On the Discrete Fréchet Distance in a Graph

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Abstract

The Fréchet distance is a well-studied similarity measure between curves that is widely used throughout computer science. Motivated by applications where curves stem from paths and walks on an underlying graph (such as a road network), we define and study the Fréchet distance for paths and walks on graphs. When provided with a distance oracle of G with $O(1)$ query time, the classical quadratic-time dynamic program can compute the Fréchet distance between two walks P and Q in a graph G in $O(|P| \cdot |Q|)$ time. We show that there are situations where the graph structure helps with computing Fréchet distance: when the graph G is planar, we apply existing (approximate) distance oracles to compute a $(1+\varepsilon)$-approximation of the Fréchet distance between any shortest path P and any walk Q in $O(|G| \log |G| / \sqrt{\varepsilon} + |P| + \frac{|Q|}{\varepsilon})$ time. We generalise this result to near-shortest paths, i.e. κ-straight paths, as we show how to compute a $(1+\varepsilon)$-approximation between a κ-straight path P and any walk Q in $O(|G| \log |G| / \sqrt{\varepsilon} + |P| + \frac{\kappa|Q|}{\varepsilon})$ time. Our algorithmic results hold for both the strong and the weak discrete Fréchet distance over the shortest path metric in G.

Finally, we show that additional assumptions on the input, such as our assumption on path straightness, are indeed necessary to obtain truly subquadratic running time. We provide a conditional lower bound showing that the Fréchet distance, or even its 1.01-approximation, between arbitrary *paths* in a weighted planar graph cannot be computed in $O((|P| \cdot |Q|)^{1-\delta})$ time for any $\delta > 0$ unless the Orthogonal Vector Hypothesis fails. For walks, this lower bound holds even when G is planar, unit-weight and has $O(1)$ vertices.

2012 ACM Subject Classification Theory of computation → Design and analysis of algorithms

Keywords and phrases Fréchet, graphs, planar, complexity analysis

Digital Object Identifier 10.4230/LIPIcs.SoCG.2022.36

Related Version *Full Version*: `https://arxiv.org/abs/2201.02121`

Funding Partially supported by Independent Research Fund Denmark grants 2020-2023 (9131-00044B) "Dynamic Network Analysis".

Acknowledgements We thank David Goeckede and Petra Mutzel for useful discussions.

1 Introduction

The Fréchet distance is a popular metric for measuring the similarity between (polygonal) curves. The Fréchet distance is often intuitively defined through the following metaphor: suppose that we have two curves that are traversed by a person and their dog. Over all possible traversals by both the person and the dog, what is the minimum length of their connecting leash? The Fréchet distance has many applications; in particular in the analysis and visualization of movement data [10, 14, 31, 44]. It is a versatile distance measure that can be used for a variety of objects, such as handwriting [38], coastlines [34], outlines of geometric shapes in geographic information systems [20], trajectories of moving objects,

38th International Symposium on Computational Geometry (SoCG 2022).
Editors: Xavier Goaoc and Michael Kerber; Article No. 36; pp. 36:1–36:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

such as vehicles, animals or sports players [37, 39, 6, 14], air traffic [5] and also protein structures [28]. There are many variants of the Fréchet distance, some of which we also discuss further below. The two most-studied variants are the *continuous* and *discrete* Fréchet distance (based on whether the entities traverse a curve continuously or vertex-by-vertex).

Alt and Godau [2] were the first to study the Fréchet distance from a computational perspective. They studied how to compute the continuous Fréchet distance between two polygonal curves of n and m vertices each in $O(mn \log(n + m))$ time. Recently, this running time was improved by Buchin *et al.* [11] to $O(n^2 \sqrt{\log n}(\log \log n)^{3/2})$ on a real-valued pointer machine and $O(n^2 \log \log n)$ on a word RAM with word size $\Omega(\log n)$. Eiter and Manila [23] showed how to compute the discrete Fréchet distance between two polygonal curves in $O(nm)$ time, which was later improved to $O(nm(\log \log nm)/ \log nm)$ by Buchin *et al.* [11].

Conditional lower bounds for the Fréchet distance. The above (near-) quadratic upper bound algorithms are accompanied by a series of conditional lower bounds for computing the Fréchet distance or a constant factor approximation. All these results assume the Orthogonal Vector Hypothesis (OVH) or, by extension, the strong exponential time hypothesis (SETH) [42]. Bringmann [7] shows that there is no $O(n^{2-\delta})$ algorithm, for any $\delta > 0$, for computing the (discrete or continuous) Fréchet distance between two polygonal curves of n vertices each. The statement also holds for approximation algorithms with small constant approximation factor. Bringmann's original proof uses self-intersecting curves in the plane. Later, Bringmann and Mulzer [9] showed the same conditional lower bound for intersecting curves in $\mathbb{R}^1$. Bringmann [7] also showed the following conditional lower bound tailored to the unbalanced setting where the two input curves have different complexities: given two polygonal curves of n and m vertices each, there is no $O((nm)^{1-\delta})$ time algorithm for computing the Fréchet distance. Recently Buchin, Ophelders and Speckmann [13] showed that (assuming OVH) there can be no $O((nm)^{1-\delta})$ time algorithm that computes anything better than a 3-approximation of the Fréchet distance for pairwise disjoint planar curves in $\mathbb{R}^2$ and intersecting curves in $\mathbb{R}^1$.

Avoiding lower bounds. These lower bounds can be circumvented whenever the input curves come from well-behaved classes of curves, such as c-packed curves [22, 8], ϕ-low density curves [22], and κ-straight curves [3, 4], and in special cases when the edges of the input curves are long [26]. Another way to avoid the quadratic complexity is to allow relatively large approximation factors. Bringmann and Mulzer [9] presented an α-approximation algorithm for the discrete Fréchet distance, that runs in time $O(n \log n + n^2/\alpha)$, for any α in $[1, n]$. This was recently improved by Chan and Rahmati [16] to $O(n \log n + n^2/\alpha^2)$ for any α in $[1, n/ \log n]$. For the continuous Fréchet distance a weaker result was presented by Colombe and Fox [19]. They show an $O(\alpha)$-approximation algorithm for any α in $[\sqrt{n}, n]$ that runs in time $O((n^3/\alpha^2) \log n)$. For general polygonal curves, without further input assumptions, the best-known approximation factors with near-linear running times are still quite high, $\alpha \approx n$ for the continuous Fréchet distance and $\alpha \approx \sqrt{n}$ for the discrete case.

Fréchet distance variants. Variants of the Fréchet distance include those that model partial similarity by allowing straight-line shortcuts along a curve [21], or by maximizing the portions of the curves that a matched to each other within a fixed distance [12]. Other variants constrain the class of mappings by applying speed constraints [33] or topological constraints [15], or model the distance metric to the geodesics inside a simple polygon [27]. Even other variants extend the class of mappings, such as the weak Fréchet distance, which

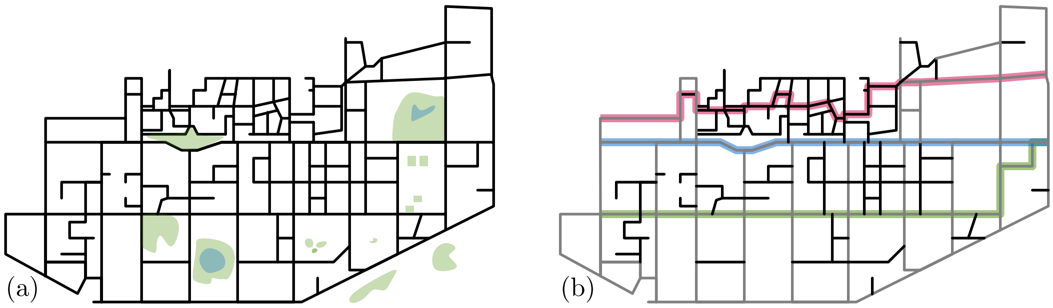


Figure 1 (a) A road network can be represented as a graph G. (b) Edges in G can be weighted, e.g. depending on whether traffic flows fast (grey) or slow (black). Under the shortest path metric, the Fréchet distance between blue and green may be smaller than the distance between red and black; even though under the Euclidean metric, the red-black Fréchet distance is smaller.

was already studied by Alt and Godau [2]. Strikingly, the Fréchet distance has not been studied in the context of graphs. Edge-weighted graphs with their shortest-path metric are commonly used to model discrete metric spaces [35], and the Fréchet distance can be derived from the underlying distance metric (Figure 2). In this paper, we intend to initiate a study of the computational complexity of the discrete Fréchet distance between paths in a planar graph, where distances between nodes are measured by their shortest path metric in this graph. This is a natural model when, for example, measuring the similarity of two trajectories in the same street network (Figure 1).

Contribution and organisation. This is the first paper that considers computing the Fréchet distance in the graph domain.[1] Section 2 contains the preliminaries where we present an overview of distance oracles and the problem statement. Section 3 serves as an introduction to our setting and techniques. We assume that P is a κ-straight path and that Q is a walk in a planar weighted graph G. We use an exact distance oracle with $O(\log^{2+o(1)}|G|)$ query time to compute a $(\kappa+1)$-approximation of $D_{\mathcal{F}}(P,Q)$. This is the first nontrivial algorithm for computing the (approximate) Fréchet distance in a planar graph. In Section 4 we extend our results. We use a $(1+\alpha)$-stretch distance oracle to compute a $(1+\varepsilon)$-approximation of $D_{\mathcal{F}}(P,Q)$. The full version contains the analogous result for the weak Fréchet distance. Finally, we show in Section 5 a conditional lower bound for computing the Fréchet distance. Specifically, assuming the Orthogonal Vector Hypothesis (OVH), we show that if G is an integer-weighted planar graph, P and Q are paths in G and $m = n^{\gamma}$ for some constant $\gamma > 0$, then for every $\delta > 0$ there can be no algorithm that computes $D_{\mathcal{F}}(P,Q)$ (or a 1.01-approximation) in $O((nm)^{1-\delta})$ time unless OVH fails. In the full version we consider walks P and Q in a planar unit-weight graph with a constant number of vertices.

