Tight Lower Bounds for Approximate & Exact k-Center in $\mathbb{R}^d$

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Abstract

In the discrete k-Center problem, we are given a metric space $(P, \texttt{dist})$ where $|P| = n$ and the goal is to select a set $C \subseteq P$ of k centers which minimizes the maximum distance of a point in P from its nearest center. For any $\epsilon > 0$, Agarwal and Procopiuc [SODA '98, Algorithmica '02] designed an $(1+\epsilon)$-approximation algorithm[1] for this problem in d-dimensional Euclidean space[2] which runs in $O(dn \log k) + \left(\frac{k}{\epsilon}\right)^{O\left(k^{1-1/d}\right)} \cdot n^{O(1)}$ time. In this paper we show that their algorithm is essentially optimal: if for some $d \geq 2$ and some computable function f, there is an $f(k) \cdot \left(\frac{1}{\epsilon}\right)^{o\left(k^{1-1/d}\right)} \cdot n^{o\left(k^{1-1/d}\right)}$ time algorithm for $(1+\epsilon)$-approximating the discrete k-Center on n points in d-dimensional Euclidean space then the Exponential Time Hypothesis (ETH) fails.

We obtain our lower bound by designing a gap reduction from a d-dimensional constraint satisfaction problem (CSP) to discrete d-dimensional k-Center. This reduction has the property that there is a fixed value ϵ (depending on the CSP) such that the optimal radius of k-Center instances corresponding to satisfiable and unsatisfiable instances of the CSP is < 1 and $\geq (1+\epsilon)$ respectively. Our claimed lower bound on the running time for approximating discrete k-Center in d-dimensions then follows from the lower bound due to Marx and Sidiropoulos [SoCG '14] for checking the satisfiability of the aforementioned d-dimensional CSP.

As a byproduct of our reduction, we also obtain that the exact algorithm of Agarwal and Procopiuc [SODA '98, Algorithmica '02] which runs in $n^{O\left(d \cdot k^{1-1/d}\right)}$ time for discrete k-Center on n points in d-dimensional Euclidean space is asymptotically optimal. Formally, we show that if for some $d \geq 2$ and some computable function f, there is an $f(k) \cdot n^{o\left(k^{1-1/d}\right)}$ time exact algorithm for the discrete k-Center problem on n points in d-dimensional Euclidean space then the Exponential Time Hypothesis (ETH) fails. Previously, such a lower bound was only known for $d = 2$ and was implicit in the work of Marx [IWPEC '06].

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1 Introduction

The k-Center problem is a classical problem in theoretical computer science and was first formulated by Hakimi [22] in 1964. In this problem, given a metric space $(P, \texttt{dist})$ and an integer $k \leq |P|$ the goal is to select a set C of k centers which minimizes the maximum

[1] The algorithm of Agarwal and Procopiuc [2] also works for the non-discrete, i.e., continuous, version of the problem when C need not be a subset of P, but our lower bounds only hold for the discrete version.

[2] The algorithm of Agarwal and Procopiuc [2] also works for other metrics such as ℓ_∞ or ℓ_q metric for $q \geq 1$. Our construction also works for ℓ_∞ (in fact, some of the bounds are simpler to derive!) but we present only the proof for ℓ_2 to keep the presentation simple.

distance of a point in P from its nearest center, i.e., select a set C which minimizes the quantity $\max_{p \in P} \min_{c \in C} \texttt{dist}(p, c)$. A geometric way to view the k-Center problem is to find the minimum radius r such that k closed balls of radius r located at each of the points in C cover all the points in P. In most applications, we require that $C \subseteq P$ and this is known as the discrete version of the problem.

As an example, one can consider the set P to be important locations in a city and solving the k-Center problem (where k is upper bounded by budget constraints) establishes the locations of fire stations which minimize the response time in event of a fire. In addition to other applications in facility location, transportation networks, etc. an important application of k-Center is in clustering. With the advent of massive data sets, the problem of efficiently and effectively summarizing this data is crucial. A standard approach for this is via centroid-based clustering algorithms of which k-Center is a special case. Clustering using k-Center has found applications in text summarization, robotics, bioinformatics, pattern recognition, etc. [41, 20, 23, 30].

1.1 Prior work on exact & approximate algorithms for discrete k-Center

The discrete[3] k-Center problem is NP-hard [44], and admits a 2-approximation [24, 21] in $n^{O(1)}$ time where n is the number of points. This approximation ratio is tight and the k-Center problem is NP-hard to approximate in polynomial time to a factor $(2 - \epsilon)$ for any constant $\epsilon > 0$ [25, 21]. Given this intractability, research was aimed at designing parameterized algorithms [10] and parameterized approximation algorithms for k-center. The k-Center problem is W[2]-hard to approximate to factor better than 2 even when allowing running times of the form $f(k) \cdot n^{O(1)}$ for any computable function f [15, 13]. The k-Center problem remains W[2]-hard even if we combine the parameter k with other structural parameters such as size of vertex cover or size of feedback vertex set [31]. Agarwal and Procopiuc [2] designed an algorithm for discrete k-Center on n points in d-dimensional Euclidean space which runs in $n^{O\left(d \cdot k^{1-1/d}\right)}$ time.

The paradigm of combining parameterized algorithms & approximation algorithms has been successful in designing algorithms for k-center in special topologies such as d-dimensional Euclidean space [2], planar graphs [19], metrics of bounded doubling dimensions [16], graphs of bounded highway dimension [15, 4], etc. Of particular relevance to this paper is the $(1 + \epsilon)$-approximation algorithm[4] of Agarwal and Procopiuc [2] which runs in $O(dn \log k) + \left(\frac{k}{\epsilon}\right)^{O\left(k^{1-1/d}\right)} \cdot n^{O(1)}$ time. This was generalized by Feldmann and Marx [16] who designed an $(1 + \epsilon)$-approximation algorithm running in $\left(\frac{k^k}{\epsilon^{O(kD)}}\right) \cdot n^{O(1)}$ time for discrete k-Center in metric spaces of doubling dimension D.

1.2 From 2-dimensions to higher dimensions

Square root phenomenon for planar graphs and geometric problems in the plane. For a wide range of problems on planar graphs or geometric problems in the plane, a certain *square root phenomenon* is observed for a wide range of algorithmic problems: the exponent of the

[3] Here we mention the known results only for the discrete version of k-Center. A discussion about results for the continuous version of the problem is given in Section 1.4.

[4] This is also known as an efficient parameterized approximation scheme (EPAS) as the running time is a function of the type $f(k, \epsilon, d) \cdot n^{O(1)}$.

running time can be improved from $O(\ell)$ to $O(\sqrt{\ell})$ where ℓ is the parameter, or from $O(n)$ to $O(\sqrt{n})$ where n is in the input size, and lower bounds indicate that this improvement is essentially best possible. There is an ever increasing list of such problems known for planar graphs [8, 37, 32, 38, 33, 14, 42, 39, 34, 1, 18] and in the plane [39, 36, 17, 3, 43, 27, 26]

Bounds for higher dimensional Euclidean spaces. Unlike the situation on planar graphs and in two-dimensions, the program of obtaining tight bounds for higher dimensions is still quite nascent with relatively fewer results [9, 40, 5, 12, 11]. Marx and Sidiropoulos [40] showed that for some problems there is a *limited blessing of low dimensionality*: that is, for d-dimensions the running time can be improved from n^{ℓ} to $n^{\ell^{1-1/d}}$ or from 2^n to $2^{n^{1-1/d}}$ where ℓ is a parameter and n is the input size. In contrast, Cohen-Addad et al. [9] showed that the two problems of k-Median and k-Means suffer from the *curse of low dimensionality*: even for 4-dimensional Euclidean space, assuming the Exponential Time Hypothesis[5] (ETH), there is no $f(k) \cdot n^{o(k)}$ time algorithm, i.e., the brute force algorithm which runs in $n^{O(k)}$ time is asymptotically optimal.

