Erdős–Szekeres-Type Problems in the Real Projective Plane

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Abstract

We consider point sets in the real projective plane $\mathbb{RP}^2$ and explore variants of classical extremal problems about planar point sets in this setting, with a main focus on Erdős–Szekeres-type problems.

We provide asymptotically tight bounds for a variant of the Erdős–Szekeres theorem about point sets in convex position in $\mathbb{RP}^2$, which was initiated by Harborth and Möller in 1994. The notion of convex position in $\mathbb{RP}^2$ agrees with the definition of convex sets introduced by Steinitz in 1913.

For $k \geq 3$, an *(affine) k-hole* in a finite set $S \subseteq \mathbb{R}^2$ is a set of k points from S in convex position with no point of S in the interior of their convex hull. After introducing a new notion of k-holes for points sets from $\mathbb{RP}^2$, called *projective k-holes*, we find arbitrarily large finite sets of points from $\mathbb{RP}^2$ with no projective 8-holes, providing an analogue of a classical result by Horton from 1983. We also prove that they contain only quadratically many projective k-holes for $k \leq 7$. On the other hand, we show that the number of k-holes can be substantially larger in $\mathbb{RP}^2$ than in $\mathbb{R}^2$ by constructing, for every $k \in \{3, \dots, 6\}$, sets of n points from $\mathbb{R}^2 \subset \mathbb{RP}^2$ with $\Omega(n^{3-3/5k})$ projective k-holes and only $O(n^2)$ affine k-holes. Last but not least, we prove several other results, for example about projective holes in random point sets in $\mathbb{RP}^2$ and about some algorithmic aspects.

The study of extremal problems about point sets in $\mathbb{RP}^2$ opens a new area of research, which we support by posing several open problems.

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1 Introduction

1.1 Erdős-Szekeres-type results in the Euclidean plane

Throughout the whole paper, we consider each set S of points from the Euclidean plane $\mathbb{R}^2$ to be finite and in *general position*, that is, no three points of S lie on a common line. We say that a set S of k points in the Euclidean plane is in *convex position* if S forms the vertex set of a convex polygon, which we call a *k-gon* or an *affine k-gon*.

In 1935, Erdős and Szekeres [16] showed that, for every integer $k \geq 3$, there is a smallest positive integer $ES(k)$ such that every finite set of at least $ES(k)$ points in the plane in general position contains a subset of k points in convex position. This result, known as the *Erdős–Szekeres theorem*, was one of the starting points of both discrete geometry and Ramsey theory. It motivated various lines of research that led to several important results as well as to many difficult open problems. For example, there were many efforts to determine the growth rate of the function $ES(k)$. Erdős and Szekeres [16] showed $ES(k) \leq \binom{2k-4}{k-2} + 1$ and conjectured that $ES(k) = 2^{k-2} + 1$ for every $k \geq 2$. This conjecture, known as the *Erdős–Szekeres conjecture*, was later supported by Erdős and Szekeres [17], who proved the matching lower bound $ES(k) \geq 2^{k-2} + 1$. The Erdős–Szekeres conjecture was verified for $k \leq 6$ [37] (see also [29, 33]), but is still open for $k \geq 7$. In fact, Erdős even offered \$500 reward for its solution. The currently best upper bound $ES(k) \leq 2^{k+O(\sqrt{k \log k})}$ is due to Holmsen, Mojarrad, Pach, and Tardos [25], who improved an earlier breakthrough by Suk [36] who showed $ES(k) \leq 2^{k+O(k^{2/3} \log k)}$. Altogether, these estimates give, for every $k \geq 2$,

$$2^{k-2} + 1 \leq ES(k) \leq 2^{k+O(\sqrt{k \log k})}. \tag{1}$$

Several variations of the Erdős–Szekeres theorem have been studied in the literature. In the 1970s, Erdős [15] asked whether there is a smallest positive integer $h(k)$ such that every set S of at least $h(k)$ points in the plane in general position contains an *(affine) k-hole*, which is a convex polygon spanned by a subset of k points from S that does not contain any point from S in its interior. In other words, a k-hole in a finite points set S in the plane in general position is a k-gon which is *empty* in S, that is, its interior does not contain any point from S. After Horton [26] constructed arbitrarily large point sets with no 7-hole, it took more than 20 years until Gerken [21] and Nicolas [31] independently showed that every sufficiently large set of points contains a 6-hole. Therefore, $h(k)$ is finite if and only if $k \leq 6$.

Estimating the minimum number of k-holes is another example of a classical Erdős–Szekeres-type problem. For a fixed integer $k \geq 3$ and a positive integer n, let $h_k(n)$ be the minimum number of k-holes in any finite set of n points in the plane. The growth rate of the function $h_k(n)$ was also studied extensively. Horton's result implies $h_k(n) = 0$ for $k \geq 7$. The minimum numbers of 3- and 4-holes are known to be quadratic in n, but we only have the bounds $\Omega(n \log^{4/5} n) \leq h_5(n) \leq O(n^2)$ and $\Omega(n) \leq h_6(n) \leq O(n^2)$ [3, 9] for 5- and 6-holes, respectively. However, it is widely conjectured that h_5 and h_6 are also both quadratic in n.

In this paper, we consider analogous Erdős–Szekeres-type problems in the real projective plane $\mathbb{R}\mathcal{P}^2$. We define notions of convex position, k-gons, and k-holes in $\mathbb{R}\mathcal{P}^2$ and study the corresponding extremal problems, providing several new results as well as numerous open problems in this new line of research.

1.2 Convex sets in the real projective plane

As in the planar case, we consider only sets P of points from the real projective plane $\mathbb{R}\mathcal{P}^2$ that are finite and in *general position*, that is, no three points from P lie on a common projective line. We say that P is in *projective convex position* if it is a set in convex position in some Euclidean plane $\rho \subset \mathbb{RP}^2$. Recall that by removing a projective line from $\mathbb{RP}^2$ one obtains a Euclidean plane. Following the notation introduced by Steinitz [35], we say that a subset X of $\mathbb{RP}^2$ is *semiconvex* if any two points of X can be joined by a line segment fully contained in X. The set X is *convex* if it is semiconvex and does not contain some projective line, that is, X is contained in a plane $\rho \subset \mathbb{RP}^2$; see also [13]. A *projective convex hull* of a set $Y \subset \mathbb{RP}^2$ is an inclusion-wise minimal convex subset of $\mathbb{RP}^2$ containing Y. We note that, unlike the situation in the plane, a projective convex hull of Y does not have to be determined uniquely; see Figure 1.

