# Persistent Cup-Length

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## Abstract

Cohomological ideas have recently been injected into persistent homology and have for example been used for accelerating the calculation of persistence diagrams by the software Ripser.

The cup product operation which is available at cohomology level gives rise to a graded ring structure that extends the usual vector space structure and is therefore able to extract and encode additional rich information. The maximum number of cocycles having non-zero cup product yields an invariant, the cup-length, which is useful for discriminating spaces.

In this paper, we lift the cup-length into the persistent cup-length function for the purpose of capturing ring-theoretic information about the evolution of the cohomology (ring) structure across a filtration. We show that the persistent cup-length function can be computed from a family of representative cocycles and devise a polynomial time algorithm for its computation. We furthermore show that this invariant is stable under suitable interleaving-type distances.

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## 1 Introduction

*Persistent Homology* [20, 21, 33, 40, 10, 18, 8, 9], one of the main techniques in *Topological Data Analysis (TDA)*, studies the evolution of homology classes across a filtration. This produces a collection of birth-death pairs which is called the *barcode* or *persistence diagram* of the filtration.

In the case of *cohomology*, which is dual to that of homology, one studies linear maps from the vector space of simplicial chains into the field $K$, known as *cochains*. Cochains are naturally endowed with a product operation, called the *cup product*, which induces a bilinear operation on cohomology and is denoted by $\smile: \mathbf{H}^p(\mathbb{X}) \times \mathbf{H}^q(\mathbb{X}) \to \mathbf{H}^{p+q}(\mathbb{X})$ for a space $\mathbb{X}$ and dimensions $p, q \geq 0$. With the cup product operation, the collection of cohomology vector spaces can be given the structure of a *graded ring*, called the *cohomology ring*; see [31, § 48 and § 68] and [23, Ch. 3, §3.D]. This makes cohomology a richer structure than homology.

*Persistent cohomology* has been studied in [15, 16, 17, 5, 27], without exploiting the ring structure induced by the cup product. Works which do attempt to exploit this ring structure include [22, 26] in the static case and [25, 39, 3, 29, 24, 6, 12] at the persistent level.

In this paper, we continue this line of work and tackle the question of quantifying the *evolution* of the cup product structure across a filtration through introducing a *polynomial-time computable* invariant which is induced from the *cup-length*: the maximal number of cocycles (in dimensions 1 and above) having non-zero cup product. We call this invariant the *persistent cup-length function*, and identify a tool - the *persistent cup-length diagram (associated to a family of representative cocycles $\boldsymbol{\sigma}$ of the barcode)* to compute it. (see Fig. 1).

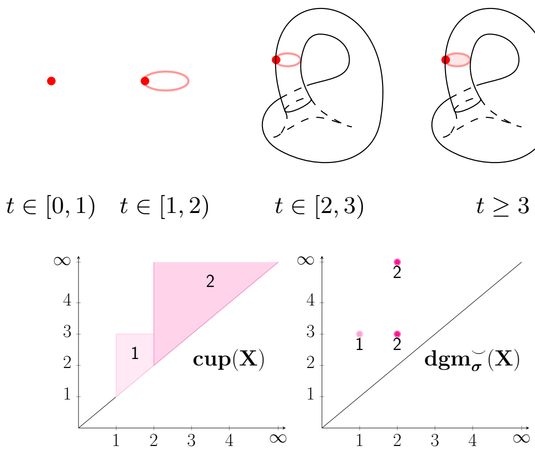



**Figure 1** A filtration $\mathbf{X}$ of the pinched Klein bottle, its persistent cup-length function $\mathbf{cup}(\mathbf{X})$ (see Ex.12) and its persistent cup-length diagram $\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})$ (see Ex. 17).

**Some invariants related to the cup product.** In standard topology, an *invariant* is a quantity associated to a given topological space which remains invariant under a certain class of maps. This invariance helps in discovering, studying and classifying properties of spaces. Beyond *Betti numbers*, examples of classical invariants are: the *Lusternik-Schnirelmann category (LS-category)* of a space $\mathbb{X}$, defined as the minimal integer $k \geq 1$ such that there is an open cover $\{U_i\}_{i=1}^k$ of $\mathbb{X}$ such that each inclusion map $U_i \hookrightarrow \mathbb{X}$ is null-homotopic, and the *cup-length invariant*, which is the maximum number of positive-dimensional cocycles having non-zero cup product. While being relatively more informative, the LS-category is difficult to compute [13], and with rational coefficients this computation is known to be NP-hard [2]. The cup-length invariant, as a lower bound of the LS-category [34, 35], serves as a computable estimate for the LS-category. Another well known invariant which can be estimated through the cup-length is the so-called *topological complexity* [38, 19, 36].

### Our contributions

Let **Top** denotes the category of (compactly generated weak Hausdorff) topological spaces.[1] Throughout the paper, by a (topological) space we refer to an object in **Top**, and by a persistent space we mean a functor from the poset category $(\mathbb{R}, \leq)$ to **Top**. A filtration (of spaces) is an example of a persistent space where the transition maps are given by inclusions. This paper considers only persistent spaces with a discrete set of critical values. In addition, all (co)homology groups are assumed to be taken over a field $K$. We denote by $\mathbf{Int}_\omega$ the set of intervals of type $\omega$, where $\omega$ can be any one of the four types: open-open, open-closed, closed-open and closed-closed. The type $\omega$ will be omitted when the results apply to all four situations and intervals are written in the form of $\langle a, b\rangle$.

We introduce the invariant, the persistent cup-length function of general persistent spaces, by lifting the standard cup-length invariant into the persistent setting. Let $\mathbf{X} : (\mathbb{R}, \leq) \to \mathbf{Top}$ be a persistent space with $t \mapsto \mathbb{X}_t$. The **persistent cup-length function** $\mathbf{cup}(\mathbf{X}) : \mathbf{Int} \to \mathbb{N}$ of $\mathbf{X}$, see Defn. 7, is defined as the function from the set **Int** to the set $\mathbb{N}$ of non-negative integers, which assigns to each interval $\langle a, b\rangle$ the cup-length of the image ring[2] $\mathbf{Im}\big(\mathbf{H}^*(\mathbf{X})\langle a, b\rangle\big)$, which is the ring $\mathbf{Im}\big(\mathbf{H}^*(\mathbb{X}_b) \to \mathbf{H}^*(\mathbb{X}_a)\big)$ when $\langle a, b\rangle$ is a closed interval (in other cases, there is some subtlety, see Rmk. 8). Note that the persistent cup-length function is a generalization of the cup-length of spaces, since $\mathbf{cup}(\mathbf{X})([a, a])$ reduces to the cup-length of the space $\mathbb{X}_a$.

In the case when $\mathbf{X}$ is a filtration, we define a notion of a diagram to compute the persistent cup-length function (see Thm. 1): the **persistent cup-length diagram** $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X}) : \mathbf{Int} \to \mathbb{N}$ (Defn. 16). We first assign a representative cocycle to every interval in the barcode of $\mathbf{X}$, and denote the family of representative cocycles by $\boldsymbol{\sigma}$. Then, the persistent cup-length diagram of an interval $\langle a, b\rangle$ is defined to be the maximum number of representative cocycles in $\mathbf{X}$ that have a nonzero cup product over $\langle a, b\rangle$. It is worth noticing that the persistent cup-length diagram depends on the choice of representative cocycles; see Ex. 18.

▶ **Theorem 1.** *Let $\mathbf{X}$ be a filtration, and let $\boldsymbol{\sigma}$ be a family of representative cocycles for the barcodes of $\mathbf{X}$. The persistent cup-length function $\mathbf{cup}(\mathbf{X})$ can be retrieved from the persistent cup-length diagram $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})$: for any $\langle a, b\rangle \in \mathbf{Int}$,*

$$\mathbf{cup}(\mathbf{X})(\langle a, b\rangle) = \max_{\langle c,d\rangle \supseteq \langle a,b\rangle} \mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})(\langle c, d\rangle). \tag{1}$$

The persistent cup-length functions do not supersede the standard persistence diagrams, partly because they do not take $\mathbf{H}^0$ classes into account. However, it effectively augments the standard diagram in the sense that there are situations in which it can successfully capture information that standard persistence diagrams neglect (see Fig. 5). Our work therefore provides additional *computable* persistence-like invariants enriching the TDA toolset which can be used in applications requiring discriminating between different hypotheses such as in shape classification or machine learning. For example, [27] mentions that cup product could provide additional evidence when recovering the structure of animal trajectories.