1.3 Motivation & Our Results

In two-dimensional Euclidean space there is an $n^{O(\sqrt{k})}$ algorithm [2, 27, 26], and a matching lower bound of $f(k) \cdot n^{o(\sqrt{k})}$ under Exponential Time Hypothesis (ETH) for any computable function f [36]. Our motivation in this paper is to investigate what is the *correct* complexity of exact and approximate algorithms for the discrete k-Center for higher dimensional Euclidean spaces. In particular, we aim to answer the following two questions:

(Question 1) Can the running time of the $(1+\epsilon)$-approximation algorithm of [2] be improved from $O(dn\log k) + \left(\frac{k}{\epsilon}\right)^{O\left(k^{1-1/d}\right)} \cdot n^{O(1)}$, or is there a (close to) matching lower bound?

(Question 2) The $n^{O\left(d \cdot k^{1-1/d}\right)}$ algorithm of [2] for d-dimensional Euclidean space shows that there is a *limited blessing of low dimensionality* for k-Center. But can the term $k^{1-1/d}$ in the exponent be improved, or is it asymptotically tight?

We make progress towards answering both these questions by showing the following theorem:

▶ **Theorem 1.** *For any $d \geq 2$, under the Exponential Time Hypothesis (ETH), the discrete k-Center problem in d-dimensional Euclidean space*

- ***(Inapproximability result)** does not admit an $(1+\epsilon)$-approximation in $f(k) \cdot \left(\frac{1}{\epsilon}\right)^{o\left(k^{1-1/d}\right)} \cdot n^{o\left(k^{1-1/d}\right)}$ time where f is any computable function and n is the number of points.*
- ***(Lower bound for exact algorithm)** cannot be solved in $f(k) \cdot n^{o\left(k^{1-1/d}\right)}$ time where f is any computable function and n is the number of points.*

Theorem 1 answers Question 1 by showing that the running time of the $(1+\epsilon)$-approximation algorithm of Agarwal and Procopiuc [2] is essentially tight, i.e., the dependence on ϵ cannot be improved even if we allow a larger dependence on both k and n. Theorem 1 answers Question 2 by showing that the running time of the exact algorithm of Agarwal and Procopiuc [2] is asymptotically tight, i.e., the exponent of $k^{1-1/d}$ cannot be asymptotically improved even if we allow a larger dependence on k.

[5] Recall that the Exponential Time Hypothesis (ETH) has the consequence that n-variable 3-SAT cannot be solved in $2^{o(n)}$ time [28, 29].

1.4 Discussion of the continuous k-Center problem

In the continuous version of the k-CENTER problem, the centers are not required to be picked from the original set of input points. The $n^{O\left(d\cdot k^{1-1/d}\right)}$ algorithm of Agarwal and Procopiuc [2] also works for this continuous version of the k-CENTER problem in $\mathbb{R}^d$. Marx [35] showed the W[1]-hardness of k-CENTER in $(\mathbb{R}^2, \ell_\infty)$ parameterized by k. Cabello et al. [6] studied the complexity of this problem parameterized by the dimension, and showed the W[1]-hardness of 4-CENTER in $(\mathbb{R}^d, \ell_\infty)$ parameterized by d. Additionally, they also obtained the W[1]-hardness of 2-CENTER in $(\mathbb{R}^d, \ell_2)$ parameterized by d; this reduction also rules out existence of $n^{o(d)}$ algorithms for this problem under the Exponential Time Hypothesis (ETH). It is an interesting open question whether the $n^{O\left(d\cdot k^{1-1/d}\right)}$ algorithm of Agarwal and Procopiuc [2] is also asymptotically tight for the continuous version of the problem: one way to possibly prove this would be to extend the W[1]-hardness reduction of Marx [35] for continuous k-CENTER in $\mathbb{R}^2$ (parameterized by k) to higher dimensions using the framework of Marx and Sidiropoulos [40]. Our reduction in this paper does not extend to the continuous version.

1.5 Notation

The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. All vectors considered in this paper have length d. If $\mathbf{a}$ is a vector then for each $i \in [d]$ its i-th coordinate is denoted by $\mathbf{a}[i]$. Addition and subtraction of vectors is denoted by $\oplus$ and $\ominus$ respectively. The i-th unit vector is denoted by $\mathbf{e}_i$ and has $\mathbf{e}_i[i] = 1$ and $\mathbf{e}_i[j] = 0$ for each $j \neq i$. The d-dimensional vector whose every coordinate equals 1 is denoted by $\mathbf{1}^d$. If u is a point and X is a set of points then $\texttt{dist}(u, X) = \min_{x\in X} \texttt{dist}(u, x)$. We will sometimes abuse notation slightly and use x to denote both the name and location of the point x.

2 Lower bounds for exact & approximate k-Center in d-dimensional Euclidean space

The goal of this section is to prove Theorem 1 which is restated below:

▶ **Theorem 1.** *For any $d \geq 2$, under the Exponential Time Hypothesis (ETH), the discrete k-CENTER problem in d-dimensional Euclidean space*

- ***(Inapproximability result)*** *does not admit an $(1+\epsilon)$-approximation in $f(k) \cdot \left(\frac{1}{\epsilon}\right)^{o\left(k^{1-1/d}\right)} \cdot n^{o\left(k^{1-1/d}\right)}$ time where f is any computable function and n is the number of points.*
- ***(Lower bound for exact algorithm)*** *cannot be solved in $f(k) \cdot n^{o\left(k^{1-1/d}\right)}$ time where f is any computable function and n is the number of points.*

Roadmap to prove Theorem 1. To prove Theorem 1, we design a gap reduction (described in Section 2.2) from a constraint satisfaction problem (CSP) to the k-CENTER problem. The definition and statement of the lower bound for the CSP due to Marx and Sidiropoulos [40] is given in Section 2.1. The correctness of the reduction is shown in Section 2.4 and Section 2.3. Finally, everything is tied together in Section 2.5 which contains the proof of Theorem 1.

2.1 Lower bound for d-dimensional geometric $\geq$-CSP [40]

This section introduces the d-dimensional geometric $\geq$-CSP problem of Marx and Sidiropoulos [40]. First we start with some definitions before stating the formal lower bound (Theorem 5) that will be used to prove Theorem 1. Constraint Satisfaction Problems (CSPs) are a general way to represent several important problems in theoretical computer science. In this paper, we will only need a subclass of CSPs called binary CSPs which we define below.

▶ **Definition 2.** *An instance of a binary constraint satisfaction problem (CSP) is a triple $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ where $\mathcal{V}$ is a set of variables, $\mathcal{D}$ is a domain of values and $\mathcal{C}$ is a set of constraints. There are two types of constraints:*

- ***Unary constraints**: For some $v \in \mathcal{V}$ there is a unary constraint $\langle v, R_v \rangle$ where $R_v \subseteq \mathcal{D}$.*
- ***Binary constraints**: For some $u, v \in \mathcal{V}$, $u \neq v$, there is a binary constraint $\big\langle (u, v), R_{u,v} \big\rangle$ where $R_{u,v} \subseteq \mathcal{D} \times \mathcal{D}$.*

Solving a given CSP instance $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ is to check whether there exists a satisfying assignment for it, i.e., a function $f : \mathcal{V} \to \mathcal{D}$ such that all the constraints are satisfied. For a binary CSP, a satisfying assignment f has the property that for each unary constraint $\langle v, R_v \rangle$ we have $f(v) \in R_v$ and for each binary constraint $\big\langle (u, v), R_{u,v} \big\rangle$ we have $(f(u), f(v)) \in R_{u,v}$.

The constraint graph of a given CSP instance $\mathcal{I} = (V, D, C)$ is an undirected graph $G_{\mathcal{I}}$ whose vertex set is V and the adjacency relation is defined as follows: two vertices $u, v \in V$ are adjacent in $G_{\mathcal{I}}$ if there is a constraint in $\mathcal{I}$ which contains both u and v. Marx and Sidiropoulos [40] observed that binary CSPs whose primal graph is a subgraph of the d-dimensional grid is useful in showing lower bounds for geometric problems in d-dimensions.

