

Non-Uniform Robust Network Design in Planar Graphs

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Abstract

Robust optimization is concerned with constructing solutions that remain feasible also when a limited number of resources is removed from the solution. Most studies of robust combinatorial optimization to date made the assumption that every resource is equally vulnerable, and that the set of scenarios is implicitly given by a single budget constraint. This paper studies a robustness model of a different kind. We focus on **bulk-robustness**, a model recently introduced [3] for addressing the need to model non-uniform failure patterns in systems.

We significantly extend the techniques used in [3] to design approximation algorithm for bulk-robust network design problems in planar graphs. Our techniques use an augmentation framework, combined with linear programming (LP) rounding that depends on a planar embedding of the input graph. A connection to cut covering problems and the dominating set problem in circle graphs is established. Our methods use few of the specifics of bulk-robust optimization, hence it is conceivable that they can be adapted to solve other robust network design problems.

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1 Introduction

Robust optimization is concerned with finding solutions that perform well in any one of a given set of scenarios. Many paradigms were proposed for robust optimization in the last decades. Some models assume uncertainty in the cost structure of the optimization problem. Such robust models typically have as scenarios different cost structures for the resources in the system, and ask to find a solution whose worst-case cost is as small as possible. Another kind of robustness postulates uncertainty in the feasible set of the optimization problem. Typically, in such models scenarios correspond to different realizations of the feasible set. A minimum-cost solution is then sought that is feasible in any possible realization of the feasible set.

This paper deals with the latter class of robust models, i.e. ones that incorporate uncertainty in the feasible set. Concretely, we are interested in **robust network design problems**, that are generally defined as follows. The input specifies a graph $G = (V, E)$, a set of **failure scenarios** Ω , consisting of subsets of the nodes and edges of G , and some connectivity requirement. The goal is to find a minimum-cost subgraph of G satisfying the connectivity requirement, even when the elements in any one single scenario are removed from the solution. Different problems are obtained for different types of connectivity requirement and when different representations of the scenario set are assumed.



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Most existing models of robust network design assume **uniform scenario sets**. Given interdiction costs for the resources, and a bound $B \in \mathbb{Z}_{\geq 0}$, such models assume that the adversary can remove **any** subset of resources of interdiction cost at most B . In fact, unit interdiction costs are almost always assumed. While such uniform robust network design problems often enjoy good algorithms, they also often do not reflect realistically the uncertainty in the modeled system, which feature highly non-uniform failure patterns.

In a recent paper, Adjashvili, Stiller and Zenklusen [3] introduced a new model for robust network design called **bulk-robustness**, specifically designed to model such highly non-uniform failure patterns. In bulk-robust optimization failure scenarios are given explicitly, as a list of subsets of the resources. These subsets may be arbitrary, and in particular, they are allowed to vary in size. The goal, as in robust network design problems, is to find a minimum-cost set of resources that contains a feasible solution, even when the resources in any one of the scenarios are removed.

The authors justify the model by bringing many example from health care optimization, computer systems, digitally controlled systems, military applications, financial systems and more. For example, in computer systems, different components of the network rely on the different resources, such a databases, power sources etc. At down-times of such resources, the components that can not operate properly are exactly those that depend of the downed resource. While no uniform failure model can capture such failure patterns, bulk-robustness seems to be a suitable choice.

In [3] the authors study a number of problems in the bulk-robust model, including the s - t connection problem, and the spanning tree problem. In particular, the approximability of these problems is studied in general graphs. The goal of this paper is to extend the existing tool set available for designing approximation algorithm for robust network design in this model. In this paper we focus on the important special case of **planar graphs**. We show a widely-applicable method for computing approximate solutions to bulk-robust network design problem in planar graphs.

1.1 Results and methods

For an integer $r \in \mathbb{Z}_{\geq 0}$ we let $[r] = \{1, \dots, r\}$ and $[r]_0 = \{0, 1, \dots, r\}$. The **bulk-robust network design problem** is defined as follows. Given an undirected graph $G = (V, E)$, a weight function $w : E \rightarrow \mathbb{Z}_{\geq 0}$, a connectivity requirement \mathcal{C} and a set of m scenarios F_1, \dots, F_m , each comprising a set of edges $F_i \subseteq E$, find a minimum-cost set of edges $S \subseteq E$, such that $(V, S \setminus F_i)$ satisfies \mathcal{C} for every $i \in [m]$. When \mathcal{C} is the requirement that two specific nodes $s, t \in V$ are to be connected we obtain the **bulk-robust s - t connection problem**. When \mathcal{C} is the requirement that all nodes are pair-wise connected, we obtain the **bulk-robust spanning tree problem**. Other bulk-robust problems such as **bulk-robust Steiner tree** and **bulk-robust survivable network design** are obtained analogously, by choosing the appropriate \mathcal{C} . We let $n = |V|$, and $k = \max_{i \in [m]} |F_i|$ denote the maximum size of a scenario. The parameter k is called **the diameter** of the instance. Adjashvili et. al. [3] proved the following theorem.

► **Theorem 1** (Adjashvili et. al. [3]). *The bulk-robust s - t connection problem admits a polynomial 13-approximation algorithm in the case $k = 2$. The bulk-robust spanning tree problem admits an $(\log n + \log m)$ -approximation algorithm.*

On the complexity side, the authors prove set cover hardness for all considered bulk-robust counterparts, implying a conditional $\log m$ lower bound on the approximation in general graphs. In terms of the parameter k , the authors show that in general graphs a

sub-exponential approximation factor is likely not achievable for certain variants of the bulk-robust s - t connection problem.

1.1.1 Contribution

Our goal is to prove a significant strengthening of Theorem 1 for the special case where the input graph is planar. Concretely, we prove the following theorem.

► **Theorem 2.** *The bulk-robust s - t connection and the bulk-robust spanning tree problems admit polynomial $O(k^2)$ -approximation algorithms on planar graphs.*

The latter result implies constant-factor approximation algorithms for the case of fixed k . In light of the results in [3] this is qualitatively best possible. To complement our algorithmic result we also prove the following stronger inapproximability result.

► **Theorem 3.** *For some fixed constant $c > 0$ it is NP-hard to approximate the bulk-robust s - t connection and bulk-robust spanning tree problems within a factor of ck , even when the input graphs are restricted to series-parallel graphs.*

The expression ck in the latter theorem can be replaced with the concrete expression $\frac{1}{2}k - 1 + \frac{1}{2k} - \epsilon$. Theorem 3 suggests that the dependence of the approximation factor on k is necessary.

For concreteness and clarity of the exposition we prove Theorem 2 for the s - t connection problem. We discuss the necessary minor adaptation needed for the spanning tree problem later. Furthermore, the methods we employ use very little of the particularities of bulk-robust optimization, and are thus likely to be adaptable to other robust problems on planar graphs.

1.1.2 Our methods

Our algorithm is a combination of combinatorial and LP-based techniques. On the top level, our algorithm employs an augmentation framework, which constructs a feasible solution by solving a sequence of relaxations of the problem. The lowest level corresponds to a simple polynomial problem, while the last level corresponds to the original instance. The idea of augmentation is well known in the literature of network design (see e.g. [26, 14]). We use here the variant of the augmentation framework defined for bulk-robust optimization in [3].