**A polynomial time algorithm.** We develop a poly-time algorithm (Alg. 3) to compute the persistent cup-length diagram of a filtration $\mathbf{X}$ of a simplicial complex $\mathbb{X}$ of dimension $(k+1)$. This algorithm is output sensitive, and it has complexity bounded above by

[1] We are following the convention from [7].
[2] For $f : R \to S$ a graded ring morphism, we denote the graded ring $f(R)$ by $\mathbf{Im}(f)$.

$O((m_k)^2 \cdot q_1 \cdot q_{k-1} \cdot \max\{c_k, q_1\}) \leq O((m_k)^{k+3})$ (cf. Thm. 20), with $m_k$ being the cardinality of $\mathbb{X}$, $q_1$ being the cardinality of the barcode, and parameters $q_{k-1}$ ($\leq q_1^{k-1}$) and $c_k$ ($\leq m_k$) which we describe in §3.2 on page 13. In the case of the Vietoris-Rips filtration of an $n$-point metric space, this complexity is improved to $O((m_k)^2 \cdot q_1^2 \cdot q_{k-1})$, which can be upper bounded by $O(n^{k^2+5k+6})$.

**Gromov-Hausdorff stability and discriminating power.** In Thm. 2 we prove that the persistent cup-length function is stable to perturbations of the involved filtrations (in a suitable sense involving weak homotopy equivalences). Below, $d_{\mathrm{E}}, d_{\mathrm{HI}}$ and $d_{\mathrm{GH}}$ denote the erosion, homotopy-interleaving and Gromov-Hausdorff distances, respectively. See [11, §D] for details.

In general, the Gromov-Hausdorff distance is NP-hard to compute [37] whereas the erosion distance is computable in polynomial time (see [28, Thm. 5.4]) and thus, in combination with Thm. 2, provides a computable estimate for the Gromov-Hausdorff distance.

▶ **Theorem 2** (Homotopical stability). *For two persistent spaces* $\mathbf{X}, \mathbf{Y} : (\mathbb{R}, \leq) \to \mathbf{Top}$,

$$d_{\mathrm{E}}(\mathbf{cup}(\mathbf{X}), \mathbf{cup}(\mathbf{Y})) \leq d_{\mathrm{HI}}(\mathbf{X}, \mathbf{Y}). \tag{2}$$

*For the Vietoris-Rips filtrations* $\mathbf{VR}(X)$ *and* $\mathbf{VR}(Y)$ *of compact metric spaces* $X$ *and* $Y$,

$$d_{\mathrm{E}}\left(\mathbf{cup}\left(\mathbf{VR}(X)\right), \mathbf{cup}\left(\mathbf{VR}(Y)\right)\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y). \tag{3}$$

Through several examples, we show that the persistent cup-length function helps in discriminating filtrations when the persistent homology fails to or has a relatively weak performance in doing so. Ex. 13 is a situation when two filtrations have identical persistent homology but induce different persistent cup-length functions. In addition, in [11, Ex. 54] by specifying suitable metrics on the torus $\mathbb{T}^2$ and on the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$, we compute the erosion distance between their persistent cup-length functions (see Fig. 2) and apply Thm. 2 to obtain a lower bound $\frac{\pi}{3}$ for the Gromov-Hausdorff distance between them $\mathbb{T}^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ (see [11, Prop. 55]):

$$\frac{\pi}{3} = d_{\mathrm{E}}\left(\mathbf{cup}(\mathbf{VR}(\mathbb{T}^2)), \mathbf{cup}(\mathbf{VR}(\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1))\right) \leq 2 \cdot d_{\mathrm{GH}}\left(\mathbb{T}^2, \mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1\right).$$

We also verify that the interleaving distance between the persistent homology of these two spaces is at most $\frac{3}{5}$ of the bound obtained from persistent cup-length functions (a fact which we also establish). See [11, Rmk. 56].

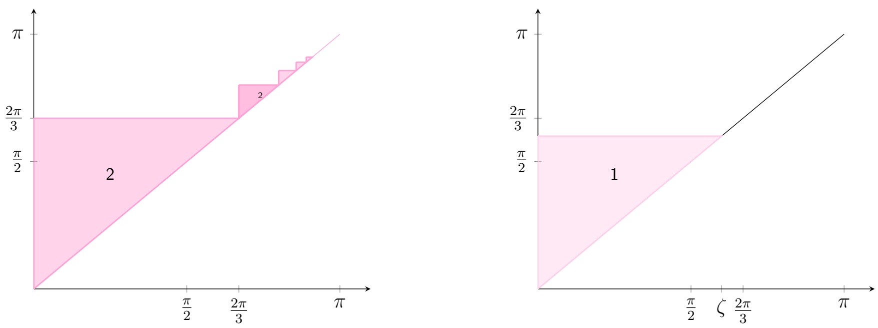



**Figure 2** The persistent cup-length functions $\mathbf{cup}(\mathbf{VR}(\mathbb{T}^2))$ (left) and $\mathbf{cup}(\mathbf{VR}(\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1))|_{(0,\zeta)}$ (right), respectively. Here, $\zeta = \arccos(-\frac{1}{3}) \approx 0.61\pi$.

Proofs of all the theorems and results mentioned above are available in the appendix of the full version [11].

## 2 Persistent cup-length function

In the standard setting of persistent homology, one considers a *filtration* of spaces, i.e. a collection of spaces $\mathbf{X} = \{\mathbb{X}_t\}_{t\in\mathbb{R}}$ such that $\mathbb{X}_t \subset \mathbb{X}_s$ for all $t \leq s$, and studies the *$p$-th persistent homology* for any given dimension $p$, defined as the functor $\mathbf{H}_p(\mathbf{X}) : (\mathbb{R}, \leq) \to \mathbf{Vec}$ which sends each $t$ to the $p$-th homology $\mathbf{H}_p(\mathbb{X}_t)$ of $\mathbb{X}_t$, see [18, 8]. Here $\mathbf{Vec}$ denotes the category of vector spaces. The $p$-th persistent homology encodes the lifespans, represented by intervals, of the $p$-dimensional holes ($p$-cycles that are not $p$-boundaries) in $\mathbf{X}$. The collection $\mathcal{B}_p(\mathbf{X})$ of these intervals is called *the $p$-th barcode of* $\mathbf{X}$, and its elements are named *bars*. The *$p$-th persistent cohomology* $\mathbf{H}^p(\mathbf{X})$ and its corresponding barcode are defined dually. Since persistent homology and persistent cohomology have the same barcode [15], we will denote both barcodes by $\mathcal{B}_p(\mathbf{X})$ for dimension $p$. We call the disjoint union $\mathcal{B}(\mathbf{X}) := \sqcup_{p\in\mathbb{N}}\mathcal{B}_p(\mathbf{X})$ *the total barcode of* $\mathbf{X}$, and assume bars in $\mathcal{B}(X)$ are of the same interval type[3].

By considering the *cup product* operation on cocycles, the persistent cohomology is naturally enriched with the structure of a *persistent graded ring*, which carries additional information and leads to invariants stronger than standard barcodes in cases like Ex. 13.

In §2.1 we recall the cup product operation, as well as the notion and properties of the cup-length invariant of cohomology rings. In §2.2 we lift the cup-length invariant to a persistent invariant, called the persistent cup-length function, and examine some examples that highlight its strength. Proofs and details are available in [11, §B].

### 2.1 Cohomology rings and the cup-length invariant

For a topological space $\mathbb{X}$ and a dimension $p \in \mathbb{N}$, denote by $C_p(\mathbb{X})$ and $C^p(\mathbb{X})$ the spaces of singular $p$-chains and $p$-cochains, respectively. For a cocycle $\sigma$, denote by $[\sigma]$ the cohomology class of $\sigma$. If $\mathbb{X}$ is given by the geometric realization of some simplicial complex, then we consider its simplicial cohomology, by assuming an ordering on the vertex set of $\mathbb{X}$ and considering its simplices to be sets of ordered vertices.

