_s(v) \neq \emptyset$, and it follows that

$$\frac{\mathrm{vol}(u_i \cap B_{ls}(v))}{\mathrm{vol}(B_{ls}(v))} \ge \frac{\mathrm{vol}(C_a \cap B_{ls}(v))}{\mathrm{vol}(B_{ls}(v))} \ge \tfrac{1}{2} - \tfrac{1}{5t}.$$

Since S is t-thin and n is an integer, this implies $n \le 2t$, verifying the part (ii) of the property $P(p \mapsto t(2p+3)^d, a, 2t)$. ◀

To bound the weak coloring numbers, we need the following result about graphs of bounded pathwidth which appears in a stronger form (for treewidth) in van den Heuvel et al. [14]. For us, it is convenient to state the result as follows (without explicitly defining pathwidth), and thus we include the proof for completeness. A path $P = v_1v_2\ldots v_m$ in a graph G with a linear ordering $\prec$ of vertices is *decreasing* if $v_1 \succ v_2 \succ \cdots \succ v_m$. For each $v \in V(G)$, we define $\mathrm{decr}_{G,\prec,k}(v)$ as the set of vertices reachable from v by decreasing paths of length at most k.

▶ **Lemma 6.** *Let k and w be non-negative integers. Let $\prec$ be a linear ordering of the vertices of a graph G. If for every $x \in V(G)$, at most w vertices $y \prec x$ have a neighbor $y' \succeq x$, then $|\mathrm{decr}_{G,\prec,k}(v)| \le \binom{k+w}{w}$ for every $v \in V(G)$.*

Proof. Without loss of generality, we assume that if $yy' \in E(G)$ and $y \prec y'$, then y is also adjacent to all vertices x such that $y \prec x \prec y'$. Indeed, adding such an edge yx does not violate the assumptions and can only increase $|\mathrm{decr}_{G,\prec,k}(v)|$.

The proof is by induction on $k+w$. Note that $|\mathrm{decr}_{G,\prec,0}(v)| = 1$, and thus we can assume $k \ge 1$. If no neighbor of v is smaller than v, then $|\mathrm{decr}_{G,\prec,k}(v)| = 1$, and thus the claim of the lemma holds. Hence, we can assume v has such a neighbor, and in particular $w \ge 1$. Let z be the smallest neighbor of v. Let G' be the subgraph of G induced by the vertices greater than z and smaller or equal to v. Since z is adjacent to all the vertices of G', then for each $x \in V(G')$, at most $w-1$ vertices $y \prec x$ of G' have a neighbor $y' \succeq x$ in G'.

Consider now a vertex $u \in \mathrm{decr}_{G,\prec,k}(v)$, and let Q be a decreasing path of length at most k from v to u. If $z \prec u$, then Q is also a decreasing path in G', and thus $u \in \mathrm{decr}_{G',\prec,k}(v)$. Note that $|\mathrm{decr}_{G',\prec,k}(v)| \le \binom{k+w-1}{w-1}$ by the induction hypothesis. If $u \prec z$, consider the edge $u'z'$ of Q such that $u' \prec z$ and $z \preceq z'$. Note that u' is not adjacent to v by the minimality of z, and thus $z' \neq v$. Moreover, by the assumption made in the first paragraph, $u'z \in E(G)$. Hence, u is reachable from v by the decreasing path of length at most k starting with vzu' and continuing along Q, and thus $u \in \mathrm{decr}_{G,\prec,k-1}(z)$. If $u = z$, then we also have $u \in \mathrm{decr}_{G,\prec,k-1}(z)$. By the induction hypothesis, we have $|\mathrm{decr}_{G,\prec,k-1}(z)| \le \binom{k+w-1}{w}$.

Therefore,

$$\begin{aligned}|\mathrm{decr}_{G,\prec,k}(v)| &= |\mathrm{decr}_{G',\prec,k}(v)| + |\mathrm{decr}_{G,\prec,k-1}(z)| \\ &\le \binom{k+w-1}{w-1} + \binom{k+w-1}{w} = \binom{k+w}{w}.\end{aligned}$$ ◀

We use the following corollary, obtained by applying Lemma 6 to the graph obtained by contracting each interval to a single vertex.