2 Preliminaries

Let $G = (V,E)$ be a planar undirected weighted graph with N vertices, where every edge e_i has some corresponding integer weight ω_i and all weights can be expressed in a word of $\Theta(\log N)$ bits. For any two vertices $v_1, v_2 \in V$ their distance, denoted by $d(v_1,v_2)$, is the

[1] Similar ideas were used in the master's thesis of David Goeckede [24]. In particular, the approach we use in Section 3 and a lower bound construction for walks was used there.

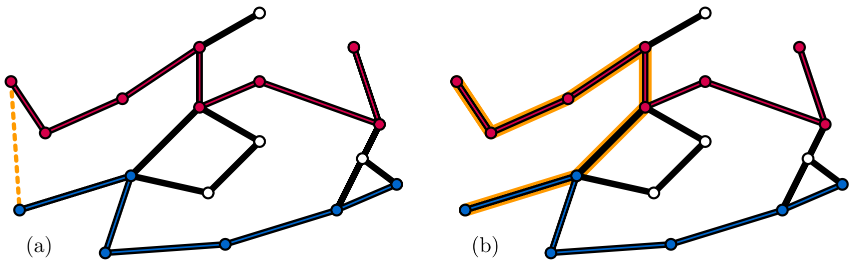


Figure 2 The Fréchet distance may be derived from the Euclidean or the shortest path metric.

length of the shortest path from v_1 to v_2 in G. A walk in G is any sequence of vertices where every subsequent pair of vertices is connected by an edge in E. A path in G is a walk where no vertex appears twice in the sequence. Let P be any walk in G, represented by an ordered set of vertices $P = (p_1, p_2, \ldots p_n)$. We denote by $|P| = n$ the number of vertices in P and by $[n]$ the set $(1, 2, \ldots, n)$. We denote the walk $Q = (q_1, q_2, \ldots q_m)$, $|Q|$ and $[m]$ analogously.

Discrete Fréchet distance. Given two walks P and Q in G, we denote by $[n] \times [m] \subset \mathbb{N} \times \mathbb{N}$ the integer lattice of n by m integers. We say that an ordered sequence F of points in $[n] \times [m]$ is a *discrete walk* if for every consecutive pair $(i, j), (k, l) \in F$, we have $k \in \{i-1, i, i+1\}$ and $l \in \{j-1, j, j+1\}$. It is furthermore *xy-monotone* when we restrict to $k \in \{i, i+1\}$ and $l \in \{j, j+1\}$. Let F be a discrete walk from $(1, 1)$ to (n, m). The *cost* of F is the maximum over $(i, j) \in F$ of $d(p_i, q_j)$. The (weak) discrete Fréchet distance is the minimum over all (not necessarily xy-monotone) walks F from $(1, 1)$ to (n, m) of its associated cost:

$$D_{\mathcal{F}}(P, Q) := \min_F \operatorname{cost}(F) = \min_F \max_{(i,j)\in F} d(p_i, q_j).$$

The discrete free-space matrix. In this paper we show an algorithm for computing the discrete Fréchet distance between two walks P and Q in a graph G. To this end, we use what we will call a free-space matrix which can be seen as a discrete free-space diagram. Given P, Q and some real value ρ, we construct a $|P| \times |Q|$ matrix M which we call the free-space matrix M_ρ. The i'th column of M_ρ corresponds to the vertex $p_i \in P$ and the j'th row corresponds $q_j \in Q$. We assign to each matrix cell $M_\rho[i, j]$ the integer -1 if $d(p_i, q_j) \leq \rho$, and a 0 if $d(p_i, q_j) > \rho$. From our above definition of the discrete Fréchet distance, we immediately conclude the following:

▶ **Lemma 1.** *The Fréchet distance between P and Q is at most ρ, if and only if there exists a discrete (xy-monotone) walk F from $(1, 1)$ to (n, m) such that $\forall (i, j) \in F$, $M_\rho[i, j] = -1$.*

Orthogonal Vectors Hypothesis. The Orthogonal Vectors problem can be stated as follows. Given are a set A and B of d-dimensional Boolean vectors with $|A| = n$ and $|B| = m$. The goal is to identify whether there exist two vectors $a = (a_1, a_2, \ldots a_d)$ and $b = (b_1, b_2, \ldots b_d)$ with $a \in A$ and $b \in B$, such that a and b are orthogonal (i.e. $\sum_{i=1}^{d} a_i \cdot b_i = 0$). In this paper, we use the following variant of the Orthogonal Vectors hypothesis. It is implied by SETH, see Abboud and Williams [1, Section 3], and it is equivalent to the standard variant of OVH defined by Williams [42], see Bringmann [7].

▶ **Definition 2.** *The Orthogonal Vectors Hypothesis states that for every $\delta > 0$ and $1 > \gamma > 0$, there exists an $\omega > 0$ and such that the Orthogonal Vectors problem for d-dimensional vectors with $d = \omega \log n$ and $m = n^\gamma$, cannot be solved in $O((nm)^{1-\delta})$ time.*

Distance oracles. A distance oracle is a compact data structure that facilitates fast exact or approximate distance queries between vertices in a graph. A distance oracle has *stretch* S if it never underestimates the distance, and it at most overestimates by a factor S, i.e. $d(a,b) \leq d_{\text{estim.}}(a,b) \leq S \cdot d(a,b)$. For general graphs [36, 41, 43], the best possible stretch in sub-quadratic space is 3, but for planar graphs on N vertices, Thorup [40] shows that it is possible to compute $(1+\varepsilon)$-stretch distance oracles in the near-linear $O(N/\varepsilon \log N)$ time and space, and with a query-time of $O(1/\varepsilon)$. The study of distance oracles for planar graphs is an active research area [17, 18, 25, 29, 30, 32, 40]. For $(1+\varepsilon)$-stretch oracles, Gu and Xu [25] show that it is possible to achieve constant query-time *independently of* ε at the cost of an increased construction time and space of $O\left(N(\log N)^4/\varepsilon + 2^{O(1/\varepsilon)}\right)$. Even for exact distances, Charalampopoulos et al. [17] give an $O\left(N^{1+o(1)}\right)$-space and $O\left(N^{o(1)}\right)$-query time data structure. Long and Pettie [32] improve these exact queries to polylogarithmic $O\left((\log(N))^{2+o(1)}\right)$ time while maintaining the $O\left(N^{1+o(1)}\right)$-space bound.

In the following sections we use the exact distance oracle by Long and Pettie [32] and the $(1+\varepsilon)$-stretch oracle by Thorup [40]. Any distance oracle that improves the efficiency of these data structures, or any extension of them to larger classes of graphs, immediately leads to improving or extending our results correspondingly.

From distance oracles to an upper bound. Given a distance oracle with $T(G)$ query time it is straightforward to find an $O(nm \cdot T(G))$ time algorithm for computing $D_{\mathcal{F}}(P,Q)$ between two walks P and Q in G that "matches" the conditional $\Omega(nm^{1-\delta})$ lower bound. Indeed, for any pair $(p,q) \in P \times Q$ we can query their pairwise distance in G. Given such a weighted graph, we want to find an xy-monotone path from $(1,1)$ to (n,m) with minimal cost (which can be done with an $O(nm \cdot T(G))$ dynamic program as by Eiter and Manila [23]).

κ-straight paths. Alt, Knauer and Wenk [3] define κ-straight paths as a generalisation of shortest paths. A path P is κ-straight if for any two points $s,t \in P$, the length of the subpath $P[s,t]$ from s to t is at most $\kappa \cdot d(s,t)$. Shortest paths are 1-straight. When we replace the term "points" by "vertices", this definition immediately transfers to our graph setting.

3 A $(\kappa+1)$-approximation for the discrete Fréchet distance

Let $G=(V,E)$ be a planar weighted graph with N vertices and integer weights. We use the structure by Long and Pettie [32] to preprocess G, such that given two walks $P=(p_1,\ldots p_n)$ and $Q=(q_1,\ldots q_m)$, where P is a κ-straight path we can compute a $(\kappa+1)$-approximation of $D_{\mathcal{F}}(P,Q)$. In the following section we extend this approach to an algorithmic result for computing a $(1+\varepsilon)$-approximation. Recall that the decision variant of the Fréchet distance may be answered with the help of a free-space matrix M_ρ. Here, we extend its definition:

▶ **Definition 3.** *We denote by M_ρ^κ the* κ-straight free-space matrix*, which is a matrix with dimensions $n \times m$. We define the matrix $M_\rho^\kappa[i,j]$ as follows:*
- *$M_\rho^\kappa[i,j] = -1$ if the distance $d(p_i,q_j) \leq \rho$,*
- *$M_\rho^\kappa[i,j] = 1$ if the distance $d(p_i,q_j) > (\kappa+1)\rho$, or*
- *$M_\rho^\kappa[i,j] = 0$ otherwise.*

Every cell $M_\rho^\kappa[i,j]$ has a corresponding point (i,j) in the integer lattice $[n] \times [m]$. The discrete Fréchet distance is at most ρ, iff there exists a discrete walk F through $[n] \times [m]$ where for every pair $(i,j) \in F$, $M_\rho^\kappa[i,j] = -1$. Explicitly constructing M_ρ^κ takes at least $\Omega(nm)$ time. However, we show that we can use the distance oracle to implicitly traverse M_ρ^κ to find the existence of such a discrete walk. To this end, we first show the following:

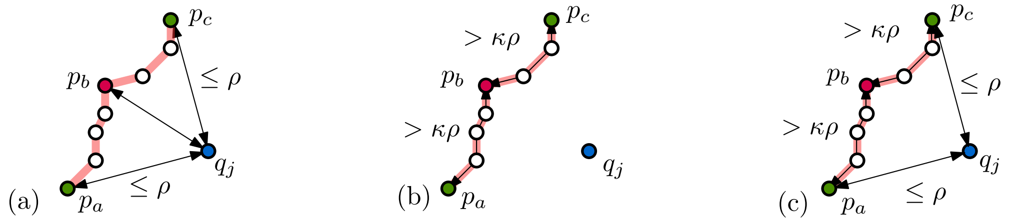


Figure 3 (a) Three vertices $p_a, p_b, p_c \in P$ and a vertex $q_j \in Q$ such that $M^\kappa_\rho[a,j] = M^\kappa_\rho[c,j] = -1$ and $M^\kappa_\rho[b,j] = 1$. (b) We show that the distance between p_a and p_b must be more than $\kappa\rho$. (c) However, this implies that P is not κ-straight, as there is a shortcut from p_a to p_c through q_j.