Let $\mathbf{X} := \{\mathbb{X}_t\}_{t\in\mathbb{R}}$ be a filtration of topological spaces, and let $I = \langle b, d\rangle \in \mathcal{B}_p(\mathbf{X})$. If $I$ is closed at its right end $d$, we denote by $\sigma_I$ a cocycle in $C^p(\mathbb{X}_d)$; if not, we denote by $\sigma_I$ a cocycle in $C^p(\mathbb{X}_{d-\delta})$ for sufficiently small $\delta > 0$. For any $t \leq d$, denote by $\sigma_I|_{C_p(\mathbb{X}_t)}$ the restriction of $\sigma_I$ to $C_p(\mathbb{X}_t)(\subset C_p(\mathbb{X}_d))$. We introduce the notation $[\sigma_I]_t$ by defining $[\sigma_I]_t$ to be $\left[\sigma_I|_{C_p(\mathbb{X}_t)}\right]$ for $t \leq d$ and 0 for $t > d$.

▶ **Definition 3** (Representative cocycles). *Let $\boldsymbol{\sigma}^p := \{\sigma_I\}_{I\in\mathcal{B}_p(\mathbf{X})}$ be a $\mathcal{B}_p(\mathbf{X})$-indexed collection of $p$-cocycles in $\mathbf{X}$. The collection $\boldsymbol{\sigma}^p$ is called* a family of representative $p$-cocycles for $\mathbf{H}^p(\mathbf{X})$, *if for any $t \in \mathbb{R}$, the set $\{[\sigma_I]_t\}_{t\in I\in\mathcal{B}_p(\mathbf{X})}$ forms a linear basis for $\mathbf{H}^p(\mathbb{X}_t)$. In this case, each $\sigma_I$ is called* a representative cocycle associated to the interval $I$. *The disjoint union $\boldsymbol{\sigma} := \sqcup_{p\in\mathbb{N}}\boldsymbol{\sigma}^p$ is called* a family of representative cocycles for $\mathbf{H}^*(\mathbf{X})$.

The existence of a family of representative cocycles for $\mathbf{H}^*(\mathbf{X})$ (assuming that the filtration $\mathbf{X}$ has finite critical values and finite-dimensional cohomology point-wise) is guaranteed by the interval decomposition theorem of point-wise finite dimensional persistence modules (see [14]) and the axiom of choice. Software programs are available to compute the total barcode and return a family of representative cocycles, such as Ripser (see [5]), Java-Plex (see [1]), Dionysus (see [16]), and Gudhi (see [30]). These cocycles are naturally equipped with the cup product operation, which we recall as follows.

[3] In TDA it is often the case that bars are of a fixed interval type, usually in closed-open form [32].

**Cup product.** We recall the cup product operation in the setting of simplicial cohomology. Let $\mathbb{X}$ be a simplicial complex with an ordered vertex set $\{x_1 < \cdots < x_n\}$. For any non-negative integer $p$, we denote a $p$-simplex by $\alpha := [\alpha_0, \ldots, \alpha_p]$ where $\alpha_0 < \cdots < \alpha_p$ are ordered vertices in $\mathbb{X}$, and by $\alpha^* : C_p(\mathbb{X}) \to K$ the dual of $\alpha$, where $\alpha^*(\alpha) = 1$ and $\alpha^*(\tau) = 0$ for any $p$-simplex $\tau \neq \alpha$. Here $K$ is the base field as before, and $\alpha^*$ is also called a $p$-cosimplex. Let $\beta := [\beta_0, \ldots, \beta_q]$ be a $q$-simplex for some integer $q \geq 0$. The *cup product* $\alpha^* \smile \beta^*$ is defined as the linear map $C_{p+q}(\mathbb{X}) \to K$ such that for any $(p+q)$-simplex $\tau = [\tau_0, \ldots, \tau_{p+q}]$,

$$\alpha^* \smile \beta^*(\tau) := \alpha^*([\tau_0, \ldots, \tau_p]) \cdot \beta^*([\tau_p, \ldots, \tau_{p+q}]).$$

Equivalently, we have that $\alpha^* \smile \beta^*$ is $[\alpha_0, \ldots, \alpha_p, \beta_1, \ldots, \beta_q]^*$ if $\alpha_p = \beta_0$, and 0 otherwise. By a *$p$-cochain* we mean a finite linear sum $\sigma = \sum_{j=1}^{h} \lambda_j \alpha^{j*}$, where each $\alpha^j$ is a $p$-simplex in $\mathbb{X}$ and $\lambda_j \in K$. The *cup product* of a $p$-cochain $\sigma = \sum_{j=1}^{h} \lambda_j \alpha^{j*}$ and a $q$-cochain $\sigma' = \sum_{j'=1}^{h'} \mu_{j'} \beta^{j'*}$ is defined as $\sigma \smile \sigma' := \sum_{j,j'} \lambda_j \mu_{j'} \left(\alpha^{j*} \smile \beta^{j'*}\right)$.

In our algorithms, $K$ is taken to be $\mathbb{Z}_2$ and every $p$-simplex $\alpha = [x_{i_0}, \ldots, x_{i_p}]$ is represented by the ordered list $[i_0, \ldots, i_p]$. We assume a total order (e.g. the order given in [5]) on the simplices in $\mathbb{X}$. Since coefficients are either 0 or 1, a $p$-cochain can be written as $\sigma = \sum_{j=1}^{h} \alpha^{j*}$ for some $\alpha^j = [x_{i_0^j}, \ldots, x_{i_p^j}]$ and will be represented by the list $\left[[i_0^1, \ldots, i_p^1], \ldots, [i_0^h, \ldots, i_p^h]\right]$. We call $h$ the size of $\sigma$. Let $\mathbb{X}_p \subset \mathbb{X}$ be the set of $p$-simplices. Alg. 1 computes the cup product of two cochains over $\mathbb{Z}_2$.

**Algorithm 1** CupProduct$(\sigma_1, \sigma_2, \mathbb{X})$.

**Input :** Two cochains $\sigma_1$ and $\sigma_2$, and the simplicial complex $\mathbb{X}$.
**Output :** The cup product $\sigma = \sigma_1 \smile \sigma_2$, at cochain level.

$\sigma \leftarrow [\,]$;
**if** $\dim(\sigma_1) + \dim(\sigma_2) \leq \dim(\mathbb{X})$ **then**
  **for** $i \leq \mathrm{size}(\sigma_1)$ *and* $j \leq \mathrm{size}(\sigma_2)$ **do**
    $a \leftarrow \sigma_1(i)$ and $b \leftarrow \sigma_2(j)$;
    **if** $a[\mathrm{end}] == b[\mathrm{first}]$ **then**
      $c \leftarrow a.\mathrm{append}(b[\mathrm{second} : \mathrm{end}])$;
      **if** $c \in \mathbb{X}_{\dim(\sigma_1)+\dim(\sigma_2)}$ **then**
        Append $c$ to $\sigma$;
**return** $\sigma$.

▶ Remark 4 (Complexity of Alg. 1). Let $c$ be the complexity of checking whether a simplex is in the simplicial complex, and let $m := \mathrm{card}(\mathbb{X})$ be the number of simplices. For $\mathbb{Z}_2$-coefficients, cocycles are in one-to-one correspondence with the subsets of $\mathbb{X}$, so the size of a cocycle is at most $m$. Thus, the complexity for Alg. 1 is $O(\mathrm{size}(\sigma_1) \cdot \mathrm{size}(\sigma_2) \cdot c) \leq O(m^2 \cdot c)$.

**Cohomology ring and cup-length.** For a given space $\mathbb{X}$, the cup product yields a bilinear map $\smile : \mathbf{H}^p(\mathbb{X}) \times \mathbf{H}^q(\mathbb{X}) \to \mathbf{H}^{p+q}(\mathbb{X})$ of vector spaces. In particular, it turns the total cohomology vector space $\mathbf{H}^*(\mathbb{X}) := \bigoplus_{p \in \mathbb{N}} \mathbf{H}^p(\mathbb{X})$ into a graded ring $(\mathbf{H}^*(\mathbb{X}), +, \smile)$ (see [11, §B] for the explicit definition of a graded ring). The *cohomology ring map* $\mathbb{X} \mapsto \mathbf{H}^*(\mathbb{X})$ defines a contravariant functor from the category of spaces, **Top**, to the category of graded rings, **GRing** (see [23, §3.2]). To avoid the difficulty of describing and comparing ring structures in a computer, we study a computable invariant of the graded cohomology ring, called the *cup-length*. See [11, §A.2] for the general notion of *invariants*. For a category $\mathcal{C}$, denote by $\mathbf{Ob}(\mathcal{C})$ the set of objects in $\mathcal{C}$.