▶ **Corollary 7.** *Let w, k, and m be non-negative integers. Let $\prec$ be a linear ordering of vertices of a graph H, and let $\mathcal{I} = \{L_i \colon i = 0, 1, \ldots\}$ be a partition of $V(H)$ into consecutive intervals in this ordering, where for every $i < j$, $u \in L_i$, and $v \in L_j$, we have $u \succ v$ (note* *the reverse ordering of the indices). Suppose that for each $i \geq 0$, we have $|L_i| \leq m$ and there are at most w indices $j > i$ such that a vertex of L_j has a neighbor in $L_0 \cup L_1 \cup \cdots \cup L_i$. Then $|\operatorname{decr}_{H,\prec,k}(v)| \leq m\binom{k+w}{w}$ for each $v \in V(H)$.*

Theorem 3 now follows from Lemma 5 and the following theorem.

▶ **Theorem 8.** *Let $f\colon \mathbb{Z}_0^+ \to \mathbb{Z}^+$ be a function and let a and e be positive integers. For a graph G and a function $r\colon V(G) \to \mathbb{R}^+$, let $\prec$ be a linear ordering of $V(G)$ such that if $u \prec v$, then $r(u) \geq r(v)$. If (G, r) has the property $P(f, a, e)$, then*

$$\operatorname{wcol}_{\prec,k}(G) \leq \max(1, \lceil \log_2 k \rceil) f(2k-2) \binom{k+e+2}{e+2}$$

for every integer $k \geq a$.

Proof. Consider any integer $k \geq a$ and a vertex $v \in V(G)$; we are going to bound the number of vertices weakly k-reachable from v. Note that for $k = 1$, $\operatorname{wreach}_{G,\prec,1}(v)$ consists of the vertices $x \in V(G)$ such that $r(x) \geq r(v)$ and $\lambda_r(v, x) = 0$, and thus $|\operatorname{wreach}_{G,\prec,1}| \leq f(0)$ by the part (i) of the property $P(f, a, e)$ with $s = 1$ and $p = 0$. Hence, we can assume that $k \geq 2$.

Let H be the graph with the vertex set $\operatorname{wreach}_{G,\prec,k}(v)$, such that for $x, y \in V(H)$ with $x \prec y$, we have $xy \in E(H)$ if and only if there exists a path Q of length at most k in G from v to x such that $y \in V(Q)$ and all the internal vertices of the subpath of Q between x and y are greater than y. Let $\ell(xy)$ denote the minimum length of the subpath between x and y over all paths Q satisfying these conditions. Observe that, by the definition of $V(H)$ and $\ell(xy)$, for every edge e' of H, there exists a decreasing path D from v in H containing the edge e' such that $\sum_{e \in E(D)} \ell(e) \leq k$. Moreover, $V(H) = \operatorname{decr}_{H,\prec,k}(v)$.

For $i \geq 0$, let L_i consist of the vertices $x \in V(H)$ such that $k^i r(v) \leq r(x) < k^{i+1} r(v)$; in particular, $v \in L_0$. Let $c = \lceil \log_2 k \rceil$ and further partition L_i into $L_{i,1}, \ldots, L_{i,c}$, where $L_{i,b}$ consists of the vertices $x \in L_i$ with $2^{b-1} k^i r(v) \leq r(x) < 2^b k^i r(v)$ for $b = 1, \ldots, c$. Consider any vertex $x \in L_{i,b}$. Since x is weakly k-reachable from v and $r(x) < 2^b k^i r(v)$, we have $\lambda_r(v, x) < (k-1) 2^b k^i r(v)$. Moreover, $r(x) \geq 2^{b-1} k^i r(v)$, and thus by the part (i) of the property $P(f, a, e)$ with $s = 2^{b-1} k^i$ and $p = 2(k-1)$, we conclude $|L_{i,b}| \leq f(2k-2)$ for each $b \in \{1, \ldots, c\}$. Hence, we have $|L_i| = |L_{i,1}| + \cdots + |L_{i,c}| \leq c f(2k-2) = \lceil \log_2 k \rceil f(2k-2)$.