We solve each stage of the augmentation problem by considering a suitable set cover problem, the analysis of which comprises the core technical contribution of the paper. Using a combinatorial transformation that amounts to finding certain shortest paths in the graph, we obtain a simpler covering problem, which we call **the link covering problem**. The remainder of the algorithm relies on the analysis of the standard LP relaxation of the latter problem. Using properties of planar graphs, we show that the obtained LP has an integrality gap of $O(k)$, and that a solution of this quality can be obtained in polynomial time. Our rounding procedure relies on a decomposition according to the planar embedding of the graph, and a connection to the dominating set problem in circle graphs, for which we develop an LP-respecting constant-factor approximation algorithm. The line of our proof follows that of the proof in [3]. Our main technical contribution can hence be seen in the additional techniques developed to deal with planar graphs. As we mentioned before, these new techniques seem more general than the bulk-robust model, and are likely to be applicable to other network design problems in planar graphs.

The proof of Theorem 3 relies on a reduction from the minimum vertex cover problem in m -uniform, m -partite hypergraphs.

1.1.3 Organization

In the remainder of this section we review related work. In Section 2 we present the algorithm for the bulk-robust s - t connection problem, and prove Theorem 2 for this case. The required modification for the bulk-robust spanning tree problem and possible extensions of our results are discussed in Section 3. The proof of Theorem 3 is brought in Appendix B.

1.2 Related work

For a comprehensive survey on general models for robust optimization we refer the reader to the paper of Bertsimas, Brown and Caramanis [7].

Robustness discrete optimization with cost uncertainty was initially studied by Kouvelis and Yu [22] and Yu and Yang [27]. These works mainly consider the min-max model, where the goal is to find a solution that minimizes the worst-case cost according to the given set of cost functions. See the paper of Aissi, Bazgan and Vanderpooten [5] for a survey. A closely related class of multi-budgeted problem has received considerable attention recently (see e.g. [25, 24, 8, 16] and references therein).

An interesting class of problems with uncertainty in the feasible set was introduced by Dhamdhere, Goyal, Ravi and Singh [11]. In this two-stage models the feasibility condition is only fully revealed in the second stage. While resources can be bought in both stages, they are cheaper in the first stage, in which only partial information about the feasible set is available. This model was subsequently studied by several other authors (see [15, 12, 21]). Different two-stage model was proposed in [4, 2] for the shortest path problem. Several other important network design problems are motivated by robust optimization. Such problems include the minimum k -edge connected spanning subgraph problem [9, 13] and the survivable network design problem [19, 20]. Various other robust variants of classical combinatorial optimization problems were proposed. For a survey of these results we refer the reader to the theses of Adjashvili [1] and Olver [23].

2 Bulk-robust s - t connection in planar graphs

In this section we are concerned with the **bulk-robust s - t connection problem**, which given an undirected graph $G = (V, E)$, a weight function $w : E \rightarrow \mathbb{Z}_{\geq 0}$, two terminals $s, t \in V$ and a set of m scenarios $F_1, \dots, F_m \subseteq E$, asks to find minimum-cost set of edges S , such that $S \setminus F_i$ contains an s - t path for every $i \in [m]$.

The remainder of the section is organized as follows. First we explain the augmentation framework in general. Then we define the set cover problem for the i -th augmentation step and analyze its properties. Finally, we propose a LP-based approximation algorithm for the set cover problem.

2.1 The augmentation framework

Consider the following sequence of relaxations of the given instance of the bulk-robust s - t connection problem. For an integer $i \in [k]_0$ define Ω_i to be the collection of subsets of cardinality at most i of the failure scenarios F_1, \dots, F_m , i.e.

$$\Omega_i = \{F \subseteq E \mid \exists j \in [m] F \subseteq F_j \wedge |F| \leq i\}.$$

Now, define the i -th **level relaxation** P_i of our instance to be the instance where Ω is replaced by Ω_i . Clearly P_0 is simply the shortest path problem, as $\Omega_0 = \{\emptyset\}$, and P_k is the

original instance. Furthermore, we indeed obtained a sequence of relaxations, as any feasible solution for P_i is feasible for P_j if $i \geq j$.

The augmentation framework constructs the solution for the given instance by iteratively adding additional edges to the solution. The solution X_{i-1} obtained until the beginning of the i -th augmentation step is feasible for P_{i-1} . The **i -th augmentation problem** is to augment X_{i-1} with additional edges A_i of minimum cost so that $X_{i-1} \cup A_i$ is feasible for P_i . We denote by AUG_i the optimal value the i -th augmentation problem.

2.2 The i -th augmentation problem

In the first iteration, the problem P_0 becomes the shortest s - t path problem, and is solved in polynomial time by any shortest path algorithm. We denote by $X_{i-1} \subseteq E$ the set of edges presented to the i -augmentation problem, and let $G_{i-1} = (V, X_{i-1})$.

Consider the i -th augmentation problem for some $i \geq 1$. Since X_{i-1} is a feasible solution of P_{i-1} , we know that any scenario Ω_j for $j < i$ does not disconnect s from t in G_{i-1} . The same may hold true for some scenarios in Ω_i . If this holds for all scenarios in Ω_i , then X_{i-1} is already feasible for P_i , and we can set $X_i = X_{i-1}$. In the other case, some scenarios in Ω_i are still relevant, i.e. they disconnect s from t in G_{i-1} . We abuse notation and let Ω_i denote this set of relevant scenarios.

Let us formulate the i -th augmentation problem as a set cover problem. To this end we let $E_i = E \setminus X_{i-1}$ denote the set of edges not yet chosen to be included in the solution. Let $\bar{V} = V[X_{i-1}]$ be the set of nodes incident to X_{i-1} . Let us define the following useful notion of links.

► **Definition 4.** Let $u, v \in \bar{V}$ be distinct nodes. Define the u - v **link** $L_{u,v}$ to be any shortest u - v path in (V, E_i) . Let $\ell_{u,v} = w(L_{u,v})$ denote the length of this path.

Consider any optimal solution A^* to the i -th augmentation problem. It is easy to see that A^* is acyclic, i.e. it forms a forest in (V, E_i) . Instead of looking for forests, however, we would like to restrict our search to collections of links.

The advantage of using links is twofold. On the one hand, it is possible to compute all links using a shortest path algorithm in polynomial time. On the other hand, using links will allow us to decompose the augmentation problem in a later stage. Let us define the notion of covering with links next.

► **Definition 5.** A link $L_{u,v}$ is said to **cover** $F \in \Omega_i$ if its endpoints u and v lie on different sides of the cut formed by F .

It is easy to see that a union of links forms a feasible solution to the augmentation problem if and only if for every set $F \in \Omega_i$, at least one of the links in the union covers F . This formulation naturally gives rise to our desired set cover problem, defined next.

► **Definition 6.** The **i -th link covering problem** asks to find a collection of links of minimum total cost, covering every scenario $F \in \Omega_i$.

We also know that feasible solutions to the i -th links covering problem correspond to feasible solutions of the i -th augmentation problem with the same objective function value, or better. The following lemma from [3] states that any feasible solution to the i -th augmentation problem corresponds to a feasible solution to the i -th links covering problem of at most twice the cost, thus by solving the link covering problem we lose at most a factor of 2.