▶ **Lemma 4.** *Let P be a κ-straight path and Q a walk in G, ρ be some fixed value and $j \leq m$ some integer. For any two integers a, c such that $M^\kappa_\rho[a,j] = -1$ and $M^\kappa_\rho[c,j] = -1$, there cannot be an integer $b \in [a,c]$ for which $M^\kappa_\rho[b,j] = 1$.*

Proof. Suppose for the sake of contradiction that there are three integers a, b, c with $b \in [a,c]$, $M^\kappa_\rho[a,j] = -1$ and $M^\kappa_\rho[c,j] = -1$ and $M^\kappa_\rho[b,j] = 1$. It cannot be that $b = a$ or $b = c$, so there are three vertices $p_a, p_b, p_c \in P$ with $d(p_a, q_j) \leq \rho$, $d(p_c, q_j) \leq \rho$ and $d(p_b, q_j) > (\kappa+1)\rho$ (Figure 3). Moreover, p_b lies on the κ-straight subpath $P[p_a, p_c]$. It follows that the length of the subtrajectory $P[p_a, p_b]$ is more than $\kappa\rho$ (otherwise, the distance between p_b and q_j is at most $(\kappa+1)\rho$ by the path through p_a to q_j). We can apply a symmetric argument to $P[p_b, p_c]$. Thus, the length of $P[p_a, p_c]$ is more than $2\kappa\rho$. At the same time, there exists a path in G from p_a to p_b through q_j of length at most 2ρ. This contradicts that P is κ-straight. ◀

A consequence of the above lemma is the following: let (i,j) be a lattice point for which $M^\kappa_\rho[i,j] = -1$. For the nearest lattice point (l,j) left of (i,j) for which $M^\kappa_\rho[l,j] = 1$, there can be no lattice point left of (l,j) for which the matrix evaluates to -1. A symmetrical statement holds for the nearest such point right of (i,j). This leads to the following algorithm to conclude if $D_\mathcal{F}(P,Q) \leq (\kappa+1)\rho$ or $D_\mathcal{F}(P,Q) > \rho$, where we construct a discrete walk F':

We compute the distance oracle in $O(N^{1+o(1)})$ time. If $M^\kappa_\rho[1,1] > -1$ then our algorithm terminates and concludes that $D_\mathcal{F}(P,Q) > \rho$. We iteratively perform the following procedure, to construct a path F'. Let (i,j) be the latest point added to F', then:

1. If $(i,j) = (n,m)$ the algorithm terminates and concludes that $D_\mathcal{F}(P,Q) \leq (\kappa+1)\rho$.
2. If $(j+1) > m$, go to the last step.
3. Otherwise, we use two distance queries to check $M^\kappa_\rho[i,j+1]$ and $M^\kappa_\rho[i+1,j+1]$:
   - **(i)** If $M^\kappa_\rho[i,j+1] = -1$, add $(i,j+1)$ to F'.
   - **(ii)** Else if $M^\kappa_\rho[i+1,j+1] = -1$, add $(i+1,j+1)$ to F'.
4. Otherwise, we use a distance query to check if $M^\kappa_\rho[i+1,j]$:
   - **(i)** If $(i+1) > n$ or $M^\kappa_\rho[i+1,j] = 1$,
     we terminate the procedure and conclude that $D_\mathcal{F}(P,Q) > \rho$.
   - **(ii)** Otherwise, we add $(i+1,j)$ to F'.

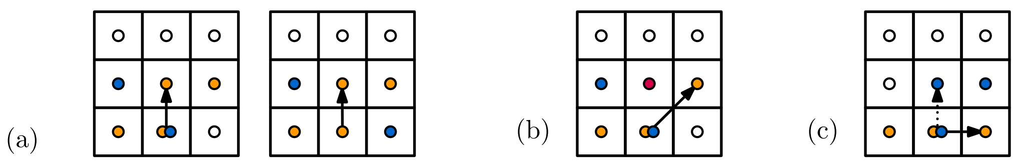


Figure 4 Lattice points to prove Lemma 5. Blue $\in F$. Orange $\in F'$ and Red $\notin F$.

▶ **Lemma 5.** *Let P be κ-straight in G, Q be any walk and $D_{\mathcal{F}}(P,Q) < \rho$. Denote by F an xy-monotone path over the lattice $[n] \times [m]$ such that for all $(i,j) \in F$, $M[i,j] = -1$. All lattice points in our constructed path F' are either in F or lie to the left of a point of F.*

Proof. Consider for the sake of contradiction the first iteration where the algorithm would add a lattice point (c,d) right of a point in F. Let $(a,b) \in F'$ be the point preceding (c,d). We make a case distinction based on whether (c,d) was added through step 3(i), 3(ii) or 4(ii). The three cases are illustrated by Figure 4, (a) (b) and (c) respectively.

First suppose that $(c,d) = (a,b+1)$. Since (c,d) is the first point right of F, it must be that F contains either (a,b) or a point right of (a,b). Moreover (since (c,d) is right of F), F also contains a point left of $(a,b+1)$. This implies that F is not xy-monotone, contradiction.

Now suppose that $(c,d) = (a+1,b+1)$. Because we reached step 3(ii), we know that $M_\rho^\kappa[a,b+1] > -1$ and thus $(a,b+1) \notin F$. However, since (c,d) is the first point right of F, F either contains (a,b) or a point right of (a,b), and a point strictly left of $(a,b+1)$. This implies that F is not xy-monotone which is a contradiction.

Finally, suppose that $(c,d) = (a+1,b)$. Since (c,d) is the first point right of F, it must be that $(a,b) \in F$. However, consider now the successor of (a,b) in F. Since F is xy-monotone, this successor is either $(a,b+1)$ or $(a+1,b+1)$, as it cannot be $(a+1,b) = (c,d)$. However, this implies that either $M_\rho^\kappa[a,b+1] = -1$ or $M_\rho^\kappa[a+1,b+1] = -1$, which contradicts the assumption that we have reached step 4 of the algorithm. ◀

With these two observations, we are ready to prove our main theorem:

▶ **Theorem 6.** *We can preprocess a planar graph G with N vertices in $O(N^{1+o(1)})$ time and space s.t: for any κ-straight path $P = (p_1, \dots, p_n)$, walk $Q = (q_1, \dots, q_m)$ and $\rho \in \mathbb{R}$, we can conclude either $D_{\mathcal{F}}(P,Q) > \rho$ or $D_{\mathcal{F}}(P,Q) \leq (\kappa+1)\rho$ in $O((n+m)\log^{2+o(1)} N)$ time.*

Proof. We first preprocess G to construct a distance oracle using $O(N^{1+o(1)})$ time and space. Given ρ, our algorithm spends at most $n+m$ iterations before it either reaches (n,m) or step 4(i) and terminates. At each iteration we perform at most three distance queries. We prove that if $D_{\mathcal{F}}(P,Q) \leq \rho$, we always conclude that $D_{\mathcal{F}}(P,Q) \leq (\kappa+1)\rho$. Indeed, suppose that $D_{\mathcal{F}}(P,Q) \leq \rho$ then there exists a discrete walk F such that for every $(i,j) \in F$, $M_\rho^\kappa[i,j] = -1$ and F is xy-monotone. Per construction, the path F' is xy-monotone and for all $(i,j) \in F'$, $M[i,j] < 1$. What remains to show is that F' is from $(1,1)$ to (n,m). Suppose for the sake of contradiction that F' does not reach (n,m) and let (i,j) be the last element added to F' before the algorithm terminated in step 4. Since we reached step 4 it must be that:

$$M_\rho^\kappa[i,j+1] > -1 \text{ and } M_\rho^\kappa[i+1,j+1] > -1 \qquad (\text{or } (j+1 \leq m)).$$

Let $\ell \leq i$ be the lowest integer such that $M_\rho^\kappa[\ell,j] = -1$. Such an ℓ must always exist, since we only enter the j'th row through a point (k,j) for which $M_\rho^\kappa[k,j] = -1$ (step 3(i) or 3(ii)). Since we arrived in step 4(i), it must be that either $M_\rho^\kappa[i+1,j] = 1$ or $(i+1) > n$. However, this implies that $(i,j) \in F$ (indeed, by Lemma 5 there exists a point equal to or to the right of (i,j) in F. However, given Lemma 4 and (ℓ,i), there is no a point in F right of (i,j)). Because if F is xy-monotone, the successor of $(i,j) \in F$ is either $(i+1,j+1)$, $(i+1,j)$ or $(i,j+1)$. Since we terminated, none of these elements can be in F, contradiction. ◀

The following corollary is a direct result of the assumption that edge weights each fit in a constant number of words (thus, the range of values for $D_{\mathcal{F}}(P,Q)$ is polynomial in N).