Let $j_{-1} < j_0 < j_1 < \cdots < j_{w-2}$ be all indices such that $j_{-1} > i$ and for each $m \in \{-1, \ldots, w-2\}$, a vertex $u_m \in L_{j,m}$ has a neighbor $y_m \in L_0 \cup \cdots \cup L_i$ for each m. For $m = 1, \ldots, w-2$, since there exists a decreasing path D from v containing the edge $u_m y_m$ such that $\sum_{e \in E(D)} \ell(e) \leq k$, there exists a path Q in G from v to u_m of length at most k such that $r(x) \leq r(y_m) < k^{i+1} r(v)$ for every internal vertex x of Q. Consequently, we have $\lambda_r(v, u_m) \leq (k-1) k^{i+1} r(v) \leq s r(v)$ for $s = k^{i+2}$. Moreover, note that $j_m \geq i + 2 + m$, and thus $r(u_m) \geq k^{i+2+m} r(v) \geq a^m s r(v)$. By part (ii) of the property $P(f, a, e)$, we conclude that $w \leq e + 2$.

Hence, Corollary 7 implies that

$$|\operatorname{wreach}_{G,\prec,k}(v)| = |\operatorname{decr}_{H,\prec,k}(v)| \leq \lceil \log_2 k \rceil f(2k-2) \binom{k+e+2}{e+2}$$

for each $v \in V(G)$. ◀

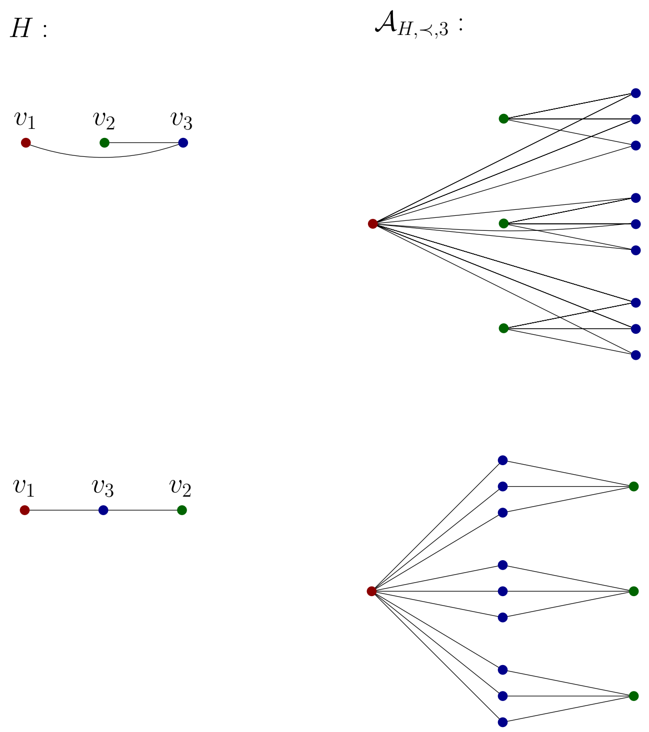

Figure 2 The graph $\mathcal{A}_{H,\prec,3}$ depicted in two ways, the first respecting the ordering and the second is easier to translate into a geometric setting.

4 Lower bounds

It is relatively easy to construct intersection graphs with large weak coloring numbers with respect to a fixed ordering. The following construction (illustrated in Figure 2) enables us to turn such graphs into graphs that have large weak coloring numbers with respect to every ordering. Let H be a graph and $\prec$ a linear ordering of its vertices. Let $v_1 \prec \cdots \prec v_n$ be the vertices of H. Let m be a positive integer and let T be the complete rooted m-ary tree of depth $n-1$. For $i \in \{1, \ldots, n\}$, let $T(v_i)$ be the set of vertices of T at distance exactly $i-1$ from the root. The graph $\mathcal{A}_{H,\prec,m}$ has vertex set $V(T)$, with vertices $x \in T(v_i)$ and $y \in T(v_j)$ adjacent if and only if $i \neq j$, $v_iv_j \in E(H)$, and x is an ancestor of y in T or vice versa. We say that T is the *scaffolding* of $\mathcal{A}_{H,\prec,m}$.