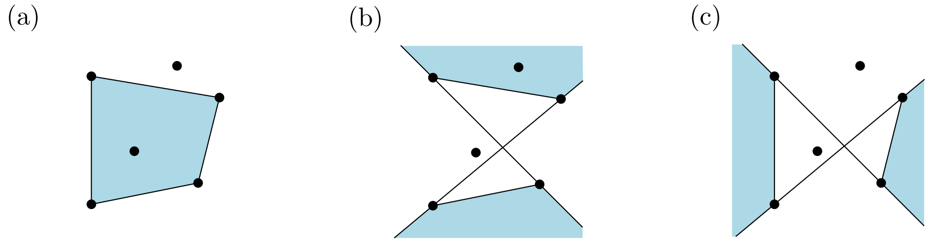

Figure 1 An example of three projective 4-gons determined by the same subset of four points from a set P of six points in $\mathbb{RP}^2$. The projective 4-gons in (a) and (b) are not projective 4-holes in P, but the projective 4-gon in (c) is a projective 4-hole in P.

▶ **Definition 1** (A projective k-gon). *For a positive integer k and a finite set P of points from $\mathbb{RP}^2$ in general position, a* projective k-gon *determined by P is a projective convex hull of a set I of k points from P which contains all points of I on its boundary; see Figure 1.*

The notion "projective k-gon" in $\mathbb{RP}^2$ is a natural analogue of the notion "affine k-gon" in $\mathbb{R}^2$, since projective k-gons in $\mathbb{RP}^2$ are exactly those subsets of $\mathbb{RP}^2$ which are convex k-gons in some of the planes contained in $\mathbb{RP}^2$.

Since a projective convex hull is not determined uniquely, a set of k points in $\mathbb{RP}^2$ can determine several projective k-gons. In particular, it is not difficult to verify that

(i) any three points in general position in $\mathbb{RP}^2$ determine four projective 3-gons,
(ii) any four points in general position in $\mathbb{RP}^2$ determine three projective 4-gons,
(iii) any five points in general position in $\mathbb{RP}^2$ determine exactly one projective 5-gon, and
(iv) any $k \geq 6$ points in general position in $\mathbb{RP}^2$ determine at most one projective k-gon.

We also introduce the following natural analogue of holes in the real projective plane.

▶ **Definition 2** (A projective k-hole). *For an integer $k \geq 3$ and a finite set P of points from $\mathbb{RP}^2$ in general position, a* projective k-hole *in P is a projective k-gon determined by points from P that does not contain any point from P in its interior; see Figure 1.*

The notion of a "projective k-hole" in $\mathbb{RP}^2$ is a natural analogue of the notion of an "(affine) k-hole" in $\mathbb{R}^2$, since projective k-holes in $\mathbb{RP}^2$ are exactly those subsets of $\mathbb{RP}^2$ which are (affine) k-holes in some of the planes contained in $\mathbb{RP}^2$.

We note that, again, a single set of $k \in \{3, 4\}$ points in general position in $\mathbb{RP}^2$ can determine several different projective k-holes. Also note that, if H is a projective k-hole in a finite set P of points from $\mathbb{RP}^2$ in general position, then in every affine plane $\rho \subset \mathbb{RP}^2$ containing H, the set H is an affine k-hole. A subset of $\mathbb{RP}^2$ is a *projective hole* in P if it is a projective k-hole in P for some integer $k \geq 3$.

We also describe the following alternative view on projective k-gons and k-holes via planar point sets. A *double chain* [27] is a set $S = A \cup B$ of k points from $\mathbb{R}^2$ with $A = \{s_1, \ldots, s_m\}$ and $B = \{s_{m+1}, \ldots, s_k\}$ for some m with $1 \leq m \leq k-1$ such that, for every $i = 1, \ldots, k$, the line $\overline{s_i s_{i+1}}$ separates $A \setminus \{s_i, s_{i+1}\}$ from $B \setminus \{s_i, s_{i+1}\}$ (indices modulo k); see Figure 2. The sets A and B are the *chains* of the double chain. For a line ℓ not separating A, let H_ℓ^A be the closed half-plane bounded by ℓ that contains A and we similarly define H_ℓ^B. The *double chain k-wedge* of S is the union $W_A \cup W_B$ where $W_A = \bigcap_{i=0}^{m} H^A_{\overline{s_i s_{i+1}}}$ and $W_B = \bigcap_{i=m}^{k} H^B_{\overline{s_i s_{i+1}}}$.

▶ **Observation 3.** *Let P be a set of k points from $\mathbb{RP}^2$ in general position and let $\rho \subset \mathbb{RP}^2$ be an affine plane containing P. A convex set G in $\mathbb{RP}^2$ is a projective k-gon determined by P if and only if, in ρ, G is either a convex polygon with k vertices (that is, an affine k-gon) or a double chain k-wedge.* ◀

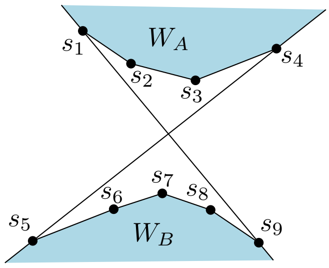

Figure 2 A double chain S on 9 points and the corresponding double chain 9-wedge.

▶ **Observation 4.** *Let P be a set of k points from $\mathbb{RP}^2$ in general position and let $\rho \subset \mathbb{RP}^2$ be an affine plane containing P. A convex set H in $\mathbb{RP}^2$ is a projective k-hole in P if and only if, in ρ, H is either a convex polygon with k vertices that is empty in P (that is, an affine k-hole) or a double chain k-wedge that is empty in P.* ◀

Convex sets in the real projective plane were considered by many authors [10, 13, 14, 23, 28] and their study goes back more than 100 years to Steinitz [35]. Besides the article of Harborth and Möller [24], which introduced the notion of projective k-gons, we are not aware of any further literature on projective k-gons or projective k-holes. Thus, our goal is to conduct a first extensive study of extremal properties of point sets in $\mathbb{RP}^2$.

2 Our results

First, we consider an analogue of the Erdős–Szekeres theorem in the real projective plane. For an integer $k \geq 2$, let $ES^p(k)$ be the minimum positive integer N such that every set of at least N points in $\mathbb{RP}^2$ in general position contains k points in projective convex position. Interestingly, due to Observation 3, $ES^p(k)$ equals the minimum positive integer such that every set of at least $ES^p(K)$ points in $\mathbb{R}^2$ in general position contains either k points in convex position or a double chain of size k. As already noted in [24], one immediately gets $ES^p(k) \leq ES(k)$. On the other hand, $ES^p(k) \geq ES(\lceil k/2 \rceil)$, since the largest chain of a double chain of size k has at least $\lceil k/2 \rceil$ points. Thus, by (1), we have $2^{\lceil k/2 \rceil - 2} + 1 \leq ES^p(k) \leq 2^{k+O(\sqrt{k \log k})}$ for every $k \geq 2$ and, in particular, the numbers $ES^p(k)$ are finite. As our first result, we prove an almost matching lower bound on $ES^p(k)$.