▶ **Corollary 7.** *We can preprocess a planar graph G with N vertices in $O(N^{1+o(1)})$ time such that: for any κ-straight path $P = (p_1, \dots, p_n)$ and walk $Q = (q_1, \dots, q_m)$, we can compute a $(\kappa+1)$-approximation of $\mathcal{D}(G)(P,Q)$ in $O((n+m)\log^{3+o(1)} N)$ time.*

4 A $(1+\varepsilon)$-approximation for Fréchet distance

We present a more involved approach to compute a $(1+\varepsilon)$ approximation of $D_{\mathcal{F}}(P,Q)$. Specifically, we choose $(1+\varepsilon) = (1+\alpha)(1+\alpha+\beta)$ for some α and β. We show for any ρ how to correctly conclude either $D_{\mathcal{F}}(P,Q) \leq (1+\alpha)(1+\alpha+\beta)\rho$ or $D_{\mathcal{F}}(P,Q) > \rho$.

To obtain this result, we use two data structures. A Voronoi diagram of P in G marks every vertex v in G with the closest vertex $p \in P$ (and the exact distance $d(v,p)$). For completeness, we prove in the full version the following (folklore) result:

▶ **Theorem 8.** *For any planar weighted graph $G = (V,E)$ and any vertex set $P \subseteq V$, it is possible to construct the Voronoi diagram of P in G in $O(|V|\log|V|)$ time.*

Additionally, we use the $(1+\alpha)$-stretch distance oracle $\mathcal{D}(G)$ by Thorup [40]. We differentiate between the distance $d(p_i,q_j)$ and what we call the *preceived* distance between p_i and q_j. For any two vertices p_i, q_j we denote by $d_o(p_i,q_j)$ their *perceived* distance (the result of the distance query of $\mathcal{D}(G)$). Per definition $d(p_i,q_j) \leq d_o(p_i,q_j) \leq (1+\alpha)\cdot d(p_i,q_j)$.

▶ **Definition 9.** *For a given value $\rho \in \mathbb{R}$ we denote by M_ρ^β the approximate free-space matrix, which is a matrix with dimensions $n \times m$ where:*

- *$M_\rho^\beta[i,j] = -1$ if the perceived distance $d_o(p_i,q_j) \leq (1+\alpha)\rho$,*
- *$M_\rho^\beta[i,j] = 1$ if the perceived distance $d_o(p_i,q_j) > (1+\alpha)(1+\alpha+\beta)\rho$, or*
- *$M_\rho^\beta[i,j] = 0$ otherwise.*

β-compression. Given a κ-straight path P and real values (ρ,β) we define the β-compression P^β as an ordered set that is obtained in three steps (Figure 5):

- The first step is a greedy iterative process where:
  - we remove (consecutive) p_x where the length of $P[p_1,p_x]$ is fewer than $\beta\rho$.
  - the first such vertex p_i that does not meet this criterion is added to P^β. Then, we remove (consecutive) p_x where the length of $P[p_i,p_x]$ is fewer than $\beta\rho$. and so forth.
- In the second step we add for every vertex in P^β its preceding vertex in P.
- In the third step we add p_n.

The result of this procedure is that we have an ordered set P^β with $n' \leq n$ vertices. We create a map $\pi : [n'] \hookrightarrow [n]$ that maps every vertex in P^β to its corresponding vertex in P (i.e. the k'th element of P^β is denoted by $p_{\pi(k)}) \in P$) and we observe:

- $\pi(1) = 1$ and $\pi(n') = n$,
- for all i, the length of $P[p_{\pi(i)}, p_{\pi(i+3)}]$ is greater than $\beta\rho$ and
- for all $x \in [\pi(i), \pi(i+1)]$, the exact distance $d(p_{\pi(i)}, p_x) < \beta\rho$ and $d(p_{\pi(i+1)}, p_x) < \beta\rho$.

We denote $P^\beta = \big(p_{\pi(1)}, p_{\pi(2)}, \ldots p_{\pi(n')}\big)$. The global approach is to approximate the Fréchet distance between P^β and Q instead. We first note the following three properties of P^β:

▶ **Lemma 10.** *For every two integers i and j, if $M_\rho^\beta[\pi(i),j] = -1$, then for all integers $x \in (\pi(i-1), \pi(i+1))$ it must be that $M_\rho^\beta[i,j] \leq 1$.*

Proof. Either $p_{\pi(i-1)}$ and $p_{\pi(i)}$ are consecutive in P (thus, the set $(\pi(i-1),\pi(i))$ is empty) or per construction the length of $P[p_{\pi(i-1)}, p_{\pi(i)}]$ is less than $\beta\rho$.

Thus, if the perceived distance $d_o(p_{\pi(i)}, q_j) \leq (1+\alpha)\rho$, then for all points p_x with $x \in (\pi(i-1),\pi(i))$, the exact distance $d(p_x,q_j) \leq (1+\alpha+\beta)\rho$ by traversing through $p_{\pi(i)}$. Thus, the perceived distance $d_o(p_x,q_j) \leq (1+\alpha)(1+\alpha+\beta)\rho$. A symmetrical argument holds for all $x \in (\pi(i),\pi(i+1))$. ◀

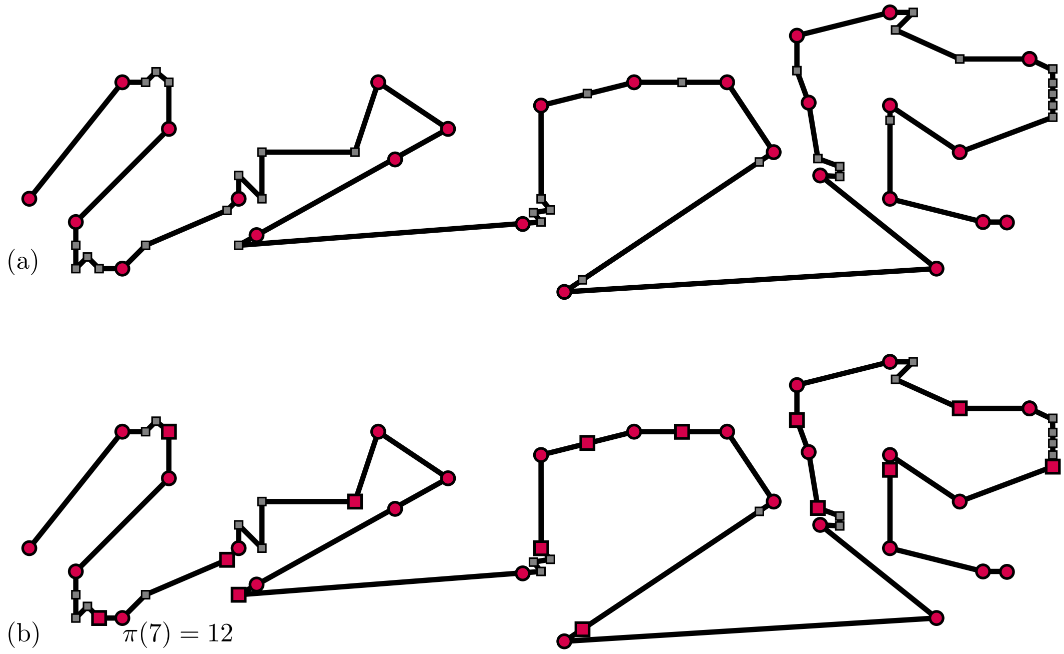


Figure 5 A planar path where the edge weights correspond to their length. (a) We greedily add vertices to P^β such that for all vertices $p_x \in P$ with preceding vertex $p_i \in P^\beta$ the length of $P[p_i, p_x]$ is at most $\beta\rho$. (b) For every vertex in P^β, we subsequently add its preceding vertex in P to P^β.

▶ **Lemma 11.** *For all i and j, if there exists an integer $x \in (\pi(i), \pi(i+1))$ such that $M^\beta_\rho[x, j] = -1$, then $M^\beta_\rho[\pi(i), j] \leq 1$ and $M^\beta_\rho[\pi(i+1), j] \leq 1$.*

Proof. As in Lemma 10, $d(p_x, p_{\pi(i)}) \leq \beta\rho$ and $d(p_x, p_{\pi(i+1)}) \leq \beta\rho$ implies the lemma. ◀

▶ **Lemma 12.** *For any j, let i be an integer such that there exists an $x \in [\pi(i), \pi(i+1)]$ with $M^\beta_\rho[x, j] = -1$. Denote $a = i - \lceil \frac{9\kappa}{\beta} \rceil$ and $b = i + \lceil \frac{9\kappa}{\beta} \rceil$. There can be no integer $y \notin [\pi(a), \pi(b)]$ such that $M^\beta_\rho[y, j] = -1$.*

Proof. For all i, the length of $P[p_{\pi(i)}, p_{\pi(i+3)}]$ is greater than $\beta\rho$. It follows that the length of the subpath $P[p_{\pi(a)}, p_x]$ is more than: $\sum_{t=1}^{\lceil 3\kappa/\beta \rceil} \beta\rho = \frac{3\kappa}{\beta}\beta\rho = 3\kappa\rho$ (Figure 6). Suppose for the sake of contradiction that there exists an integer $y < \pi(a)$ such that $d_o(p_y, q_j) \leq (1+\alpha)\rho$. Then the exact distance $d(p_y, p_x)$ is at most $2(1+\alpha)\rho$ through traversing from p_y to q_j to p_x.