▶ **Lemma 9.** *Let k and m be positive integers. Let H be a graph and $\prec$ a linear ordering of its vertices. Suppose that for each $v \in V(H)$, the graph $H[\{u \in V(H) : v \preceq u\}]$ is connected and has diameter at most k. Then*

$$\mathrm{wcol}_k(\mathcal{A}_{H,\prec,m}) \geq \min\bigl(m, \mathrm{wcol}_{\prec,k}(H)\bigr).$$

Proof. Consider any linear ordering $\lhd$ of the vertices of $\mathcal{A}_{H,\prec,m}$. Let T be the scaffolding of $\mathcal{A}_{H,\prec,m}$ and suppose first that there exists a non-leaf vertex $z \in V(T)$ such that all children $z_1, \ldots, z_m$ of z in T are smaller than z in the ordering $\lhd$. For $i = 1, \ldots, m$, let A_i be the subgraph of $\mathcal{A}_{H,\prec,m}$ induced by z, z_i, and all descendants of z_i in T. Let v be the vertex of

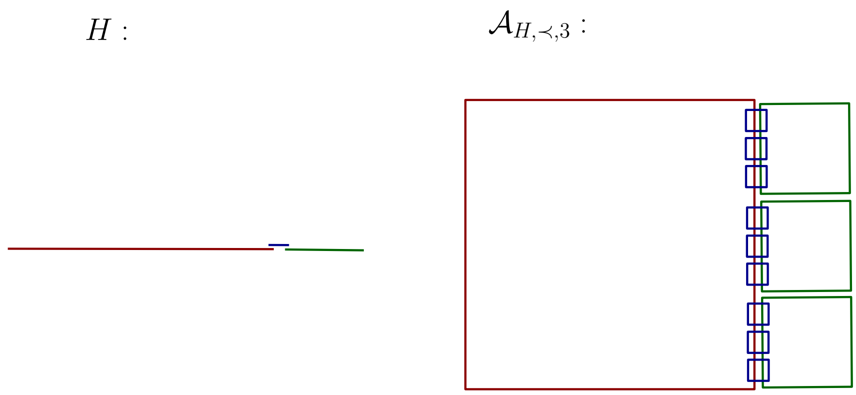

Figure 3 Representation of the graphs H and $\mathcal{A}_{H,\prec,3}$ in Figure 2 as intersection graphs of intervals and squares.

H such that $z \in T(v)$; since the graph $H[\{u \in V(H)\colon v \preceq u\}]$ has diameter at most k, every vertex of A_i is at distance at most k from z. Since $z_i \lhd z$, we conclude that a vertex of A_i distinct from z is weakly k-reachable from z. Since this is the case for each $i \in \{1, \dots, m\}$ and the subgraphs $A_1, \dots, A_m$ intersect only in z, it follows that

$$\operatorname{wcol}_{\lhd,k}(\mathcal{A}_{H,\prec,m}) \geq |\operatorname{wreach}_{\mathcal{A}_{H,\prec,m},\lhd,k}(z)| \geq m.$$

Hence, we can assume that each non-leaf vertex z of T has a child which is greater than z in the ordering $\lhd$. Consequently, T contains a path $u_1 u_2 \dots u_n$ from the root to a leaf such that $u_1 \lhd \dots \lhd u_n$. The subgraph A of $\mathcal{A}_{H,\prec,m}$ induced by $\{u_1, \dots, u_n\}$ with ordering $\lhd$ is isomorphic to H with ordering $\prec$, and thus

$$\operatorname{wcol}_{\lhd,k}(\mathcal{A}_{H,\prec,m}) \geq \operatorname{wcol}_{\lhd,k}(A) = \operatorname{wcol}_{\prec,k}(H). \blacktriangleleft$$

Moreover, assuming H has a sufficiently generic representation by comparable axis-aligned boxes, we can also find such a representation for $\mathcal{A}_{H,\prec,m}$. Given an axis-aligned box v in $\mathbb{R}^d$ and $i \in \{1, \dots, d\}$, let $\ell_i(v)$ denote the length of v in the i-th coordinate. We say that a sequence $v_1, \dots, v_n$ of axis-aligned boxes is *m-shrinking* if $\ell_d(v_i) > m\ell_d(v_{i+1})$ holds for $1 \leq i \leq n-1$. See Figure 3 for an illustration of the following construction.