▶ **Definition 5** (Length and cup-length). *The* ***length*** *of a graded ring $R = \bigoplus_{p\in\mathbb{N}} R_p$ is the largest non-negative integer $\ell$ such that there exist positive-dimension homogeneous elements $\eta_1, \ldots, \eta_\ell \in R$ (i.e. $\eta_1, \ldots, \eta_\ell \in \bigcup_{p\geq 1} R_p$) with $\eta_1 \bullet \cdots \bullet \eta_\ell \neq 0$. If $\bigcup_{p\geq 1} R_p = \emptyset$, then we define the length of $R$ to be zero. We denote the length of a graded ring $R$ by $\mathbf{len}(R)$, and call the following map the* ***length invariant****:*

$$\mathbf{len} : \mathbf{Ob}(\mathbf{GRing}) \to \mathbb{N}, \text{ with } R \mapsto \mathbf{len}(R).$$

*When $R = (\mathbf{H}^*(\mathbb{X}), +, \smile)$ for some space $\mathbb{X}$, we denote $\mathbf{cup}(\mathbb{X}) := \mathbf{len}(\mathbf{H}^*(\mathbb{X}))$ and call it the* ***cup-length of*** *$\mathbb{X}$ And we call the following map the* ***cup-length invariant****:*

$$\mathbf{cup} : \mathbf{Ob}(\mathbf{Top}) \to \mathbb{N}, \text{ with } X \mapsto \mathbf{cup}(\mathbb{X}).$$

▶ Remark 6 (About the strength of the cup-length invariant). In some cases, cup-length captures more information than homology. One well-known example is given by the torus $\mathbb{T}^2$ v.s. the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$, where despite having the same homology groups, these two spaces have *different* cup-length. By specifying suitable metrics and considering the Vietoris-Rips filtrations of the two spaces, the strength of cup-length persists in the setting of persistence (see [11, Ex. 54]). It is also worth noticing that cup-length is not a complete invariant for graded cohomology rings. For instance, after taking the wedge sum of $\mathbb{T}^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ with $\mathbb{T}^2$ respectively, the resulted spaces $\mathbb{T}^2 \vee \mathbb{T}^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{T}^2$ still have different ring structures, but they have the same cup-length (since cup length takes the "maximum").

An important fact about the cup-length is that it can be computed using a linear basis for the cohomology vector space. In [11, Prop. 36] we show that if $B_p$ is a linear basis for $\mathbf{H}^p(\mathbb{X})$ for each $p \geq 1$ and $B := \bigcup_{p\geq 1} B_p$, then $\mathbf{cup}(\mathbb{X}) = \sup\left\{\ell \geq 1 \mid B^{\smile \ell} \neq \{0\}\right\}$.

## 2.2 Persistent cohomology rings and persistent cup-length functions

We study the persistent cohomology ring of a filtration and the associated notion of persistent cup-length invariant. We examine several examples of this persistent invariant and establish a way to visualize it in the half-plane above the diagonal. See [11, §B.2] for the proofs of our results in this section.

Filtrations of spaces are special cases of *persistent spaces*. In general, for any category $\mathcal{C}$, one can define the notion of a *persistent object* in $\mathcal{C}$, as a functor from the poset $(\mathbb{R}, \leq)$ (viewed as a category) to the category $\mathcal{C}$. For instance, a functor $\mathbf{R} : (\mathbb{R}, \leq) \to \mathbf{GRing}$ is called a **persistent graded ring**. Recall the contravariant cohomology ring functor $\mathbf{H}^* : \mathbf{Top} \to \mathbf{GRing}$. Given a persistent space $\mathbf{X} : (\mathbb{R}, \leq) \to \mathbf{Top}$, the composition $\mathbf{H}^*(\mathbf{X}) : (\mathbb{R}, \leq) \to \mathbf{GRing}$ is called the **persistent cohomology ring of X**. Due to the contravariance of $\mathbf{H}^*$, we consider only contravariant persistent graded rings in this paper.

▶ **Definition 7.** *We define the* ***persistent cup-length function*** *of a persistent space $\mathbf{X}$ as the function $\mathbf{cup}(\mathbf{X}) : \mathbf{Int} \to \mathbb{N}$ given by $\langle t, s\rangle \mapsto \mathbf{len}\left(\mathbf{Im}(\mathbf{H}^*(\mathbf{X})(\langle t, s\rangle)\right)$.*

▶ Remark 8 (Notation for image ring). $\mathbf{Im}(\mathbf{H}^*(\mathbf{X})(\langle t, s\rangle)$ is defined as the image ring $\mathbf{Im}(\mathbf{H}^*(\mathbf{X})([t-\delta, s+\delta])) = \mathbf{Im}(\mathbf{H}^*(\mathbb{X}_{s+\delta}) \to \mathbf{H}^*(\mathbb{X}_{t-\delta}))$ for sufficiently small $\delta > 0$, when $\langle t, s\rangle = (t, s)$, and is defined similarly for the cases when $\langle t, s\rangle = (t, s]$ or $[t, s)$.

▶ Remark 9. It follows from [11, Prop. 38] that the cup-length invariant **cup** is non-increasing under surjective morphisms and non-decreasing under injective morphisms, which we call an *inj-surj invariant*. As a consequence, for any persistent space $\mathbf{X}$, the persistent cup-length function $\mathbf{cup}(\mathbf{X})$ defines a *functor* from $(\mathbf{Int}, \leq)$ to $(\mathbb{N}, \geq)$.

Prop. 10 below allows us to compute the cohomology images of a persistent cohomology ring from representative cocycles, which will be applied to compute persistent cup-length functions in Ex. 12 and prove Thm. 1 in the full version of this paper, see [11, page 24]. Prop. 11 allows us to simplify the calculation of persistent cup-length functions in certain cases, such as the Vietoris-Rips filtration of products or wedge sums of metric spaces.

▶ **Proposition 10** (Persistent image ring). *Let $\mathbf{X} = \{\mathbb{X}_t\}_{t\in\mathbb{R}}$ be a filtration, together with a family of representative cocycles $\boldsymbol{\sigma} = \{\sigma_I\}_{I\in\mathcal{B}(\mathbf{X})}$ for $\mathbf{H}^*(\mathbf{X})$. Let $t \leq s$ in $\mathbb{R}$. Then* $\mathbf{Im}(\mathbf{H}^*(\mathbb{X}_s) \to \mathbf{H}^*(\mathbb{X}_t)) = \langle [\sigma_I]_t : [t,s] \subset I \in \mathcal{B}(\mathbf{X})\rangle$*, generated as a graded ring.*

▶ **Proposition 11.** *Let $\mathbf{X}, \mathbf{Y} : (\mathbb{R}, \leq) \to \mathbf{Top}$ be two persistent spaces. Then:*
- $\mathbf{cup}\,(\mathbf{X} \times \mathbf{Y}) = \mathbf{cup}(\mathbf{X}) + \mathbf{cup}(\mathbf{Y})$,
- $\mathbf{cup}\,(\mathbf{X} \amalg \mathbf{Y}) = \max\{\mathbf{cup}(\mathbf{X}), \mathbf{cup}(\mathbf{Y})\}$, *and*
- $\mathbf{cup}\,(\mathbf{X} \vee \mathbf{Y}) = \max\{\mathbf{cup}(\mathbf{X}), \mathbf{cup}(\mathbf{Y})\}$.

*Here $\times, \amalg$ and $\vee$ denote point-wise product, disjoint union, and wedge sum, respectively.*

**Examples and visualization.** Each interval $\langle a, b\rangle$ in $\mathbf{Int}$ is visualized as a point $(a, b)$ in the half-plane above the diagonal (see Fig. 3). To visualize the persistent cup-length function of a filtration $\mathbf{X}$, we assign to each point $(a, b)$ the integer value $\mathbf{cup}(\mathbf{X})(\langle a, b\rangle)$, if it is positive. If $\mathbf{cup}(\mathbf{X})(\langle a, b\rangle) = 0$ we do not assign any value. We present an example to demonstrate how persistent cup-length functions are visualized in the upper-diagonal plane (see Fig. 1).

