Communication with Partial Noiseless Feedback

Bernhard Haeupler[1], Pritish Kamath[2], and Ameya Velingker[1]

1 Computer Science Department
Carnegie Mellon University, USA
{haeupler,avelingk}@cs.cmu.edu

2 Computer Science and Artificial Intelligence Laboratory
Massachusetts Institute of Technology, USA
pritish@mit.edu

Abstract

We introduce the notion of *one-way communication schemes with partial noiseless feedback*. In this setting, Alice wishes to communicate a message to Bob by using a communication scheme that involves sending a sequence of bits over a channel while receiving feedback bits from Bob for δ fraction of the transmissions. An adversary is allowed to corrupt up to a constant fraction of Alice's transmissions, while the feedback is always uncorrupted. Motivated by questions related to coding for interactive communication, we seek to determine the maximum error rate, as a function of $0 \leq \delta \leq 1$, such that Alice can send a message to Bob via some protocol with δ fraction of noiseless feedback. The case $\delta = 1$ corresponds to *full feedback*, in which the result of [1] implies that the maximum tolerable error rate is $1/3$, while the case $\delta = 0$ corresponds to *no feedback*, in which the maximum tolerable error rate is $1/4$, achievable by use of a binary error-correcting code.

In this work, we show that for any $\delta \in (0, 1]$ and $\gamma \in [0, 1/3)$, there exists a *randomized* communication scheme with noiseless δ-feedback, such that the probability of miscommunication is low, as long as no more than a γ fraction of the rounds are corrupted. Moreover, we show that for any $\delta \in (0, 1]$ and $\gamma < f(\delta)$, there exists a *deterministic* communication scheme with noiseless δ-feedback that always decodes correctly as long as no more than a γ fraction of rounds are corrupted. Here f is a monotonically increasing, piecewise linear, continuous function with $f(0) = 1/4$ and $f(1) = 1/3$. Also, the rate of communication in both cases is constant (dependent on δ and γ but independent of the input length).

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases Communication with feedback, Interactive communication, Coding theory

Digital Object Identifier 10.4230/LIPIcs.APPROX-RANDOM.2015.881

1 Introduction

Motivated by questions in interactive coding, we introduce the model of *communication with partial noiseless feedback*. Alice wishes to communicate a message, say in $\{0, 1\}^k$, to Bob. Alice sends a total of N bits to Bob, and she receives δN bits of feedback from Bob for some fixed $\delta > 0$. We have an adversary who can corrupt γN of the bits sent by Alice, but the feedback bits are left uncorrupted. We wish to find the maximal tolerable error fraction γ (on Alice's transmissions) under which we can guarantee that Bob is able to receive Alice's message correctly. This problem is summarized in Figure 1.

We introduce this problem as a generalization of the problem of communication with complete noiseless feedback (corresponds to $\delta = 1$ above), where after each bit sent by Alice,

18th Int'l Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX'15) / 19th Int'l Workshop on Randomization and Computation (RANDOM'15).
Editors: Naveen Garg, Klaus Jansen, Anup Rao, and José D. P. Rolim; pp. 881–897
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

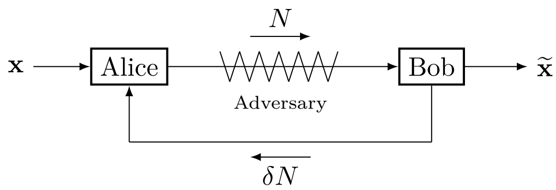


Figure 1 Communication with partial noiseless feedback.

Bob sends the received bit as feedback, so that Alice can use her knowledge of exactly what Bob has received thus far to possibly adapt her future transmissions. Indeed, Berlekamp showed that in this setting, one can tolerate any error rate less than $1/3$ with non-vanishing communication rate if we require that any possible error pattern of up to the error rate be corrected [1], and moreover, this is the maximal error fraction one can hope to correct with constant rate. This bound also follows from the game of questions with liars [20].

On the other extreme, if there were no feedback at all (i.e. $\delta = 0$), then this is equivalent to error correcting codes, for which it is known that one can tolerate up to $1/4$ error fraction while still achieving positive communication rate [12, 21]. And an error fraction of $\geq 1/4$ necessarily results in zero asymptotic rate, due to the Plotkin bound [15]. Thus, feedback increases the set of achievable communication rates in the adversarial error model. This is in contrast to the random-error setting in which random error patterns need to be corrected only with high probability, as Shannon showed that feedback does not increase the capacity of a discrete memoryless channel [19].

1.1 Coding for interactive communication

The problem of communication with noiseless feedback has garnered further interest recently in the context of coding for interactive communication. In this setting, Alice and Bob are given inputs $\mathbf{x}$ and $\mathbf{y}$, respectively, and they are required to compute some function $f(\mathbf{x}, \mathbf{y})$ by exchanging messages over a noisy channel. In particular, up to a γ fraction of the total transmitted bits may get flipped by the channel, and one requires a coding scheme that allows successful computation even in the presence of noise, preferably with a only constant blow up in communication. Schulman was the first to investigate the problem, and in a series of works, he gave the first constant rate scheme that could tolerate an error fraction of up to $\gamma = 1/240$ [16, 17, 18]. Subsequently, in an influential work, Braverman and Rao [6] showed a coding scheme that works for any error rate $\gamma < 1/4$, and moreover, they showed that any error rate of $\geq 1/4$ cannot be tolerated as long as the encoded protocol is *non-adaptive* (meaning that whose turn it is to speak during each round of communication is predetermined). There have been a lot of subsequent works since, which deal with computational efficiency [2, 4, 3, 9], allowing adaptivity [11], list-decoding [11, 10, 5], interactive channel capacity under random noise [14] and adversarial noise [13] etc.

Approaching an error fraction of $1/4$ in the non-adaptive setting requires communicating symbols from a growing alphabet size. If we restrict ourselves to communicating bits, then the coding scheme of Braverman and Rao [6] tolerates any error fraction $\gamma < 1/8$. Determining the maximum tolerable noise for interactive coding with symbols from a binary alphabet is still an open question. However, the optimality of $1/3$ as the maximal tolerable error fraction in the noiseless feedback problem can be used to establish an upper bound of $1/6$ for the maximal tolerable error fraction for interactive coding over binary alphabets! Furthermore, Efremenko, Gelles and Haeupler [7] show a coding scheme over binary alphabets that tolerates any error fraction $\gamma < 1/6$ if noiseless feedback is allowed in the interactive setting as well. Also, Gelles and Haeupler [8] show that for an error fraction of ε, any *alternating* interactive protocol can be encoded with rate $1 - \Theta(H(\varepsilon))$ over channels with noiseless feedback as well as erasure channels.

1.2 Our results

In this work, we show that for any $\delta \in (0, 1]$ and $\gamma \in [0, 1/3)$, there exists a *randomized* communication scheme with noiseless δ-feedback, such that the probability of miscommunication is low, as long as no more than a γ fraction of the rounds are corrupted. Moreover, we show that for any $\delta \in (0, 1]$ and $\gamma < f(\delta)$, there exists a *deterministic* communication scheme with noiseless δ-feedback that always decodes correctly as long as no more than a γ fraction of rounds are corrupted. Here f is a monotonically increasing, piecewise linear, continuous function with $f(0) = 1/4$ and $f(1) = 1/3$. Also, the rate of communication in both cases is constant (dependent on δ and γ but independent of the input length).