▶ **Theorem 5.** *There are constants $c, c' > 0$ such that, for every integer $k \geq 2$,*

$$2^{k - c \log k} \leq ES^p(k) \leq 2^{k + c' \sqrt{k \log k}}.$$

The precise value of $ES^p(k)$ is known for small values of k. For $k \leq 5$, all sets of k points from $\mathbb{RP}^2$ determine a projective k-gon by properties (i)–(iii) below Definition 1 and thus $ES^p(k) = k$. Using SAT-solver-based computations, we have also verified the value

$ES^p(6) = 9$, which was determined by Harborth and Möller [24]. This value can also be verified with an exhaustive search, or by using the database of order types of planar point sets [1, 2] or the database of (acyclic) oriented matroids [19, 20]. We also found sets of 17 points from $\mathbb{RP}^2$ with no projective 7-gon, witnessing $ES^p(7) \geq 18$.

Now, we focus on extremal problems about holes in the real projective plane. As our first result, we show that the existence of projective 8-holes is not guaranteed in large point sets in $\mathbb{RP}^2$, proving an analogue of the result by Horton [26].

▶ **Theorem 6.** *For every $n \in \mathbb{N}$, there exist sets of n points from $\mathbb{RP}^2$ in general position with no projective 8-hole.*

We recall that Theorem 6 implies that there are arbitrarily large finite sets of points from $\mathbb{RP}^2$ in general position with no projective k-holes for any $k \geq 8$. The proof of Theorem 6 uses *Horton sets* defined by Valtr [38] as a generalization of a construction of Horton [26] of an arbitrarily large planar point set in general position (so-called *perfect Horton set*) with no 7-hole; see Section 5 for the definition of Horton sets. Horton sets contain no affine 7-holes in $\mathbb{R}^2$ and we actually show that, if they are embedded in $\mathbb{RP}^2$, they contain no projective 8-holes. Moreover, we show quadratic bounds on the number of projective k-holes in Horton sets for $k \leq 7$.

▶ **Theorem 7.** *Let H be a Horton set of size n in $\mathbb{R}^2 \subset \mathbb{RP}^2$. Then H has $\Theta(n^2)$ projective k-holes for every $k \leq 7$. Moreover, if H is the perfect Horton set of size $n = 2^z$, then the number of projective 3-holes in H equals*

$$4.25 \cdot 2^{2z} + 2^z(-3z^2/2 - z/2 - 5.5) - 4z + 2 = 4.25n^2 - 1.5n\log^2 n - \Theta(n \log n).$$

For positive integers $k \geq 3$ and n, let $h_k^p(n)$ be the minimum number of projective k-holes in any set of n points in $\mathbb{RP}^2$ in general position. Theorem 7 gives $h_k^p(n) \leq O(n^2)$ for every $k \leq 7$ and Theorem 6 gives $h_k^p(n) = 0$ for every $k > 7$.

In contrast to the planar case, each sufficiently large Horton set in $\mathbb{RP}^2$ contains a projective 7-hole. We do not have examples of large point sets in $\mathbb{RP}^2$ without projective 7-holes, thus it is natural to ask whether there are projective 7-holes in every sufficiently large point set in $\mathbb{RP}^2$. We believe this to be the case; see Subsection 3 for more open problems.

We also prove that every set of at least 7 points in $\mathbb{RP}^2$ contains a projective 5-hole while there are sets of 6 points in $\mathbb{RP}^2$ with no projective 5-hole. Interestingly, every set of 5 points in $\mathbb{RP}^2$ contains a projective 5-hole. This is in contrast with the situation in the plane, where we have $h_k(n) \leq h_k(n+1)$ for every k and n, which can be seen by removing a vertex of the convex hull of a set S of $n+1$ points from $\mathbb{R}^2$ with $h_k(n+1)$ affine k-holes.

▶ **Proposition 8.** *Every set of at least 7 points in general position in $\mathbb{RP}^2$ contains a projective 5-hole. Also, $h_5^p(5) = 1$ and $h_5^p(6) = 0$.*

The proof of Proposition 8 can be found in [7]. The following theorem shows that for some point sets the number of holes is substantially larger in $\mathbb{RP}^2$ than in $\mathbb{R}^2$.

▶ **Theorem 9.** *For every $k \in \{3, \ldots, 6\}$ and every positive integer n, there is a set $S_k(n)$ of n points in general position in $\mathbb{R}^2 \subset \mathbb{RP}^2$ such that $S_k(n)$ has $O(n^2)$ affine k-holes in $\mathbb{R}^2$ and $\Omega(n^{3-\frac{5}{3k}})$ projective k-holes.*

More generally, for every $k \in \{3, \ldots, 6\}$, every real number $\alpha \in [0, k-2]$, and each positive integer n, there is a set $S_k^\alpha(n)$ of n points in general position in $\mathbb{R}^2 \subset \mathbb{RP}^2$ such that $S_k^\alpha(n)$ has $O(n^{2+\alpha})$ affine k-holes in $\mathbb{R}^2$ and $\Omega(n^{2+\beta})$ projective k-holes, where

$$\beta := \begin{cases} 1 - \frac{5}{3k} + \alpha \cdot \frac{k-1}{k} & \text{if } 0 \leq \alpha \leq \frac{2k-5}{3}, \\ (1+\alpha)\frac{k-2}{k-1} & \text{if } \frac{2k-5}{3} < \alpha \leq k-2. \end{cases}$$

The following result shows a significant difference between the number of holes of all sizes in the plane and in the real projective plane.

▶ **Theorem 10.** *For any two positive integers n and x with $x \leq 2^{n/2}$, there is a set $S(n,x)$ of n points in general position in $\mathbb{R}^2 \subset \mathbb{RP}^2$ containing at most $O(x+n^2)$ affine holes in $\mathbb{R}^2$ and at least $\Omega(x^2)$ projective holes.*

In general, we can show that every set P of n points from $\mathbb{R}^2 \subset \mathbb{RP}^2$ contains at least quadratically many projective holes which are not affine holes in $\mathbb{R}^2$.

▶ **Proposition 11.** *Let P be a set of n points in $\mathbb{R}^2 \subset \mathbb{RP}^2$ in general position, and let $h_k^p(P)$ be the number of projective k-holes in P. Then,*

$$h_3^p(P) \geq h_3(P) + \frac{1}{3}\binom{n}{2} \qquad \text{and} \qquad h_4^p(P) \geq h_4(P) + \frac{1}{2}\left(\binom{n}{2} - 3n + 3\right),$$

where $h_k(P)$ is the number of affine k-holes in P in the plane $\mathbb{R}^2$.