However, the subpath $P[p_y, p_x]$ is longer than $P[p_{\pi(a)}, p_x]$ and thus longer than $3\kappa\rho$. For $\alpha < 0.5$, this contradicts the assumption that P is κ-straight.

A symmetrical argument holds for $y > \pi(b)$. ◀

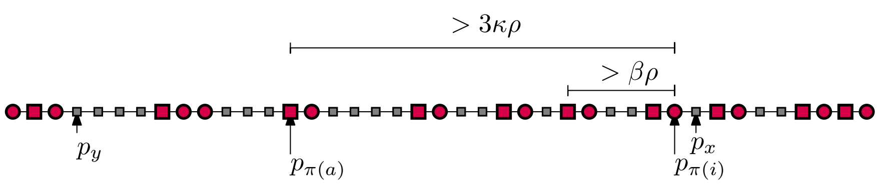


Figure 6 A schematic representation of P^β. For any i as in Lemma 12, we consider an integer $a = i - \lceil \frac{9\kappa}{\beta} \rceil$ and some p_y preceding $p_{\pi(a)}$.

Defining β-windows. Now, we use two lattices: $[n] \times [m]$ and the smaller lattice $[n'] \times [m]$. Points on the first lattice will be denoted by (x, j) and (y, j). Points on the second lattice will be denoted by (i, j) or (a, j) or (b, j). Intuitively, Lemma 12 shows for every integer j a "horizontal window" in $[n'] \times [m]$ (of width $O(\frac{\kappa}{\beta})$) that bounds the subpath of P of vertices that *may* have perceived distance fewer than $(1 + \alpha)\rho$ to the vertex $q_j \in Q$. We formalise this intuition by defining β-windows (denoted by $W_1, W_2, \ldots W_m$, see Figure 7):

- Let for an index j, p_x be any vertex in P with minimal distance to q_j in the graph G.
- Let i be the integer such that $p_{\pi(i)}$ is the point in P^β that precedes p_x.
- We distinguish two cases:
  1. If the exact distance $d(p_x, q_j) > \rho$ then: W_j is empty.
  2. Otherwise: $W_j = [i - \lceil \frac{9\kappa}{\beta} \rceil, i + \lceil \frac{9\kappa}{\beta} \rceil] \times \{j\} \subset [n'] \times [m]$.

The high-level approach. We first construct the Voronoi diagram of P in G in $O(N \log N)$ time. For every $q_j \in Q$, we obtain from the diagram the vertex $p_x \in P$ that is closest to q_j and the *exact* distance $d(p_i, q_j)$ in $O(1)$ time. With q_j, we construct W_j in $O(\frac{\kappa}{\beta})$ time. For every point $(a, j) \in W_j$ we compute $d(p_{\pi(a)}, j)$ in $O(\frac{1}{\alpha})$ time. Any lattice walk that realises a distance $D_{\mathcal{F}}(P, Q) \le (1 + \alpha)(1 + \alpha + \beta)\rho$ must be contained in the grid: $A = \cup_j W_j$ which has $O(m \cdot \frac{\kappa}{\beta})$ complexity. We compute a minimal cost path in time linear in the size of A.

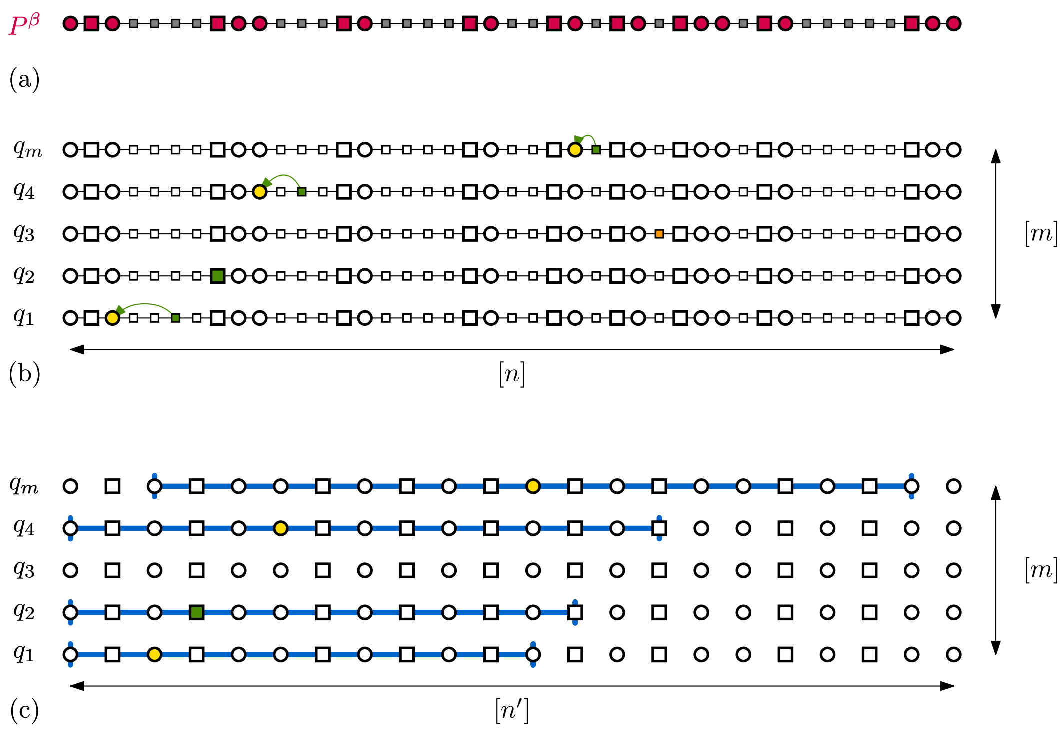


Figure 7 (a) a schematic representation of a path P with P^β in red. (b) For every $j \in [m]$, we observe the closest point p_x. If $d(p_x, q_j) \le \rho$ we color it green. Otherwise, we color it orange. In addition, if $p_x \notin P^\beta$ we color its predecessor in P^β yellow. (c) For every yellow or green vertex in $[n'] \times [m]$, we create a horizontal window in blue. We show the window for $\kappa = \beta = 1$.

▶ **Theorem 13.** *Let G be a planar graph with N vertices, $P = (p_1, \ldots, p_n)$ a κ-straight path and $Q = (q_1, \ldots, q_m)$ be any walk in G. Given a value $\rho \in \mathbb{R}$ and some β and $\alpha \le 0.5$, we correctly conclude either $D_{\mathcal{F}}(P, Q) > \rho$ or $D_{\mathcal{F}}(P, Q) \le (1 + \alpha)(1 + \alpha + \beta)\rho$ in $O(N \log N/\alpha + n + \frac{\kappa}{\alpha\beta} m))$ time using $O(N \log N/\alpha)$ space.*

Proof. We construct the approximate distance oracle $\mathcal{D}(G)$ using $O(N \log N/\alpha)$ time and space. Given P and Q, we construct the β-compressed path P^β in $O(n)$ time. We supply every point in $P \backslash P^\beta$ with a pointer to the point in P^β that precedes it. We construct the Voronoi diagram of P in the graph G in $O(N \log N)$ time. Given P^β, we construct for every integer $j \in [m]$ the window W_j in $O(\frac{\kappa}{\beta})$ time. Specifically, for any point q_j we obtain the point p_x that is closest to q_j. If $d(p_x, q_j) \leq \rho$ then we obtain the point $p_{\pi(i)}$ in P^β that precedes p_x in constant time through the pre-stored pointer and we set: $W_j = [i - \lceil \frac{9\kappa}{\beta} \rceil, i + \lceil \frac{9\kappa}{\beta} \rceil] \times \{j\}$.

The union of windows ($A = \cup_j W_j$) is a grid in $[n'] \times [m]$ of at most $O(m \cdot \frac{\kappa}{\beta})$ lattice points. For each $(a, j) \in A$ we query $\mathcal{D}(G)$ in $O(\frac{1}{\alpha})$ time to determine the value $M_\rho^\beta[\pi(a), j]$ in $O(m \frac{\kappa}{\alpha\beta})$ total time. Given this grid, we construct a directed grid graph where there is:

- a vertical edge from (a, j) to $(a, j + 1)$ if $M_\rho^\beta[\pi(a), j] < 1$ and $M_\rho^\beta[\pi(a), j + 1] < 1$,
- a horizontal edge from (a, j) to $(a + 1, j)$ if $M_\rho^\beta[\pi(a), j] < 1$ and $M_\rho^\beta[\pi(a + 1), j] = -1$,
- diagonal edge from (a, j) to $(a+1, j+1)$ if $M_\rho^\beta[\pi(a), j] < 1$ and $M_\rho^\beta[\pi(a+1), j+1] = -1$.

We can determine if there exists a path in A from $(1, 1)$ to (n', m) in $O(\frac{m\kappa}{\beta})$ time.

If such a path F^* exists. we claim that $D_\mathcal{F}(P, Q) \leq (1 + \alpha)(1 + \alpha + \beta)\rho$. Indeed, we transform F^* into a path over $[n] \times [m]$ as follows: for all $(a, j) \in F^*$ we add $(\pi(a), j)$. Note that per construction of the grid graph, for all points in F^* it must be that $M_\rho^\beta[\pi(a), j] < 1$ and thus $d_o(\pi(a), j) \leq (1+\alpha)(1+\alpha+\beta)\rho$. For every two consecutive points (a, j), $(a+1, j')$ in F^*, per construction, $M_\rho^\beta[\pi(a+1), j'] = -1$. We add all points (x, j') with $x \in [\pi(a), \pi(b)]$. By Lemma 10, for all these points (x, j') it must be that $M_\rho^\beta[x, j'] < 1$. Thus, we found a walk F from $(1, 1)$ to (n, m) where for every $(i, j) \in F$, $M_\rho^\beta[i, j] < 1$ and the Fréchet distance between P and Q is at most $(1 + \alpha)(1 + \alpha + \beta)\rho$.