1.3 Organization of this paper

In Section 2, we give some of the basic definitions and notations as well as the statements of the two main theorems in this work. In Section 3, we describe a simple deterministic communication scheme with 1-feedback that tolerates up to $1/3$ fraction errors. In Section 4, we describe our randomized communication scheme with partial noiseless feedback. In Section 5, we describe our deterministic communication schemes which comes about by a de-randomization of the randomized communication scheme. Finally, we give a summary of our results and suggest possible future directions in Section 6.

2 Preliminaries and results

In this section, we describe our problem set-up and state our results.

2.1 One-way communication schemes with partial noiseless feedback

We consider the problem of *one-way communication with partial noiseless feedback*, which we define as follows:

▶ **Definition 1.** For all $\delta \in [0, 1]$, a *"one-way communication scheme with noiseless δ-feedback"* is defined as follows (summarized in Figure 1):

- Alice wishes to send a message in $\mathbf{x} \in \Sigma^k$ to Bob.
- Alice and Bob engage in a communication protocol of length $N + \delta N$, out of which N symbols are sent by Alice (forward rounds) and δN symbols are sent by Bob (feedback rounds). At most one of the parties can send a symbol in any round.
- The adversary can corrupt at most γN of the forward rounds, but none of the feedback rounds.
- At the end of the protocol, Bob is required to decode the message $\mathbf{x}$ from the transcript of the protocol.

We call N as the *length* of the communication scheme. The *'rate'* of the scheme is k/N, and γ is the error fraction tolerated. (We will often drop the words 'one-way' and 'noiseless'. All feedback in this paper will be assumed to be noiseless.)

The protocol can be *deterministic* or *randomized* (but with only private randomness). In deterministic schemes, we require that Bob is always able to recover $\mathbf{x}$ correctly. In randomized schemes, we require that Bob is able to recover $\mathbf{x}$ correctly with probability at least $1 - o_k(1)$, where the probability is over the private randomness of Alice and Bob.

Note. All the results in this paper will only be for $\Sigma = \{0, 1\}$. Thus, for the rest of the paper, we will work with communication schemes over a binary alphabet.

An important remark about randomized communication schemes is that the adversary is not aware of the random bits being used by the parties in advance. The adversary can only infer the random bits after they are used. We emphasize that it is this remark that makes *de-randomizing* such communication schemes very challenging!

In this work, we wish to fix δ and find an infinite sequence of communication schemes with noiseless δ-feedback, for increasing values of k, where the communication schemes have length $N(k)$. The *asymptotic rate* of the sequence of communication schemes is defined to be $\lim_{k\to\infty} k/N(k)$. For the remainder of the paper, we will often say "communication scheme" with a particular "rate" as shorthand for an *infinite sequence* of communication schemes for increasing message lengths with a particular *asymptotic* rate.

The main question we seek to answer in this work is: For a fixed $\delta \in [0, 1]$, what is the largest error fraction that can be tolerated by an infinite sequence of communication schemes with feedback fraction at most δ? We can ask this question for both deterministic as well as randomized communication schemes.

▶ **Definition 2.** For any $\delta \in [0, 1]$, we define $\Gamma^{\text{det}}(\delta)$ and $\Gamma^{\text{rand}}(\delta)$ as follows:

- $\Gamma^{\text{det}}(\delta)$ is the supremum over γ such that there exists a deterministic communication scheme with δ-feedback that tolerates an error fraction of γ and has constant rate.[1]
- $\Gamma^{\text{rand}}(\delta)$ is the supremum over γ such that there exists a randomized communication scheme with δ-feedback that tolerates an error fraction of γ and has constant rate.

We know that error correcting codes with distance $1/2 - \varepsilon$ exist for all $\varepsilon > 0$. Thus, we get that an error fraction of $\gamma = 1/4 - \varepsilon$ can be tolerated even without having any feedback, and thus, $\Gamma^{\text{rand}}(\delta) \geq \Gamma^{\text{det}}(\delta) \geq 1/4$ for all $\delta \geq 0$.

2.2 Upper bounds on the tolerable error fraction

It is known that for $\delta = 1$, if the communication scheme uses the *mirror feedback* structure, then no communication scheme can tolerate a $1/3$ error fraction for arbitrarily large message length [1]. The mirror feedback structure means that the communication protocol consists of alternating forward and feedback rounds, where each feedback bit sent by Bob is simply the bit that he has received from Alice in the preceding round.

▶ **Observation 3.** *If $\delta = 1$, we can assume without loss of generality that any deterministic one-way communication scheme with noiseless δ-feedback has only* mirror feedback*, namely, after every bit sent by Alice, Bob simply sends back the (potentially corrupted) bit he received.*

The observation follows because if Bob were to send back the precise bits that he receives from Alice, then Alice can compute any deterministic function of the same and thus any 1-feedback protocol can be simulated by using only mirror feedback. Combined with the

[1] rate that can depend on γ and δ, but not on the length of the input $\mathbf{x}$.

upper limit of $1/3$ from [1], we get that for any $\delta > 0$, no *deterministic* communication scheme with δ-feedback can tolerate an error fraction of $1/3$. For completeness, we give a proof of this result in Appendix A.

▶ **Theorem 4.** *For any $\delta \geq 0$, we have that $\Gamma^{\text{det}}(\delta) \leq 1/3$.*

2.3 Main results

We prove two results that provide lower bounds on $\Gamma^{\text{rand}}(\delta)$ and $\Gamma^{\text{det}}(\delta)$, respectively. Our first result presents a randomized communication scheme that tolerates any error fraction $\gamma < 1/3$ for any $\delta > 0$.

▶ **Theorem 5.** *For any $\delta > 0$, we have that $\Gamma^{\text{rand}}(\delta) \geq 1/3$. Namely, for any $\delta > 0$ and for all $\varepsilon > 0$, and $\gamma = 1/3 - \varepsilon$, there is a randomized communication scheme with noiseless δ-feedback that tolerates an error fraction of γ. Furthermore, one can achieve a rate of communication of $\Omega(\varepsilon\delta)$ with failure probability $\exp(-\Omega(k))$, where k is the length of the message being transmitted.*

Our second result presents a 'derandomization' of the underlying randomized communication scheme of Theorem 5 that beats the $1/4$ bound achieved by error correcting codes for all $\delta > 0$. The tolerable error fraction becomes $1/3$ for $\delta \geq 2/3$, which is optimal.

▶ **Theorem 6.** *Define $f : (0,1] \to \mathbb{R}$ as follows:*

$$f(\delta) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{3}, & \text{if } \frac{2}{3} \leq \delta \leq 1 \\ \max\left\{\frac{\delta(r+1)}{2}, \frac{r+2}{4r+7}\right\}, & \text{if } 0 < \delta < \frac{2}{3} \end{cases} \qquad \text{where } r = r(\delta) \stackrel{\text{def}}{=} \left\lfloor \frac{1}{2\delta} - \frac{3}{4} \right\rfloor$$

Then, for any $\delta \in (0,1]$, $\Gamma^{\text{det}}(\delta) \geq f(\delta)$. Namely, for any $\delta > 0$ and for all $\epsilon > 0$, there is a deterministic communication scheme with δ-feedback that tolerates an error fraction of $\gamma = f(\delta) - \varepsilon$ such that the rate of communication is $\Omega(\varepsilon\delta)$.