▶ **Definition 3.** *The d-dimensional grid $\mathrm{R}[N, d]$ is an undirected graph with vertex set $[N]^d$ and the adjacency relation is as follows: two vertices $(a_1, a_2, \ldots, a_d)$ and $(b_1, b_2, \ldots, b_d)$ have an edge between them if and only if $\sum_{i=1}^{d} |a_i - b_i| = 1$.*

▶ **Definition 4.** *A d-dimensional geometric $\geq$-CSP $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ is a binary CSP whose*

- *set of variables $\mathcal{V}$ is a subset of $\mathrm{R}[N, d]$ for some $N \geq 1$,*
- *domain is $[\delta]^d$ for some integer $\delta \geq 1$,*
- *constraint graph $G_{\mathcal{I}}$ is an* induced *subgraph of $\mathrm{R}[N, d]$,*
- *unary constraints are arbitrary, and*
- *binary constraints are of the following type: if $\mathbf{a}, \mathbf{a}' \in \mathcal{V}$ such that $\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i$ for some $i \in [d]$ then there is a binary constraint $\big\langle (\mathbf{a}, \mathbf{a}'), R_{\mathbf{a},\mathbf{a}'} \big\rangle$ where $R_{\mathbf{a},\mathbf{a}'} = \{(\mathbf{x}, \mathbf{y}) \in R_{\mathbf{a}} \times R_{\mathbf{a}'} \mid \mathbf{x}[i] \geq \mathbf{y}[i]\}$.*

Observe that the set of unary constraints of a d-dimensional geometric $\geq$-CSP is sufficient to completely define it. The size $|\mathcal{I}|$ of a binary CSP $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ is the combined size of the variables, domain and the constraints. With appropriate preprocessing (e.g., combining different constraints on the same variables) we can assume that $|\mathcal{I}| = (|\mathcal{V}| + |\mathcal{D}|)^{O(1)}$. We now state the result of Marx and Sidiropoulos [40] which gives a lower bound on the complexity of checking whether a given d-dimensional geometric $\geq$-CSP has a satisfying assignment.

▶ **Theorem 5** ([40, Theorem 2.10])**.** *If for some fixed $d \geq 2$, there is an $f(|\mathcal{V}|) \cdot |\mathcal{I}|^{o\left(|\mathcal{V}|^{1-1/d}\right)}$ time algorithm for solving a d-dimensional geometric $\geq$-CSP $\mathcal{I}$ for some computable function f, then the Exponential Time Hypothesis (ETH) fails.*

▶ Remark 6. The problem defined by Marx and Sidiropoulos [40] is actually d-dimensional geometric $\leq$-CSP which has $\leq$-constraints instead of the $\geq$-constraints. However, for each $\mathbf{a} \in \mathcal{V}$ by replacing each unary constraint $\mathbf{x} \in R_{\mathbf{a}}$ by $\mathbf{y}$ such that $\mathbf{y}[i] = N + 1 - \mathbf{x}[i]$ for each $i \in [d]$, it is easy to see that d-dimensional geometric $\leq$-CSP and d-dimensional geometric $\geq$-CSP are equivalent.

2.2 Reduction from d-dimensional geometric $\geq$-CSP to k-Center in $\mathbb{R}^d$

We are now ready to describe our reduction from d-dimensional geometric $\geq$-CSP to k-Center in $\mathbb{R}^d$. Fix any $d \geq 2$. Let $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ be a d-dimensional geometric $\geq$-CSP instance on variables $\mathcal{V}$ and domain $[\delta]^d$ for some integer $\delta \geq 1$. We fix[6] the following two quantities:

$$r := \frac{1}{4} \quad \text{and} \quad \epsilon := \frac{r^2}{(d-1)\delta^2} = \frac{1}{16(d-1)\delta^2}. \tag{1}$$

Since $d \geq 2$ and $\delta \geq 1$, we obtain the following bounds from Equation 1,

$$0 < \epsilon \leq \epsilon\delta \leq \epsilon\delta^2 \leq \epsilon\delta^2(d-1) = r^2 = \frac{1}{16}. \tag{2}$$

Given an instance $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ of d-dimensional geometric $\geq$-CSP, we add a set $\mathcal{U}$ of points in $\mathbb{R}^d$ as described in Table 1 and Table 2. These set of points are the input for the instance of the $|\mathcal{V}|$-Center problem.

Table 1 The set $\mathcal{U}$ of points in $\mathbb{R}^d$ (which gives an instance of k-Center) constructed from an instance $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ of d-dimensional geometric $\geq$-CSP.

(1) Corresponding to variables: If $\mathbf{a} \in \mathcal{V}$ then we add the following set of points which are collectively called as Border$[\mathbf{a}]$
- For each $i \in [d]$, the point $B_{\mathbf{a}}^{+i}$ which is located at $\mathbf{a} \oplus \mathbf{e}_i \cdot r(1-\epsilon) \oplus (\mathbf{1}^d - \mathbf{e}_i) \cdot 2\epsilon\delta$.
- For each $i \in [d]$, the point $B_{\mathbf{a}}^{-i}$ which is located at $\mathbf{a} \ominus \mathbf{e}_i \cdot r(1-\epsilon) \ominus (\mathbf{1}^d - \mathbf{e}_i) \cdot 2\epsilon\delta$.

This set of points are referred to as *border* points.

(2) Corresponding to unary constraints: If $\mathbf{a} \in \mathcal{V}$ and $\langle(\mathbf{a}), R_{\mathbf{a}}\rangle$ is the unary constraint on $\mathbf{a}$, then we add the following set of points which are collectively called as Core$[\mathbf{a}]$:
- for each $\mathbf{x} \in R_{\mathbf{a}} \subseteq [\delta]^d$ we add a point called $C_{\mathbf{a}}^{\mathbf{x}}$ located at $\mathbf{a} \oplus \epsilon \cdot \mathbf{x}$.

This set of points are referred to as *core* points.

(3) Corresponding to adjacencies in $G_{\mathcal{I}}$: For every edge $(\mathbf{a}, \mathbf{a}')$ in $G_{\mathcal{I}}$ we add a collection of δ points denoted by $\mathcal{S}_{\{\mathbf{a},\mathbf{a}'\}}$. Assume, without loss of generality, that $\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i$ for some $i \in [d]$. Then the set of points $\mathcal{S}_{\{\mathbf{a},\mathbf{a}'\}}$ is defined as follows:
- for each $\ell \in [\delta]$ we add a point $S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}$ which is located at $\mathbf{a} \oplus \mathbf{e}_i \cdot ((1-\epsilon)2r + \epsilon\ell)$.

This set of points are referred to as *secondary* points.

Note that we add at most $|\mathcal{V}| \cdot 2d$ many border points, at most $|\mathcal{C}|$ many core points, and at most $|\mathcal{V}|^2 \cdot \delta$ many secondary points. Hence, the total number of points n in the instance $\mathcal{U}$ is $\leq |\mathcal{V}| \cdot 2d + |\mathcal{C}| + |\mathcal{V}|^2 \cdot \delta = |\mathcal{I}|^{O(1)}$ where $|\mathcal{I}| = |\mathcal{V}| + |\mathcal{D}| + |\mathcal{C}|$. We now prove some preliminary lemmas to be later used in Section 2.4 and Section 2.3.

2.2.1 Preliminary lemmas

▶ **Lemma 7.** *For each $\mathbf{a} \in \mathcal{V}$ and $i \in [d]$, we have* $\mathtt{dist}\left(B_{\mathbf{a}}^{+i}, B_{\mathbf{a}}^{-i}\right) \geq 2r(1+\epsilon)$.

[6] For simplicity of presentation, we choose $r = 1/4$ instead of $r = 1$: by scaling the result holds for $r = 1$.