The proof of Proposition 11 can be found in [7]. Together with the best known lower bounds on $h_3(n)$ and $h_4(n)$ by Aichholzer et al. [3], the estimates from Proposition 11 give

$$h_3^p(n) \geq \frac{7}{6}n^2 + \Omega(n \log^{2/3} n) \qquad \text{and} \qquad h_4^p(n) \geq \frac{3}{2}n^2 + \Omega(n \log^{3/4} n).$$

We also discuss random point sets in the real projective plane and provide the following analogue to results for random point sets in the plane [8, 40]. This gives an alternative proof of the upper bound $h_3^p(n) \leq O(n^2)$. The proof of Theorem 12 can be found in [7].

▶ **Theorem 12.** *Let K be a compact convex subset in $\mathbb{R}^2$ of unit area. If P is a set of n points chosen uniformly and independently at random from $K \subset \mathbb{R}^2 \subset \mathbb{RP}^2$, then the expected number of projective 3-holes in P is in $\Theta(n^2)$. Moreover, the expected number of projective holes in P, which are not affine holes in $\mathbb{R}^2$, is in $\Theta(n^2)$.*

Last but not least, we discuss the computational complexity of determining the number of k-gons and k-holes in a given point set. Mitchell et al. [30] gave an $O(mn^3)$ time algorithm to compute, for all $k = 3, \ldots, m$, the number of k-gons and k-holes in a given set S of n points in the Euclidean plane. Their algorithm also counts k-islands in $O(k^2n^4)$ time. Here, an *(affine) k-island* in a finite point set S in the plane in general position is the convex hull of a k-tuple I of points from S that does not contain any point from $S \setminus I$. Note that a convex set in $\mathbb{R}^2$ is a k-hole in S if and only if it is a k-gon and a k-island in S.

Here, we consider the algorithmic aspects of the analogous problems in the real projective plane. By modifying the algorithm by Mitchell et al. [30], we can efficiently compute the number of projective k-gons, k-holes, and k-islands of a finite set in the real projective plane. Here, a *projective k-island* in a finite set P of points from $\mathbb{RP}^2$ in general position is a projective convex hull of a k-tuple I of points from P that does not contain any point from $P \setminus I$. Note that, similarly as in the affine case, a convex set in $\mathbb{RP}^2$ is a projective k-hole in P if and only if it is a projective k-gon and a projective k-island in P.

▶ **Theorem 13.** *Let P be a set of n points in $\mathbb{R}^2 \subset \mathbb{RP}^2$ in general position. Assuming a RAM model of computation which can perform arithmetic operations on integers in constant time, we can compute the total number of projective k-gons and k-holes in P for $k = 3, \ldots, m$ in $O(mn^4)$ time and $O(mn^2)$ space. The number of projective k-islands in P for $k = 3, \ldots, m$ can be computed in $O(m^2n^5)$ time and $O(m^2n^3)$ space.*

3 Discussion

The study of extremal questions about finite point sets in $\mathbb{RP}^2$ suggests a wealth of interesting open problems and topics one can consider. Here, we draw attention to some of them.

By Theorem 6, there are arbitrarily large finite point sets in $\mathbb{RP}^2$ that avoid k-holes for any $k \geq 8$. On the other hand, the result by Gerken [21] and Nicolas [31] implies that every sufficiently large finite subset of $\mathbb{RP}^2$ contains a projective k-hole for any $k \leq 6$, as an analogous statement is true already in the affine setting. The existence of projective 7-holes in sufficiently large finite subsets of $\mathbb{RP}^2$ remains an intriguing open problem and we believe that projective 7-holes can be always found in large points sets in $\mathbb{RP}^2$.

▶ **Conjecture 14.** *Every sufficiently large point set in $\mathbb{RP}^2$ contains a projective 7-hole.*

As we already mentioned, point sets in the plane satisfy $h_k(n) \leq h_k(n+1)$ for all k and n. By Proposition 8, this is no longer true in the real projective plane. However, we do not know any other example violating this inequality except of the single case for 5-holes in $\mathbb{RP}^2$. Thus, it is natural to ask the following question.

▶ **Problem 15.** *Is it true that for every integer $k \geq 3$ there is $n_0 = n_0(k)$ such that $h^p_k(n+1) \geq h^p_k(n)$ for every $n \geq n_0$?*

We have shown in Theorem 7 that Horton sets only contain $\Theta(n^2)$ projective k-holes. Since Horton sets only contain $\Theta(n^2)$ affine k-islands [18], which is asymptotically minimal, we wonder whether the same bound applies to projective k-islands.

▶ **Problem 16.** *For every fixed integer $k \geq 3$, is the minimum number of projective k-islands among all sets of n points from $\mathbb{RP}^2$ in general position in $\Theta(n^2)$?*

We have shown in Theorem 12 that the expected number of 3-holes in random sets of n points from $\mathbb{RP}^2$ is in $\Theta(n^2)$. In the plane, we know that the expected number of k-holes and k-islands is in $\Theta(n^2)$ for any fixed k [5, 6]. Can analogous estimates be obtained also in the real projective plane? We note that the lower bound $\Omega(n^2)$ follows from the planar case.

▶ **Problem 17.** *Let K be a compact convex subset in $\mathbb{R}^2$ of unit area and let $k \geq 3$. Is the expected number of projective k-holes and k-islands in a set of n points, which is chosen uniformly and independently at random from $K \subset \mathbb{R}^2 \subset \mathbb{RP}^2$, in $\Theta(n^2)$?*

Besides all these Erdős–Szekeres-type problems related to k-gons, k-holes and k-islands, many other classical problems have natural analogues in the projective plane. In the following, we discuss the problem of *crossing families*. Let P be a finite set of points in the plane. For a positive integer n, let $T(n)$ be the largest number such that any set of n points in general position in the plane determines at least $T(n)$ pairwise crossing segments. The problem of estimating $T(n)$ was introduced in the 1990s by Erdős et al. [4] who proved $T(n) \geq \Omega(\sqrt{n})$. Since then it was widely conjectured that $T(n) \in \Theta(n)$. However, nobody has been able to improve the lower bound from [4] until a recent breakthrough by Pach, Rubin, and Tardos [32] who showed $T(n) \geq n^{1-o(1)}$.

In $\mathbb{RP}^2$, every pair of points determines a projective line that can be divided into two projective line segments. Given $2n$ points $p_1, \ldots, p_k, q_1, \ldots, q_k$ from $\mathbb{RP}^2$, we say that they form *projective crossing family* of size k if, for each i, we can choose a projective line segment s_i between p_i and q_i such that for any pair i, j with $1 \leq i < j \leq k$ the projective line segments s_i and s_j intersect. We can then ask about the maximum size $T^p(n)$ of a projective crossing family in a set P of n points from $\mathbb{RP}^2$. Note that any set of k pairwise crossing segments of P, which live in a plane $\rho \subset \mathbb{RP}^2$, gives a projective crossing family of size k in P. Thus, proving a linear lower bound might be simpler for $T^p(n)$ than for $T(n)$.