▶ **Lemma 10.** *Let d, t and m be positive integers. Let S be a t-thin finite set of comparable axis-aligned boxes in $\mathbb{R}^d$ and let H be the intersection graph of S. Let T be the scaffolding of $\mathcal{A}_{H,\prec,m}$. Let $\prec$ be a sizewise linear ordering of S and let $v_1, \dots, v_n$ be the sequence of vertices of H in this order. If this sequence is m-shrinking, then $\mathcal{A}_{H,\prec,m}$ is the intersection graph of a t-thin set of comparable axis-aligned boxes in $\mathbb{R}^{d+1}$, where for $v \in V(H)$ and $u \in T(v)$, u is the product of v with an interval of length $\ell_d(v)$.*

Proof. Let $\varepsilon > 0$ be small enough so that $\ell_d(v_i) \geq m(\ell_d(v_{i+1}) + \varepsilon)$ holds for $1 \leq i \leq n-1$. For each non-leaf vertex z of T, assign labels $0, \dots, m-1$ to the edges from z to the children of z in any order; let $l(e)$ denote the label assigned to the edge e. For a vertex y of T, if $y_1 y_2 \dots y_c$ is the path in T from the root to y, then let $l(y) = (l(y_1y_2), l(y_2y_3), \dots, l(y_{c-1}y_c))$. Note that y is an ancestor of a vertex x in T if and only if $l(y)$ is a prefix of $l(x)$. Let $s(y) = \sum_{i=1}^{c-1} (l(y))_i (\ell_d(v_{i+1}) + \varepsilon)$, and let $I(y)$ be the interval $[s(y), s(y) + \ell_d(v_c)]$. Observe that if y is an ancestor of a vertex x in T, then $I(x) \subset I(y)$, and if x is neither an ancestor nor a descendant of y in T, then $I(x) \cap I(y) = \emptyset$.

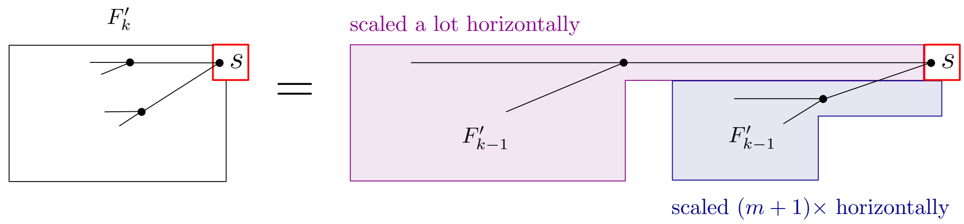

Figure 4 The construction from Lemma 12.

Hence, letting each vertex y at distance $c-1$ from the root of T be represented by the box $v_c \times I(y)$ in $\mathbb{R}^{d+1}$, we obtain a t-thin intersection representation of $\mathcal{A}_{H,\prec,m}$ as described in the statement of the lemma. ◀

To verify the assumptions of Lemma 9, the following concept is useful. Let $\prec$ be a linear ordering of vertices of a graph G. A *decreasing spanning tree* is a spanning tree T of G rooted in the maximum vertex such that any path in T starting in the root is decreasing.

▶ **Lemma 11.** *Let $k \geq 0$ be an integer. Let $\prec$ be a linear ordering of vertices of a graph G. If G has a decreasing spanning tree T of depth at most k, then $\mathrm{wcol}_{\prec,k}(G) = |V(G)|$, and for each $v \in V(G)$, the graph $G[\{u \in V(H) \colon v \preceq u\}]$ is connected and has diameter at most $2k$.*

Proof. Let z be the maximum vertex of G. Since T is decreasing and has depth at most k, we have $\mathrm{wcol}_{\prec,k}(G) \geq |\,\mathrm{wreach}_{G,\prec,k}(z)| = |V(G)|$. Moreover, for each $v \in V(G)$, letting $C_v = \{u \in V(H) \colon v \preceq u\}$, observe that for each $x \in C_v$, all ancestors of x also belong to C_v. Hence, $T[C_v]$ is a spanning tree of $G[C_v]$ of depth at most k, and thus $G[C_v]$ is connected and has diameter at most $2k$. ◀

We now find some basic graphs to which we can apply the construction.