► **Lemma 7** (Adjashvili et. al. [3]). *There exists a collection $Q_1, \dots, Q_r \subseteq E_i$ of paths, such that for each $F \in \Omega_i$, the collection contains at least one path covering F and $\sum_{j=1}^r w(Q_j) \leq 2\text{AUG}_i$.*

Before proposing an approximation algorithm for the link covering problem let us make the following additional assumption. We assume that **every** edge $e \in X_{i-1}$ appears in at least one scenario from Ω_i . This assumption does not compromise generality, as any edge not satisfying the latter condition can be safely contracted for the solution of the i -th augmentation problem.

2.3 Approximating the link covering problem

We focus next on approximating the i -th link covering problem. For simplicity we drop the index i from our notation in this section and use X, Ω and P for X_{i-1}, Ω_i and P_i , respectively. The case $i = 1$ is particularly simple and is treated as follows. In the case $i = 1$ the set X simply corresponds to an s - t path. This path can be seen as a line and links can be seen as intervals on this line. Scenarios $F \in \Omega$ are singletons corresponding to edges on this path, and are interpreted as points on the line. The link covering problem now becomes an interval covering problem that can be solved exactly in polynomial time using various algorithms.

In the case $i \geq 2$, which we henceforth assume, the situation is much more complex. Consider next the following standard linear programming relaxation of the link covering problem. We include a variable $x_{u,v} \in [0, 1]$ for each link $L_{u,v}$, where $x_{u,v} = 1$ is interpreted as including the link $L_{u,v}$. Furthermore, we denote by $\text{cover}(F)$ all pairs $\{u, v\} \subseteq \bar{V} \times \bar{V}$ such that the link $L_{u,v}$ covers F .

$$\min \left\{ \ell(x) : x_{u,v} \geq 0 \ \forall \{u, v\} \in \bar{V} \times \bar{V}, \quad \sum_{\{u,v\} \in \text{cover}(F)} x_{u,v} \geq 1 \quad \forall F \in \Omega \right\}$$

It is well-known that in general, the latter LP has an integrality gap as large as $\log N$, where N is the size of the ground set of the set cover problem. Our goal here is to show that in the case of the link covering problem and when the input graph is required to be **planar**, a stronger bound can be proved. Concretely, we will show that a fractional solution x^* to the LP can be rounded in polynomial time to an integral solution with cost at most $8i\ell(x^*)$, thus also proving a bound of $8i$ on the integrality gap.

2.3.1 Solving the LP

Before we turn to our rounding algorithm, let us discuss the problem of solving the latter LP. Clearly, if k is a fixed constant, the size of the LP is polynomial, and any polynomial time LP algorithm can be used. In the other case, when the diameter k is not bounded by a constant, the sets Ω might have exponential size, as they potentially contain all subsets of cardinality $i \leq k$ of sets of cardinality k . It is however not difficult to design a polynomial-time separation procedure for the latter LP as follows. Given a fractional vector x , we can check if it is feasible for the LP by checking for every one of the polynomially many failure scenarios F_1, \dots, F_m , if it contains a subset F of size i that is both an s - t cut in $\bar{G} = (V, X)$, and

$$\sum_{\{u,v\} \in \bar{V} \times \bar{V} \text{ covers } F} x_{u,v} < 1.$$

Let us call a set F of the latter type **violating**. This can be achieved as follows. Let F_j be the scenario from the family of input scenarios that we would like to test. Let $H = (\bar{V}, Y)$ be the graph obtained from \bar{G} by adding the direct edge $\{u, v\}$ for every pair of distinct nodes $u, v \in \bar{V}$. The new edge $\{u, v\}$ represents the link $L_{u,v}$. Define an edge capacity vector $c : Y \rightarrow \mathbb{R}_{\geq 0}$ on the new edge set Y setting $c_j(e) = 1$ if $e \in F_j$, $c_j(e) = \infty$ if $e \in X \setminus F_j$ and $c_j(e) = x_e$ if $e \in Y \setminus X$. It is now easy to verify that a violating set $F \subseteq F_j$ exists if and only if the capacity of the minimum s - t cut in H with capacity vector c is strictly below $i + 1$. Furthermore, if such a cut exists, the set F can be chosen to be all edges of F_j crossing the minimum cut. Polynomiality of the latter transformation and the minimum s - t cut problem now imply that the Ellipsoid algorithm can be used to solve the LP in polynomial time.

2.3.2 Rounding the LP

Let x^* denote an optimal solution to our LP. We describe our rounding procedure next.

Our rounding technique heavily exploits the planarity of the input graph G . Let us henceforth assume that G is presented with a planar embedding Γ . Such an embedding can be computed in polynomial time. We let $\psi^1, \dots, \psi^{\bar{q}}$ denote the faces of the embedding of \bar{G} , induced by the embedding of G .

► **Definition 8.** We say that link $L_{u,v}$ is of **type** j if it connects two nodes u, v on ψ^j , and if $L_{u,v}$ is completely contained in the face ψ^j . We call a link **typed** if it is of type j for some $j \in [\bar{q}]$. Links that are not typed are called **untyped**.

For what follows it will be convenient to assume that x^* is **clean**, i.e. that $x_{u,v}^* = 0$ holds for every untyped link $L_{u,v}$. This assumption does not compromise generality, as we state and prove in the following lemma.

► **Lemma 9.** *Restricting the solutions of the LP to satisfy $x_{u,v} = 0$ for every untyped link $L_{u,v}$ does not change the optimal value of the LP.*

Proof. Assume that x^* is an optimal solution to the LP with minimum possible weight assigned to untyped links

$$\sum_{\{u,v\} \in \bar{V} \times \bar{V} \text{ untyped}} x_{u,v}^*.$$

Assume towards contradiction that $x_{u,v}^* > 0$ holds for some untyped link $L_{u,v}$. Since $L_{u,v}$ is untyped, it forms a shortest path between the nodes u and v , composed of edges contained in several faces of \bar{G} . Let $u = v_1, \dots, v_p = v$ be nodes on $L_{u,v}$ with the following properties.

- The nodes appear in this order on $L_{u,v}$, when it is traversed from u to v .
- For every $i \in [p-1]$, it holds that $L_{v_i, v_{i+1}}$ is a typed link, i.e. it holds that $v_i, v_{i+1} \in \bar{V}$ and the sub-path of $L_{u,v}$ between v_i and v_{i+1} is completely contained in some face ψ^{j_i} .

Now, consider the LP solution y where

- $y_{u,v} = 0$,
- $y_{v_i, v_{i+1}} = \min\{1, x_{v_i, v_{i+1}}^* + x_{u,v}^*\}$ for every $i \in [p-1]$, and
- $y_{w,z} = x_{w,z}^*$ everywhere else.

Since all links are shortest paths we have $w(L_{u,v}) = \sum_{i \in [p-1]} w(L_{v_i, v_{i+1}})$, and thus $\ell_{u,v}(y) = \sum_{i \in [p-1]} \ell_{v_i, v_{i+1}}(y)$. This implies that $\ell(y) \leq \ell(x^*)$.