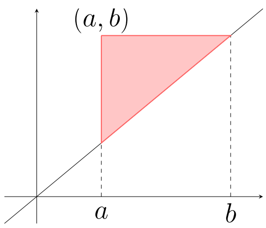



**Figure 3** The interval $\langle a, b\rangle$ in $\mathbf{Int}$ corresponds to the point $(a, b)$ in $\mathbb{R}^2$.

▶ **Example 12** (Visualization of $\mathbf{cup}(\cdot)$). Recall the filtration $\mathbf{X} = \{\mathbb{X}_t\}_{t\geq 0}$ of a Klein bottle with a 2-cell attached, defined in Fig. 1. Consider the persistent cohomology $\mathbf{H}^*(\mathbf{X})$ in $\mathbb{Z}_2$-coefficients. Let $v$ be the 0-cocycle born at $t = 0$, let $\alpha$ be the 1-cocycle born at $t = 1$ and died at $t = 3$, and let $\beta$ be the 1-cocycle born at time $t = 2$. Let $\gamma := \beta \smile \beta$, which is then a non-trivial 2-cocycle born at time $t = 2$, like $\beta$. Then the barcodes of $\mathbf{X}$ are: $\mathcal{B}_0(\mathbf{X}) = \{[0, \infty)\}$, $\mathcal{B}_1(\mathbf{X}) = \{[1, 3), [2, \infty)\}$, and $\mathcal{B}_2(\mathbf{X}) = \{[2, \infty)\}$. See Fig. 4.

Using the formula in Prop. 10, for any $t \leq s$, we have

$$\mathbf{Im}(\mathbf{H}^*(\mathbb{X}_s) \to \mathbf{H}^*(\mathbb{X}_t)) = \begin{cases} \langle [v]_t, [\beta]_t, [\gamma]_t\rangle, & \text{if } 2 \leq t < 3 \text{ and } s \geq 3 \\ \langle [v]_t, [\alpha]_t, [\beta]_t, [\gamma]_t\rangle, & \text{if } 2 \leq t \leq s < 3 \\ \langle [v]_t, [\alpha]_t\rangle, & \text{if } 1 \leq t < 2 \text{ and } s < 3 \\ \langle [v]_t\rangle, & \text{otherwise.} \end{cases}$$

The persistent cup-length function of $\mathbf{X}$ is computed as follows and visualized in Fig. 1.

$$\mathbf{cup}(\mathbf{X})([t, s]) = \begin{cases} 2, & \text{if } t \geq 2 \\ 1, & \text{if } 1 \leq t < 2 \text{ and } s < 3 \\ 0, & \textit{otherwise.} \end{cases}$$

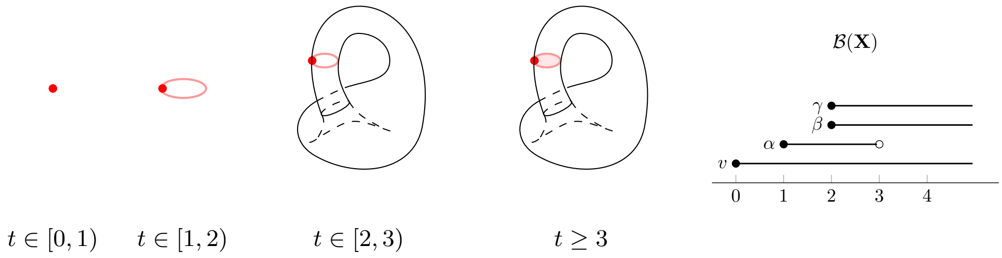



**Figure 4** The filtration $\mathbf{X}$ given in Fig. 1 and its barcode $\mathcal{B}(\mathbf{X})$, see Ex. 12.

We end this section by presenting an example, Ex. 13, where the persistent cup-length function distinguishes a pair of filtrations which the total barcode is not able to. A similar example is available in [11, §D.2], where we will also give a quantitative measure via the erosion distance on the difference between persistent cup-length functions of different filtrations.

▶ **Example 13** ($\mathbf{cup}(\cdot)$ better than standard barcode)**.** Consider the filtration $\mathbf{X} = \{\mathbb{X}_t\}_{t\geq 0}$ of a 2-torus $\mathbb{T}^2$ and the filtration $\mathbf{Y} = \{\mathbb{Y}_t\}_{t\geq 0}$ of the space $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ as shown in Fig. 5. Knowing that $\mathbb{X}_3 = \mathbb{T}^2$ and $\mathbb{Y}_3 = \mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ have the same (co)homology vector spaces in all dimensions, one can directly check that the persistent (co)homology vector spaces associated to $\mathbf{X}$ and $\mathbf{Y}$ are the same. However, the cohomology ring structure of $\mathbb{X}_3$ is different from that of $\mathbb{Y}_3$: there are two 1-cocycles in $\mathbb{X}_3$ with a non-zero product (indeed the product is a 2-cocycle), whereas all 1-cocycles in $\mathbb{Y}_3$ have zero product. This difference between the cohomology ring structures of these two filtration is quantified by their persistent cup-length functions, see Fig. 5. Also, see [11, Ex. 54] for a more geometric example, which considers the Vietoris-Rips filtrations of $\mathbb{T}^2$ and $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$.

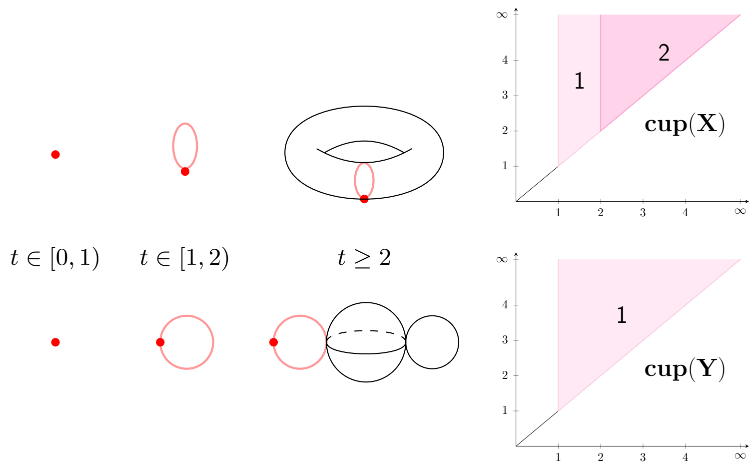



**Figure 5** Top: A filtration $\mathbf{X}$ of $\mathbb{T}^2$ and its persistent cup-length function $\mathbf{cup}(\mathbf{X})$. Bottom: A filtration $\mathbf{Y}$ of the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$ and its persistent cup-length function $\mathbf{cup}(\mathbf{Y})$. See Ex. 13.

# 3 The persistent cup-length diagram

In this section, we introduce the notion of the *persistent cup-length diagram of a filtration*, by using the cup product operation on cocycles. In §3.1, we show how the persistent cup-length diagram is used to compute the persistent cup-length function (see Thm. 1). In §3.2, we develop an algorithm (see Alg. 3) to compute the persistent cup-length diagram, and study its complexity. Proofs, details and extra examples are available in [11, §C].

## 3.1 Persistent cup-length diagram

We define the persistent cup-length diagram using a family of representative cocycles. In Thm. 1 we show that the persistent cup-length function can be retrieved from the persistent cup-length diagram. See [11, §C.1] for the proofs of Thm. 1 and details for this section.

▶ **Definition 14** ($\ell$-fold $*_{\boldsymbol{\sigma}}$-product). *Let $\boldsymbol{\sigma}$ be a family of representative cocycles for $\mathbf{H}^*(\mathbf{X})$. Let $\ell \in \mathbb{N}^*$ and let $I_1, \ldots, I_\ell$ be a sequence of elements in $\mathcal{B}(\mathbf{X})$ with representative cocycles $\sigma_{I_1}, \ldots, \sigma_{I_\ell} \in \boldsymbol{\sigma}$, respectively. We define the **$\ell$-fold $*_{\boldsymbol{\sigma}}$-product of $I_1, \cdots, I_\ell$** to be*

$$I_1 *_{\boldsymbol{\sigma}} \cdots *_{\boldsymbol{\sigma}} I_\ell := \{t \in \mathbb{R} \mid [\sigma_{I_1}]_t \smile \cdots \smile [\sigma_{I_\ell}]_t \neq [0]_t\}, \tag{4}$$

*associated with the formal representative cocycle $\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}$. We also call the right-hand side of Eq. (4) the **support** of $\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}$, and denote it by $\mathrm{supp}(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell})$.*

The support of a product of representative cocycles is always an interval:

▶ **Proposition 15** (Support is an interval). *With the same assumption and notation in Defn. 14, let $I := \mathrm{supp}(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell})$. If $I \neq \emptyset$, then $I$ is an interval $\langle b, d \rangle$, where $b \leq d$ are such that $d$ is the right end of $\cap_{1 \leq i \leq \ell} I_i$ and $b$ is the left end of some $J \in \mathcal{B}(\mathbf{X})$ ($J$ is not necessarily one of the $I_i$).*

The $*_{\boldsymbol{\sigma}}$-product is associative and invariant under permutations. Let $\mathcal{B}_{\geq 1}(\mathbf{X}) := \sqcup_{p \geq 1} \mathcal{B}_p(\mathbf{X})$. Let $\mathcal{B}_{\geq 1}(\mathbf{X})^{*_{\boldsymbol{\sigma}} \ell}$ be the set of $I_1 *_{\boldsymbol{\sigma}} \cdots *_{\boldsymbol{\sigma}} I_\ell$ where each $I_i \in \mathcal{B}_{\geq 1}(\mathbf{X})$. For the simplicity of notation, we often write $\mathcal{B}_{\geq 1}(\mathbf{X})^{*_{\boldsymbol{\sigma}} \ell}$ as $\mathcal{B}(\mathbf{X})^{*_{\boldsymbol{\sigma}} \ell}$.