▶ **Problem 18.** *Is the maximum size $T^p(n)$ of a projective crossing family in a set of n points from $\mathbb{R}\mathcal{P}^2$ in general position in $\Theta(n)$?*

All the notions we discussed (general position, convex position, k-gons, k-holes, k-islands, crossing families, and various others) naturally extend to higher dimensional Euclidean spaces and also to higher dimensional projective spaces. In fact, k-gons and k-holes in higher dimensional Euclidean spaces are currently quite actively studied:

- One central open problem in higher dimensions is to determine the largest value $H(d)$ such that every sufficiently large set in $\mathbb{R}^d$ contains an $H(d)$-hole. While $H(2) = 6$ is known, the gap between the upper and the lower bound for $H(d)$ remains huge for $d \geq 3$. [11, 12, 34, 39]
- For sets of n points sampled independently and uniformly at random from a unit-volume convex body in $\mathbb{R}^d$, the expected number of k-holes and k-islands is in $\Theta(n^d)$. [5, 6]
- While the k-gons and k-holes can be counted efficiently in the Euclidean plane, determining the size of the largest gon or hole is NP-hard already in $\mathbb{R}^3$. [22]

These analogues in $\mathbb{R}\mathcal{P}^2$ and in high dimensional projective spaces are interesting by themselves, but they might also shed new light on the original problems. We plan to address further such analogues and we hope to also motivate some readers for this line of research.

4 Proof of Theorem 5

Here, we show, for every integer $k \geq 2$, almost matching bounds on the minimum size $ES^p(k)$ that guarantees the existence of a projective k-gon in every set of at least $ES^p(k)$ points from $\mathbb{R}\mathcal{P}^2$. More precisely, we prove that there are constants $c, c' > 0$ such that

$$2^{k-c\log k} \leq ES^p(k) \leq 2^{k+c'\sqrt{k\log k}}.$$

The upper bound follows from (1), thus it remains to prove the lower bound on $ES^p(k)$. To do so, we construct sets of $2^{k-c\log k}$ points in $\mathbb{R}\mathcal{P}^2$ with no projective k-gon. By Observation 3, it suffices to show that S contains no k points in convex position and no double chain of size k. To obtain such sets, we employ a recursive construction by Erdős and Szekeres [16]. By choosing c sufficiently large, we can assume $k \geq 7$.

Let X and Y be finite sets of points in the Euclidean plane. We say that X *lies deep below* Y and Y *lies high above* X if each point of X lies below every line through two points of Y, and each point of Y lies above every line through two points of X. For $k \geq 2$, we say that a set C of k points in the plane is a *k-cup* if its points lie on the graph of a convex function and we call C a *k-cap* if its points lie on the graph of a concave function.

We now construct the set S inductively as follows. For $a \leq 2$ or $u \leq 2$, let $S_{a,u}$ be a set consisting of a single point from $\mathbb{R}^2$ and note that $S_{a,u}$ then does not contain a 2-cap nor a 2-cup. For integers $a, u \geq 3$, we let $S_{a,u}$ be a set obtained by placing a copy of $S_{a,u-1}$ to the left and deep below a copy of $S_{a-1,u}$. It follows by induction that $|S_{a,u}| = \binom{a+u-4}{a-2} = \binom{a+u-4}{u-2}$ and that $S_{a,u}$ does not contain an a-cap nor a u-cup; see [16]. Finally, we let $S = S_{\lfloor k/2\rfloor-1,\lfloor k/2\rfloor-1}$. Since $k \geq 7$, we have $\lfloor k/2\rfloor - 1 \geq 2$ and thus the set S is well-defined.

Note that $|S| = \binom{\lfloor k/2\rfloor+\lfloor k/2\rfloor-4}{\lfloor k/2\rfloor-2} \geq 2^{k-c\log k}$ for some constant $c > 0$. The set S does not contain k points in convex position, as such a k-tuple contains either a $(\lfloor k/2\rfloor - 1)$-cap or a $(\lfloor k/2\rfloor - 1)$-cup. Thus, it remains to show that S does not contain a double chain of size k.

Suppose for contradiction that W is a double chain k-wedge with $A \cup B$ in S with $A = \{s_1, \ldots, s_m\}$ and $B = \{r_1, \ldots, r_{k-m}\}$ for some m with $1 \leq m \leq k-1$; using the notation from Subsection 1.2. We let ℓ_1 be the line $\overline{s_1 r_{k-m}}$ and ℓ_2 be the line $\overline{s_m r_1}$. Let

$a \leq \lfloor k/2 \rfloor - 1$ and $u \leq \lfloor k/2 \rfloor - 1$ be two numbers such that W has all vertices in $S_{a,u}$ but it does not have all vertices in $S_{a-1,u}$ nor in $S_{a,u-1}$. Let D and U be the copies of $S_{a-1,u}$ and $S_{a,u-1}$, respectively, forming $S_{a,u}$. We can assume without loss of generality that $|\{s_1, s_m, r_1, r_{k-m}\} \cap D| \geq 2$, as the other case $|\{s_1, s_m, r_1, r_{k-m}\} \cap U| \geq 2$ is treated analogously. We distinguish the following two cases.

Case 1. Assume $|\{s_1, s_m, r_1, r_{k-m}\} \cap D| = 2$. Then two points from $\{s_1, s_m, r_1, r_{k-m}\}$ are in D and the other two points are in U. By symmetry, we can assume $s_1 \in U$. We distinguish the following two subcases, which are shown in Figure 3. Note that, since the line segments $s_1 r_{k-m}$ and $s_m r_1$ cross, the cases $s_1, r_{k-m} \in U$ and $r_1, s_m \in D$ cannot occur.

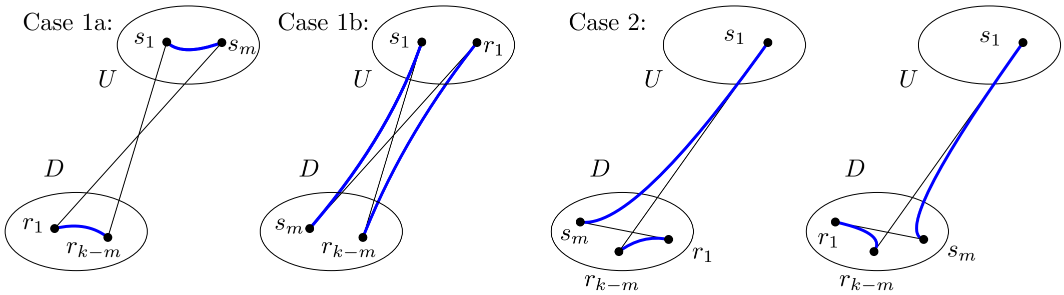

Figure 3 The cases in the proof of Theorem 5.