▶ **Lemma 12.** *For all integers $k \geq 0$ and $m \geq 1$, there exists a graph F'_k with $2^{k+1}-1$ vertices represented as the touching graph of an m-shrinking sequence of comparable axis-aligned rectangles in $\mathbb{R}^2$, such that F'_k has a spanning tree of depth at most k decreasing in the sizewise ordering.*

Proof. We proceed by induction on k. For each k, we construct a representation of F'_k where the last vertex is represented by a unit square s and the rest of the representation is contained in the lower left quadrant starting from the middle of the upper side of s. The second coordinate (relevant for the definition of an m-shrinking sequence) is the horizontal one. In the vertical coordinate, all rectangles have length 1. See Figure 4 for an illustration of the construction.

The graph F'_0 is a single vertex represented by s. For $k \geq 1$, to obtain a representation of F'_k, we scale the representation of F'_{k-1} in the horizontal direction by the factor of $m+1$ and place it so that its upper right corner is the middle of the lower side of s. Then we add another copy of a representation of F'_{k-1}, scaled in the horizontal direction so that all its rectangles are more than m times longer than the already placed ones and so that when we place its upper right corner at the upper left corner of s, their interiors are disjoint from the already placed rectangles.

Observe that F'_k contains a spanning complete binary tree of depth k rooted in s, with the vertices along each path from the root increasing in size, and thus decreasing in the sizewise ordering. ◀

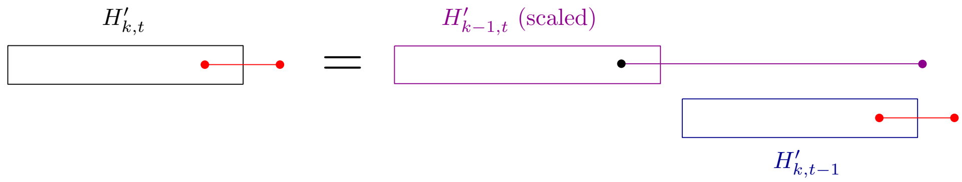

Figure 5 The construction from Lemma 13.

▶ **Lemma 13.** *For all integers $k \geq 0$ and $m, t \geq 1$, there exists a graph $H'_{k,t}$ with $\binom{k+t}{t}$ vertices represented by a t-thin m-shrinking sequence of intervals in $\mathbb{R}$, such that $H'_{k,t}$ has a spanning tree of depth at most k decreasing in the sizewise ordering. Furthermore, $H'_{k,t}$ is properly $(t+1)$-colorable.*

Proof. We construct a representation of $H'_{k,t}$ with the additional property that the right end of the smallest interval is the strictly rightmost point of the whole representation. See Figure 5 for an illustration of the construction.

We proceed by the induction on $k+t$. If $k = 0$, the representation of $H'_{k,t}$ consists of a single unit interval. If $t = 1$, then the representation consists of an m-shrinking sequence of $k+1$ intervals intersecting only in endpoints. Hence, suppose that $k \geq 1$ and $t \geq 2$. Then the representation consists of the representation A of $H'_{k,t-1}$ and of the representation B of $H'_{k-1,t}$ scaled so that all its intervals are more than m times longer than all intervals in A and so that when we place the rightmost point of B slightly to the left of the rightmost point of A, only the smallest interval of B intersects all intervals of A.

Observe that $H'_{k,t}$ has a spanning tree of depth k rooted in the smallest vertex, with the vertices along each path from the root increasing in size, and thus decreasing in the sizewise ordering. Finally, note that $H'_{k,t}$ is an interval graph with clique number at most $t+1$. Since interval graphs are perfect, $H'_{k,t}$ is properly $(t+1)$-colorable. ◀

As a final ingredient, we note that we can trade thinness for dimension.