▶ Remark. The function f defined in Theorem 6 is a monotonically increasing piecewise linear function that is continuous on the interval in which it is defined (see Figure 2). Moreover $\lim_{\delta\to 0^+} f(\delta) = 1/4$, and one can easily tolerate any error fraction less than $\frac{1}{4}$ with zero feedback by simply using a binary error-correcting code with relative distance of twice the desired error fraction. For the other extremal case, namely $\delta = 1$, the protocol $\pi_1^{\text{det}}(\gamma)$ (Figure 3 adapted from [7]) can be used to tolerate any error fraction less than $f(1) = 1/3$. The main contribution of this work is to establish the achievability of error fractions up to $f(\delta)$ for *intermediate* values of $\delta \in (0,1)$. This shows that any non-zero feedback fraction allows us to beat the $1/4$ limit on the tolerable error fraction in the presence of no feedback.

Note that $\Gamma^{\text{det}}(\delta) \leq 1/3$ for all $\delta \in [0,1]$ (as implied by Theorem 4), and Theorem 5 shows us that randomized communication schemes are able to get arbitrarily close to this limit for *all* non-zero feedback fractions. We do not know whether the same is true for deterministic communication schemes, as the positive result of Theorem 6 exhibits a gap to the $1/3$ upper bound for $0 < \delta < 2/3$. We leave the question of whether $\Gamma^{\text{det}}(\delta) = 1/3$ for all δ as an open problem. The error fractions tolerated by our communication schemes are summarized in Figure 2.

3 Deterministic communication scheme with full feedback

For completeness, we first present a communication scheme $\pi_1^{\text{det}}(\gamma)$ with 1-feedback that tolerates an error fraction of $\gamma = 1/3 - \varepsilon$ (where $\varepsilon > 0$). Such a protocol was obtained

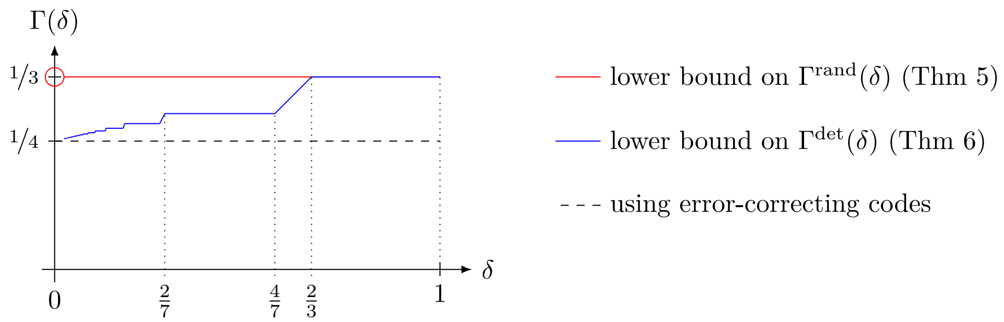


Figure 2 Maximum error fraction tolerated as function of δ.

previously by Berlekamp [1]. We present a very simple scheme (in Figure 3) that was implicit in [7]. It is easy to see that the communication scheme presented has rate $\Theta(\varepsilon)$.

Correctness of $\pi_1^{\text{det}}(\gamma)$

We first introduce a couple of notations. Firstly, for all $s \in \{0,1\}^*$, define $\text{len}(s)$ to be the length of s. Next, for strings s which do not contain consecutive 0's, we define the *weight* of s as follows.

▶ **Definition 7** (Weight of a string). Given string $s \in \{0,1\}^*$, such that s has no consecutive 0's, we define the *weight* of s as follows: Suppose s breaks into a 0's, b 1's and c 10's, with the smallest number of pieces. We define $\text{wt}(s) \stackrel{\text{def}}{=} 2a + 2b + c$.

For example,

- $\text{wt}(\text{‘0110101’}) = 2 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 = 8$, since ‘0110101’ = ‘0’ + ‘1’ + ‘10’ + ‘10’ + ‘1’.
- $\text{wt}(\text{‘111010’}) = 2 \cdot 0 + 2 \cdot 2 + 1 \cdot 2 = 6$, since ‘111010’ = ‘1’ + ‘1’ + ‘10’ + ‘10’.

Note that since s does not contain consecutive 0's, it follows that $a = 1$ if s starts with a ‘0’ and $a = 0$ otherwise.

To prove that the communication scheme $\pi_1^{\text{det}}(\gamma)$ in Figure 3 tolerates an error fraction of γ, we define a potential function as $\Phi = \Phi(T) \stackrel{\text{def}}{=} \text{len}(T_{\text{right}}) - \text{wt}(T_{\text{wrong}})$. Note that the scheme in Figure 3 ensures that T_{wrong} never has consecutive 0's, and thus $\text{wt}(T_{\text{wrong}})$ is always well defined.

The following easy proposition shows how Φ changes after each round of communication. The proof appears in Appendix B.

▶ **Proposition 8.** *After each round of communication, if Alice's bit is received correctly by Bob, then Φ increases by at least 1. On the other hand, if Alice's bit is received incorrectly by Bob, then Φ decreases by at most 2.*

As an easy corollary of Proposition 8, we get a lower bound on Φ at the end of the protocol, as a function of the error fraction.

▶ **Corollary 9.** *If γ fraction of Alice's transmissions in communication scheme $\pi_1^{\text{det}}(\gamma)$ (given in Figure 3) are corrupted, then at the end of the protocol, $\Phi \geq (1 - 3\gamma)\,N$.*

Proof. We have that there are $(1-\gamma)N$ forward rounds of the protocol which are not corrupted in which Φ increases by 1, while γN rounds which are corrupted in which Φ

Alice

Parameters: $\varepsilon = \frac{1}{3} - \gamma$

Bob

Input: Message $\mathbf{x} \in \{0,1\}^k$

Initialization:
- ▷ $\mathbf{y} \leftarrow E(\mathbf{x}) \in \{0,1\}^n$ . . . (1)
- ▷ $N \leftarrow \lceil n/3\varepsilon \rceil$ . . . (2)
- ▷ $T, T_{\text{right}}, T_{\text{wrong}} \leftarrow \emptyset$ . . . (3)

Initialization: (Bob)
- ▷ $N \leftarrow \lceil n/3\varepsilon \rceil$ . . . (2)
- ▷ $T \leftarrow \emptyset$ . . . (3)

Repeat N times

Alice:

if $T_{\text{wrong}} = \emptyset$ **then**
  $b \leftarrow y(\text{len}(T_{\text{right}}) + 1)$
else
  $b \leftarrow$ '0'
end if

b → $\widetilde{b}$

Bob:

$T \leftarrow T \circ \widetilde{b}$
if T ends in '00' **then**
  Backtrack last 3 bits in T
end if

$\widetilde{b}$ → $\widetilde{b}$

Alice:

$T \leftarrow T \circ \widetilde{b}$
if T ends in '00' **then**
  Backtrack last 3 bits in T
end if
Set $T_{\text{right}}, T_{\text{wrong}}$ appropriately

End of repeat

Output: $E^{-1}(T[1, \cdots, n])$

(1) $E(\mathbf{x})$ is a simple encoding of $\mathbf{x}$ such that $E(\mathbf{x})$ does not contain any consecutive '0's. One way to do this: add a '1' between two consecutive bits, making $n = 2k$. We will refer to the bits of $\mathbf{y}$ as $y(i)$. For $i > n$, we will assume $y(i) =$ '1'.