Case 1a. Assume $s_1, s_m \in U$ and $r_1, r_{k-m} \in L$. We assume that s_1 is to the left of s_m, otherwise we reverse the order of the elements in A and B which, in particular, exchanges the roles of s_1 and s_m. Since U is high above D, the line $\overline{s_1 r_{k-m}}$ is almost vertical and separates s_m from r_1, where s_1 is to the left of s_m and r_1 is to the left of r_{k-m}. All points of $A \setminus \{s_1\}$ lie to the right of $\overline{s_1 r_{k-m}}$ and to the left of $\overline{s_m r_{k-m}}$. Since D is deep below U, no point of D satisfies these two conditions. Hence all points of A lie in U. An analogous argument shows that all points of B lie in D. Since A forms an m-cup in U and B forms a $(k-m)$-cap in D, we have $m \leq u - 1$ and $k - m \leq a - 1$. Consequently, $k = m + (m - k) \leq a + u - 2 \leq \lfloor k/2 \rfloor + \lfloor k/2 \rfloor - 4 < k$, which is impossible.

Case 1b. Assume $s_1, r_1 \in U$ and $s_m, r_{k-m} \in L$. We assume that s_1 is to the left of r_1, as otherwise we exchange the roles of A and B which, in particular, exchanges the roles of s_1 and r_1. Since U is high above D, the line $\overline{s_1 r_{k-m}}$ is almost vertical and separates s_m from r_1 and s_m is to the left of r_{k-m}. All points of $A \setminus \{s_1\}$ lie to the left of the almost vertical line $\overline{s_1 r_{k-m}}$ and to the right of the almost vertical line $\overline{s_1 s_m}$. Hence, $A \cap U = \{s_1\}$ and all points from $A \setminus \{s_1\}$ lie in D. The set $A \setminus \{s_1\}$ forms an $(m-1)$-cup in D and thus $m - 1 \leq u - 1$. An analogous argument shows that $B \setminus \{r_1\}$ forms a $(k-m-1)$-cap in D and thus $(k-m) - 1 \leq a - 1$. In total, we obtain $k = (m-1) + (k-m-1) + 2 \leq (u-1) + (a-1) + 2 \leq \lfloor k/2 \rfloor + \lfloor k/2 \rfloor - 2 < k$, which is again impossible.

Case 2. Assume $|\{s_1, s_m, r_1, r_{k-m}\} \cap D| = 3$. We can assume that either s_1 or s_m lies in U, as otherwise we exchange the roles of A and B. Furthermore, we can assume that $s_1 \in U$, as otherwise we reverse the order of the elements in A and B. Since U is high above D, the line $\overline{s_1 r_{k-m}}$ is almost vertical and separates r_1 and s_m. Since all vertices of W lie either to the left of the almost vertical line $\overline{s_1 s_m}$ and to the right of the almost vertical line $\overline{s_1 r_1}$ or to the right of $\overline{s_1 s_m}$ and to the left of $\overline{s_1 r_1}$, the point s_1 is the only vertex of W in U. Hence, the points $S \setminus \{s_1\}$ lie in D and form an $(m-1)$-cup in D. Thus, $m-1 \leq u$. The points of B all lie in D and form a $(k-m)$-cap in D. Thus, $k-m \leq a-1$. Altogether, we have $k = (m-1)+1+(k-m) \leq u+1+a-1 \leq \lfloor k/2 \rfloor + \lfloor k/2 \rfloor - 2 < k$, which is impossible.

Since there is no case left, we have a contradiction with the assumption that W is a double chain k-wedge with vertices in S. This completes the proof of Theorem 5.

5 Sketch of the proofs of Theorem 6 and Theorem 7

Here, we sketch the proof of the fact that there are arbitrarily large finite sets of points from $\mathbb{RP}^2$ in general position with no projective 8-hole and with only quadratically many projective k-holes for every $k \leq 7$. For the full proof see [7].

The construction uses so-called *Horton sets* defined by Valtr [38]. Let H be a set of n points $p_1, \dots, p_n$ from $\mathbb{R}^2$, sorted according to increasing x-coordinates. Let H_0 be the set of points p_i with odd i and let H_1 be the set of points p_i with even i. The set H is *Horton* if either $|H| \leq 1$ or if $|H| \geq 2$, H_0 and H_1 are both Horton and H_0 lies deep below or high above H_1. In the second case, we call H_0 and H_1 the *layers* of H. As in Section 4, we say that H_0 *lies deep below* H_1 and H_1 *lies high above* H_0 if each point of H_0 lies below every line spanned by two points of H_1, and each point of H_1 lies above every line spanned by two points of H_0. For a nonempty subset A of H, we define the *base* of A in H as the smallest recursive layer of H containing A.

As in Section 4, we use the terms k*-cup* and k*-cap*. A *cap* is a set that is a k-cap for some integer k and, analogously, a *cup* is a set that is a k-cup for some k. A cap C is *open* in a set $S \subseteq \mathbb{R}^2$ if there is no point of S below C, that is, for each pair of points c_1, c_2 from C, no point of S has its coordinate between $x(c_1)$ and $x(c_2)$ and lies below the line $\overline{c_1 c_2}$. Analogously, a cup in S is *open* in S if there is no point of S above it.

5.1 Quadratic upper bounds on the number of k-holes

We show that any Horton set on n points embedded in the real projective plane does not contain 8-holes and that H has at most $O(n^2)$ k-holes for every $k \in \{3, \dots, 7\}$. By Observation 4, it suffices to show that any Horton set H on n points in the plane does not contain 8-holes nor an empty double chain 8-wedge and that, for every $k \in \{3, \dots, 7\}$, H contains only at most $O(n^2)$ k-holes and empty double chain k-wedges. Valtr [38] showed that any Horton set in the plane does not contain 7-holes and that it does not contain any open 4-cap nor an open 4-cup. Bárány and Valtr [9] showed that the number of k-holes in any Horton set of size n is at most $O(n^2)$ for every $k \in \{3, \dots, 6\}$. Thus, it suffices to estimate the number of double chain k-wedges in Horton sets.

Let H be a Horton set with n points in the plane. We first show that the number of open caps in every Horton set H with n points in the plane is at most $O(n)$ and that analogous statement is true for open cups. To prove this claim, it suffices to consider only open 2-caps and 3-caps, as H does not contain open 4-caps.