▶ **Definition 16** (persistent cup-length diagram). *Let $\mathbf{X}$ be a filtration and let $\mathcal{B}_{\geq 1}(\mathbf{X})$ be its barcode over positive dimensions. Let $\boldsymbol{\sigma} = \{\sigma_I\}_{I \in \mathcal{B}_{\geq 1}(\mathbf{X})}$ be a family of representative cocycles for $\mathbf{H}^{\geq 1}(\mathbf{X})$. The **persistent cup-length diagram of $\mathbf{X}$ (associated to $\boldsymbol{\sigma}$)** is defined to be the map $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X}) : \mathbf{Int} \to \mathbb{N}$, given by:*

$$\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})(I) := \max\{\ell \in \mathbb{N}^* \mid I = I_1 *_{\boldsymbol{\sigma}} \cdots *_{\boldsymbol{\sigma}} I_\ell, \text{ where each } I_i \in \mathcal{B}_{\geq 1}(\mathbf{X})\},$$

*with the convention that* $\max \emptyset = 0$.

Recall Thm. 1, which states the relation between the persistent cup-length function $\mathbf{cup}(X)$ and the persistent cup-length diagram $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})$: for any interval $\langle a, b \rangle$, the $\mathbf{cup}(\mathbf{X})(\langle a, b \rangle)$ attains the **maximum** value of $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})(\langle c, d \rangle)$ over all intervals $\langle c, d \rangle \supseteq \langle a, b \rangle$. This is in the same spirit as in [32] where the rank function can be reconstructed from the persistence diagram by replacing "max" operation with the **sum** operation.

▶ **Example 17** (Example of $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\cdot)$ and Thm. 1). Recall the filtration $\mathbf{X} = \{\mathbb{X}_t\}_{t \geq 0}$ of the pinched Klein bottle defined in Fig. 1, and its persistent cup-length function and the representative cocycles $\{\alpha, \beta, \gamma\} =: \boldsymbol{\sigma}$ from Ex. 12. Because $\mathbf{H}^*(\mathbf{X})$ is non-trivial up to dimension 2, $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})(I) \leq 2$ for any $I$. It follows from $[\alpha \smile \alpha] = 0$ that

$\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})([1,3)) = 1$, and from $[\alpha \smile \beta] = [\gamma]$ that $[2,3) = [1,3) *_{\boldsymbol{\sigma}} [2,\infty)$, implying $\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})([2,3)) = 2$. A similar argument holds for $[2,\infty)$, using the fact that $[\beta \smile \beta] = [\gamma]$. Thus, we obtain the persistent cup-length diagram $\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})$ as below (see the right-most figure in Fig. 1 for its visualization):

$$\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})(I) = \begin{cases} 1, & \text{if } I = [1,3) \\ 2, & \text{if } I = [2,3) \text{ or } I = [2,\infty) \\ 0, & \text{otherwise.} \end{cases}$$

Applying Thm. 1, we obtain the persistent cup-length function $\mathbf{cup}(\mathbf{X})$ shown in the middle figure of Fig. 1.

See [11, §C.1] for the proof of Thm. 1 and more examples of persistent cup-length diagrams. It is worth noticing that the persistent cup-length diagram depends on the choice of the family of representative cocycles $\boldsymbol{\sigma}$, see Ex. 18 below.

▶ **Example 18 ($\mathbf{dgm}_{\boldsymbol{\sigma}}^{\smile}(\mathbf{X})$ depends on $\boldsymbol{\sigma}$).** Let $\mathbb{RP}^2$ be the real projective plane. Consider the filtration $\mathbf{X}$ of the 2-skeleton $S_2(\mathbb{RP}^2 \times \mathbb{RP}^2)$ of the product space $\mathbb{RP}^2 \times \mathbb{RP}^2$, given by:

$$\mathbf{X}: \quad \underset{t\in[0,1)}{\bullet} \hookrightarrow \underset{t\in[1,2)}{\mathbb{RP}^2} \hookrightarrow \underset{t\in[2,3)}{\mathbb{RP}^2 \vee \mathbb{RP}^2} \hookrightarrow \underset{t\geq 3}{S_2(\mathbb{RP}^2 \times \mathbb{RP}^2)}$$

Let $\alpha$ be the 1-cocycle born at $t = 1$, and $\beta$ be the 1-cocycles born at $t = 2$ when the second copy of $\mathbb{RP}^2$ appears. See Fig. 6 for two choices of representative cocycles $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ for $\mathcal{B}_{\geq 1}(\mathbf{X})$, where these two choices only differ by the first dimensional cocycles associated with the bar $[1,\infty)$. For a detailed explanation of the cohomology rings of the above spaces, see [11, §C.1].

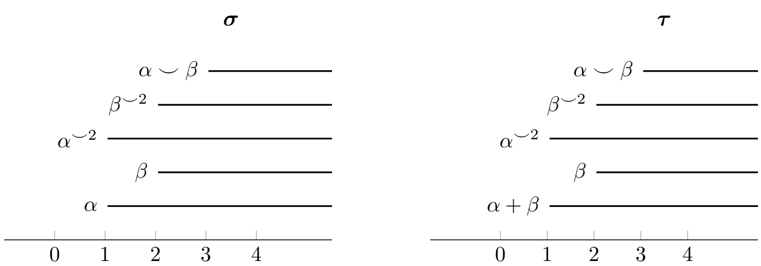
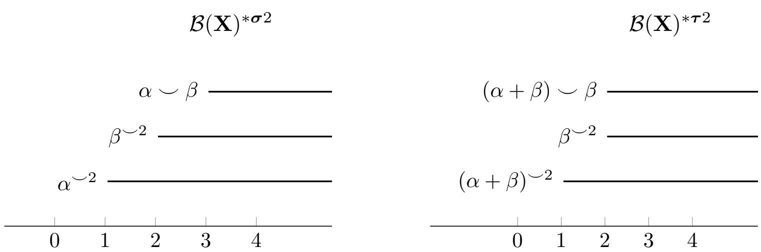



**Figure 6** Two choices of representative cocycles for the filtration $\mathbf{X}$ given by Ex. 18.

To obtain the cup-length diagram, we first compute $\mathcal{B}(\mathbf{X})^{*\boldsymbol{\sigma}2}$ and $\mathcal{B}(\mathbf{X})^{*\boldsymbol{\tau}2}$:




By Defn. 16, the persistent cup-length diagram associated to $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are (see Fig. 7):

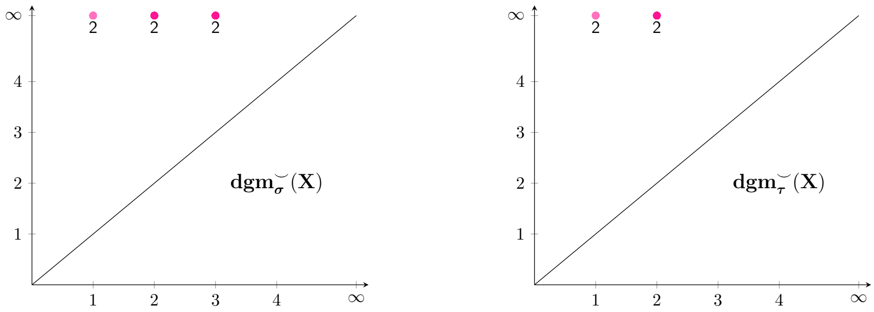



**Figure 7** The persistent cup-length diagrams $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})$ (left) and $\mathbf{dgm}^{\smile}_{\boldsymbol{\tau}}(\mathbf{X})$ (right), see Ex. 18.