(2) N is the number of rounds

(3) T is the transcript as maintained by Bob. However, Alice is able to decompose T as $T = T_{\text{right}} \circ T_{\text{wrong}}$, where T_{right} is the largest prefix of T which exactly matches the prefix of $\mathbf{y}$ of the same length, and T_{wrong} is the remaining part in T. Basically, T_{wrong} is the part of the transcript which starts with an incorrectly received bit, and so all the following bits have to be erased before proceeding further.

Figure 3 Communication scheme with complete feedback : $\pi_1^{\text{det}}(\gamma)$.

decreases by at most 2. Thus, by Proposition 8, we see that at the end of the protocol,

$$\Phi(T) \geq 1 \cdot (1-\gamma)N - 2 \cdot \gamma N = (1-3\gamma)\,N$$

◀

Thus, by the above Corollary, if $\gamma = 1/3 - \varepsilon$ fraction of the forward rounds of protocol $\pi_1^{\text{det}}(\gamma)$ are corrupted, then at the end of the protocol we will have,

$$\text{len}(T_{\text{right}}) \geq \Phi(T) \geq 3\varepsilon N$$

By our choice of N, we have that $3\varepsilon N \geq n$. Therefore, at the end of the protocol, $\text{len}(T_{\text{right}}) \geq n$, meaning that the first n bits of T are exactly $\mathbf{y}$. Hence, Bob is able to decode $\mathbf{x}$ correctly by just applying E^{-1} on the first n bits of T.

4 Randomized communication scheme with partial feedback

In this section, we prove Theorem 5 by giving a randomized protocol $\pi_\delta^{\text{rand}}(\gamma)$ with δ-feedback (for any $\delta \in (0,1]$), that tolerates an error fraction of $\gamma = 1/3 - \varepsilon$ (where $\varepsilon > 0$) and has a rate of $\Theta(\varepsilon\delta)$. The full details of $\pi_\delta^{\text{rand}}(\gamma)$ can be found in Figure 4. The main idea is as follows:

We wish to simulate $\pi_1^{\text{det}}(\gamma)$ with a smaller feedback fraction. We break the protocol into $N_0 = \lceil 2n/3\varepsilon \rceil$ iterations, where each iteration roughly corresponds to one forward and feedback round of $\pi_1^{\text{det}}(\gamma)$. In each iteration, Alice sends $D = \lceil 2/\delta \rceil$ bits, namely $\mathbf{c} = (c_1, \cdots, c_D) = b^D$ (i.e. D copies of bit b she would have sent in π_1^{det}). Bob receives a set of symbols $\widetilde{\mathbf{c}} = (\widetilde{c}_1, \cdots, \widetilde{c}_D)$. He uses a 'soft decoding' scheme to obtain $\widetilde{b}$ which is his interpretation of what b must have been. Let m be the number of '1's present in $\widetilde{\mathbf{c}}$. If $m \leq D/2$, then he interprets $\widetilde{b}$ as '0' with probability $1 - 2m/D$, and if $m > D/2$, then he interprets $\widetilde{b}$ to be '1' with probability $2m/D - 1$. In other cases, Bob interprets $\widetilde{b}$ to be '?'. If $\widetilde{b} \neq$ '?', then Bob then makes appropriate progress on the protocol $\pi_1^{\text{det}}(\gamma)$. As feedback, Bob sends back the value of $\widetilde{b}$ (which takes 2 bits of feedback). Thus, the entire protocol uses $N = \lceil 2/\delta \rceil N_0$ number of forward bits of communication and $2N_0$ bits of feedback, and thus it uses $(2/\lceil 2/\delta \rceil)$-feedback (which is at most δ-feedback), and has rate $\Theta(\varepsilon\delta)$. All that remains to show now is that this protocol tolerates an error fraction of $\gamma = 1/3 - \varepsilon$.

Correctness of $\pi_\delta^{\text{rand}}(\gamma)$

We show that for any $\delta \in (0,1]$ and $\gamma < 1/3$, the communication scheme $\pi_\delta^{\text{rand}}(\gamma)$ tolerates an error fraction of γ with constant rate. This immediately implies Theorem 5.

▶ **Theorem 10.** *For any $\delta \in (0,1]$ and $\gamma = 1/3 - \varepsilon$ (where $\varepsilon > 0$), the communication scheme $\pi_\delta^{\text{rand}}(\gamma)$ (from Figure 4) tolerates an error fraction of γ with rate being $\Theta(\varepsilon\delta)$.*

Proof. Suppose that an adversary corrupts at most $\gamma N = (1/3 - \varepsilon)\,N$ of Alice's transmissions in $\pi_\delta^{\text{rand}}(\gamma)$ (recall that $N = \lceil 2/\delta \rceil N_0$). We will show that for any fixed error pattern, Bob can successfully recover Alice's message with high probability.

Consider the potential function Φ that was used for proving the correctness of protocol $\pi_1^{\text{det}}(\gamma)$. In any iteration $1 \leq i \leq N_0$ of the simulating protocol $\pi_\delta^{\text{rand}}(\gamma)$ (in Figure 4) above, with e fraction of errors (i.e. with eD errors), the potential function Φ changes by an amount X_i given as follows (see Proposition 8):

- *if* $e_i \leq 1/2$, then with probability $1 - 2e_i$, Φ increases by at least $X_i = 1$, and with probability $2e_i$ the potential function remains unchanged ($X_i = 0$). In expectation, Φ increases by at least $\mathbb{E}[X_i] = (1 - 2e_i)$.

Alice

Parameters: $\varepsilon = \frac{1}{3} - \gamma$

Bob

Alice

Input: Message $\mathbf{x} \in \{0,1\}^k$

Initialization:

- $\triangleright$ $\mathbf{y} \leftarrow E(\mathbf{x}) \in \{0,1\}^n$ . . . (1)
- $\triangleright$ $N_0 \leftarrow \lceil 2n/3\varepsilon \rceil$ . . . (2)
- $\triangleright$ $T, T_{\text{right}}, T_{\text{wrong}} \leftarrow \emptyset$ . . . (3)
- $\triangleright$ $D \leftarrow \lceil 2/\delta \rceil$ . . . (4)

Bob

Initialization:

- $\triangleright$ $N_0 \leftarrow \lceil 2n/3\varepsilon \rceil$ . . . (2)
- $\triangleright$ $T \leftarrow \emptyset$ . . . (3)
- $\triangleright$ $D \leftarrow \lceil 2/\delta \rceil$ . . . (4)

——— **Repeat N_0 times** ———

Alice:

if $T_{\text{wrong}} = \emptyset$ **then**
  $b \leftarrow y(\text{len}(T_{\text{right}}) + 1)$
else
  $b \leftarrow$ '0'
end if
$\mathbf{c} \leftarrow b^D$

Alice → Bob: $\mathbf{c}$ → $\widetilde{\mathbf{c}}$

Bob:

$m \leftarrow \left|\{i : \widetilde{c}_i = 1\}\right|$
if $m \le D/2$ **then**
  $\widetilde{b} \leftarrow \begin{cases} \text{‘0’} & \text{w.p. } 1 - 2m/D \\ \text{‘?’} & \text{w.p. } 2m/D \end{cases}$
else
  $\widetilde{b} \leftarrow \begin{cases} \text{‘1’} & \text{w.p. } 2m/D - 1 \\ \text{‘?’} & \text{w.p. } 2 - 2m/D \end{cases}$
end if
if $\widetilde{b} \neq$ '?' **then**
  $T \leftarrow T \circ \widetilde{b}$
  if T ends in '00' **then**
    Backtrack last 3 bits in T
  end if
  Set T_{right}, T_{wrong} appropriately
end if

Bob → Alice: $\widetilde{b}$ → $\widetilde{b}$

Alice:

if $\widetilde{b} \neq$ '?' **then**
  $T \leftarrow T \circ \widetilde{b}$
  if T ends in '00' **then**
    Backtrack last 3 bits in T
  end if
  Set T_{right}, T_{wrong} appropriately
end if

——— **End of repeat** ———

Bob: **Output:** $E^{-1}(T[1, \cdots, n])$

(1), (2) and (3) as defined in the scheme $\pi_1^{\text{det}}(\gamma)$ (Figure 3)

(4) D is the number of bits sent in every iteration.