Table 2 Notation for some special subsets of points from $\mathcal{U}$. Note that a primary point is either a core point or a border point.

$$\text{For each } \mathbf{a} \in \mathcal{V}, \text{ let } \mathcal{D}[\mathbf{a}] := \textsc{Core}[\mathbf{a}] \bigcup \textsc{Border}[\mathbf{a}]. \tag{3}$$

$$\text{The set of primary points is } \textsc{Primary} := \bigcup_{\mathbf{a} \in \mathcal{V}} \mathcal{D}[\mathbf{a}]. \tag{4}$$

$$\text{The set of secondary points is } \textsc{Secondary} := \bigcup_{\mathbf{a} \text{ \& } \mathbf{a}' \text{ forms an edge in } G_{\mathcal{I}}} \mathcal{S}_{\{\mathbf{a},\mathbf{a}'\}}. \tag{5}$$

$$\text{The final collection of points is } \mathcal{U} := \textsc{Primary} \bigcup \textsc{Secondary}. \tag{6}$$

Proof. Fix any $\mathbf{a} \in \mathcal{V}$ and $i \in [d]$. By Table 1, the points $B_{\mathbf{a}}^{+i}$ and $B_{\mathbf{a}}^{-i}$ are located at $\mathbf{a} \oplus \mathbf{e}_i \cdot r(1-\epsilon) \oplus (\mathbf{1}^d - \mathbf{e}_i) \cdot 2\epsilon\delta$ and $\mathbf{a} \ominus \mathbf{e}_i \cdot r(1-\epsilon) \ominus (\mathbf{1}^d - \mathbf{e}_i) \cdot 2\epsilon\delta$ respectively. Hence, we have that

$$\begin{aligned}
\texttt{dist}\left(B_{\mathbf{a}}^{+i}, B_{\mathbf{a}}^{-i}\right)^2 &= (2r(1-\epsilon))^2 + (d-1)\cdot(4\epsilon\delta)^2 = (2r(1-\epsilon))^2 + 16\epsilon\cdot(d-1)\epsilon\delta^2, \\
&= (2r(1-\epsilon))^2 + 16\epsilon\cdot r^2, && \text{(by definition of } \epsilon \text{ in Equation 1)} \\
&= (2r)^2[(1-\epsilon)^2 + 4\epsilon] = (2r(1+\epsilon))^2.
\end{aligned}$$

◀

▶ **Lemma 8.** *For each $\mathbf{a} \in \mathcal{V}$, the distance between any two points in $\textsc{Core}[\mathbf{a}]$ is $< r$.*

Proof. Fix any $\mathbf{a} \in \mathcal{V}$. Consider any two points in $\textsc{Core}[\mathbf{a}]$, say $C_{\mathbf{a}}^{\mathbf{x}}$ and $C_{\mathbf{a}}^{\mathbf{y}}$, for some $\mathbf{x} \neq \mathbf{y}$. By Table 1, these points are located at $\mathbf{a} \oplus \epsilon \cdot \mathbf{x}$ and $\mathbf{a} \oplus \epsilon \cdot \mathbf{y}$ respectively. Hence, we have

$$\begin{aligned}
\texttt{dist}\left(C_{\mathbf{a}}^{\mathbf{x}}, C_{\mathbf{a}}^{\mathbf{y}}\right)^2 &= (\epsilon \cdot \texttt{dist}(\mathbf{x},\mathbf{y}))^2, \\
&\leq \epsilon^2 \cdot d \cdot (\delta-1)^2, && \text{(since } \mathbf{x},\mathbf{y} \in R_{\mathbf{a}} \subseteq [\delta]^d) \\
&= \frac{d(\delta-1)^2}{(d-1)^2\delta^4} \cdot r^4, && \text{(by definition of } \epsilon \text{ in Equation 1)} \\
&\leq \frac{1}{8} \cdot r^4 < r. && \text{(since } d \geq 2 \text{ and } \delta \geq 1)
\end{aligned}$$

◀

▶ **Lemma 9.** *For each $\mathbf{a} \in \mathcal{V}$, the distance of any point from $\textsc{Core}[\mathbf{a}]$ to any point from $\textsc{Border}[\mathbf{a}]$ is $< 2r$.*

Proof. Fix any $\mathbf{a} \in \mathcal{V}$ and consider any point $C_{\mathbf{a}}^{\mathbf{x}} \in \textsc{Core}[\mathbf{a}]$ where $\mathbf{x} \in R_{\mathbf{a}} \subseteq [\delta]^d$. We prove this lemma by showing that, for each $i \in [d]$, the point $C_{\mathbf{a}}^{\mathbf{x}}$ is at distance $< 2r$ from both the points $B_{\mathbf{a}}^{+i}$ and $B_{\mathbf{a}}^{-i}$. Fix some $i \in [d]$.

(i) By Table 1, the points $C_{\mathbf{a}}^{\mathbf{x}}$ and $B_{\mathbf{a}}^{+i}$ are located at $\mathbf{a} \oplus \epsilon \cdot \mathbf{x}$ and $\mathbf{a} \oplus \mathbf{e}_i \cdot r(1-\epsilon) \oplus (\mathbf{1}^d - \mathbf{e}_i) \cdot 2\epsilon\delta$ respectively. Hence, we have

$$\begin{aligned}
\texttt{dist}\left(C_{\mathbf{a}}^{\mathbf{x}}, B_{\mathbf{a}}^{+i}\right)^2 &= (r(1-\epsilon) - \epsilon \cdot \mathbf{x}[i])^2 + \sum_{j=1:\, j \neq i}^{d} (2\epsilon\delta - \epsilon \cdot \mathbf{x}[j])^2, \\
&\leq (r(1-\epsilon))^2 + (d-1)(2\epsilon\delta)^2, && \text{(since } \mathbf{x}[i], \mathbf{x}[j] \geq 1) \\
&= (r(1-\epsilon))^2 + 4\epsilon r^2, && \text{(by definition of } \epsilon \text{ in Equation 1)} \\
&= (r(1+\epsilon))^2 < (2r)^2. && \text{(since } \epsilon < 1)
\end{aligned}$$

(ii) By Table 1, the points $C_{\mathbf{a}}^{\mathbf{x}}$ and $B_{\mathbf{a}}^{-i}$ are located at $\mathbf{a}\oplus\epsilon\cdot\mathbf{x}$ and $\mathbf{a}\ominus\mathbf{e}_i\cdot r(1-\epsilon)\ominus(\mathbf{1}^d-\mathbf{e}_i)\cdot 2\epsilon\delta$ respectively. Hence, we have

$$\begin{aligned}
\texttt{dist}\left(C_{\mathbf{a}}^{\mathbf{x}}, B_{\mathbf{a}}^{-i}\right)^2 &= (r(1-\epsilon)+\epsilon\cdot\mathbf{x}[i])^2 + \sum_{j=1:\ j\neq i}^{d}(\epsilon\cdot\mathbf{x}[j]+2\epsilon\delta)^2, \\
&\le (r(1-\epsilon)+\epsilon\delta)^2 + (d-1)(3\epsilon\delta)^2, && (\text{since } \mathbf{x}[i],\mathbf{x}[j]\le\delta) \\
&= (r(1-\epsilon)+\epsilon\delta)^2 + 9\epsilon r^2, && (\text{by definition of } \epsilon) \\
&\le 2r^2(1-\epsilon)^2 + 2\epsilon^2\delta^2 + 9\epsilon r^2, && (\text{since } (\alpha+\beta)^2\le 2\alpha^2+2\beta^2) \\
&\le 2r^2(1-\epsilon)^2 + 11\epsilon r^2, && (\text{since } \epsilon\delta^2\le r^2) \\
&= 2r^2((1-\epsilon)^2+5.5\epsilon) < 2r^2(1+1.75\epsilon)^2 < (2r)^2. && (\text{since } \epsilon\le 1/16)
\end{aligned}$$

◀

▶ **Lemma 10.** *For each $\mathbf{a}\in\mathcal{V}$, the distance of $\mathbf{a}$ to any point in* Border$[\mathbf{a}]$ *is $r(1+\epsilon)$.*