The new solution is also a feasible LP solution. To see this we only need to verify that for every $F \in \Omega$, the constraint

$$\sum_{\{z,w\} \in \text{cover}(F)} y_{z,w} \geq 1$$

holds. If $L_{u,v}$ does not cover F , this is obvious from feasibility of x^* , since $y_{z,w} \geq x_{z,w}^*$ for all links except $L_{u,v}$.

In the remaining case $L_{u,v}$ covers F . Now, since the union of links $\cup_{i \in [p-1]} L_{v_i, v_{i+1}}$ contains a u - v path, clearly at least one of these links, say $L_{v_{i^*}, v_{i^*+1}}$, also covers F . If $y_{v_{i^*}, v_{i^*+1}} = 1$ we are clearly done. In the other case

$$y_{v_{i^*}, v_{i^*+1}} = x_{v_{i^*}, v_{i^*+1}}^* + x_{u,v}^*,$$

and thus what is lost by reducing $x_{u,v}^*$ is compensated by increasing $x_{v_{i^*}, v_{i^*+1}}^*$, and the constraint is also satisfied.

Finally, we obtained a new optimal solution y with a lower weight assigned to untyped links, as all the links of the form $L_{v_i, v_{i+1}}$ are typed links, and the link $L_{u,v}$ is untyped. This contradicts the choice of x^* . ◀

A set of links S is **clean** if it only contains typed links. The following lemma proves certain useful connections between the planar embedding of G and the link covering problem, which we later use to round the LP solution. We say that an edge is **on the boundary of a face** if both of its endpoints lie on the face.

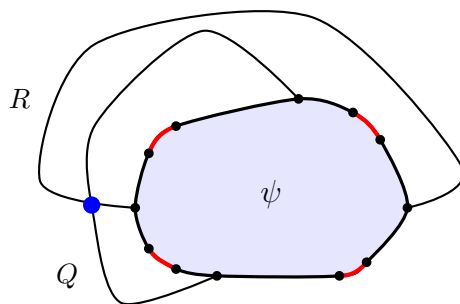
► **Lemma 10.** *Let $F \in \Omega$ be some failure scenario and let $\psi \in \{\psi^1, \dots, \psi^{\bar{q}}\}$ be some face. Then, if $i \geq 2$, the number of edges of F that lie on the boundary of ψ is either zero or two. Furthermore, the number of faces $\{\psi^1, \dots, \psi^{\bar{q}}\}$ that contain two edges of F on their boundary is exactly i .*

Proof. Since $F \in \Omega$ we know that F is an s - t cut in \bar{G} . Observe that $(\bar{V}, X \setminus F)$ contains exactly two connected components, one $C^s(F)$ containing s and one $C^t(F)$ containing t . This holds since, by definition of Ω and the augmentation problem, the set X is feasible for P_{i-1} , and thus any subset of F is **not** an s - t cut in \bar{G} .

This implies that all edges in F can be directed unambiguously from the node in $C^s(F)$ to the node in $C^t(F)$. Now consider any edge $e \in F$ and any face $\psi \in \{\psi^1, \dots, \psi^{\bar{q}}\}$ which contains e on its boundary. Since ψ corresponds to a cycle in \bar{G} , the number of edges of F on its boundary cannot be odd, as an odd number of such edges would imply the existence of path in \bar{G} connecting $C^s(F)$ to $C^t(F)$, and containing no edge of F .

Next we prove that this number must be two, i.e. that ψ contains exactly one more edge of F . Assume towards contradiction that there are at least four such edges. By traversing the cycle in \bar{G} , forming the face ψ , the cut defined by F is crossed every time an edge of F is crossed. In particular, there are some four nodes u_1, v_1, u_2, v_2 appearing in this order on the face, and such that u_1, u_2 belong to $C^s(F)$ and v_1, v_2 belong to $C^t(F)$. Let Q and R be a u_1 - u_2 path in $C^s(F)$ and a v_1 - v_2 path in $C^t(F)$, respectively. Since ψ is a face, the embedding of both Q and R is disjoint from the interior of ψ . Now, since Q and R form continuous curves in the plane, and are connected to alternating nodes on the boundary of a face, they must intersect at some point, contradicting the fact that $C^s(F)$ and $C^t(F)$ are different connected components in $(\bar{V}, X \setminus F)$. Figure 1 illustrates this argument.

Finally, since every edge $e \in F$ belongs to the boundary of exactly two faces from $\psi^1, \dots, \psi^{\bar{q}}$, and since every face containing some edge of F on the boundary contains exactly two such edges, we conclude that there are exactly $|F| = i$ faces containing some edge of F on the boundary. In the first assertion we assumed there are no cut edges in \bar{G} . For $i \geq 2$ this can be assumed without loss of generality, as cut edges are either contracted in the pre-processing stage before the augmentation step, or, they are redundant, and can be removed from X . ◀



■ **Figure 1** The situation in the proof of Lemma 10.

For simplicity we say that a **scenario** $F \in \Omega$ is **contained in a face** $\psi \in \{\psi^1, \dots, \psi^{\bar{q}}\}$, if two edges of F lie on the boundary of ψ . Lemma 10 implies that a clean set S of links is feasible if and only if for every $F \in \Omega$, there exists a face ψ^j containing F , and a link $L_{u,v} \in S$ of type j with u and v on different sides of the cut defined by F . With this criterion we are ready to prove the main lemma of this section.

► **Lemma 11.** *Let x be a clean feasible solution to the LP. Then, there exists a feasible set S of links with total cost at most $8il(x)$.*

Proof. We construct the desired set of links in two steps. First, we partition the set of scenarios Ω into \bar{q} parts, one for each face of \bar{G} . In the second stage, we process the faces of \bar{G} one by one, and for each face we use the part of the LP solution x corresponding to the face to construct a set of links that cover the scenarios assigned to that face.

Consider any scenario $F \in \Omega$. Since x is feasible we have

$$\sum_{\{u,v\} \in \text{cover}(F)} x_{u,v} \geq 1.$$

Let $\psi^{p_1}, \dots, \psi^{p_i} \in \{\psi^1, \dots, \psi^{\bar{q}}\}$ be the set of faces that contain F . According to Lemma 10, there are exactly i such faces. Now, since x is clean, the latter sum can be decomposed as follows.

$$\sum_{\{u,v\} \in \text{cover}(F)} x_{u,v} = \sum_{j=1}^i \sum_{\substack{\{u,v\} \in \text{cover}(F) \\ L_{u,v} \text{ type } p_j}} x_{u,v}$$

Let us denote the second sum on the right hand side by $\sigma^j[F]$, i.e. let

$$\sigma^j[F] = \sum_{\substack{\{u,v\} \in \text{cover}(F) \\ L_{u,v} \text{ type } p_j}} x_{u,v}.$$

Now since $\sum_{j=1}^i \sigma^j[F] \geq 1$, there exists at least one index $j \in [i]$ such that $\sigma^j[F] \geq \frac{1}{i}$. We let $j[F] \in [i]$ be one such index. If several indices $j \in [i]$ satisfy the latter condition, one is chosen arbitrarily. Note that the index $j[F]$ is chosen in such a way that in the LP solution x , the total weight of links of type $p^{j[F]}$ that cover F is at least $\frac{1}{i}$.