If no such path F^* exists. we claim that $D_\mathcal{F}(P, Q) > \rho$. Suppose for the sake of contradiction that $D_\mathcal{F}(P, Q) \leq \rho$ then there exists an xy-monotone path F from $(1, 1)$ to (n, m) where for all $(i, j) \in F$, $d(p_i, q_j) \leq \rho$. We use F to construct a path F^* from $(1, 1)$ to (n', m) in our grid graph. Specifically, for every element $(x, j) \in F$ we check if p_x has been removed during compression.

- If p_x has an equivalent in P^β then there exists an integer a such that $p_{\pi(a)} = p_x$ and we add the lattice point $(a, j) \in [n'] \times [m]$ to F^*. Per definition of F, $M_\rho^\beta[\pi(a), j] = -1$.
- Otherwise, we identify the index i such that $\pi(i)$ is the vertex of P^β preceding p_x and we add the point $(i, j) \in [n'] \times [m]$ to F^*. By Lemma 11, $M_\rho^\beta[\pi(i), j] < 1$.

Since F is a connected xy-monotone path from $(1, 1)$ to (n, m), we obtain an xy-monotone path F^* from $(1, 1)$ to (n', m). Moreover, whenever this path traverses a horizontal or diagonal edge to a point (a, j) it must be that $(\pi(a), j) \in F$ and thus $M_\rho^\beta[\pi(a), j] = -1$. Thus, F^* is a path from $(1, 1)$ to (n', m) in our grid graph which contradicts the earlier assumption that no such path exists. ◀

This corollary follows immediately from choosing $\alpha = \beta = 0.25(\sqrt{8\varepsilon + 9} - 3)$.

▶ **Corollary 14.** *Let G be a planar graph with N vertices, $P = (p_1, \ldots, p_n)$ a κ-straight path and $Q = (q_1, \ldots, q_m)$ be any walk in G. Given a value $\rho \in \mathbb{R}$ and some $\varepsilon > 0$ we correctly conclude either $D_\mathcal{F}(P, Q) > \rho$ or $D_\mathcal{F}(P, Q) \leq (1 + \varepsilon)\rho$ in $O(N \log N/\sqrt{\varepsilon} + n + \frac{\kappa}{\varepsilon} m))$ time.*

5 A conditional lower bound for computing the Fréchet distance

We show that for every $\delta > 0$ there is no $O((nm)^{1-\delta})$ algorithm for computing for the discrete Fréchet distance between two paths in a planar graph (unless OVH fails). We show this using a planar graph $G = (V, E)$ where the edges have integer weights in $\{0.001, 0.35, 0.6, 0.65, 1, 2, 3\}$.

In the full version we prove a similar statement for walks in a constant-complexity unit-weight graph. Throughout this section, we fix some $\delta > 0$ and $\gamma > 0$ and consider two sets A and B of d-dimensional Boolean vectors (with $d = \omega \log n$ where the constant ω depends on δ). In addition, we assume that A and B contain n' and m' vectors respectively with $n' = (m')^\gamma$. Using A and B, we reduce from Orthogonal Vectors using what we call a *vector gadget*. We construct a graph G and two paths P and Q where $D_{\mathcal{F}}(P,Q) < 3$ if and only if there exists $(a,b) \in A \times B$ such that a and b are orthogonal.

Proof notation. Throughout this section, we label vertices to represent an equivalence class. We construct a graph where we label "blue" vertices with a label in $\{x, y, z, B^{\{0\}}, B^{\{1\}}, B\}$ and "red" vertices with a label in $\{\alpha, \alpha^*, \beta, \beta^*, \gamma, A^{\{0\}}, A^{\{1\}}, A\}$. Ideally, we would construct a graph where for every red-blue pair of labels, all red-blue vertices with those two labels have the same distance. We maintain a slightly weaker property: consider any red-blue pair of vertices b, r with $\text{LABEL}(b) \in \{x, y, z, B^{\{0\}}, B^{\{1\}}, B\}$ and $\text{LABEL}(r) \in \{\alpha, \alpha^*, \beta, \beta^*, \gamma, A^{\{0\}}, A^{\{1\}}, A\}$. We demand the following: if $d(b,r) < 3$ then for all (b', r') with $\text{LABEL}(b') = \text{LABEL}(b)$ and $\text{LABEL}(r') = \text{LABEL}(r)$ it must be that $d(b', r') < 3$.

We construct for every vector in A (and B) a vector gadget. This gadget resembles the gadget used in the conditional lower bound for the Fréchet distance in the Euclidean plane by Bringmann [7]. The path P will traverse all vector gadgets of A in sequence (and Q will traverse gadgets of B). We connect all gadgets of A to all gadgets of B via "star" vertices (grey triangles or diamonds). These stars ensure that there can be a matching between every pair of gadgets (vectors). Finally, we add "park" vertices (square vertices) which are vertices of A (or B) that are close to all vertices of B (or A). The intuition is, that during a traversal (reparametrization) of P and Q an entity can remain stationary at a park vertex, whilst the other entity traverses their corresponding path until the appropriate gadgets can be matched.

Vector gadget. We illustrate the vector gadget for vectors $b \in B$ (see Figure 8). The "core" of this subgraph is vertex y connected to the following construction (repeated d times): there are two *Boolean vertices* ($B^{\{0\}}, B^{\{1\}}$), followed by an *intermediary* vertex B. This core will allow us to model a d-dimensional Boolean vector. We connect the core to two park vertices x and z where we add an edge (x, y) and (B, z) of weight 3. Finally, we add two star vertices where every vertex B, y and x get connected to the top star vertex, and every vertex $x, B^{\{0\}}, z$ get connected to the bottom star vertex. For every vector in A, the corresponding vector gadget is nearly identical. Most crucially, this subgraph is vertically mirrored and the edges attached to star vertices have different weights.

From gadgets to a graph. Given our instance of OV, we construct $(n+m)$ vector gadgets. Next, we combine the gadgets (Figure 9). We highlight the important steps: all the vector gadgets of B (and A) are placed horizontally adjacent to each other.
The vertices $\{s^\downarrow, z, \sigma^\uparrow\}$ get connected via a star vertex in the centre of the graph. Each vertex $s^\uparrow$ gets connected to a star vertex at the top of the graph. Each vertex $\sigma^\downarrow$ gets connected to a star vertex at the bottom of the graph. These two stars get connected via an edge with weight 2. Given this graph G, we say that a red vertex r is *close* to a blue vertex b if $d(r,b) < 3$. For every blue label, we observe the set of close red labels (Table 1):

Constructing the paths P and Q. Given G, A and B, we construct a path P consisting of $n = O(n' \cdot d)$ vertices and a path Q consisting of $m = O(m' \cdot d)$ vertices (refer to Figure 9). The path P starts in α and then moves to α^*. Then, P traverses every vector gadget of A in

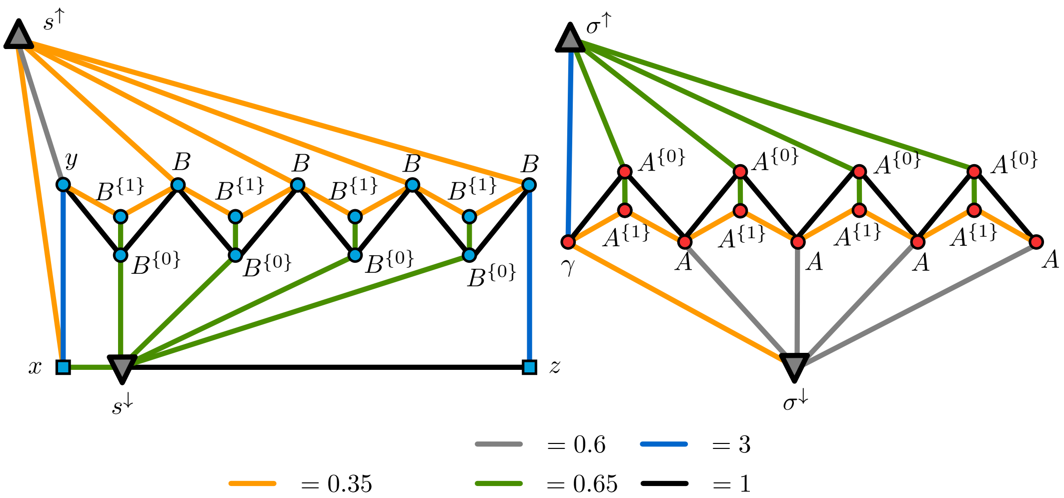


Figure 8 The gadgets for vectors in B and in A. The path corresponding to B will traverse blue vertices, the path corresponding to A red.

Table 1 The shortest distance between vertices with a label in $\{\alpha, \alpha^*, \beta, \beta^*, \gamma, A^{\{0\}}, A^{\{1\}}, A\}$ and in { x, y, z, $B^{\{0\}}$, $B^{\{1\}}$, B }, showing far and near pairs of labels.