Proof. Let p be any point in Border$[\mathbf{a}]$. Then we have two choices for p, namely $p = B_{\mathbf{a}}^{+i}$ or $p = B_{\mathbf{a}}^{-i}$. In both cases, we have

$$\texttt{dist}(p,\mathbf{a})^2 = (r(1-\epsilon))^2 + (d-1)(2\epsilon\delta)^2 = r^2(1-\epsilon)^2 + 4\epsilon r^2 = (r(1+\epsilon))^2,$$

where the second equality is obtained by the definition of ϵ (Equation 1). ◀

▶ **Lemma 11.** *For each $\mathbf{a}\in\mathcal{V}$ and each $i\in[d]$,*
- *If $w\in\mathcal{U}$ such that $\texttt{dist}\left(w, B_{\mathbf{a}}^{+i}\right) < 2r(1+\epsilon)$ then $w\in\left(\mathcal{D}[\mathbf{a}]\ \bigcup\ \mathcal{S}_{\{\mathbf{a},\mathbf{a}\oplus\mathbf{e}_i\}}\right)$.*
- *If $w\in\mathcal{U}$ such that $\texttt{dist}\left(w, B_{\mathbf{a}}^{-i}\right) < 2r(1+\epsilon)$ then $w\in\left(\mathcal{D}[\mathbf{a}]\ \bigcup\ \mathcal{S}_{\{\mathbf{a},\mathbf{a}\ominus\mathbf{e}_i\}}\right)$.*

Proof. The proof of this lemma is deferred to the full version [7]. ◀

▶ Remark 12. Lemma 11 gives a necessary but not sufficient condition. Also, it might be the case that for some $\mathbf{a}\in\mathcal{V}$ and $i\in[d]$ the vector $\mathbf{a}\oplus\mathbf{e}_i\notin\mathcal{V}$ (resp., $\mathbf{a}\ominus\mathbf{e}_i\notin\mathcal{V}$) in which case the set $\mathcal{S}_{\{\mathbf{a},\mathbf{a}\oplus\mathbf{e}_i\}}$ (resp., $\mathcal{S}_{\{\mathbf{a},\mathbf{a}\ominus\mathbf{e}_i\}}$) is empty.

▶ **Lemma 13.** *Let $\mathbf{a}\in\mathcal{V}$ and $i\in[d]$ be such that $\mathbf{a}' := (\mathbf{a}\oplus\mathbf{e}_i)\in\mathcal{V}$. For each $\ell\in[\delta]$,*
(1) *If $\mathbf{x}\in R_{\mathbf{a}}$ and $\ell\le\mathbf{x}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}}^{\mathbf{x}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right) < 2r$.*
(2) *If $\mathbf{x}\in R_{\mathbf{a}}$ and $\ell>\mathbf{x}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}}^{\mathbf{x}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right) \ge 2r(1+\epsilon)$.*
(3) *If $\mathbf{y}\in R_{\mathbf{a}'}$ and $\ell>\mathbf{y}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}'}^{\mathbf{y}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right) < 2r$.*
(4) *If $\mathbf{y}\in R_{\mathbf{a}'}$ and $\ell\le\mathbf{y}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}'}^{\mathbf{y}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right) \ge 2r(1+\epsilon)$.*

Proof. Recall from Table 1 that the points $C_{\mathbf{a}}^{\mathbf{x}}$ and $S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}$ are located at $\mathbf{a}\oplus\epsilon\cdot\mathbf{x}$ and $\mathbf{a}\oplus\mathbf{e}_i\cdot((1-\epsilon)2r+\epsilon\ell)$ respectively.
(1) If $\ell\le\mathbf{x}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}}^{\mathbf{x}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right)^2$

$$\begin{aligned}
&= (2r(1-\epsilon)+\epsilon(\ell-\mathbf{x}[i]))^2 + \sum_{j=1:\ j\neq i}^{d}(\epsilon\cdot\mathbf{x}[j])^2, \\
&\le (2r(1-\epsilon))^2 + (d-1)\epsilon^2\delta^2 = (2r(1-\epsilon))^2 + \epsilon r^2 && (\text{since } \ell\le\mathbf{x}[i] \text{ and } \mathbf{x}[j]\le\delta) \\
&= (2r)^2\left((1-\epsilon)^2+\frac{\epsilon}{4}\right) < (2r)^2. && (\text{since } 0<\epsilon<1)
\end{aligned}$$

(2) If $\ell > \mathbf{x}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}}^{\mathbf{x}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right)^2$

$$
\begin{aligned}
&= (2r(1-\epsilon) + \epsilon(\ell - \mathbf{x}[i]))^2 + \sum_{j=1:\ j\neq i}^{d} (\epsilon \cdot \mathbf{x}[j])^2, \\
&\geq (2r(1-\epsilon) + \epsilon)^2 = (2r(1-\epsilon) + 4r\epsilon)^2 = (2r(1+\epsilon))^2. \quad \text{(since } \ell > \mathbf{x}[i] \text{ and } 4r = 1\text{)}
\end{aligned}
$$

We now show the remaining two claims: recall from Table 1 that the points $C_{\mathbf{a}'}^{\mathbf{y}}$ and $S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}$ are located at $(\mathbf{a}' \oplus \epsilon \cdot \mathbf{y}) = \mathbf{a} \oplus \mathbf{e}_i \oplus \epsilon \cdot \mathbf{y}$ and $\mathbf{a} \oplus \mathbf{e}_i \cdot ((1-\epsilon)2r + \epsilon\ell)$ respectively.

(3) If $\ell > \mathbf{y}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}'}^{\mathbf{y}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right)^2$

$$
\begin{aligned}
&= (1 + \epsilon \cdot \mathbf{y}[i] - (1-\epsilon)2r - \epsilon\ell)^2 + \sum_{j=1:\ j\neq i}^{d} (\epsilon \cdot \mathbf{y}[j])^2, \\
&\leq (4r + \epsilon \cdot \mathbf{y}[i] - (1-\epsilon)2r - \epsilon\ell)^2 + (d-1)\epsilon^2\delta^2, && \text{(since } 4r = 1 \text{ and } \mathbf{y}[j] \leq \delta\text{)} \\
&= (2r(1+\epsilon) - \epsilon(\ell - \mathbf{y}[i]))^2 + \epsilon r^2, && \text{(since } (d-1)\epsilon\delta^2 = r^2\text{)} \\
&\leq (2r(1+\epsilon) - \epsilon)^2 + \epsilon r^2, && \text{(since } \ell > \mathbf{y}[i]\text{)} \\
&= (2r(1-\epsilon))^2 + \epsilon r^2, && \text{(since } 4r = 1\text{)} \\
&= (2r)^2 \left((1-\epsilon)^2 + \frac{\epsilon}{4}\right) < (2r)^2. && \text{(since } 0 < \epsilon < 1\text{)}
\end{aligned}
$$

(4) If $\ell \leq \mathbf{y}[i]$, then $\texttt{dist}\left(C_{\mathbf{a}'}^{\mathbf{y}}, S_{\{\mathbf{a},\mathbf{a}'\}}^{\ell}\right)^2$

$$
\begin{aligned}
&= (1 + \epsilon \cdot \mathbf{y}[i] - (1-\epsilon)2r - \epsilon\ell)^2 + \sum_{j=1:\ j\neq i}^{d} (\epsilon \cdot \mathbf{y}[j])^2, \\
&\geq (2r(1+\epsilon) + \epsilon(\mathbf{y}[i] - \ell))^2, && \text{(since } 4r = 1\text{)} \\
&\geq (2r(1+\epsilon))^2. && \text{(since } \mathbf{y}[i] \geq \ell\text{)}
\end{aligned}
$$

◀

▶ **Lemma 14.** *Let $\mathbf{a} \in \mathcal{V}$ and $i \in [d]$ be such that $\mathbf{a}' := (\mathbf{a} \oplus \mathbf{e}_i) \in \mathcal{V}$. If $\mathbf{a}'' \notin \{\mathbf{a}, \mathbf{a}'\}$ then the distance between any point in* $\textsc{Core}[\mathbf{a}'']$ *and any point in $\mathcal{S}_{\mathbf{a},\mathbf{a}'}$ is at least $2r(1+\epsilon)$.*

Proof. Let $\mathbf{p}$ and $\mathbf{q}$ be two arbitrary points from $\textsc{Core}[\mathbf{a}'']$ and $\mathcal{S}_{\mathbf{a},\mathbf{a}'}$, respectively. By Table 1, $\mathbf{p}$ is located at $\mathbf{a}'' \oplus \epsilon \cdot \mathbf{x}$ for some $\mathbf{x} \in R_{\mathbf{a}} \subseteq [\delta]^d$ and $\mathbf{q}$ is located at $\mathbf{a} \oplus \mathbf{e}_i \cdot ((1-\epsilon)2r + \epsilon\ell)$ for some $\ell \in [\delta]$.