We proceed by induction on $\log_2 n$ and show that the number $t_2(H)$ of open 2-caps equals $2n - \log_2(n) - 2$ and that the number $t_3(H)$ of open 3-caps in H equals $n - \log_2(n) - 1$ if n is a power of 2. Both expressions hold for $n = 1$ and thus we assume $n \geq 2$. Let $p_1, \dots, p_n$ be the points of H ordered according to increasing coordinates and let $H_0 = L(H)$ and $H_1 = U(H)$ be the sets that partition H such that H_0 is deep below H_1. Every line segment $p_i p_{i+1}$ forms an open 2-cap in H and there is no other open 2-cap in H with points in H_0

and H_1, as there is a point of H_1 above any such line segment p_ip_j with $j > i+1$. Since no two points from H_1 form an open 2-cap in H, we have $t_2(H_0) + n - 1$ open 2-caps in H. By the induction hypothesis, it follows $t_2(H) = 2n - \log_2(n) - 2$.

To determine the number of open 3-caps in H, note that every triple $p_ip_{i+1}p_{i+2}$ with odd i forms an open 3-cap in H. In fact, there is no other open 3-cap in H with a point in H_0 and also in H_1, as there is a point of H_1 above any such line segment p_ip_j with $j > i+1$. Since no three points in H_1 form an open 3-cap in H, we obtain $t_3(H_0) + n/2 - 1$ open 3-caps in H. The induction hypothesis then gives $t_3(H) = n - \log_2(n) - 1$.

If n is not a power of two, we consider a Horton set H' of size m instead, where m is as the smallest power of 2 larger than n, and denote its leftmost n points by H''. Since H'' is also a Horton set of n points and contains the same open caps as H, we obtain $t_2(H) \le t_2(H') < 4n$ and $t_3(H) \le t_3(H') < 2n$. Overall, the number of open caps in H is at most $O(n)$. With an analogous argument we obtain the same upper bound on the number of open cups in H.

We now proceed with the proof by induction on n. Clearly, the claims about the double chain k-wedges are true in any Horton set with one or two points, so we assume $n \ge 3$. For some integer $k \ge 3$, let $W \subseteq H$ be a double chain k-wedge that is empty in H. We will show that $k \le 7$ and estimate the number of such double chain k-wedges for each $k \in \{3, \dots, 7\}$.

If W is contained in H_0 or in H_1, then $k \le 7$ by the induction hypothesis. Thus, we assume that W contains a point from H_0 and also from H_1. An elaborate case analysis shows that H contains no double chain 8-wedge that is empty in H and that has points in H_0 and H_1; see [7]. By the induction hypothesis, the sets H_0 and H_1 do not contain any double chain 8-wedge that is empty in H_0 and in H_1, respectively. Since every double chain 8-wedge that is contained in H_i and is empty in H is also empty in H_i for every $i \in \{0, 1\}$, we see that there is no double chain 8-wedge in H that is empty in H. This completes the proof of Theorem 6.

Let $k \in \{3, \dots, 7\}$. For the quadratic upper bounds, it can be shown that there is a constant c such that H contains at most cn^2 double chain k-wedges that are empty in H and that have points in H_0 and H_1 (again, see [7]). Altogether, the number $w_k(H)$ of empty double chain k-wedges in H satisfies $w_k(H) \le w_k(H_0) + w_k(H_1) + cn^2$. Solving this linear recurrence with the initial condition $w_k(H') = 0$ for any set H' with $|H'| = 1$ gives $w_k(H) \le O(n^2)$. This completes the proof of the first part of Theorem 7.

6 Outline of the construction giving Theorems 9 and 10

Here we outline the construction giving Theorems 9 and 10. For the full proof, see [7].

We are given a $k \in \{3, \dots, 6\}$ and a positive integer n. Our construction uses two integer parameters $a, b \ge 2$ satisfying $a \le n^{1/3}$ and $ab \le n$. In the proof of Theorem 9, these parameters depend on the value of the parameter α in the theorem. For the proof of Theorem 10, where we are given an integer parameter x, we choose $a := 2$ and $b \approx \log_2(x)$.

Assuming $\sqrt{n}$ is an integer, we start the construction with the $\sqrt{n} \times \sqrt{n}$ integer lattice in the plane, denoted by $L(\sqrt{n} \times \sqrt{n})$, and we fix a subset C_3 of $\Theta(n^{1/3})$ points in convex position in $L(\sqrt{n} \times \sqrt{n})$. We then perturb the lattice to get a so-called *random squared Horton set*, denoted by $H(\sqrt{n} \times \sqrt{n})$, which is a randomized version [9] of the lattice version of so-called Horton sets [38], which generalize the famous construction of Horton [26] of planar point sets in general position with no 7-holes. The random squared Horton set is described in [9, Section 2] and denoted by Λ^* there.

We consider the $|C_3|$-element subset C_3^H of $H(\sqrt{n} \times \sqrt{n})$ corresponding to C_3. Since C_3 is in convex position, the set C_3^H is also in convex position. We fix an a-element subset C of C_3^H, where a is the above mentioned parameter. For each $c \in C$, we take a set S_c of b

points lying in a very small neighborhood of c and on a unit circle touching the polygon $\operatorname{conv} C_3^H$ in the point c. Since the points of S_c are placed very close together on a unit circle, they are almost collinear. We consider the set $H(\sqrt{n} \times \sqrt{n}) \cap \operatorname{conv} C_3^H$, and denote its union with the sets $S_c, c \in C$, by $T = T(a, b)$; see Figure 4. The set T has at most $n + ab \leq 2n$ points, and it is just a little technicality to adjust its size to n at the right place in the proof.

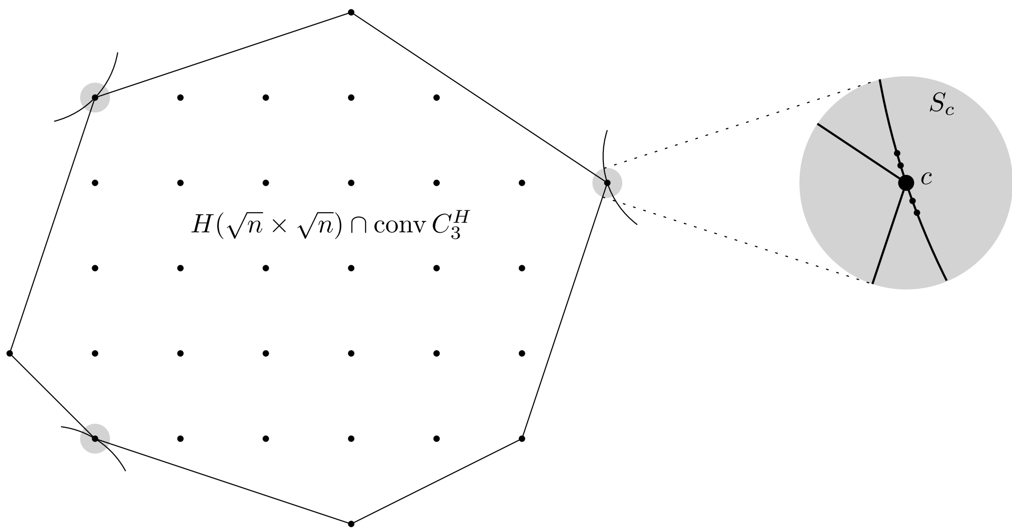

Figure 4 An illustration of the set $T(a, b)$ for $a = 3$ and $b = 5$ (we assume each c lies in S_c).