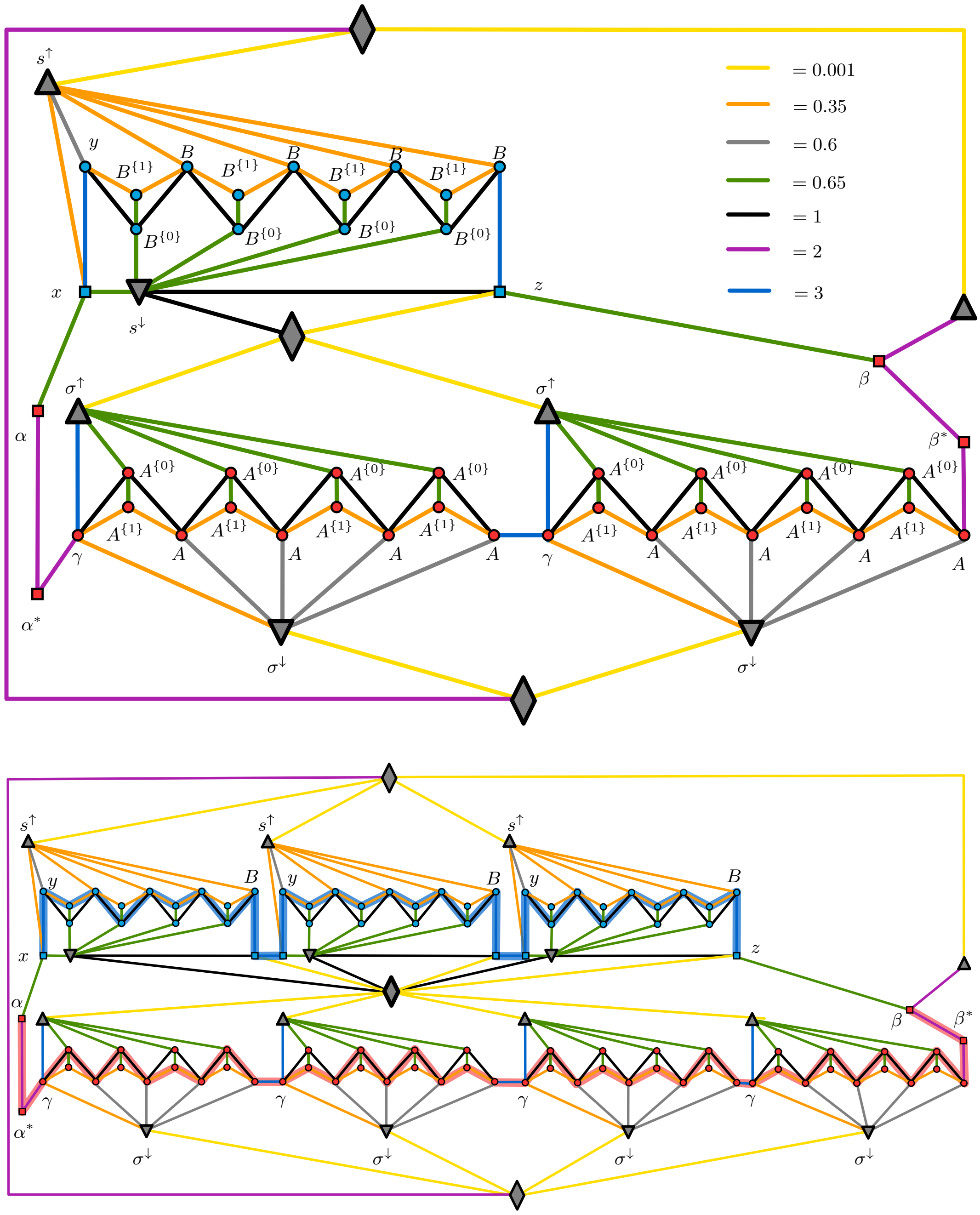
dist.	α	α^*	β	β^*	γ	$A^{\{0\}}$	$A^{\{1\}}$	A
x	.65	2.65	2.3	4.3	2.702	2.301	2.951	2.952
y	1.6	3.6	2.601	4.601	2.952	3.251	3.302	3.202
z	2.3	4.3	.65	2.65	1.652	0.652	1.302	1.652
$B^{\{0\}}$	1.95	3.95	2.3	4.3	3.301	2.301	2.951	3.301
$B^{\{1\}}$	1.7	3.7	2.701	4.701	3.051	2.951	3.402	3.302
B	1.35	3.35	2.351	4.351	2.702	3.001	3.051	2.951

sequence. Let v be the first vector in A. The path P arrives at y and traverses the Boolean vertices and intermediate vertices in an alternating manner (where P traverses $A^{\{0\}}$ if the corresponding Boolean in v is false and $A^{\{1\}}$ if the corresponding Boolean is true). Having traversed every vector gadget, P moves through β^* to β. The path Q traverses every vector gadget of B in sequence. Let a gadget correspond to a vector $v' \in B$:

The path Q starts at the vector x in the gadget and then traverses the Boolean vertices and intermediate vertices in an alternating manner (where Q traverses $B^{\{0\}}$ if the corresponding Boolean in v' is false and $B^{\{1\}}$ if the corresponding Boolean is true). The path Q ends at the vector z, and continues to the next gadget.

▶ **Theorem 15.** *Let G be a planar, integer-weighted graph, P and Q be two paths in G with n and m vertices and $n = m^\gamma$ for some constant $0 < \gamma \leq 1$. For all $\delta > 0$, there can be no algorithm that computes (a 1.01-approximation of $D_{\mathcal{F}}(P, Q)$) in $O((nm)^{1-\delta})$ time.*

Proof. For any given A and B of n' and m' vectors, we construct two paths P and G with $n = O(n' \log n')$ and $m = O(m' \log m')$ vertices respectively. OVH postulates that there exists no algorithm that can conclude if there exists two orthogonal vectors $(a, b) \in A \times B$ in $O((nm)^{1-\delta})$ time, for any $\delta > 0$. We prove this theorem by showing that there are two such vectors if and only if $D_{\mathcal{F}}(P, Q) < 3$. We observe that in our graph for all red/blue vertices r and b either $d(r, b) \leq 2.96$ or $d(r, b) \geq 3$ (which implies this proof for the 1.01-approximation).



$$B = \{(1, 0, 1, 0), (1, 1, 1, 0), (0, 0, 1, 1)\}$$

$$A = \{(0, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (1, 0, 0, 0)\}$$

Figure 9 Top: we show how pairwise gadgets get connected. Bottom: given a set A of four and B of three vectors, we construct the corresponding graph and path.

We show that if there exist two orthogonal vectors $(a, b) \in A \times B$ then $D_{\mathcal{F}}(P, Q) < 3$. We construct a traversal of P and Q where the red entity (henceforth "Red") traversing P remains close to the blue entity ("Blue") traversing Q. First, Red is stationary at the park vertex α, whilst Blue traverses B until it reaches the vector gadget corresponding to $b \in B$. Then, whilst Blue remains stationary at the park vertex x, Red traverses P until it reaches the vector gadget corresponding to $a \in A$. At this point, Blue moves to y as Red moves to γ. Both entities simultaneously traverse their vector gadgets. During this traversal (since a and b are orthogonal) the entities remain close. Then, Blue remains stationary at z, whilst Red traverses the rest of P. Finally, Red remains at β whilst Blue traverses the rest of Q.

We show that if $D_{\mathcal{F}}(P, Q) < 3$ then there exists a pair of vectors $(a, b) \in A \times B$ such that a and b are orthogonal. Indeed, fix any traversal of P and Q that realises the Fréchet distance. When Red is at α^*, Blue must be at some vertex x.

Consider now the time when Blue moves from x to y (where y lies in a gadget corresponding to some vector $b \in B$). At this time, Red cannot be at the park vertex α because α precedes α^*. Similarly, Red cannot be at the park vertex β because β^* precedes β (and β^* is not close to x). Since $\textsc{close}(y) = \{\gamma, \alpha, \beta\}$, it must be that Blue is at some vertex γ (corresponding to some vector $a \in A$). Now consider the next time step, when we assume that Red moves to $\{A^{\{0\}}, A^{\{1\}}\}$ (the argument for when Blue moves to $\{B^{\{0\}}, B^{\{1\}}\}$ is symmetrical). If Red moves to $A^{\{0\}}$ then, via the same argument as above, Blue has to simultaneously move to $B^{\{0\}}$ or $B^{\{1\}}$. If Red moves to $A^{\{1\}}$ then Blue must move to $B^{\{0\}}$. For the next time step, via the same argument, both entities must move to A and B. We can continue this same argument, which shows that the two vectors a and b must be orthogonal. ◀

6 Concluding remarks

This paper is the first to study the natural question of computing the Fréchet distance between walks P and Q in graphs. Our algorithmic results (including the Voronoi diagram construction) do not depend on the planarity of G; we rely only on a distance oracle. Hence, our result immediately holds for other classes of graphs where it is possible to efficiently construct distance oracles or in computational models where the distance oracle is provided. Given a distance oracle, our $(\kappa + 1)$ approximation is obtained in time (near-) linear in $(|P| + |Q|)$. In other words, our result in Section 3 allows us to pre-process a graph G in time nearly linear to its vertices, in order to efficiently facilitate Fréchet distance queries between two any two walks in (as long as one of the two walks is κ straight for some query constant κ). This is not true for our $(1 + \varepsilon)$-approximation algorithm, which currently requires the construction of a Voronoi diagram of P in G and thus, for every pair of walks, must spend near-linear time in G.

References

1 Amir Abboud and Virginia Vassilevska Williams. Popular conjectures imply strong lower bounds for dynamic problems. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 434–443. IEEE, 2014.

2 Helmut Alt and Michael Godau. Computing the Fréchet distance between two polygonal curves. *International Journal of Computational Geometry & Applications*, 5(01n02):75–91, 1995.

3 Helmut Alt, Christian Knauer, and Carola Wenk. Comparison of distance measures for planar curves. *Algorithmica*, 38(1):45–58, 2004.

4 Boris Aronov, Sariel Har-Peled, Christian Knauer, Yusu Wang, and Carola Wenk. Fréchet distance for curves, revisited. In *European symposium on algorithms*, pages 52–63. Springer, 2006.