We are now ready to define the partition of Ω into \bar{q} parts, corresponding to the \bar{q} faces of \bar{G} . For $j \in [\bar{q}]$ we let

$$\Omega^{(j)} = \left\{ F \in \Omega \mid p^{j[F]} = j \right\}.$$

Clearly, $\Omega = \cup_{j \in [\bar{q}]} \Omega^{(j)}$ is a partition of Ω . To conclude the first stage of the our procedure, it remains to define a corresponding decomposition $x = \sum_{j \in [\bar{q}]} x^{(j)}$ of the LP solution x . The vector $x^{(j)}$ is defined by setting $x_{u,v}^{(j)} = x_{u,v}$ if $L_{u,v}$ is of type j , and $x_{u,v}^{(j)} = 0$ otherwise. This concludes the first step of the rounding procedure.

In the second step we construct for every $j \in [\bar{q}]$, a set of links $S^{(j)}$ of total cost $8il(x^{(j)})$ that covers all scenarios in $\Omega^{(j)}$. By doing so we clearly conclude the proof of the lemma, since by taking $\cup_{j \in [\bar{q}]} S^{(j)}$ we obtain a feasible solution with total cost of at most $\sum_{j \in [\bar{q}]} 8il(x^{(j)}) = 8il(x)$, as desired.

It remains to show how a single set $S^{(j)}$ can be constructed. Our plan is the following. First, we observe that, by construction, $ix^{(j)}$ is an LP solution that fractionally covers all scenarios in $\Omega^{(j)}$. Then, we observe that the link covering problem restricted to links of type j , and to scenarios in $\Omega^{(j)}$ essentially becomes a variant of the **dominating set problem on circle graphs**. We explain the required transformation next, and conclude by proving that the integrality gap of the standard LP relaxation for the latter problem is constant, and that the corresponding integral solution can be found in polynomial time.

Recall that a **circle graph** is an intersection graph of the set of chords in a circle. The dominating set problem in circle graphs hence corresponds to finding a minimum-cost collection of chords that intersect every chord of the graph. We are interested in a variant of this problem, where chords are partitioned into two groups called **demand chords** and **covering chords**, and the goal is to find a minimum-cost set of covering chords that dominates all the demand chords. We call this problem the **restricted dominating set problem in circle graphs**.

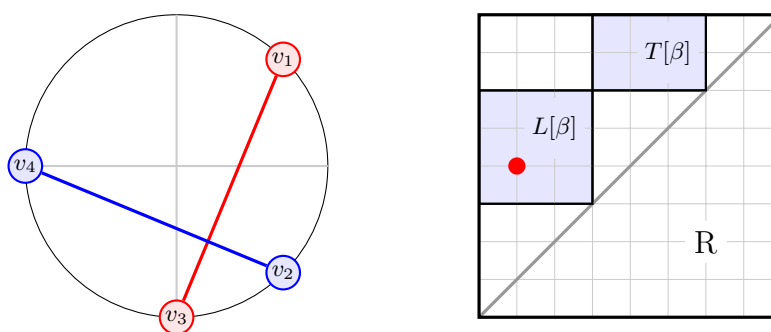
The link covering problem restricted to a the face ψ can now be seen as a dominating set problem on circle graphs as follows. Let $v_0, v_1, \dots, v_d = v_0$ be the nodes on the boundary of ψ . We subdivide each edge $\{v_j, v_{j+1}\}$ for $j \in [d]$ by adding the node w_j . This new cycle corresponds to the circle of the circle graph we construct. Let us define the chords of the graph, and their corresponding weights, next. For every scenario F contained in ψ we add the demand chord α_F connecting w_{j_1} to w_{j_2} , where $\{u_{j_1}, u_{j_1+1}\} \in F$ and $\{u_{j_2}, u_{j_2+1}\} \in F$ (recall from Lemma 10 that there are exactly two such edges). Next, for every link of the form L_{v_l, v_r} , we add the covering chord β_{v_l, v_r} connecting v_l with v_r . The cost of this chord is set to ℓ_{v_l, v_r} , i.e. we set $c(\beta_{v_l, v_r}) = \ell_{v_l, v_r}$. This concludes the transformation.

To see that the latter problem indeed models the desired link covering problem it suffices to make the following simple observation. Sets of chords corresponding to links that form a restricted dominating set in the circle graph are in one-to-one correspondence with sets of links that cover all scenarios, with identical costs. This is true, since a link L_{v_l, v_r} covers a scenario F if and only if the chords α_F and β_{v_l, v_r} intersect.

We can now naturally interpret the solution $y^{(j)} = ix^{(j)}$ as a feasible fractional solution to the standard LP relaxation of the restricted dominating set problem on the obtained circle graph. In the following claim we show that the integrality gap of the latter LP is constant. The proof of the claim uses a connection to a special case of the **axes-parallel rectangle covering problem**, for which Bansal and Pruhs [6] provided an LP-respecting 2-approximation with the natural LP.

► **Lemma 12.** *The integrality gap of the standard LP relaxation of the restricted dominating set problem on circle graphs is bounded by 8.*

Proof. Let $H = (V^d \cup V^c, E)$ be the given circle graph with V^d and V^c corresponding to the demand chords and the covering chords, respectively. Let $g : V^c \rightarrow \mathbb{Z}_{\geq 0}$ denote the cost function for the covering chords. Let $p_0, \dots, p_m = p_0$ be all the points on the circle to which



■ **Figure 2** An illustration of the transformation. The chords $\alpha = (v_1, v_3)$ and $\beta = (v_2, v_4)$ are demand and covering chords, respectively. The ordering of the points on the circle is clockwise starting from the highest point.

chords are connected, in the order that they appear when the circle is traversed in some arbitrary direction. For a chord $\alpha \in V^d \cup V^c$ we write $\alpha = (p_l, p_r)$ with $l \leq r$ to indicate the endpoints of the chord in the circle.

We interpret the restricted dominating set problem as a kind of **point covering problem by axis-aligned rectangles** as follows. Construct a large square R with side length m . The points in R are indexed by pairs of points on the circle, with (p_0, p_0) and (p_{m-1}, p_{m-1}) being, respectively, the lower-left corner of R and the upper-right corner of R . For four points $p_{l_1}, p_{l_2}, p_{r_1}, p_{r_2}$ with $l_1 \leq l_2$ and $r_1 \leq r_2$ we denote by $[p_{l_1}, p_{r_1}] \times [p_{l_2}, p_{r_2}] \subseteq R$ the rectangle contained in R with lower-left point and upper-right point (p_{l_1}, p_{r_1}) and (p_{l_2}, p_{r_2}) , respectively.

Demand chords are interpreted as **points in R** . The chord $\alpha = (p_l, p_r) \in V^d$ is interpreted as the point $Q[\alpha] = (p_l, p_r)$ inside R . Observe that $Q[\alpha]$ is contained above the main diagonal in R , that is the line connecting (p_0, p_0) and (p_{m-1}, p_{m-1}) , as $p_l \leq p_r$.

Covering chords are interpreted as **pairs of rectangles** contained in R . The chord $\beta = (p_l, p_r) \in V^d$ is interpreted as the pair of rectangles

$$L[\beta] = [p_0, p_l] \times [p_l, p_r] \text{ and } T[\beta] = [p_l, p_r] \times [p_r, p_{m-1}].$$

Observe the following property. $L[\beta]$ intersects the left side of R and $T[\beta]$ intersects the top side of R .