We now sketch a proof that the set T satisfies Theorems 9 and 10 for properly chosen parameters a and b. The random squared Horton set of size n has $O(n^2)$ affine holes [9, 38]. Likewise, using the condition $ab \leq n$ and two additional facts, it can be argued that the set T has at most $O(n^2)$ affine holes that do not lie completely in some S_c. The two additional facts are that (i) the expected number of affine holes containing a fixed point of C is at most $O(n)$ and (ii) the expected number of affine holes containing a fixed pair of points of C is at most $O(n)$. The number of affine k-holes that lie completely in one of the sets S_c is clearly $a\binom{b}{k} < ab^k$. Thus, the total number of affine k-holes in $T = T(a, b)$ is at most $O(n^2 + ab^k)$.

Due to the construction, any $(k-1)$-element subset of any set S_c, together with any point of $T \setminus S_c$, forms a projective k-hole. There are a sets S_c and each of them has size b. Thus, there are at least $a \cdot \binom{b}{k-1} \cdot (|T| - b) = \Theta(ab^{k-1}n)$ projective k-holes in T.

Now, Theorem 9 is obtained from the above construction by setting the parameters a, b carefully with respect to α. Namely, for $\alpha \in [0, \frac{2k-5}{3}]$ we set $a \approx n^{1/3}$ and $b := n^{(5/3+\alpha)/k}$, and for $\alpha \in (\frac{2k-5}{3}, k-2]$ we set $a \approx n^{1-(1+\alpha)/(k-1)}$ and $b := n^{(1+\alpha)/(k-1)}$. We remark that in the range $\alpha \in [0, \frac{2k-5}{3}]$, the parameter a corresponds to its maximum possible size which is the maximum size of a subset in the lattice $L(\sqrt{n} \times \sqrt{n})$ in convex position, and the parameter b grows with α, since increased α allows bigger affine holes. In the range $\alpha \in (\frac{2k-5}{3}, k-2]$, the parameter b continues to grow with α but a is decreasing to keep the size ab of S below n.

To obtain Theorem 10 from the above construction, we set $a := 2$ and $b \approx \log_2 x$. Then the number of affine holes contained in one of the two sets S_c is $\approx a2^b = \Theta(x)$ and the number of other affine holes in T is again in $O(n^2)$. Any subset of the $(ab =)2b$-element union of the two sets S_c is in convex position or is a double chain, determining a projective hole. Thus, $T = T(2, b)$ has at least $\Theta(2^{2b}) = \Theta(x^2)$ projective holes. Theorem 10 follows.

7 Proof of Theorem 13

Let S be a set of n points in the Euclidean plane in general position. Mitchell et al. [30] use a dynamic programming approach to determine, for every point $p \in S$, the number of k-gons and k-holes for $k = 3, \ldots, m$, which have p as the bottom-most point. The algorithm performs in $O(mn^2)$ time and space. They also determine the number of k-islands in S, which have p as the bottom-most point, in $O(m^2n^3)$ time and space. Note that the bottom-most point is unique without loss of generality, as otherwise we perform an affine transformation which does not affect the number of k-gons, k-holes, and k-islands.

Here, we introduce an algorithm that efficiently computes the number of projective k-gons, k-holes, and k-islands of a finite set P of n points from $\mathbb{R}^2 \subset \mathbb{RP}^2$. First, we discuss how to determine the number of projective k-gons in P.

Let G be a projective k-gon with $k \geq 3$ and let p_1, p_2 be two vertices that are consecutive on the boundary of G. If we start at p_1 and trace the boundary of G in the direction of p_2, we obtain a unique cyclic permutation $p_1, \ldots, p_k$ of the vertices of G. By starting at p_2 and tracing in the direction of p_1, we obtain the reversed cyclic permutation. It is crucial that, independently from the starting point and the direction, only the k pairs $\{p_i, p_{i+1}\}$ for $i = 1, \ldots, k$ (indices modulo k) appear as consecutive vertices along the boundary of G.

For every pair of points $\{s, t\} \in P$, the algorithm will count (with multiplicities) the number of projective k-gons in P, which have s and t as consecutive vertices on the boundary. Since each projective k-gon is counted exactly k times, we can then derive the number projective k-gons in P by a simple division by k.

For a pair $\{s, t\}$ of distinct points from P, we can choose a line $\ell^+_{s,t}$ ($\ell^-_{s,t}$) which is parallel to the line $\overline{st}$ and lies very close and to the left (right) of $\overline{st}$. By removing $\ell^+_{s,t}$ and $\ell^-_{s,t}$, respectively, from $\mathbb{RP}^2$, we obtain two planes $\rho^+_{s,t} \subset \mathbb{RP}^2$ and $\rho^-_{s,t} \subset \mathbb{RP}^2$. Now, every projective k-gon G of P, which has s and t as consecutive vertices on its boundary, is a convex k-gon either in $\rho^+_{s,t}$ or in $\rho^-_{s,t}$, but not in both. Note that in both planes $\rho^+_{s,t}$ and $\rho^-_{s,t}$, s and t lie on the boundary of the convex hull of P. Moreover, we can assume that s is the bottom-most point in both planes $\rho^+_{s,t}$ and $\rho^-_{s,t}$, as otherwise we apply a suitable rotation.

For each of the $\binom{n}{2}$ pairs $\{s, t\}$ of distinct points from P, we now count the number of convex k-gons in the planes $\rho^+_{s,t}$ and $\rho^-_{s,t}$, which have s and t as consecutive vertices on the boundary. This counting can be done in $O(mn^2)$ time and space by using the algorithm of Mitchell et al. [30] with the slight modification that, in the initial phase, we only count 3-gons of the form $p_1 = s, p_2 = t, p_3$; see equation (3) in [30]. Since each projective k-gon G is now counted precisely k times, once for each pair of consecutive vertices along the boundary of G, this completes the argument for projective k-gons.

Similarly, we count projective k-holes and k-islands. The time and space requirements of the algorithm from [30] for counting projective k-holes, which are incident to the bottom-most point, are the same as for projective k-gons. For counting projective k-islands, which are incident to the bottom-most point, the algorithm from [30] uses $O(m^2n^3)$ time and space.

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