▶ **Lemma 14.** *For a positive integer d, let $S = \{v_1, \ldots, v_n\}$ be a finite set of hypercubes in $\mathbb{R}^d$, and let G be the intersection graph of S. For any set $Y \subseteq \{1, \ldots, n\}$, there exists a set $\{u_1, \ldots, u_n\}$ of hypercubes in $\mathbb{R}^{d+1}$ whose intersection graph is isomorphic to G via the isomorphism mapping u_i to v_i for each i, such that*

- *for $1 \leq i < j \leq n$, if v_i and v_j have disjoint interiors, then u_i and u_j have disjoint interiors, and*
- *for $i \in Y$ and $j \in \{1, \ldots, n\} \setminus Y$, the hypercubes u_i and u_j have disjoint interiors.*

Proof. For $i \in Y$, we set $u_i = v_i \times [0, \ell_1(v_i)]$. For $i \in \{1, \ldots, n\} \setminus Y$, we set $u_i = v_i \times [0, -\ell_1(v_i)]$. Note that the intersection of the representation with the hyperplane defined by the last coordinate being 0 is equal to S, and thus indeed the intersection graph of S' is isomorphic to G as described. ◀

▶ **Corollary 15.** *Let $c \geq 0$ and $d \geq 1$ be integers. If G is a graph of chromatic number at most 2^c representable as an intersection graph of hypercubes in $\mathbb{R}^d$, then G is also representable as a touching graph of hypercubes in $\mathbb{R}^{d+c}$.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$, and let $\varphi \colon V(G) \to \{0,1\}^c$ be a proper coloring of G. By repeatedly applying Lemma 14 for sets $Y_1, \ldots, Y_c$, where $Y_b = \{i \in \{1, \ldots, n\} \colon \varphi(v_i)_b = 0\}$ for $b \in \{1, \ldots, c\}$, we obtain a representation of G as an intersection graph of hypercubes $u_1, \ldots, u_n$ in $\mathbb{R}^{d+c}$ with the property that for $1 \leq i < j \leq n$, if $\varphi(v_i) \neq \varphi(v_j)$, then u_i

and u_j have disjoint interiors. If $\varphi(v_i) = \varphi(v_j)$, then since φ is a proper coloring, we have $v_iv_j \notin E(G)$, and thus the hypercubes u_i and u_j are disjoint. Consequently, the hypercubes $u_1, \ldots, u_n$ have pairwise disjoint interiors. ◀

We are now ready to give the lower bounds.

Proof of Theorem 4. We prove each point separately:

(i) Let F'_k be the graph obtained in Lemma 12, represented as a touching graph of an m-shrinking sequence of axis-aligned rectangles for $m = 2^{k+1} - 1$. Let $\prec$ be the sizewise ordering of F'_k. By Lemma 11, we have $\mathrm{wcol}_{\prec,k}(F'_k) = |V(F'_k)| = 2^{k+1} - 1$. Letting $F_k = \mathcal{A}_{F'_k,\prec,m}$, Lemma 9 implies $\mathrm{wcol}_{2k}(F_k) \geq 2^{k+1} - 1$. Moreover, by Lemma 10, F_k is a touching graph of comparable axis-aligned boxes in $\mathbb{R}^3$.

(ii) Let $H'_{k,t}$ be the graph obtained in Lemma 13, represented as the intersection graph of a t-thin m-shrinking sequence of intervals for $m = \binom{k+t}{t}$. Let $\prec$ be the sizewise ordering of $H'_{k,t}$. By Lemma 11, we have $\mathrm{wcol}_{\prec,k}(H'_{k,t}) = |V(H'_{k,t})| = \binom{k+t}{t}$. Letting $H_{k,t} = \mathcal{A}_{H'_{k,t},\prec,m}$, Lemma 9 implies $\mathrm{wcol}_{2k}(H_{k,t}) \geq \binom{k+t}{t}$. Moreover, by Lemma 10, $H_{k,t}$ is the intersection graph of a t-thin set of axis-aligned squares in $\mathbb{R}^2$.