It is now straightforward to verify that a covering chord β dominates a demand chord α if and only if

$$Q[\alpha] \in L[\beta] \cup T[\beta].$$

Finally, we arrived at the desired covering problem, namely the problem of selecting a minimum cost set of rectangles pairs $L[\beta] \cup T[\beta]$ in R , corresponding to covering chords, so as to cover every point $Q[\alpha]$, corresponding to demand chords. The cost of a rectangle pair is simply the cost of the corresponding covering chord. Figure 2 illustrates the transformation.

The standard LP relaxation for this covering problem reads

$$\min \left\{ g(z) \mid z_\beta \geq 0 \quad \forall \beta \in V^c, \quad \sum_{\beta: Q[\alpha] \in L[\beta] \cup T[\beta]} z_\beta \geq 1 \quad \forall \alpha \in V^d \right\}.$$

Let z be a fractional feasible solution to the latter LP. We construct an integral solution as follows. First, observe that for every demand chord $\alpha \in V^d$, at least one of the following holds due to feasibility of z :

- $\sum_{\beta: Q[\alpha] \in L[\beta]} z_\beta \geq \frac{1}{2}$
- $\sum_{\beta: Q[\alpha] \in T[\beta]} z_\beta \geq \frac{1}{2}$

Let $V_L^d \subseteq V^d$ be the set of all $\alpha \in V^d$ for which the first condition holds. Let $V_T^d = V^d \setminus V_L^d$ be all other demand chords. We show how to construct an integral solution of cost at most $4g(z)$ that dominates all chords in V_L^d . From symmetry, this implies that another integral solution can be constructed for V_T^d with cost at most $4g(z)$. This will then prove the claim, as the union of both solutions is an integral feasible solution of cost at most $8g(z)$.

To this end observe that $2z$ is a fractional feasible solution to the LP

$$\min \left\{ g(z) \mid z_\beta \geq 0 \quad \forall \beta \in V^c, \quad \sum_{\beta: Q[\alpha] \in L[\beta]} z_\beta \geq 1 \quad \forall \alpha \in V_L^d \right\}.$$

Now, it remains to observe that the latter LP is the natural LP relaxation of an ordinary rectangle covering problem. The rectangles $\{L[\beta] \mid \beta \in V^c\}$ also have the additional property that their left side lies on the left side of R . This restricted variant of the rectangle covering problem was studied by Bansal and Pruhs [6], who proved that the standard LP relaxation of the problem has integrality gap of 2. This implies that there exists an integral solution covering V_L^d with cost $4g(z)$. This solution can also be constructed in polynomial time. This concludes the proof of the claim. ◀

Lemma 12 concludes the proof. We note that Lemma 12 provides an 8-approximation algorithms for the dominating set problem in circle graphs based on the natural LP relaxation. While constant factor approximations were known for this problem (see e.g. [10]), to best of our knowledge, none of the existing algorithms use LP rounding. ◀

2.3.3 Putting it all together

We are ready to prove Theorem 2.

Proof. The feasibility of the solution obtained after the final augmentation step is obvious. It remains to compute the approximation guarantee. Let ALG denote the cost of the solution returned by the algorithm. Clearly, $\text{AUG}_i \leq 2\text{OPT}$ holds for every $i \in [k]$, as any optimal solution is feasible for any augmentation problem, and Lemma 7 asserts that by using unions of paths we lose a factor of at most 2. According to Lemma 11, an $8i$ -approximation can be obtained for the i -th augmentation problem in polynomial time. Also, the shortest path comprising the solution of P_0 has cost of at most OPT . In total, we obtain the bound $\text{ALG} \leq \text{OPT} + \sum_{i=0}^k 8i \cdot 2\text{OPT} = O(k^2)\text{OPT}$. ◀

3 Bulk-robust spanning trees and further extensions

3.1 Bulk-robust spanning trees

Let us discuss first the minor changes needed to prove Theorem 2 for the bulk-robust spanning tree problem. More details are given in Appendix A.

The augmentation procedure remains unchanged, except that at every iteration i , the sets in Ω_i are those sets of cardinality i that disconnect the graph in any way. The link covering problem and the obtained LP are essentially the same. The separation procedure of the LP now requires a polynomial minimum cut algorithm, instead of a minimum s - t cut algorithm. The rounding procedure only used the property that every set $F \in \Omega_i$ creates exactly two

connected components when removed from X_{i-1} . Finally, the augmentation algorithm for $i = 1$ is no longer an interval covering problem, but can be treated as in Lemma 11, with the outer face being the only face of G .

3.2 Further extensions

Let us conclude by discussing some further extensions and implications of our techniques. First, our techniques can clearly be applied to other bulk-robust network design problems. A treatment of the bulk-robust survivable network design problem is deferred to the full version of the paper.

Also, as we mentioned in the introduction, our methods seem to be suitable for solving other robust problems in planar graphs. Consider, for example, the uniform model with varying interdiction costs, where each edge has an interdiction cost $c(e) \in \mathbb{Z}_{\geq 0}$, and the set of scenarios is exactly the set of all edge subsets with total interdiction cost at most $B \in \mathbb{Z}_{\geq 0}$. Our methods can be used to approximate this problem provided that a suitable (approximate) separation oracle is provided for the resulting LP. In general, however, this separation problem coincides with difficult interdiction problems (see e.g. [18, 28] and references therein).

References

- 1 D. Adjiashvili. *Structural Robustness in Combinatorial Optimization*. PhD thesis, ETH Zürich, Zürich, Switzerland, 2012.
- 2 D. Adjiashvili, G. Oriolo, and M. Senatore. The online replacement path problem. In *Proceedings of 18th Annual European Symposium on Algorithms (ESA)*, pages 1–12. Springer, 2013.
- 3 D. Adjiashvili, S. Stiller, and R. Zenklusen. Bulk-robust combinatorial optimization. *Mathematical Programming*, 149(1-2):361–390, 2014.
- 4 D. Adjiashvili and R. Zenklusen. An s-t connection problem with adaptability. *Discrete Applied Mathematics*, 159:695–705, 2011.
- 5 H. Aissi, C. Bazgan, and D. Vanderpooten. Min–max and min–max regret versions of combinatorial optimization problems: A survey. *European Journal of Operational Research*, 197(2):427–438, 2009.
- 6 N. Bansal and K. Pruhs. The geometry of scheduling. In *51st Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 407–414, 2010.
- 7 D. Bertsimas, D. B. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM review*, 53:464–501, 2011.
- 8 C. Chekuri, J. Vondrák, and R. Zenklusen. Multi-budgeted matchings and matroid intersection via dependent rounding. In *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1080–1097, 2011.
- 9 J. Cheriyan and R. Thurimella. Approximating Minimum-Size k-Connected Spanning Subgraphs via Matching. *SIAM J. Comput.*, 30:292–301, 2000.
- 10 M. Damian-Iordache and S. V. Pemmaraju. A $(2 + \epsilon)$ -approximation scheme for minimum domination on circle graphs. *Journal of Algorithms*, 42(2):255–276, 2002.
- 11 K. Dhamdhere, V. Goyal, R. Ravi, and M. Singh. How to pay, come what may: approximation algorithms for demand-robust covering problems. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 367–376, October 2005.
- 12 U. Feige, K. Jain, M. Mahdian, and V. Mirrokni. Robust combinatorial optimization with exponential scenarios. In Matteo Fischetti and David Williamson, editors, *IPCO 2007*, volume 4513 of *Lecture Notes in Computer Science*, pages 439–453. Springer Berlin / Heidelberg, 2007.