Since $\mathbf{a}' = \mathbf{a} \oplus \mathbf{e}_i$ and $\mathbf{a}'' \notin \{\mathbf{a}, \mathbf{a}'\}$, we have three cases to consider:

- $\mathbf{a}''[j] = \mathbf{a}[j]$ for all $j \neq i$ and $\mathbf{a}''[i] \leq \mathbf{a}[i] - 1$: In this case, we have $\texttt{dist}(\mathbf{p}, \mathbf{q})^2$

$$
\begin{aligned}
&\geq \left((\mathbf{a}[i] + (1-\epsilon)2r + \epsilon\ell) - (\mathbf{a}''[i] + \epsilon \cdot \mathbf{x}[i])\right)^2, \\
& \qquad \text{(only considering the } i\text{-th coordinate)} \\
&= \left(\mathbf{a}[i] - \mathbf{a}''[i] + (1-\epsilon)2r + \epsilon\ell - \epsilon\mathbf{x}[i]\right)^2, \\
&\geq \left(1 + (1-\epsilon)2r + \epsilon \cdot 4r - \epsilon\delta\right)^2, && \text{(since } \mathbf{a}[i] - \mathbf{a}''[i] \geq 1,\ \ell \geq 1 = 4r \text{ and } \mathbf{x}[i] \leq \delta\text{)} \\
&> (2r(1+\epsilon))^2. && \text{(since } 1 - \epsilon\delta \geq 1 - \tfrac{1}{16} > 0\text{)}
\end{aligned}
$$

- $\mathbf{a}''[j] = \mathbf{a}[j]$ for all $j \neq i$ and $\mathbf{a}''[i] \geq \mathbf{a}[i] + 2$: In this case, we have $\mathtt{dist}(\mathbf{p}, \mathbf{q})^2$

$$
\begin{aligned}
&\geq \left((\mathbf{a}''[i] + \epsilon \cdot \mathbf{x}[i]) - (\mathbf{a}[i] + (1-\epsilon)2r + \epsilon\ell)\right)^2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(only considering the } i\text{-th coordinate)} \\
&= \left(\mathbf{a}''[i] - \mathbf{a}[i] - (1-\epsilon)2r + \epsilon \cdot \mathbf{x}[i] - \epsilon\ell\right)^2, \\
&\geq (2 - (1-\epsilon)2r + \epsilon - \epsilon\delta)^2, && \text{(since } \mathbf{a}''[i] - \mathbf{a}[i] \geq 2,\ \mathbf{x}[i] \geq 1 \text{ and } \ell \leq \delta) \\
&= (4r - (1-\epsilon)2r + 1 + \epsilon - \epsilon\delta)^2, && \text{(since } 4r = 1) \\
&> (2r(1+\epsilon))^2. && \text{(since } 1 - \epsilon\delta \geq 1 - \tfrac{1}{16} > 0)
\end{aligned}
$$

- There exists $j \neq i$ such that $\mathbf{a}''[j] \neq \mathbf{a}[j]$: In this case, we have $\mathtt{dist}(\mathbf{p}, \mathbf{q})$

$$
\begin{aligned}
&\geq |\mathbf{a}[j] - (\mathbf{a}''[j] + \epsilon \cdot \mathbf{x}[j])|, && \text{(only considering the } j\text{-th coordinate)} \\
&\geq |\mathbf{a}[j] - \mathbf{a}''[j]| - \epsilon \cdot \mathbf{x}[j], && \text{(by triangle inequality)} \\
&\geq 1 - \epsilon \cdot \delta, && \text{(since } \mathbf{a}[j] \neq \mathbf{a}''[j] \text{ and } \mathbf{x}[j] \leq \delta) \\
&\geq 2r + 2r - r^2 = 2r + 2r\left(1 - \frac{r}{2}\right), && \text{(since } 4r = 1 \text{ and } \epsilon\delta \leq r^2) \\
&> 2r(1+\epsilon). && \text{(since } 1 - \tfrac{r}{2} > \tfrac{1}{16} \geq \epsilon)
\end{aligned}
$$

◀

2.3 $\mathcal{I}$ has a satisfying assignment $\Rightarrow$ OPT for the instance $\mathcal{U}$ of $|\mathcal{V}|$-Center is $< 2r$

Suppose that the d-dimensional geometric $\geq$-CSP , $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$, has a satisfying assignment $f : \mathcal{V} \to \mathcal{D}$. Consider the set of points F given by $\left\{C_{\mathbf{a}}^{f(\mathbf{a})} : \mathbf{a} \in \mathcal{V}\right\}$. Since $f : \mathcal{V} \to \mathcal{D}$ is a satisfying assignment for $\mathcal{I}$, it follows that $f(\mathbf{a}) \in R_{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{V}$ and hence the set F is well-defined. Clearly, $|F| = |\mathcal{V}|$. We now show that

$$\mathrm{OPT}(F) := \left(\max_{u \in \mathcal{U}} \left(\min_{v \in F} \mathtt{dist}(u, v)\right)\right) < 2r.$$

This implies that OPT for the instance $\mathcal{U}$ of $|\mathcal{V}|$-Center is $< 2r$. We show $\mathrm{OPT}(F) < 2r$ by showing that $\mathtt{dist}(p, F) < 2r$ for each $p \in \mathcal{U}$. From Table 1 and Table 2, it is sufficient to consider the two cases depending on whether p is a primary point or a secondary point.

▶ **Lemma 15.** *If p is a primary point, then $\mathtt{dist}(p, F) < 2r$.*

Proof. If p is a primary point, then by Table 1 and Table 2 it follows that p is either a core point or a border point.

- ***p* is a core point**: By Table 1, $p \in \textsc{Core}[\mathbf{b}]$ for some $\mathbf{b} \in \mathcal{V}$. Then, Lemma 8 implies that $\mathtt{dist}\left(p, C_{\mathbf{b}}^{f(\mathbf{b})}\right) < r$. Since $C_{\mathbf{b}}^{f(\mathbf{b})} \in F$, we have $\mathtt{dist}(p, F) \leq \mathtt{dist}\left(p, C_{\mathbf{b}}^{f(\mathbf{b})}\right) < r$.
- ***p* is a border point**: By Table 1, $p \in \textsc{Border}[\mathbf{b}]$ for some $\mathbf{b} \in \mathcal{V}$. Then, Lemma 9 implies that $\mathtt{dist}\left(p, C_{\mathbf{b}}^{f(\mathbf{b})}\right) < 2r$. Since $C_{\mathbf{b}}^{f(\mathbf{b})} \in F$, we have $\mathtt{dist}(p, F) \leq \mathtt{dist}\left(p, C_{\mathbf{b}}^{f(\mathbf{b})}\right) < 2r$.

◀

▶ **Lemma 16.** *If p is a secondary point, then $\mathtt{dist}(p, F) < 2r$.*

Proof. If p is a secondary point, then by Table 1 and Table 2 it follows that there exists $\mathbf{a} \in \mathcal{V}, i \in [d]$ and $\ell \in [\delta]$ such that $p = S_{\{\mathbf{a}, \mathbf{a} \oplus \mathbf{e}_i\}}^{\ell}$. Note that $C_{\mathbf{a}}^{f(\mathbf{a})} \in F$ and $C_{\mathbf{a} \oplus \mathbf{e}_i}^{f(\mathbf{a} \oplus \mathbf{e}_i)} \in F$. We now prove the lemma by showing that $\min\left\{\mathtt{dist}\left(p, C_{\mathbf{a}}^{f(\mathbf{a})}\right), \mathtt{dist}\left(p, C_{\mathbf{a} \oplus \mathbf{e}_i}^{f(\mathbf{a} \oplus \mathbf{e}_i)}\right)\right\} < 2r$. Since $f : \mathcal{V} \to \mathcal{D}$ is a satisfying assignment, the binary constraint on $\mathbf{a}$ and $\mathbf{a} \oplus \mathbf{e}_i$ is satisfied, i.e., $\delta \geq f(\mathbf{a})[i] \geq f(\mathbf{a} \oplus \mathbf{e}_i)[i] \geq 1$. Since $\ell \in [\delta]$, either $\ell \leq f(\mathbf{a})[i]$ or $\ell > f(\mathbf{a} \oplus \mathbf{e}_i)[i]$. The following implications complete the proof:

- If $\ell \leq f(\mathbf{a})[i]$, then 13(1) implies that $\mathtt{dist}\left(C_{\mathbf{a}}^{f(\mathbf{a})}, p\right) < 2r$.
- If $\ell > f(\mathbf{a} \oplus \mathbf{e}_i)[i]$, then 13(3) implies that $\mathtt{dist}\left(C_{\mathbf{a}\oplus\mathbf{e}_i}^{f(\mathbf{a}\oplus\mathbf{e}_i)}, p\right) < 2r$. ◀

From Table 2, Lemma 15 and Lemma 16 it follows that OPT for the instance $\mathcal{U}$ of $|\mathcal{V}|$-Center is $< 2r$.