Figure 4 Randomized communication scheme with complete feedback : $\pi_\delta^{\text{rand}}(\gamma)$.

- *if* $e_i > 1/2$, then with probability $2e_i - 1$, Φ decreases by at most 2, i.e. increases by at least $X_i = -2$, and with probability $2 - 2e_i$ the potential function remains unchanged ($X_i = 0$). In expectation, Φ increases by at least $\mathbb{E}[X_i] = -(4e_i - 2)$.

Suppose we use N_0 phases of the above protocol and suppose that the fraction of errors that the adversary makes in each of these N_0 phases is $e_1, e_2, \cdots, e_{N_0}$, respectively. Let $S_1 = \{i : e_i \leq 1/2\}$ and $S_2 = \{i : e_i > 1/2\}$.

Thus, the expected value of the potential function at the end of N_0 phases will be,

$$\begin{aligned}
\mathbb{E}[\Phi] &\geq \sum_{i=1}^{N} \mathbb{E}[X_i] \\
&= \sum_{i \in S_1} (1 - 2e_i) - \sum_{j \in S_2} (4e_j - 2) \\
&= \sum_{i \in S_1 \cup S_2} (1 - 2e_i) - \sum_{j \in S_2} (2e_j - 1) \\
&= N_0 - \sum_{i=1}^{N_0} 2e_i - \sum_{j \in S_2} (2e_j - 1)
\end{aligned}$$

We want to bound $\mathbb{E}[\Phi]$ from below. Firstly, $\sum_{i=1}^{N_0} e_i \leq N_0\,(1/3 - \varepsilon)$. Also, since each $e_j \leq 1$, we have that $|S_2| \geq \sum_{j \in S_2} e_j$, and hence $\sum_{j \in S_2} (2e_j - 1) \leq \sum_{j \in S_2} e_j \leq \sum_{i=1}^{N_0} e_i \leq N_0\,(1/3 - \varepsilon)$. Thus, from above equations we have that,

$$\mathbb{E}[\Phi] \geq N_0 - 2N_0 \left(\frac{1}{3} - \varepsilon\right) - N_0 \left(\frac{1}{3} - \varepsilon\right) = 3\varepsilon N_0.$$

Since we choose $N_0 = \lceil 2n/3\varepsilon \rceil$, we have that $\mathbb{E}[\Phi] \geq 2n$. Also, note that either $X_i \in [0, 2]$ or $X_i \in [-2, 0]$ for all i. Thus, using Hoeffding's concentration inequality, we have

$$\begin{aligned}
\Pr[\Phi \geq n] &\geq \Pr[X_1 + X_2 + \cdots + X_{N_0} \geq n] \\
&\geq 1 - \Pr\left[\left|X_1 + X_2 + \cdots + X_{N_0} - \mathbb{E}\left[\textstyle\sum_{i=1}^{N_0} X_i\right]\right| \geq n\right] \\
&\geq 1 - 2e^{-2n^2/4N_0} \\
&\geq 1 - 2e^{-3\varepsilon n/4} \qquad [\text{putting } N_0 = \lceil 2n/3\varepsilon \rceil]
\end{aligned}$$

Recall that if $\Phi \geq n$ at the end of the protocol, then Bob is able to decode Alice's message correctly. Thus, we conclude that $\pi_\delta^{\text{rand}}(\gamma)$ works with a failure probability of at most $\exp(-\Omega(k))$ (since $n = \Theta(k)$). ◀

▶ Remark. In the application of the Hoeffding's inequality we required the error pattern to be fixed, that is, the adversary pre-commits to the error pattern (although unknown to Alice and Bob). We feel that this is only a technical difficulty and it should be generalizable to adaptive adversaries. Nevertheless, in the next section, we de-randomize this protocol, although with a smaller error fraction. For deterministic protocols, it does not matter whether the errors are adaptive or not, as we require a worst-case guarantee.

5 Deterministic communication schemes with partial feedback

In this section, we prove Theorem 6 by giving a deterministic protocol $\pi_\delta^{\text{det}}(\gamma)$ with δ-feedback (for any $\delta \in (0, 1]$). We first give a deterministic communication scheme with δ-feedback for $\delta = 2/(4r + 3)$ for any $r \in \mathbb{N}$ (this scheme is described in Section 5.1), which we obtain by a certain derandomization of π_δ^{rand}. Next, for $\frac{2}{4r+7} < \delta < \frac{2}{4r+3}$, we give a communication scheme that *interpolates* between $\pi^{\text{det}}_{2/(4r+3)}$ and $\pi^{\text{det}}_{2/(4r+7)}$ (the full details are described in Section 5.2).

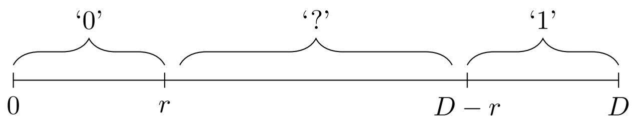
5.1 Deterministic communication scheme for $\delta = 2/(4r + 3)$

In this section, we present a deterministic communication scheme $\pi^{\text{det}}_{\delta}(\gamma)$, where $\delta = 2/(4r+3)$ for any $r \in \mathbb{N}$. We show that this scheme tolerates an error fraction of $\gamma = (r+1)/(4r+3) - \varepsilon$ and has rate $\Theta(\varepsilon\delta)$. We obtain this by instantiating the communication scheme $\pi^{\text{det}}_{\delta}(\gamma)$ in Figure 5, with $D_i = (4r + 3)$ and $r_i = r$ for all i. The main idea behind the protocol is as follows:

We will consider a protocol identical to $\pi^{\text{rand}}_{\delta}$, except that in each of the N_0 iterations, Bob chooses a value ('0,' '1,' or '?') for $\widetilde{b}$ in a *deterministic* fashion. In particular, in any iteration, Alice sends $D = (4r + 3) = 2/\delta$ bits (say $\mathbf{c}$ given by b^D), which Bob then receives as $\widetilde{\mathbf{c}}$. Let m be the number of 1's in $\widetilde{\mathbf{c}}$. Bob chooses $\widetilde{b}$ as follows:

$$\widetilde{b} \quad \leftarrow \quad \begin{cases} \text{`0'} & \text{if } m \le r \\ \text{`1'} & \text{if } m \ge D - r \\ \text{`?'} & \text{if } r < m < D - r \end{cases}$$

Thus, $r + 1$ is the minimum number of bits of $\mathbf{c}$ that an adversary must corrupt in order to force Bob to interpret the round as a '?', and $D - r$ is the minimum number of bits of $\mathbf{c}$ that must be corrupted in order to force Bob to interpret the round as opposite of the bit that Alice intended to send. The decoding strategy of Bob is summarized in the following figure:



Thus, the entire protocol uses $N = DN_0 = (2/\delta)N_0$ number of forward bits of communication and $2N_0$ bits of feedback, and thus it uses δ-feedback. We will choose $N_0 = \lceil n/3\varepsilon \rceil$, and hence the rate is $\Theta(\varepsilon\delta)$. So all that remains to show now is that this protocol tolerates an error fraction of $\gamma = (r + 1)/(4r + 3) - \varepsilon$. The following proposition (which is an analogue of Corollary 9) will be useful in proving the same.