- 13 H. N. Gabow, M. X. Goemans, É. Tardos, and D. P. Williamson. Approximating the smallest k-edge connected spanning subgraph by lp-rounding. *Networks*, 53(4):345–357, 2009.
- 14 H. N. Gabow, M.X. Goemans, and D.P. Williamson. An efficient approximation algorithm for the survivable network design problem. *Mathematical Programming, Series B*, 82(1-2):13–40, 1998.
- 15 D. Golovin, V. Goyal, and R. Ravi. Pay today for a rainy day: Improved approximation algorithms for demand-robust min-cut and shortest path problems. In Bruno Durand and Wolfgang Thomas, editors, *STACS 2006*, volume 3884 of *Lecture Notes in Computer Science*, pages 206–217. Springer Berlin / Heidelberg, 2006.
- 16 F. Grandoni, R. Ravi, M. Singh, and R. Zenklusen. New approaches to multi-objective optimization. *Mathematical Programming*, 146(1-2):525–554, 2014.
- 17 V. Guruswami, S. Sachdeva, and R. Saket. Inapproximability of minimum vertex cover on k-uniform k-partite hypergraphs. *SIAM Journal on Discrete Mathematics*, 29(1):36–58, 2015.
- 18 E. Israeli and R. K. Wood. Shortest-path network interdiction. *Networks*, 40:97–111, 2002.
- 19 K. Jain. A factor 2 approximation algorithm for the generalized steiner network problem. *Combinatorica*, 21:39–60, 2001.
- 20 H. Kerivin and A.R. Mahjoub. Design of survivable networks: A survey. *Networks*, 46(1):1–21, 2005.
- 21 R. Khandekar, G. Kortsarz, V. Mirrokni, and M. Salavatipour. Two-stage robust network design with exponential scenarios. In Dan Halperin and Kurt Mehlhorn, editors, *ESA 2008*, volume 5193 of *Lecture Notes in Computer Science*, pages 589–600. Springer Berlin / Heidelberg, 2008.
- 22 P. Kouvelis and G. Yu. *Robust Discrete Optimization and Its Applications*. Kluwer Academic Publishers, Boston., 1997.
- 23 N.-K. Olver. *Robust network design*. PhD thesis, McGill University, Montreal, Quebec, Canada, 2010.
- 24 C. H. Papadimitriou and M. Yannakakis. On the approximability of trade-offs and optimal access of web sources. In *41st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 86–92, 2000.
- 25 R. Ravi, M. V. Marathe, Ravi S. S., Rosenkrantz D. J., and Hunt H. B. Many birds with one stone: Multi-objective approximation algorithms. In *25th annual ACM Symposium on the Theory of Computing (STOC)*, pages 438–447, 1993.
- 26 D.P. Williamson, M.X. Goemans, M. Mihail, and V.V. Vazirani. A primal-dual approximation algorithm for generalized steiner network problems. *Combinatorica*, 15(3):435–454, 1995.
- 27 G. Yu and J. Yang. On the robust shortest path problem. *Computers & Operations Research*, 25(6):457–468, 1998.
- 28 R. Zenklusen. Matching interdiction. *Discrete Applied Mathematics*, 158(15):1676–1690, 2010.

A Bulk-Robust Spanning Trees

In this section we describe the proof of Theorem 2 for the bulk-robust spanning tree problem. As the changes are minor, we choose to follow the outline of the proof given in the main text, and describe the required modifications.

A.1 The augmentation framework

We use the same sets Ω_i , $i \in [k]_0$ to define the relaxations of the problem. The i -th augmentation problem P_i is to augment the set X_{i-1} of edges chosen so far to a set X_i with the property that $(V, X_i \setminus F)$ is a connected graph for all $F \in \Omega_i$.

As for the bulk-robust s - t connection problem, the optimal solution to any augmentation problem is a forest. As a consequence, Lemma 7 still applies, so we can again use unions of paths to approximate the augmentation problem, at the loss of a factor 2.

The notion of links and covering by links is defined as before, except that now cuts formed by sets $F \in \Omega_i$ are arbitrary cuts in the graph, and not just s - t cuts.

A.2 Solving the link covering problem

The approximate solution to the link covering problems are obtained in essentially the same way for the bulk-robust spanning tree problem, as for the bulk-robust s - t connection problem. The differences are minor and are explained next.

The solution for P_0 is computed by computing a minimum spanning tree in the input graph in polynomial time. The cost of this tree is clearly at most OPT.

A.2.1 The case $i = 1$

The link covering problem corresponding to P_1 is no longer equivalent to an interval covering problem, but it can be approximated as follows. Recall that the solution obtained before the first augmentation problem is a spanning tree of the graph. Each edge in this tree either belong to some failure scenario F_i , in which case it comprises a failure scenario in Ω_1 , or it is not contained in any failure scenario. In the latter case the edge can be simply contracted, so we henceforth assume that all edges of the tree form a scenario in Ω_1 .

The augmentation problem now becomes a standard **connectivity augmentation problem**, where, given a spanning tree T of a graph G , the task is to compute a minimum-cost set of edges, not in the tree, whose addition to the tree will increase the size of the minimum cut in the resulting graph to two. Indeed, on the one hand any set A of edges satisfying that the graph $(V, T \cup A)$ has no cut of size one is feasible, as the removal of any edge of T cannot disconnect this graph. On the other hand, if a set A is such that $(V, T \cup A)$ does contain a cut edge, this edge must belong to T (since T is a spanning tree of G). Since all edges of T are assumed to comprise failure scenarios in Ω_1 , this means that A is infeasible for the augmentation problem.

It remains to note that the latter connectivity augmentation problem can be efficiently approximated within a constant factor. One way to achieve this is to use the algorithm for survivable network design in [19].

A.2.2 The case $i \geq 2$

As before, we omit the index i from our notation, as we now discuss the i -th link covering problem for some arbitrary $i \geq 2$.

The set cover LP appropriate for modeling the link covering problem for the bulk-robust spanning tree problem remain exactly the same as before. There is, however, a slight difference in the design of the separation oracle for the LP. Concretely, the construction of the capacitated graph H remains the same, but now violating sets correspond to sets of edges in minimum cuts (instead of minimum s - t cuts), if the value of the minimum cut is below

$i + 1$. Since minimum cuts can be found in polynomial time, this separation procedure is also polynomial.

Finally, the rounding procedure and its analysis remain unchanged. While it may seem that the proof of Lemma 10 used the fact that $(\bar{V}, X \setminus F)$ contains exactly two connected components $C^s(F)$ and $C^t(F)$, one containing s and the other containing t , the fact that two specific nodes were separated by the cut was never used. The only property that is used is that $(\bar{V}, X \setminus F)$ contains exactly two connected components. Here we can simply use instead the fact that $(V, X \setminus F)$ contains exactly two connected components.

This concludes the description of the required modifications.