(iii) Recall that by Lemma 13, the graph $H'_{k,2^d-1}$ is properly 2^d-colorable. Let T be the scaffolding of $H_{k,2^d-1}$. For each $v \in V(H'_{k,2^d-1})$, we can assign the color of v to all vertices in $T(v)$, obtaining a proper coloring of $H_{k,2^d-1}$ by 2^d colors. Corollary 15 implies that $H_{k,2^d-1}$ can be represented as a touching graph of axis-aligned hypercubes in $\mathbb{R}^{d+2}$. ◀

5 Conclusions

In this paper we have provided upper bounds on the weak coloring number of t-thin intersection graphs of d-dimensional objects of different kinds. Our bounds are qualitatively tight in several aspects. We would like to mention a few open questions, beyond improving the proven upper and lower bounds:

- What is the asymptotic behavior of the k-th weak coloring numbers of planar graphs? It is known to be $O(k^3)$ [14] and $\Omega(k^2 \log k)$ [9].
- What is the asymptotic behavior of the k-th strong coloring numbers of touching graphs of unit balls in $\mathbb{R}^d$? It is known to be $O(k^{d-1})$ and $\Omega(k^{d/2})$.

References

1. Vida Dujmović, Pat Morin, and David R. Wood. Graph product structure for non-minor-closed classes. *arXiv*, 1907.05168, 2019. arXiv:1907.05168.
2. Zdeněk Dvořák. Constant-factor approximation of domination number in sparse graphs. *European Journal of Combinatorics*, 34:833–840, 2013.
3. Zdeněk Dvořák, Rose McCarty, and Sergey Norin. Sublinear separators in intersection graphs of convex shapes. *SIAM Journal on Discrete Mathematics*, 35(2):1149–1164, 2021. doi:10.1137/20M1311156.
4. Zdeněk Dvořák, Daniel Gonçalves, Abhiruk Lahiri, Jane Tan, and Torsten Ueckerdt. On comparable box dimension. Manuscript.
5. Kord Eickmeyer, Archontia C. Giannopoulou, Stephan Kreutzer, O-joung Kwon, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. Neighborhood complexity and kernelization for nowhere dense classes of graphs. In Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl, editors, *44th International Colloquium on Automata, Languages, and Programming (ICALP 2017)*, volume 80 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 63:1–63:14, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi:10.4230/LIPIcs.ICALP.2017.63.

6 Louis Esperet and Jean-Florent Raymond. Polynomial expansion and sublinear separators. *European Journal of Combinatorics*, 69:49–53, 2018.

7 Martin Grohe, Stephan Kreutzer, Roman Rabinovich, Sebastian Siebertz, and Konstantinos Stavropoulos. Coloring and covering nowhere dense graphs. *SIAM Journal on Discrete Mathematics*, 32:2467–2481, 2018.

8 Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 89–98. ACM, 2014.

9 Gwenaël Joret and Piotr Micek. Improved bounds for weak coloring numbers. *CoRR*, 2021. `arXiv:2102.10061`.

10 Hal A. Kierstead and Daqing Yang. Orderings on graphs and game coloring number. *Order*, 20(3):255–264, 2003.

11 Paul Koebe. Kontaktprobleme der Konformen Abbildung. *Math.-Phys. Kl.*, 88:141–164, 1936.

12 Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity (Graphs, Structures, and Algorithms)*, volume 28 of *Algorithms and Combinatorics*. Springer, 2012.

13 Felix Reidl and Blair D. Sullivan. A color-avoiding approach to subgraph counting in bounded expansion classes. *arXiv*, 2001.05236, 2020. `arXiv:2001.05236`.

14 Jan van den Heuvel, Patrice Ossona de Mendez, Daniel Quiroz, Roman Rabinovich, and Sebastian Siebertz. On the generalised colouring numbers of graphs that exclude a fixed minor. *European Journal of Combinatorics*, 66:129–144, 2017.

15 Xuding Zhu. Colouring graphs with bounded generalized colouring number. *Discrete Math.*, 309(18):5562–5568, 2009.