▶ **Proposition 11.** *For $\delta = 2/(4r + 3)$, if γ fraction of Alice's transmissions in communication scheme $\pi^{\text{det}}_{\delta}(\gamma)$ (from Figure 5) are corrupted, then at the end of the protocol, $\Phi \ge \left(1 - \frac{(4r+3)\gamma}{r+1}\right) N_0$.*

Proof. In the communication scheme $\pi^{\text{det}}_{\delta}(\gamma)$, any iteration is considered *correctly decoded* if $\widetilde{b}$ is set to be the bit that appears in $\mathbf{c}$, while it is said to be *incorrectly decoded* if $\widetilde{b}$ is set to the opposite bit. An iteration is considered *ambiguous* if $\widetilde{b}$ is set to be '?'.

Now, suppose an adversary corrupts the scheme such that AN_0, BN_0, and CN_0 are the number of iterations that are ambiguous, correctly decoded and incorrectly decoded respectively. Note that the adversary has a total budget of γDN_0 corruptions, where γ is the overall error fraction of the protocol. Thus, we have the following constraints:

$$A + B + C = 1 \qquad (1)$$

$$(r + 1)A + (D - r)C \le \gamma D \qquad (2)$$

Alice

Input: Message $\mathbf{x} \in \{0,1\}^k$

Initialization:

$\triangleright$ $\mathbf{y} \leftarrow E(\mathbf{x}) \in \{0,1\}^n$ ... (1)

$\triangleright$ $N_0 \leftarrow \Theta_{\delta,\gamma}(n)$ ... (2)

$\triangleright$ $T, T_{\text{right}}, T_{\text{wrong}} \leftarrow \emptyset$ ... (3)

$\triangleright$ $\{D_i\}_{i\in[N_0]}, \{r_i\}_{i\in[N_0]}$ ... (4)

Bob

Initialization:

$\triangleright$ $N_0 \leftarrow \Theta_{\delta,\gamma}(n)$ ... (2)

$\triangleright$ $T \leftarrow \emptyset$ ... (3)

$\triangleright$ $\{D_i\}_{i\in[n]}, \{r_i\}_{i\in[n]}$ ... (4)

Repeat N_0 times

Alice:

if $T_{\text{wrong}} = \emptyset$ **then**
  $b \leftarrow y(\text{len}(T_{\text{right}}) + 1)$
else
  $b \leftarrow$ '0'
end if
$\mathbf{c} \leftarrow b^{D_i}$

$\mathbf{c}$ → $\widetilde{\mathbf{c}}$

Bob:

$m \leftarrow \left|\{i : \widetilde{c}_i = 1\}\right|$

$$\widetilde{b} \leftarrow \begin{cases} \text{'0'} & \text{if } m \leq r_i \\ \text{'1'} & \text{if } m \geq D_i - r_i \\ \text{'?'} & \text{if } r_i < m < D_i - r_i \end{cases}$$

if $\widetilde{b} \neq$ '?' **then**
  $T \leftarrow T \circ \widetilde{b}$
  if T ends in '00' **then**
    Backtrack last 3 bits in T
  end if
  Set $T_{\text{right}}, T_{\text{wrong}}$ appropriately
end if

$\widetilde{b}$ → $\widetilde{b}$

Alice:

if $\widetilde{b} \neq$ '?' **then**
  $T \leftarrow T \circ \widetilde{b}$
  if T ends in '00' **then**
    Backtrack last 3 bits in T
  end if
  Set $T_{\text{right}}, T_{\text{wrong}}$ appropriately
end if

End of repeat

Bob: **Output:** $E^{-1}(T[1, \cdots, n])$

(1), (2) and (3) as defined in the scheme π_1^{det} (Figure 3)

(4) D_i is the number of forward bits sent in iteration i. r_i is the deterministic threshold used by Bob to interpret the received stream as either '0', '?' or '1' in iteration i. The total number of forward bits is $\sum_i D_i$, and total number of feedback bits is $2N_0$. The exact choice of D_i's and r_i's will depend on δ and γ.

Figure 5 Template for deterministic communication scheme with complete feedback : $\pi_\delta^{\text{det}}(\gamma)$.

Equation 1 follows because the total number of iterations is N_0. Inequality 2 follows because, we may assume without loss of generality, that an adversary corrupts 0 bits for an iteration that is correctly decoded, $D - r$ bits for an iteration that is incorrectly decoded, and $r + 1$ bits for an iteration that is ambiguous. This is because these are the minimum number of bits needed to be corrupted for each category, and an adversary cannot possibly gain by corrupting more than the minimum.

From Proposition 8, we have that $\Phi \geq (B - 2C)N_0$. And so, we wish to lower bound $B - 2C$. For this, we subtract $\frac{3}{D-r}$ times Inequality (2) from Equation (1) to obtain

$$B - 2C - A\left(\frac{3(r+1)}{D-r} - 1\right) \geq 1 - \frac{3\gamma D}{D-r}$$

Since $D = 4r + 3$, it follows that $\frac{3(r+1)}{D-r} = 1$, and so,

$$B - 2C \geq 1 - \frac{3\gamma D}{D-r} = 1 - \frac{(4r+3)\gamma}{r+1}$$

Thus, we get that at the end of the protocol, $\Phi \geq \left(1 - \frac{(4r+3)\gamma}{r+1}\right) N_0$. ◀

By the above Proposition, if $N_0 = \lceil n/3\varepsilon \rceil$ and $\gamma = \frac{r+1}{4r+3} - \epsilon$ fraction of Alice's transmissions in $\pi_\delta^{\text{det}}(\gamma)$ are corrupted, then at the end of the protocol we will have,

$$\text{len}(T_{\text{right}}) \geq \Phi \geq \left(1 - \frac{4r+3}{r+1}\left(\frac{r+1}{4r+3} - \epsilon\right)\right) N_0 = \frac{(4r+3)\epsilon}{r+1} N_0 \geq n$$

This guarantees that at the end of the protocol $\text{len}(T_{\text{right}}) \geq n$ meaning the first n bits of T are exactly $\mathbf{y}$ and Bob can correctly decode Alice's message by applying E^{-1} on the first n bits of T.

5.2 Deterministic communication schemes for all $\delta \in (0, 1]$

In the previous section we gave a deterministic communication scheme $\pi_\delta^{\text{det}}(\gamma)$ for $\delta = 2/(4r+3)$, and showed that one can tolerate an error fraction of up to $\frac{r+1}{4r+3}$. In the section, we give a communication scheme $\pi_\delta^{\text{det}}(\gamma)$ for any feedback fraction $\delta \in \left(\frac{2}{4r+7}, \frac{2}{4r+3}\right)$, that can tolerate an error fraction of $\gamma = \frac{\delta(r+1)}{2} - \varepsilon$ (where $\varepsilon > 0$).