See [11, §C.2] for more examples of persistent cup-length diagrams. In the next section, we develop an algorithm for computing the persistent cup-length diagram $\mathbf{dgm}^{\smile}_{\boldsymbol{\sigma}}(\mathbf{X})$, which can be used to compute the persistent cup-length function $\mathbf{cup}(\mathbf{X})$ due to Thm. 1.

## 3.2 An algorithm for computing the persistent cup-length diagram over $\mathbb{Z}_2$

Let $\mathbf{X} : \mathbb{X}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{X}_N (= \mathbb{X})$ be a finite filtration of a finite simplicial complex $\mathbb{X}$. Suppose that the barcode over positive dimensions $\mathcal{B} := \mathcal{B}_{\geq 1}(\mathbf{X})$ and a family of representative cocycles $\boldsymbol{\sigma} := \{\sigma_I\}_{I \in \mathcal{B}}$ are given. Because a finite filtration has only finitely many critical values, we assume that all intervals in the barcode are closed at the right end. If not, we replace the right end of each such interval with its closest critical value to the left. Since each interval is considered together with a representative cocycle in this section, we will abuse the notation and write $\mathcal{B}$ for the set $\{(I, \sigma_I)\}_{I \in \mathcal{B}}$ as well. Let $\mathcal{B}^{*_{\boldsymbol{\sigma}} \ell}$ be the set of $I_1 *_{\boldsymbol{\sigma}} \cdots *_{\boldsymbol{\sigma}} I_\ell$ where each $I_i \in \mathcal{B}$. We compute $\left\{\mathcal{B}^{*_{\boldsymbol{\sigma}} \ell}\right\}_{\ell \geq 1}$ using:

**Algorithm 2** Computing $\left\{\mathcal{B}^{*_{\boldsymbol{\sigma}} \ell}\right\}_{\ell \geq 1}$.

```
1 while B^{*σ ℓ} ≠ ∅ do
2   for (I_1, σ_1) ∈ B and (I_2, σ_2) ∈ B^{*σ ℓ} do
3     if I_1 *σ I_2 ≠ ∅ then
4       Append (I_1 *σ I_2, σ_1 ⌣ σ_2) to B^{*σ (ℓ+1)}
```

The **line 3** involves the computation of $\operatorname{supp}(\sigma_1 \smile \sigma_2)$ which is some interval $[b_\sigma, d_\sigma]$ for $1 \leq b_\sigma \leq d_\sigma \leq m$ such that $d_\sigma$ is simply the right end of $I_1 \cap I_2$ and $b_\sigma$ is the left end of some $I \in \mathcal{B}$, by Prop. 15. The computation of $b_\sigma$ is broken down in two steps: (1) compute the cup product (at cochain level) $\sigma := \sigma_1 \smile \sigma_2$, and (2) find $b_\sigma$ as the smallest $i \leq d_\sigma$ such that $\sigma|_{C_*(\mathbb{X}_i)}$ is not a coboundary. Step (1) is already addressed by Alg. 1 on page 6. Let us now introduce an algorithm to address Step (2).

## 3.3 Checking whether a cochain is a coboundary

As before, we assume a total order (e.g. the order given in [5]) on the simplices, and denote the ordered simplices by $S = \{\alpha_1 < \cdots < \alpha_m\}$, where $m$ is the number of simplices. We adopt the reverse ordering for the set of cosimplices $S^* := \{\alpha_m^* < \cdots < \alpha_1^*\}$. Notice that $S^*$ forms a basis for the linear space of cochains. A $p$-cochain $\sigma$ is written as a linear sum of elements in $S^*$ uniquely. If $\alpha_j^*$ appears as a summand for $\sigma$, we denote $\alpha_j \in \sigma$.

Let $A$ be the coboundary matrix associated to the ordered basis $S^*$. Assume that $\boxed{R = AV}$ is the reduced matrix of $A$ obtained from left-to-right column operations, given by the upper triangular matrix $V$. As a consequence, the pivots $\mathrm{Pivots}(R)$ of columns of $R$ are unique. Using all $i$-th row for $i \in \mathrm{Pivots}(R)$, we do bottom-to-top row reduction on $R$: $\boxed{UR = \Lambda}$, such that $\Lambda$ has at most one non-zero element in each row and column, and $U$ is an upper triangular matrix. See [11, Alg. 3] for the row reduction algorithm (with complexity $O(m^2)$), which outputs the matrix $U$. The following proposition allows us to use the row reduction matrix $U$ and $\mathrm{Pivots}(R)$ to check whether a cochain is a coboundary. For the proof of Prop. 19, see [11, §C.3].

▶ **Proposition 19.** *Given a $p$-cochain $\sigma$, let $y \in \mathbb{Z}_2^m$ be such that $\sigma = S^* \cdot y$. Let $R$ be the column reduced coboundary matrix, $U$ be the row reduction matrix of $R$. Then $\sigma$ is a coboundary, iff $\{i : \text{the } i\text{-th row of } (U \cdot y) \neq 0\} \subset \mathrm{Pivots}(R)$.*

**Using the boundary matrix.** We can also use the boundary matrix to check whether a cocycle is a coboundary, see [11, page 28]. Using the boundary matrix, only column reduction is needed, while for the coboundary matrix both column reduction and row reduction are performed. However, it has been justified in [5] that reducing the coboundary matrix is more efficient than reducing the boundary matrix. Combined with the fact that the row reduction step does not increase the computation complexity of computing the persistent cup-length diagram, we will use the coboundary matrix in this paper.

## 3.4 Main algorithm and its complexity

Motivated by the goal of obtaining a practical algorithm, and in order to control the complexity, we consider truncating filtrations up to a user specified dimension bound.

**Truncation of a filtration.** Fix a positive integer $k$. Given a filtration $\mathbf{X} : \mathbb{X}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{X}_N (= \mathbb{X})$ of a finite simplicial complex $\mathbb{X}$, let $\mathbb{X}_i^{k+1}$ be the $(k+1)$-skeleton of $\mathbb{X}_i$ for each $i$. The $(k+1)$-*dimensional truncation* of $\mathbf{X}$ is the filtration $\mathbf{X}^{k+1} : \mathbb{X}_1^{k+1} \hookrightarrow \cdots \hookrightarrow \mathbb{X}_N^{k+1}$. Since $\mathbf{H}^{\leq k}(\mathbb{Y}) \cong \mathbf{H}^{\leq k}(\mathbb{Y}^{k+1})$ (as vector spaces) for any simplicial complex $\mathbb{Y}$, we conclude that $\mathbf{H}^{\leq k}(\mathbf{X}) \cong \mathbf{H}^{\leq k}\left(\mathbf{X}^{k+1}\right)$ as persistent vector spaces. Thus, the barcode $\mathcal{B}(\mathbf{X}^{k+1})$ of $\mathbf{X}^{k+1}$ is equal to the barcode $\mathcal{B}_{\leq k}(\mathbf{X}) := \sqcup_{p \leq k} \mathcal{B}_p(\mathbf{X})$. Let $\mathcal{B}_{[1,k]}(\mathbf{X}) := \sqcup_{1 \leq p \leq k} \mathcal{B}_p(\mathbf{X})$. We introduce Alg. 3 for computing the persistent cup-length diagram for the $(k+1)$-dimensional truncation of $\mathbf{X}$ over $\mathbb{Z}_2$. The time complexity of Alg. 3 is described in terms of the variables below:

- $k$ is a dimension bound used to truncate the filtration;
- $m_k$ is the number of simplices with positive dimension in the $(k+1)$-skeleton $\mathbb{X}^{k+1}$ of $\mathbb{X}$;
- $c_k$ is the complexity of checking whether a simplex is alive at a given filtration parameter;
- $q_{k-1} := \max_{1 \leq \ell \leq k-1} \mathrm{card}\left(\left(\mathcal{B}_{[1,k]}(\mathbf{X})\right)^{*_\sigma \ell}\right)$ (see Defn. 14). In particular, $q_1 = \mathrm{card}\left(\mathcal{B}_{[1,k]}(\mathbf{X})\right)$.