2.4 $\mathcal{I}$ does not have a satisfying assignment $\Rightarrow$ OPT for the instance $\mathcal{U}$ of $|\mathcal{V}|$-Center is $\geq 2r(1+\epsilon)$

Suppose that the instance $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ of d-dimensional geometric $\geq$-CSP does not have a satisfying assignment. We want to now show that OPT for the instance $\mathcal{U}$ of $|\mathcal{V}|$-Center is $\geq 2r(1+\epsilon)$. Fix any set $Q \subseteq \mathcal{U}$ of size $|\mathcal{V}|$: it is sufficient to show that

$$\mathrm{OPT}(Q) := \left(\max_{u\in\mathcal{U}} \left(\min_{v\in Q} \mathtt{dist}(u, v)\right)\right) \geq 2r(1+\epsilon). \tag{7}$$

We consider two cases: either $\left|Q \cap \mathrm{Core}[\mathbf{a}]\right| = 1$ for each $\mathbf{a} \in \mathcal{V}$ (Lemma 17) or not (Lemma 18).

▶ **Lemma 17.** *If $\left|Q \cap \mathrm{Core}[\mathbf{a}]\right| = 1$ for each $\mathbf{a} \in \mathcal{V}$ then $OPT(Q) \geq 2r(1+\epsilon)$.*

Proof. Since $|Q| = |\mathcal{V}|$ and $|Q \cap \mathrm{Core}[\mathbf{a}]| = 1$ for each $\mathbf{a} \in \mathcal{V}$ it follows that the only points in Q are core points (see Table 1 for definition) and moreover Q contains exactly one core point corresponding to each element from $\mathcal{V}$. Let $\phi : \mathcal{V} \to [\delta]^d$ be the function such that $Q \cap \mathrm{Core}[\mathbf{a}] = C_{\mathbf{a}}^{\phi(\mathbf{a})}$. By Table 1, it follows that $\phi(a) \in R_{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{V}$.

Recall that we are assuming in this section that the instance $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ of d-dimensional geometric $\geq$-CSP does not have a satisfying assignment. Hence, in particular, the function $\phi : \mathcal{V} \to [\delta]^d$ is not a satisfying assignment for $\mathcal{I}$. All unary constraints are satisfied since $\phi(a) \in R_{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{V}$. Hence, there is some binary constraint which is not satisfied by ϕ: let this constraint be violated for the pair $\mathbf{a}, \mathbf{a} \oplus \mathbf{e}_i$ for some $\mathbf{a} \in \mathcal{V}$ and $i \in [d]$. Let us denote $\mathbf{a} \oplus \mathbf{e}_i$ by $\mathbf{a}'$. The violation of the binary constraint on $\mathbf{a}$ and $\mathbf{a} \oplus \mathbf{e}_i$ by ϕ implies that $1 \leq \phi(\mathbf{a})[i] < \phi(\mathbf{a}')[i] \leq \delta$. We now show that $\mathtt{dist}\left(Q, S_{\{\mathbf{a},\mathbf{a}'\}}^{\phi(\mathbf{a}')[i]}\right) \geq (2r(1+\epsilon)$ which, in turn, implies that $\mathrm{OPT}(Q) \geq 2r(1+\epsilon)$. The following implications complete the proof.

- 13(2) implies that $\mathtt{dist}\left(S_{\{\mathbf{a},\mathbf{a}'\}}^{\phi(\mathbf{a}')[i]}, C_{\mathbf{a}}^{\phi(\mathbf{a})}\right) \geq 2r(1+\epsilon)$.
- 13(4) implies that $\mathtt{dist}\left(S_{\{\mathbf{a},\mathbf{a}'\}}^{\phi(\mathbf{a}')[i]}, C_{\mathbf{a}'}^{\phi(\mathbf{a}')}\right) \geq 2r(1+\epsilon)$.
- Consider any point $s \in Q \setminus \left\{C_{\mathbf{a}}^{\phi(\mathbf{a})}, C_{\mathbf{a}'}^{\phi(\mathbf{a}')}\right\}$. Then $s \in \mathrm{Core}[\mathbf{a}'']$ for some $\mathbf{a}'' \notin \{\mathbf{a}, \mathbf{a}'\}$. Now Lemma 14 implies that $\mathtt{dist}\left(S_{\{\mathbf{a},\mathbf{a}'\}}^{\phi(\mathbf{a}')[i]}, s\right) \geq 2r(1+\epsilon)$. ◀

▶ **Lemma 18.** *If there exists $\mathbf{a} \in \mathcal{V}$ such that $\left|Q \cap \mathrm{Core}[\mathbf{a}]\right| \neq 1$ then $OPT(Q) \geq 2r(1+\epsilon)$.*

Proof. Suppose that $\mathrm{OPT}(Q) < 2r(1+\epsilon)$. To prove the lemma, we will now show that this implies $|Q \cap \mathrm{Core}[\mathbf{a}]| = 1$ for each $\mathbf{a} \in \mathcal{V}$. This is done via the following two claims, namely Claim 19 and Claim 20.

▷ Claim 19. $\left|Q \cap \mathcal{D}[\mathbf{a}]\right| = 1$ for each $\mathbf{a} \in \mathcal{V}$.

Proof. Define three sets I_0, I_1 and $I_{\geq 2}$ as follows:

$$I_0 := \left\{\mathbf{a} \in \mathcal{V} : \left|Q \cap \mathcal{D}[\mathbf{a}]\right| = 0\right\} \tag{8}$$
$$I_1 := \left\{\mathbf{a} \in \mathcal{V} : \left|Q \cap \mathcal{D}[\mathbf{a}]\right| = 1\right\} \tag{9}$$
$$I_{\geq 2} := \left\{\mathbf{a} \in \mathcal{V} : \left|Q \cap \mathcal{D}[\mathbf{a}]\right| \geq 2\right\} \tag{10}$$

By definition, we have

$$|I_0| + |I_1| + |I_{\geq 2}| = |\mathcal{V}| \tag{11}$$

Consider a variable $\mathbf{b} \in I_0$. Since $\texttt{dist}\left(Q, B_{\mathbf{b}}^{+i}\right)$ and $\texttt{dist}\left(Q, B_{\mathbf{b}}^{-i}\right) < 2r(1+\epsilon)$, and $Q \cap \mathcal{D}[\mathbf{b}] = \emptyset$, Lemma 11 implies that for each $i \in [d]$,

(i) Q must contain a point from $\mathcal{S}_{\{\mathbf{b}, \mathbf{b} \oplus \mathbf{e}_i\}}$, and

(ii) Q must contain a point from $\mathcal{S}_{\{\mathbf{b}, \mathbf{b} \ominus \mathbf{e}_i\}}$.