The key is to "interpolate" the protocols π_δ^{det} that we obtain for $\delta = \frac{2}{D}$ between $D = 4r+3$ and $D = 4(r+1)+3$. In particular, we will have that for the first qN_0 iterations, we use $D_i = 4r+7$ and $r_i = r+1$, and the later $(1-q)N_0$ iterations, we use $D_i = 4r+3$ and $r_i = r$. We let $N_0 \geq (r+1)\delta n/2\varepsilon$. This also gives that the rate is $\Theta(\varepsilon\delta)$ (since $\delta \geq 2/(4r+7)$).

Let A_1N_0, B_1N_0, and C_1N_0 be the number of iterations that are ambiguous, correctly decoded and incorrectly decoded respectively in the first qN_0 iterations. Similarly, let A_2N_0, B_2N_0, and C_2N_0 be the number of iterations that are ambiguous, correctly decoded and incorrectly decoded respectively in the later $(1-q)N_0$ iterations. Also note that the adversary has a total budget of $\gamma(q(4r+7) + (1-q)(4r+3))N_0$ corruptions, where γ is the overall error fraction of the protocol.

Note that $\delta = {}^2/_{(q(4r+7)+(1-q)(4r+3))} = {}^2/_{(4q+4r+3)}$ and hence $q = \frac{1}{2\delta} - \frac{(4r+3)}{4}$. We have the following constraints:

$$\begin{aligned} A_1 + B_1 + C_1 &= q && (3) \\ A_2 + B_2 + C_2 &= 1 - q && (4) \\ (r+2)A_1 + (r+1)A_2 + (3r+6)C_1 + (3r+3)C_2 &\leq \gamma(4q + (4r+3)) && (5) \end{aligned}$$

From Proposition 8, we have that the potential at the end of the protocol satisfies $\Phi \geq (B_1 + B_2 - 2C_1 - 2C_2)N_0$. Thus, we wish to lower bound $(B_1 + B_2 - 2C_1 - 2C_2)$. We add equations 3 and 4 and subtract (1/(r+1)) times inequality 5, to get,

$$B_1 + B_2 - 2C_1 - 2C_2 - \frac{A_1}{r+1} - \frac{3C_1}{r+1} \geq 1 - \frac{\gamma(4q + 4r + 3)}{r+1} = 1 - \frac{2\gamma}{(r+1)\delta}$$

Since $A_1, C_1 \geq 0$ and $\gamma = (r+1)\delta/2 - \varepsilon$, we get that,

$$\Phi \geq (B_1 + B_2 - 2C_1 - 2C_2)N_0 \geq \frac{2\varepsilon N_0}{(r+1)\delta} \geq n$$

where the last inequality follows because we chose $N_0 \geq (r+1)\delta n/2\varepsilon$. This guarantees that at the end of the protocol $\mathrm{len}(T_{\mathrm{right}}) \geq \Phi \geq n$ meaning the first n bits of T are exactly $\mathbf{y}$ and Bob can correctly decode Alice's message by applying E^{-1} on the first n bits of T.

5.3 Putting it all together

Proof of Theorem 6. In Section 5.2, we showed that when $\frac{2}{4r+7} \leq \delta \leq \frac{2}{4r+3}$, we have that $\Gamma^{\mathrm{det}}(\delta) \geq (r+1)\delta/2$. But from Section 5.1, we obtained that $\Gamma^{\mathrm{det}}\left(\frac{2}{4r+7}\right) \geq \frac{r+2}{4r+7}$, and thus by monotonicity of $\Gamma^{\mathrm{det}}(\delta)$, we have that for all $\delta \geq \frac{2}{4r+7}$, $\Gamma^{\mathrm{det}}(\delta) \geq \frac{r+2}{4r+7}$. Combining the two results we get that,

$$\forall r \in \mathbb{Z}_{\geq 0} \quad \forall \delta \in \left[\frac{2}{4r+7}, \frac{2}{4r+3}\right] \quad \Gamma^{\mathrm{det}}(\delta) \geq \max\left\{\frac{(r+1)\delta}{2}, \frac{r+2}{4r+7}\right\}$$

◀

6 Discussion

We have introduced the notion of communication schemes under partial noiseless feedback as a natural interpolation between two familiar settings, namely, the problem of transmission over a binary channel with adversarial errors (achievable by the use of error-correcting codes) as well as the problem of transmission over a binary feedback channel (achievable by the protocol in [1]). The results of this work show that the availability of a non-zero fraction of feedback, however small, allows Alice to communicate a message to Bob in a way that tolerates an adversarial error fraction of more than $1/4$, the limit for error-correcting codes. An upper bound of $1/3$ on the tolerable error fraction for a deterministic communication scheme holds for all feedback fractions $0 \leq \delta \leq 1$, and we show how to obtain a randomized communication scheme that tolerates any error fraction up to $1/3$. Furthermore, we have shown deterministic communication schemes that tolerates error fractions of up to $f(\delta)$, where f is a monotonically increasing, piecewise linear, continuous function with $f(0) = 1/4$ and $f(1) = 1/3$. In particular, we have shown that our deterministic scheme can tolerate any error fraction less than $1/3$ for all $\delta \geq 2/3$.

Our work points to several interesting directions for further investigation.

- Is the bound $\Gamma^{\mathrm{det}}(\delta) \geq f(\delta)$ provided by Theorem 6 is tight? Currently we only know that $f(\delta) \leq \Gamma^{\mathrm{det}}(\delta) \leq 1/3$ for $\delta < 2/3$. In particular, is it possible for a deterministic communication scheme to tolerate error fractions up to $1/3$ for *all* δ in the way that the randomized scheme $\pi_\delta^{\mathrm{rand}}$ can. One possible direction is to derandomize $\pi_\delta^{\mathrm{rand}}$ in a more clever way that avoids loss in the error fraction tolerance. Otherwise, is it possible to prove better upper bounds than $1/3$ on $\Gamma^{\mathrm{det}}(\delta)$?

- In this work, we have considered only protocols over binary alphabets. It will be interesting to determine the limits on the tolerable error fraction for communication schemes with partial feedback that use symbols from non-binary alphabets as well as to find explicit communication schemes in this setting. Over an alphabet of size q, we know that error correcting codes can tolerate an error fraction of $(1 - 1/q)/2$, whereas, with noiseless feedback, one can tolerate an error fraction of up to $1/2$ (see [7] for example).
- In this work we only studied the model of noiseless feedback. It will be interesting to understand what bounds could be proved for the noisy feedback model, where the adversary is allowed to corrupt the feedback as well. An immediate question is whether it is even possible to correct more than $1/4$ fraction errors in this model (for any amount of feedback - where we measure the error budget as a fraction of the length of the entire protocol and not just Alice's transmissions).

Acknowledgements. We would like to thank Madhu Sudan for many helpful discussions and comments on the draft. PK would like to thank Dana Moshkovitz for general guidance and encouragement. AV would like to thank Venkatesan Guruswami for helpful conversations.

PK was supported in part by NSF grants CCF-1218547 and CCF-1420956. Part of this work was done while AV was visiting Microsoft Research New England, Cambridge, MA. AV was supported in part by NSF grant CCF-0963975.

References

1 Elwyn R. Berlekamp. *Block Coding with Noiseless Feedback*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, USA, 1964.