B Proof of Theorem 3

Recall that a **hypergraph** is a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is a finite set of **nodes**, and $\mathcal{E} \subseteq 2^{\mathcal{V}}$ is a set of subsets of \mathcal{V} called **edges**. Let $m \in \mathbb{Z}_{\geq 0}$. We say that \mathcal{H} is **m -uniform** if $|e| = m$ for every $e \in \mathcal{E}$. Observe that 2-uniform hypergraphs are graphs. \mathcal{H} is **m -partite** if \mathcal{V} can be partitioned into m parts $\mathcal{V} = \mathcal{V}_1 \cup \dots \cup \mathcal{V}_m$ such that for every $e \in \mathcal{E}$ and for every $j \in [m]$ it holds that

$$|e \cap \mathcal{V}_j| \leq 1.$$

A **vertex cover** of \mathcal{H} is a set $S \subseteq \mathcal{V}$ of nodes that touches every edge, i.e. such that $|S \cap e| \geq 1$ holds for every $e \in \mathcal{E}$. The **hypergraph minimum vertex cover** problem is to find a vertex cover of \mathcal{H} of minimum cardinality.

Our reduction relies on the following hardness-of-approximation result of Guruswami, Sachdeva and Saket [17].

► **Theorem 13** (Guruswami et. al. [17]). *For any $\epsilon > 0$ and any $m \geq 4$ it is NP-hard to approximate the minimum hypergraph vertex cover problem within a factor $\frac{m}{2} - 1 + \frac{1}{2m} - \epsilon$, even when the hypergraph is restricted to be m -uniform and m -partite, and the m -partition is given as input.*

We show that the minimum hypergraph vertex cover problem on k -uniform and k -partite hypergraphs can be transformed to an equivalent instance of bulk-robust s - t connection with diameter k , provided that the k -partition is given as input. We then extend the argument to the bulk-robust spanning tree problem.

To this end let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k -uniform, k -partite hypergraph, and let $\mathcal{V} = \mathcal{V}_1 \cup \dots \cup \mathcal{V}_k$ be the given k -partition of \mathcal{V} . We construct the series-parallel graph to be the input of the bulk-robust s - t connection problem as follows.

Let $p = |\mathcal{E}|$ and $n_j = |\mathcal{V}_j|$ for $j \in [k]$. For every $j \in [k]$ we construct an ordering e_1^j, \dots, e_p^j of \mathcal{E} corresponding to \mathcal{V}_j . This ordering is constructed as follows. First, order the nodes in \mathcal{V}_j in an arbitrary way, say $v_1^j, \dots, v_{n_j}^j$. Now, construct the ordering of edges by first including all edges incident v_1^j in any order, then all edges incident to v_2^j in any order and so on, until all vertices are traversed. Since \mathcal{V}_j is a part in a k -partition, the latter procedure succeeds in producing an ordering of \mathcal{E} , as every edge is incident to exactly one node in \mathcal{V}_j . By design, the latter construction satisfies the following useful property that we will use later. For every node $v \in \mathcal{V}$ there exists an index $j \in [k]$ such that the set of edges $\mathcal{E}_v = \{e \in \mathcal{E} \mid v \in e\}$ incident to v appear as a sub-sequence in the j -th ordering. This index j can be chosen such that $v \in \mathcal{V}_j$.

Next, start constructing the series-parallel graph $G = (V, E)$. First, include the nodes s, t and connect them by k node-disjoint paths P^1, \dots, P^k (the nodes s and t are common to all paths). Each path P^j contains exactly p edges f_1^j, \dots, f_p^j , appearing in this order when P^j

is traversed from s to t . The edge f_l^j is associated with the edge $e \in \mathcal{E}$ of the hypergraph in the l -th position of the j -th ordering, i.e., the edge e_l^j . Clearly, every edge $e \in \mathcal{E}$ is associated with exactly one edge of G on every path P^j and thus, in total, it is associated with k edges of G .

Next, for every $j \in [k]$ and $v \in \mathcal{V}_j$ add the edge α_v to G , connecting two nodes w_v^j and z_v^j on P^j . These nodes are selected so that the set of edges between these nodes on the path P^j are exactly those that are associated with edges in \mathcal{E}_v . Since the hypergraph edge in \mathcal{E}_v appear as a sub-sequence in the order used to construct P^j , such two nodes w_v^j and z_v^j exist. It is straightforward to verify that G is series-parallel.

To complete the construction of the graph we set the weights of all edges on the paths P^j for $j \in [k]$ to zero, while the weight of edges of the type α_v for $v \in \mathcal{V}$ is set to one.

To conclude the reduction it remains to specify the scenario set Ω of the bulk-robust s - t connection problem. For $e \in \mathcal{E}$ we include in Ω a single failure scenario $F_e \subseteq E$. The set F_e contains the k edges, one from every path P^j , $j \in [k]$, that are associated with e . Since every edge $e \in \mathcal{E}$ is associated with exactly one edge on every such path, we have $|F_e| = k$, as required.

We conclude the proof by showing that the resulting instance of the bulk-robust s - t connection problem is equivalent to the hypergraph vertex cover instance. Formally, we show that a solution to the hypergraph vertex cover instance can be transformed to a solution of the bulk-robust s - t connection instance with the same cost, and vice versa.

Assume first that $S \subseteq \mathcal{V}$ is solution to the hypergraph vertex cover problem. Construct a solution $X \subseteq E$ to the the bulk-robust s - t connection problem as follows. Include in X all paths P^j for $j \in [k]$ (at zero cost), as well as all edges α_v such that $v \in S$. This solution has cost $|S|$, as required. To see that this solution is feasible, consider any $F = F_e \in \Omega$. Since S is a vertex cover, there exists some $v \in S$ such that $v \in e$. Let $j \in [k]$ be such that $v \in \mathcal{V}_j$. Observe that the path starting at s , following P^j until w_v^j , then crossing α_v and then continuing to t on P^j is contained in $X \setminus F$.

Assume next that $X \subseteq E$ is a feasible solution to the bulk-robust s - t connection instance. Let $S = \{v \in \mathcal{V} \mid \alpha_v \in X\}$. By the cost structure of the reduction we know that the cost of X is exactly $|S|$. It remains to prove that S is a vertex cover. Consider any $e \in \mathcal{E}$. Since X is feasible, there exists an s - t path $Q \subseteq X \setminus F_e$. This s - t path must use some edge of the form α_v for $v \in S$ as, by construction, every path P^j intersects every failure scenario, and these are node-disjoint s - t paths. Furthermore, we can assume that α_v is the only such edge, as from every path P^j , only one edge is contained in F_e . Now, this edge $\alpha_v \in Q$ connects some nodes w_v^j and z_v^j on some path P^j . Consequently, the unique edge in $F_e \cup P^j$ is contained on the sub-path connecting w_v^j and z_v^j and hence, by construction, $v \in e$. We conclude that S is a vertex cover, as required.

The proof of the theorem for the bulk-robust s - t connection problem now directly follows from the latter reduction and Theorem 13.

The reduction for the bulk-robust spanning tree problem is identical, and so is the correspondence between feasible solutions to the hypergraph vertex cover and the bulk-robust spanning tree problems. The paths P^1, \dots, P^k together already span the entire graphs. Since all cuts formed by failure scenarios are s - t cuts, the result readily follows from our previous arguments.

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