Since each secondary point can be "charged" to two variables in $\mathcal{V}$ (for example, the set $\mathcal{S}_{\{\mathbf{b}, \mathbf{b} \oplus \mathbf{e}_i\}}$ corresponds to both $\mathbf{b}$ and $\mathbf{b} \oplus \mathbf{e}_i$), it follows that Q contains $\geq \frac{2d}{2} = d \geq 2$ *distinct* secondary points corresponding to each variable in I_0. Therefore, we have

$$\begin{aligned}
|I_0| + |I_1| + |I_{\geq 2}| = |\mathcal{V}| = |Q|\,, & \qquad \text{(from Equation 11)}\\
\geq |Q \cap \textsc{Primary}| + |Q \cap \textsc{Secondary}|\,, & \\
& \qquad (\text{since } \textsc{Primary} \cap \textsc{Secondary} = \emptyset)\\
\geq (|I_1| + 2|I_{\geq 2}|) + |Q \cap \textsc{Secondary}|\,, & \qquad (\text{by definition of } I_1 \text{ and } I_{\geq 2})\\
\geq (|I_1| + 2|I_{\geq 2}|) + 2|I_0|, & \qquad (12)
\end{aligned}$$

where the last inequality follows because Q contains at least 2 secondary points corresponding to each variable in I_0. Hence, we have $|I_0| + |I_1| + |I_{\geq 2}| \geq 2|I_0| + |I_1| + 2|I_{\geq 2}|$ which implies $|I_0| = 0 = |I_{\geq 2}|$. From Equation 11, we get $|I_1| = |\mathcal{V}|$, i.e., $\left|Q \cap \mathcal{D}[\mathbf{a}]\right| = 1$ for each $\mathbf{a} \in \mathcal{V}$. This concludes the proof of Claim 19. ◁

Since $|Q| = \left|\mathcal{V}\right|$ and $\mathcal{D}[\mathbf{a}] \cap \mathcal{D}[\mathbf{b}] = \emptyset$ for distinct $\mathbf{a}, \mathbf{b} \in \mathcal{V}$, Claim 19 implies that

$$Q \text{ contains no secondary points.} \tag{13}$$

We now prove that Q doesn't contain border points either.

▷ Claim 20. $\left|Q \cap \textsc{Core}[\mathbf{a}]\right| = 1$ for each $\mathbf{a} \in \mathcal{V}$

Proof. Fix any $\mathbf{a} \in \mathcal{V}$. From Claim 19, we know that $\left|Q \cap \mathcal{D}[\mathbf{a}]\right| = 1$. Suppose that this unique point in $Q \cap \mathcal{D}[\mathbf{a}]$ is from $\textsc{Border}[\mathbf{a}]$. Without loss of generality, let $Q \cap \mathcal{D}[\mathbf{a}] = \left\{B_{\mathbf{a}}^{+i}\right\}$ for some $i \in [d]$. Since $\text{OPT}(Q) < 2r(1+\epsilon)$, it follows that $\texttt{dist}\left(Q, B_{\mathbf{a}}^{-i}\right) < 2r(1+\epsilon)$. Hence, Lemma 11(2) implies that $Q \cap \left(\mathcal{D}[\mathbf{a}] \bigcup \mathcal{S}_{\{\mathbf{a}, \mathbf{a} \ominus \mathbf{e}_i\}}\right) \neq \emptyset$. Since Q contains no secondary points (Equation 13), we have $Q \cap \left(\mathcal{D}[\mathbf{a}] \bigcup \mathcal{S}_{\{\mathbf{a}, \mathbf{a} \ominus \mathbf{e}_i\}}\right) = Q \cap \mathcal{D}[\mathbf{a}] = \left\{B_{\mathbf{a}}^{+i}\right\}$. But from Lemma 7 we know $\texttt{dist}\left(B_{\mathbf{a}}^{+i}, B_{\mathbf{a}}^{-i}\right) \geq 2r(1+\epsilon)$. We thus obtain a contradiction. This concludes the proof of Claim 20. ◁

Therefore, we have shown that $\text{OPT}(Q) < 2r(1+\epsilon)$ implies $\left|Q \cap \textsc{Core}[\mathbf{a}]\right| = 1$ for each $\mathbf{a} \in \mathcal{V}$. This concludes the proof of Lemma 18. ◀

2.5 Finishing the proof of Theorem 1

Finally, we are ready to prove Theorem 1 which is restated below.

▶ **Theorem 1.** *For any $d \geq 2$, under the Exponential Time Hypothesis (ETH), the discrete k-Center problem in d-dimensional Euclidean space*

- ***(Inapproximability result)*** *does not admit an $(1+\epsilon)$-approximation in $f(k) \cdot \left(\frac{1}{\epsilon}\right)^{o\left(k^{1-1/d}\right)} \cdot n^{o\left(k^{1-1/d}\right)}$ time where f is any computable function and n is the number of points.*
- ***(Lower bound for exact algorithm)*** *cannot be solved in $f(k) \cdot n^{o\left(k^{1-1/d}\right)}$ time where f is any computable function and n is the number of points.*

Proof. Given an instance $\mathcal{I} = (\mathcal{V}, \mathcal{D}, \mathcal{C})$ of a d-dimensional geometric $\geq$-CSP, we build an instance $\mathcal{U}$ of $|\mathcal{V}|$-CENTER in $\mathbb{R}^d$ given by the reduction in Section 2.2. This reduction has the property that

- if $\mathcal{I}$ does not have a satisfying assignment then OPT for the instance $\mathcal{U}$ of $|\mathcal{V}|$-CENTER is $\geq 2r(1+\epsilon^*)$ (Section 2.4), and
- if $\mathcal{I}$ has a satisfying assignment then OPT for the instance $\mathcal{U}$ of $|\mathcal{V}|$-CENTER is $< 2r$ (Section 2.3),

where $r = 1/4$ and $\epsilon^* = \dfrac{r^2}{(d-1)\delta^2} \geq \dfrac{1}{16(d-1)|\mathcal{D}|}$, since $|\mathcal{D}| = \left|[\delta]^d\right| \geq \delta^2$. Hence, any algorithm for the $|\mathcal{V}|$-center problem which has an approximation factor $\leq (1+\epsilon^*)$ can solve the d-dimensional geometric $\geq$-CSP. Note that the instance $\mathcal{U}$ of k-CENTER in $\mathbb{R}^d$ has $k = |\mathcal{V}|$ and the number of points $n \leq |\mathcal{V}| \cdot 2d + |\mathcal{C}| + |\mathcal{V}|^2 \cdot \delta = |\mathcal{I}|^{O(1)}$ where $|\mathcal{I}| = |\mathcal{V}| + |\mathcal{D}| + |\mathcal{C}|$. We now derive the two lower bounds claimed in the theorem.

- **(Inapproximability result)** Suppose that there exists $d \geq 2$ such that the k-center on n points in $\mathbb{R}^d$ admits an $(1+\epsilon)$-approximation algorithm in $f(k) \cdot \left(\dfrac{1}{\epsilon}\right)^{o\left(k^{1-1/d}\right)} \cdot n^{o\left(k^{1-1/d}\right)}$ time for some computable function f. As argued above, using a $(1+\epsilon^*)$-approximation for the k-center problem with $k = |\mathcal{V}|$ and $n = |\mathcal{I}|^{O(1)}$ points can solve the d-dimensional geometric $\geq$-CSP problem. Recall that $16(d-1)|\mathcal{I}| \geq 16(d-1)|\mathcal{D}| \geq \dfrac{1}{\epsilon^*}$ since $|I| = |\mathcal{V}| + |\mathcal{D}| + |\mathcal{C}|$, and hence we have an algorithm for the d-dimensional geometric $\geq$-CSP problem which runs in time $f\left(|\mathcal{V}|\right) \cdot (16d)^{o\left(k^{1-1/d}\right)} \cdot |\mathcal{I}|^{o\left(k^{1-1/d}\right)}$ which contradicts Theorem 5.
- **(Lower bound for exact algorithm)** Suppose that there exists $d \geq 2$ such that the k-center on n points in $\mathbb{R}^d$ admits an exact algorithm in $f(k) \cdot n^{o\left(k^{1-1/d}\right)}$ time for some computable function f. As argued above[7], solving the k center problem with $k = |\mathcal{V}|$ and $n = |\mathcal{I}|^{O(1)}$ points can solve the d-dimensional geometric $\geq$-CSP problem. Hence, we have an algorithm for the d-dimensional geometric $\geq$-CSP problem which runs in time $f\left(|\mathcal{V}|\right) \cdot |\mathcal{I}|^{o\left(k^{1-1/d}\right)}$ which again contradicts Theorem 5. ◀

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