2 Zvika Brakerski and Yael Tauman Kalai. Efficient interactive coding against adversarial noise. In *53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012*, pages 160–166, 2012.

3 Zvika Brakerski, Yael Tauman Kalai, and Moni Naor. Fast interactive coding against adversarial noise. *J. ACM*, 61(6):35, 2014.

4 Zvika Brakerski and Moni Naor. Fast algorithms for interactive coding. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2013, New Orleans, Louisiana, USA, January 6-8, 2013*, pages 443–456, 2013.

5 Mark Braverman and Klim Efremenko. List and unique coding for interactive communication in the presence of adversarial noise. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*, pages 236–245, 2014.

6 Mark Braverman and Anup Rao. Toward coding for maximum errors in interactive communication. *IEEE Transactions on Information Theory*, 60(11):7248–7255, 2014.

7 Klim Efremenko, Ran Gelles, and Bernhard Haeupler. Maximal noise in interactive communication over erasure channels and channels with feedback. In *Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science, ITCS 2015, Rehovot, Israel, January 11-13, 2015*, pages 11–20, 2015.

8 Ran Gelles and Bernhard Haeupler. Capacity of interactive communication over erasure channels and channels with feedback. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015*, pages 1296–1311, 2015.

9 Ran Gelles, Ankur Moitra, and Amit Sahai. Efficient coding for interactive communication. *IEEE Transactions on Information Theory*, 60(3):1899–1913, 2014.

10 Mohsen Ghaffari and Bernhard Haeupler. Optimal error rates for interactive coding II: efficiency and list decoding. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*, pages 394–403, 2014.
11 Mohsen Ghaffari, Bernhard Haeupler, and Madhu Sudan. Optimal error rates for interactive coding I: adaptivity and other settings. In *Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 – June 03, 2014*, pages 794–803, 2014.
12 E. N. Gilbert. A comparison of signalling alphabets. *Bell System Technical Journal*, 31:504–522, 1952.
13 Bernhard Haeupler. Interactive channel capacity revisited. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*, pages 226–235, 2014.
14 Gillat Kol and Ran Raz. Interactive channel capacity. In *Symposium on Theory of Computing Conference, STOC'13, Palo Alto, CA, USA, June 1-4, 2013*, pages 715–724, 2013.
15 Morris Plotkin. Binary codes with specified minimum distance. *IRE Transactions on Information Theory*, 6(4):445–450, 1960.
16 Leonard J. Schulman. Communication on noisy channels: A coding theorem for computation. In *33rd Annual Symposium on Foundations of Computer Science, Pittsburgh, Pennsylvania, USA, 24-27 October 1992*, pages 724–733, 1992.
17 Leonard J. Schulman. Deterministic coding for interactive communication. In *Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing, May 16-18, 1993, San Diego, CA, USA*, pages 747–756, 1993.
18 Leonard J. Schulman. Coding for interactive communication. *IEEE Transactions on Information Theory*, 42(6):1745–1756, 1996.
19 Claude E. Shannon. The zero error capacity of a noisy channel. *IRE Transactions on Information Theory*, 2(3):8–19, 1956.
20 Joel Spencer and Peter Winkler. Three thresholds for a liar. *Combinatorics, Probability & Computing*, 1:81–93, 1992.
21 R. R. Varshamov. Estimate of the number of signals in error correcting codes. *Dokl. Acad. Nauk SSSR*, 117:739–741, 1957.

A Upper bound on tolerable error fraction for mirror feedback

For completeness we give a proof of the following theorem, which was already proved by Berlekamp [1] and also later by Spencer-Winkler in the context of questions with liars [20].

▶ **Theorem 12.** *Any one-way communication scheme with noiseless feedback which uses the mirror feedback structure (that is, each feedback bit sent by Bob is simply the bit that he receives from Alice), cannot tolerate* $1/3$ *fraction of errors, as long as the input space of Alice has at least three elements.*

Proof. Let A, B and C be three possible inputs that Alice receives. Consider three parallel executions of any communication scheme with mirror feedback. We will show that there exists an adversary who can ensure that the view of Bob in two out of these three executions are the same by using only $1/3$ fraction errors in each of the executions. We describe the adversary below.

Let a_i, b_i and c_i be the bits sent by Alice in the i-th round of these three executions. Clearly at least two out of these three bits have to be the same. Thus, if one of these three bits is different from the other two, then the adversary will corrupt that bit, otherwise he will not corrupt any of the bits. This ensures that up to any round i, Bob's view of the protocol in all the three executions are the same.

Eventually, it might happen that the adversary has committed $1/3$ fraction errors on one of three executions. In this case, the adversary ignore that execution and focusses only on the other two. Suppose without loss of generality that the executions of A and B are still surviving. In the future rounds, whenever $a_i \neq b_i$, the adversary chooses to corrupt the execution where the number of error so far have been fewer.

Since the adversary makes at most one error in any round of the three executions, it is clear that the adversary never makes more than $1/3$ fraction of errors on any of the executions. Moreover, at the end of the executions, Bob will have identical views of the transcript in the executions corresponding to both A and B. ◀

B Changes in potential function

Proof of Proposition 8. We consider the following four exhaustive cases. Recall that T does not contain consecutive '0's.

Case 1 ($T_{\text{wrong}} = \emptyset$): Suppose Alice sends a bit b. In the case, that it is correctly received by Bob, $\text{len}(T_{\text{right}})$ increases by 1 and $\text{wt}(T_{\text{wrong}})$ remains 0. Thus, Φ increases by 1. Suppose it is incorrectly received by Bob. If $b = 1$ and the last bit of T_{right} is 0, then note that Bob would have received two consecutive 0's and hence will backtrack two symbols and $\text{len}(T_{\text{right}})$ decreases by 2, while T_{wrong} remains $\emptyset$. In all other cases, T_{right} remains unchanged after the transmission, while T_{wrong} is either '1' or '0', which means $\text{len}(T_{\text{right}})$ remains unchanged and $\text{wt}(T_{\text{wrong}})$ becomes 2. In either case, Φ decreases by 2.

Case 2 (T_{wrong} ends in a '1'): In this case, the scheme ensures that Alice sends a '0'. If Bob correctly receives the bit, then the unit '1' is now converted to '10'. Thus, $\text{wt}(T_{\text{wrong}})$ decreases by 1, causing Φ to increase by 1. On the other hand, if Bob does not receive the correct bit, another '1' is added to T_{wrong} which means $\text{wt}(T_{\text{wrong}})$ increases by 2, causing Φ to decrease by 2.

Case 3 (T_{wrong} is '0'): In this case, the scheme ensures that Alice sends a '0'. If Bob correctly receives the '0', then $\text{len}(T_{\text{right}})$ goes down by at most 1, but $\text{wt}(|T_{\text{wrong}}|)$ goes down by 2, implying that Φ increases by at least 1. On the other hand, if Bob incorrectly receives a '1', then $\text{wt}(T_{\text{wrong}})$ increases by 2 and thus Φ goes down by 2.

Case 4 (T_{wrong} ends in '10'): In this case, the scheme ensures that Alice sends a '0'. If Bob correctly receives the '0', then he will backtrack the '10', and thus $\text{wt}(T_{\text{wrong}})$ decreases by 1, implying that Φ increases by 1. On the other hand, if Bob receives a '1', then $\text{wt}(T_{\text{wrong}})$ increases by 2, and thus Φ decreases by 2. ◀