**Time complexity.** In Alg. 3, **line 9** runs no more than $q_1 \cdot q_{k-1}$ times, due to the definition of $q_1$ and $q_{k-1}$. The while loop in **line 14** runs no more than card(b_time) $\leq q_1$ times, and the condition of this while loop involves a matrix multiplication whose complexity is at most $O((m_k)^2)$. Combined with other comments in Alg. 3 and the fact that $k$ is a fixed constant, the total complexity is upper bounded by

$$O(k) \cdot O(q_1 \cdot q_{k-1}) \cdot O((m_k)^2 \cdot \max\{c_k, q_1\}) \leq O((m_k)^2 \cdot q_1 \cdot q_{k-1} \cdot \max\{c_k, q_1\}).$$

**Algorithm 3** Main algorithm: compute persistent cup-length diagram.

**Input :** A dimension bound $k$, the ordered list of cosimplices $S^*$ from dimension 1 to $k+1$, the column reduced coboundary matrix $R$ from dimension 1 to $k+1$, and barcodes (annotated by representative cocycles) from dimension 1 to $k$: $\mathcal{B}_{[1,k]} = \{(b_\sigma, d_\sigma, \sigma)\}_{\sigma \in \boldsymbol{\sigma}}$, where each $\sigma$ is a representative cocycle for the bar $(b_\sigma, d_\sigma)$ and $\{\sigma_1, \ldots, \sigma_{q_1}\}$ is ordered first in the increasing order of the death time and then in the increasing order of the birth time.

**Output :** A matrix representation $A_\ell$ of persistent cup-length diagram, and the lists of distinct birth times b_time and death times d_time.

$\mathrm{b\_time}, \mathrm{d\_time} \leftarrow \mathrm{unique}(\{b_\sigma\}_{\sigma\in\boldsymbol{\sigma}}), \mathrm{unique}(\{d_\sigma\}_{\sigma\in\boldsymbol{\sigma}})$;
$m_k,\ \ell,\ B_1 \leftarrow \mathrm{card}(S^*),\ 1,\ \mathcal{B}_{[1,k]}$;
$A_0 = A_1 \leftarrow \mathrm{zeros}(\mathrm{card}(\mathrm{b\_time}), \mathrm{card}(\mathrm{d\_time}))$;
$U \leftarrow \mathrm{RowReduce}(R)$ ; `// ` $O(m_k^2)$`, [11, Alg. 3]`
**for** $(b_i, b_j) \in B_1$ **do** `// ` $O(m_k)$
  $A_1(i,j) \leftarrow 1$;
**while** $A_{\ell-1} \neq A_\ell$ *and* $l \leq k-1$ **do** `// ` $O(k)$
  $B_{\ell+1} = \{\}$;
  **for** $(b_{i_1}, d_{j_1}, \sigma_1) \in B_1$ *and* $(b_{i_2}, d_{j_2}, \sigma_2) \in B_\ell$ **do** `// ` $O(q_1 \cdot q_{k-1})$
    $\sigma \leftarrow \mathrm{CupProduct}(\sigma_1, \sigma_2, S^*)$ ; `// ` $O(m_k^2 \cdot c_k)$`, Alg. 1`
    $y \leftarrow$ the vector representation of $\sigma$ in $S^*$;
    $i \leftarrow \max\{i' : b_{i'} \leq d_{\min\{j_1,j_2\}}\}$;
    $s_i \leftarrow$ number of simplices alive at $b_i$;
    **while** $\{u : (U_{m_k+1-s_i:m_k,\, m_k+1-s_i:m_k} \cdot y_{m_k+1-s_i:m_k})(u) \neq 0\} \subset \mathrm{Pivots}(R)$ **do** `// ` $O((m_k)^2 \cdot q_1)$
      $i \leftarrow i-1$;
      $s_i \leftarrow$ number of simplices alive at $b_i$;
    **if** $b_i < d_{\min\{j_1,j_2\}}$ **then**
      Append $(b_i, d_{\min\{j_1,j_2\}}, \sigma)$ to $B_{\ell+1}$;
      $A_{\ell+1}(i, \min\{j_1,j_2\}) \leftarrow \ell$
  $\ell \leftarrow \ell + 1$.
**return** $A_\ell, \mathrm{b\_time}, \mathrm{d\_time}$.

Next, we estimate $q_{k-1}$ and $c_k$ using $q_1$, $m_k$ and $k$. Since each $B_\ell$ consists of $\ell$-fold $*_{\boldsymbol{\sigma}}$-products of elements in $B_1$, we have $q_{k-1} = \max_{1\leq\ell\leq k-1} \mathrm{card}(B_\ell) \leq (q_1)^{k-1}$, which turns out to be a very coarse bound (see [11, Rmk. 45]). On the other hand, $c_k$ as the cost of checking whether a simplex is alive at a given filtration, is at most $m_k$ the number of simplices. Hence, the complexity of Alg. 3 is upper bounded by $O((m_k)^3 \cdot q_1^k)$. In addition, we have $q_1 \leq m_k$, because in the matrix reduction algorithm for computing barcodes, bars are obtained from the pivots of the column reduced coboundary matrix and each column provides at most one pivot. Thus, $O((m_k)^2 \cdot q_1 \cdot q_{k-1} \cdot \max\{c_k, q_1\}) \leq O((m_k)^3 \cdot q_1^k) \leq O((m_k)^{k+3})$.

Consider the Vietoris-Rips filtration arising from a metric space of $n$ points with the distance matrix $D$. Then **line 7** of Alg. 1, checking whether a simplex $a$ (represented by a set of at most $k+1$ indices into $[n]$) is alive at the filtration parameter value $t$, can be done by checking whether $\max(D[a,a]) \leq t$, with constant time complexity $c_k = O(k^2)$. In summary, we have the following theorem.

▶ **Theorem 20** (Complexity of Alg. 3)**.** *For an arbitrary finite filtration truncated up to dimension $(k+1)$, computing its persistent cup-length diagram via Alg. 3 has complexity at most $O((m_k)^2 \cdot q_1 \cdot q_{k-1} \cdot \max\{c_k, q_1\})$. In terms of just $m_k$, the complexity of Alg. 3 is at most $O((m_k)^{k+3})$, since $c_k \leq m_k$ and $q_{k-1} \leq (m_k)^{k-1}$.*

*For the $(k+1)$-dimensional truncation of the Vietoris-Rips filtration arising from a metric space of $n$ points, the complexity of Alg. 3 is improved to $O((m_k)^2 \cdot q_1^2 \cdot q_{k-1})$, which is at most $O((m_k)^{k+3}) \leq O(n^{k^2+5k+6})$.*

Notice that when $k = 1$, the persistent cup-length diagram simply evaluates 1 at each bar in the standard barcode, and 0 elsewhere. When $k \geq 2$, the resulting persistent cup-length diagram becomes more informative and captures certain topological features that the standard persistence diagram is not able to detect. This is reflected in [11, Ex. 42].

Although the algorithm has not been tested on datasets yet, it is a practical algorithm, given that there are available software programs, such as Ripser (see [4]), which compute the barcode and extracts representative cocycles for Vietoris-Rips filtrations. Note that, according to [5], the implementation ideas 'are also applicable to persistence computations for other filtrations as well'.

▶ Remark 21 (Estimating the parameter $q_{k-1}$). The inequality $q_{k-1} \leq (m_k)^{k-1}$ is quite coarse in general. Consider a filtration consisting of contractible spaces, where $q_{k-1}$ is always 0 but $m_k$ can be arbitrarily large. Even in the case when there is a reasonable number of cohomology classes with non-trivial cup products, $q_{k-1}$ can be much smaller than $(m_k)^{k-1}$. See [11, Rmk. 45].

▶ Remark 22 (Reducing the time complexity). Because cup products cannot live longer than their factors, discarding short bars will not result into loss of important information. In our algorithm, an extra parameter $\epsilon \geq 0$ can be added to discard all the bars in the barcode $B_1$ with length less than $\varepsilon$. By doing so, since the cardinality of $B_1$ is decreased, one expects the runtime of Alg. 3 (in particular inside the loop in **line 9**) to be significantly reduced. A similar trimming strategy can also be applied in the construction of the subsequent $B_\ell$s.

**Correctness of the algorithm.** Checking whether a cocycle is a coboundary requires local matrix reduction for the given filtration parameter $d_{\min\{j_1,j_2\}}$, but a global matrix reduction is performed in the algorithm. The reason is that the coboundary matrix $A$, the column reduction matrix $V$ and the row reduction matrix $U$ are all upper-diagonal. Therefore, reducing the ambient matrix $A$ and then taking the bottom-right submatrix to get $\bar{U}$, is equivalent to reducing the submatrix of $A$ directly.

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