5 Alessandro Bombelli, Lluis Soler, Eric Trumbauer, and Kenneth D Mease. Strategic air traffic planning with Fréchet distance aggregation and rerouting. *Journal of Guidance, Control, and Dynamics*, 40(5):1117–1129, 2017.

6 Sotiris Brakatsoulas, Dieter Pfoser, Randall Salas, and Carola Wenk. On map-matching vehicle tracking data. In *Proceedings of the 31st international conference on Very large data bases*, pages 853–864, 2005.

7 Karl Bringmann. Why walking the dog takes time: Frechet distance has no strongly subquadratic algorithms unless seth fails. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 661–670. IEEE, 2014.

8 Karl Bringmann and Marvin Künnemann. Improved approximation for Fréchet distance on c-packed curves matching conditional lower bounds. *International Journal of Computational Geometry & Applications*, 27(01n02):85–119, 2017.

9 Karl Bringmann and Wolfgang Mulzer. Approximability of the discrete Fréchet distance. *Journal of Computational Geometry*, 7(2):46–76, 2016.

10 Kevin Buchin, Maike Buchin, David Duran, Brittany Terese Fasy, Roel Jacobs, Vera Sacristan, Rodrigo I Silveira, Frank Staals, and Carola Wenk. Clustering trajectories for map construction. In *Proceedings of the 25th ACM SIGSPATIAL International Conference on Advances in Geographic Information Systems*, pages 1–10, 2017.

11 Kevin Buchin, Maike Buchin, Wouter Meulemans, and Wolfgang Mulzer. Four soviets walk the dog: Improved bounds for computing the Fréchet distance. *Discrete & Computational Geometry*, 58(1):180–216, 2017.

12 Kevin Buchin, Maike Buchin, and Yusu Wang. Exact algorithms for partial curve matching via the Fréchet distance. In *Proceedings of the twentieth annual ACM-SIAM symposium on Discrete algorithms*, pages 645–654. SIAM, 2009.

13 Kevin Buchin, Tim Ophelders, and Bettina Speckmann. Seth says: Weak Fréchet distance is faster, but only if it is continuous and in one dimension. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2887–2901. SIAM, 2019.

14 Maike Buchin, Bernhard Kilgus, and Andrea Kölzsch. Group diagrams for representing trajectories. *International Journal of Geographical Information Science*, 34(12):2401–2433, 2020.

15 Erin Wolf Chambers, Eric Colin De Verdiere, Jeff Erickson, Sylvain Lazard, Francis Lazarus, and Shripad Thite. Homotopic Fréchet distance between curves or, walking your dog in the woods in polynomial time. *Computational Geometry*, 43(3):295–311, 2010.

16 Timothy M Chan and Zahed Rahmati. An improved approximation algorithm for the discrete Fréchet distance. *Information Processing Letters*, 138:72–74, 2018.

17 Panagiotis Charalampopoulos, Pawel Gawrychowski, Shay Mozes, and Oren Weimann. Almost optimal distance oracles for planar graphs. In Moses Charikar and Edith Cohen, editors, *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019*, pages 138–151. ACM, 2019. doi:10.1145/3313276.3316316.

18 Vincent Cohen-Addad, Søren Dahlgaard, and Christian Wulff-Nilsen. Fast and compact exact distance oracle for planar graphs. In Chris Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 962–973. IEEE Computer Society, 2017. doi:10.1109/FOCS.2017.93.

19 Connor Colombe and Kyle Fox. Approximating the (continuous) Fréchet distance. In *37th International Symposium on Computational Geometry (SoCG 2021)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.

20 Thomas Devogele. A new merging process for data integration based on the discrete Fréchet distance. In *Advances in spatial data handling*, pages 167–181. Springer, 2002.

21 Anne Driemel and Sariel Har-Peled. Jaywalking your dog: computing the Fréchet distance with shortcuts. *SIAM Journal on Computing*, 42(5):1830–1866, 2013.

22 Anne Driemel, Sariel Har-Peled, and Carola Wenk. Approximating the Fréchet distance for realistic curves in near linear time. *Discret. Comput. Geom.*, 48(1):94–127, 2012. doi:10.1007/s00454-012-9402-z.

23 Thomas Eiter and Heikki Mannila. Computing discrete Fréchet distance. Technical Report CD-TR 94/64, Christian Doppler Laboratory for Expert Systems, TU Vienna, Austria, 1994.

24 David Göckede. Computing the Fréchet distance in graphs efficiently using shortest-path distance oracles. Master's thesis, Department of Computer Science, University of Bonn, 2021.

25 Qian-Ping Gu and Gengchun Xu. Constant query time $(1+\varepsilon)$-approximate distance oracle for planar graphs. *Theor. Comput. Sci.*, 761:78–88, 2019. doi:10.1016/j.tcs.2018.08.024.

26 Joachim Gudmundsson, Majid Mirzanezhad, Ali Mohades, and Carola Wenk. Fast Fréchet distance between curves with long edges. *International Journal of Computational Geometry & Applications*, 29(02):161–187, 2019.

27 Atlas F Cook IV and Carola Wenk. Geodesic Fréchet distance inside a simple polygon. *ACM Transactions on Algorithms (TALG)*, 7(1):1–19, 2010.

28 Minghui Jiang, Ying Xu, and Binhai Zhu. Protein structure–structure alignment with discrete Fréchet distance. *Journal of bioinformatics and computational biology*, 6(01):51–64, 2008.

29 Philip N. Klein. Preprocessing an undirected planar network to enable fast approximate distance queries. In David Eppstein, editor, *Proceedings of the Thirteenth Annual ACM-SIAM Symposium on Discrete Algorithms, January 6-8, 2002, San Francisco, CA, USA*, pages 820–827. ACM/SIAM, 2002. URL: http://dl.acm.org/citation.cfm?id=545381.545488.

30 Philip N. Klein. Multiple-source shortest paths in planar graphs. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2005, Vancouver, British Columbia, Canada, January 23-25, 2005*, pages 146–155. SIAM, 2005. URL: http://dl.acm.org/citation.cfm?id=1070432.1070454.

31 Maximilian Konzack, Thomas McKetterick, Tim Ophelders, Maike Buchin, Luca Giuggioli, Jed Long, Trisalyn Nelson, Michel A Westenberg, and Kevin Buchin. Visual analytics of delays and interaction in movement data. *International Journal of Geographical Information Science*, 31(2):320–345, 2017.

32 Yaowei Long and Seth Pettie. Planar distance oracles with better time-space tradeoffs. In Dániel Marx, editor, *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10 - 13, 2021*, pages 2517–2537. SIAM, 2021. doi:10.1137/1.9781611976465.149.

33 Anil Maheshwari, Jörg-Rüdiger Sack, Kaveh Shahbaz, and Hamid Zarrabi-Zadeh. Fréchet distance with speed limits. *Computational Geometry*, 44(2):110–120, 2011.

34 Ariane Mascret, Thomas Devogele, Iwan Le Berre, and Alain Hénaff. Coastline matching process based on the discrete Fréchet distance. In *Progress in Spatial Data Handling*, pages 383–400. Springer, 2006.

35 Jiri Matousek. *Lectures on discrete geometry*, volume 212. Springer Science & Business Media, 2013.

36 Liam Roditty, Mikkel Thorup, and Uri Zwick. Deterministic constructions of approximate distance oracles and spanners. In Luís Caires, Giuseppe F. Italiano, Luís Monteiro, Catuscia Palamidessi, and Moti Yung, editors, *Automata, Languages and Programming, 32nd International Colloquium, ICALP 2005, Lisbon, Portugal, July 11-15, 2005, Proceedings*, volume 3580 of *Lecture Notes in Computer Science*, pages 261–272. Springer, 2005. doi:10.1007/11523468_22.

37 Roniel S. De Sousa, Azzedine Boukerche, and Antonio A. F. Loureiro. Vehicle trajectory similarity: Models, methods, and applications. *ACM Comput. Surv.*, 53(5), September 2020. doi:10.1145/3406096.

38 E Sriraghavendra, K Karthik, and Chiranjib Bhattacharyya. Fréchet distance based approach for searching online handwritten documents. In *Ninth International Conference on Document Analysis and Recognition (ICDAR 2007)*, volume 1, pages 461–465. IEEE, 2007.

39 Han Su, Shuncheng Liu, Bolong Zheng, Xiaofang Zhou, and Kai Zheng. A survey of trajectory distance measures and performance evaluation. *The VLDB Journal*, 29(1):3–32, 2020.

40 Mikkel Thorup. Compact oracles for reachability and approximate distances in planar digraphs. *Journal of the ACM (JACM)*, 51(6):993–1024, 2004.

41 Mikkel Thorup and Uri Zwick. Approximate distance oracles. In Jeffrey Scott Vitter, Paul G. Spirakis, and Mihalis Yannakakis, editors, *Proceedings on 33rd Annual ACM Symposium on Theory of Computing, July 6-8, 2001, Heraklion, Crete, Greece*, pages 183–192. ACM, 2001. doi:10.1145/380752.380798.

42 Ryan Williams. A new algorithm for optimal 2-constraint satisfaction and its implications. *Theor. Comput. Sci.*, 348(2-3):357–365, 2005. doi:10.1016/j.tcs.2005.09.023.

43 Christian Wulff-Nilsen. Approximate distance oracles with improved query time. In *Encyclopedia of Algorithms*, pages 94–97. Springer, 2016. doi:10.1007/978-1-4939-2864-4_568.

44 Dong Xie, Feifei Li, and Jeff M Phillips. Distributed trajectory similarity search. *Proceedings of the VLDB Endowment*, 10(11):1478–1489, 2017.