How to Tame Rectangles: Solving Independent Set and Coloring of Rectangles via Shrinking

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Abstract

In the Maximum Weight Independent Set of Rectangles (MWISR) problem, we are given a collection of weighted axis-parallel rectangles in the plane. Our goal is to compute a maximum weight subset of pairwise non-overlapping rectangles. Due to its various applications, as well as connections to many other problems in computer science, MWISR has received a lot of attention from the computational geometry and the approximation algorithms community. However, despite being extensively studied, MWISR remains not very well understood in terms of polynomial time approximation algorithms, as there is a large gap between the upper and lower bounds, i.e., $O(\log n/\log\log n)$ v.s. NP-hardness. Another important, poorly understood question is whether one can color rectangles with at most $O(\omega(\mathcal{R}))$ colors where $\omega(\mathcal{R})$ is the size of a maximum clique in the intersection graph of a set of input rectangles $\mathcal{R}$. Asplund and Grünbaum obtained an upper bound of $O(\omega(\mathcal{R})^2)$ about 50 years ago, and the result has remained asymptotically best. This question is strongly related to the integrality gap of the canonical LP for MWISR.

In this paper, we settle above three open problems in a relaxed model where we are allowed to shrink the rectangles by a tiny bit (rescaling them by a factor of $(1-\delta)$ for an arbitrarily small constant $\delta > 0$.) Namely, in this model, we show (i) a PTAS for MWISR and (ii) a coloring with $O(\omega(\mathcal{R}))$ colors which implies a constant upper bound on the integrality gap of the canonical LP.

For some applications of MWISR the possibility to shrink the rectangles has a natural, well-motivated meaning. Our results can be seen as an evidence that the shrinking model is a promising way to relax a geometric problem for the purpose of better algorithmic results.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases Approximation algorithms, independent set, resource augmentation, rectangle intersection graphs, PTAS

Digital Object Identifier 10.4230/LIPIcs.APPROX-RANDOM.2015.43

1 Introduction

The main motivation of this paper is to study barriers in designing approximation algorithms for the Maximum Weight Independent Set of Rectangles (MWISR) problem and propose a way to break them. In this problem, we are given a collection of weighted axis-parallel rectangles in the plane, and our goal is to select a maximum weight subset of pairwise non-overlapping rectangles. Besides being a special case of Maximum Independent Set, which has been one of the most extensively studied problems in combinatorial optimization, MWISR is a fundamental geometric problem in itself. The problem arises in multiple applications and has connections to other problems in various areas of computer science, such as map labeling [4], data mining [19], networking [26], and pricing [14]. Therefore, it is not surprising that MWISR has received a significant amount of attention from researchers in both computational geometry and approximation algorithms communities.

While for Maximum Independent Set in general it is NP-hard to obtain approximation ratios of $n^{1-\epsilon}$ for any $\epsilon > 0$ [22, 28], much better approximation ratios are possible for MWISR. Agarwal, van Kreveld and Suri first proposed the problem with tentative applications in map labeling, where they showed the first $O(\log n)$-approximation algorithm [4]. Since then, a significant amount of research has been done in various directions: (i) Proposing $O(\log n)$ approximation algorithms with faster running times [7, 12, 27] or (ii) Showing approximation schemes or constant factor approximation algorithms for special cases, when the input rectangles are squares [16, 11], unit-height rectangles [24], or have restricted positions [26, 15]. Currently the best result for the general case is a $O(\log n/\log\log n)$-approximation by Chan and Har-Peled [13]. When all rectangles have unit weights, Chalermsook and Chuzhoy [10] present an $O(\log\log n)$-approximation algorithm. A much better approximation is possible for super-polynomial time algorithms. Recently, Adamaszek and Wiese [1] showed a quasi-polynomial time approximation scheme for MWISR, thus showing that the problem cannot be APX-hard unless $\mathsf{NP} \subseteq \mathsf{DTIME}(2^{\mathrm{poly}(\log n)})$.

Despite extensive effort of various groups of researchers, the approximability status of MWISR has so far remained elusive. On one hand, the existence of the recent QPTAS suggests that a PTAS is possible, but on the other hand, even a sub-logarithmic approximation has not been obtained for two decades. No substantial progress in the lower bound has been made, and even for the integrality gap of the natural LP relaxation we only have a lower bound of 2!

Closely related to MWISR (and notoriously hard) is the question of rectangle coloring. In this problem, we are given a collection of axis-parallel rectangles in the plane, and the goal is to color the rectangles so that intersecting rectangles have different colors, while minimizing the number of colors used. In 1948 [8] Bielecki asked whether one can bound the number of colors in such a coloring by the clique size of the intersection graph of the input rectangles. Denote the clique size of the intersection graph of $\mathcal{R}$ by $\omega(\mathcal{R})$. In 1960, Asplund and Grunbaum [6] showed that at most $O(\omega(\mathcal{R})^2)$ colors are needed. This status has not changed for half a century. The upper bound of $O(\omega(\mathcal{R})^2)$ is still asymptotically the best known result, while the best known lower bound is $3\omega(\mathcal{R})$ [6]. Closing this gap is seen as a challenging open problem in discrete mathematics (see, e.g., a survey by Kostochka [25]).

The state of the art of these two problems gives convincing evidence that rectangle problems are hard to deal with, and clearly new insights are needed.

1.1 A Relaxed Model: Shrinkable Rectangles

Motivated by the barriers of designing approximation algorithms for MWISR, we study a slight relaxation of the problem. Instead of computing a set of pairwise non-overlapping rectangles, we allow our algorithm to output a subset of rectangles that is *almost feasible* in the following sense. The subset of the rectangles must be pairwise non-overlapping *after* we shrink each rectangle by a multiplicative factor of $1-\delta$ for some small constant $\delta > 0$. Formally, this means that a rectangle $(a, a+x) \times (b, b+y)$ will become $(a+\frac{\delta}{2}x, a+(1-\frac{\delta}{2})x) \times (b+\frac{\delta}{2}y, b+(1-\frac{\delta}{2})y)$. We compare the value of our (almost feasible) solution to the value of an optimal feasible solution. We call this problem *δ-MWISR*. Observe that δ-MWISR remains NP-hard (see Appendix A for a proof). We remark that similar models have been studied before. In particular, Har-Peled and Lee showed approximation algorithms for geometric set cover problems for fat objects when the input objects are allowed to expand slightly [21]. In fact, this relaxed model still serves the purposes of many applications such as map labeling where it is tolerable to slightly shrink the rectangles without losing much benefit.

1.2 Our Contributions

We solve three long-standing open problems in the domain of rectangle intersection graphs in our new model. First, we give a polynomial time approximation scheme for δ-MWISR while, as mentioned above, the best known polynomial time algorithm in the ordinary setting has a superconstant approximation ratio of $O(\log n/\log\log n)$.

▶ **Theorem 1.** *Let $\epsilon, \delta > 0$ be any constants. There is a $(1+\epsilon)$ approximation algorithm for δ-MWISR that runs in time $n^{(\frac{1}{\epsilon\delta})^{O(1/\epsilon)}}$.*

The core of this result is a plane cutting procedure that follows the framework of [1]. The high-level idea is that we recursively partition the input plane into a collection of axis-parallel polygons. Rectangles overlapping the boundaries of the partition are lost. In [1], it has been shown that for any set of pairwise non-overlapping rectangles there exists such a cutting sequence where only an ϵ-fraction of all rectangles (or rectangles of small total weight) is cut and the maximum complexity of a polygon arising in this sequence is bounded by $(\log n/\epsilon)^{O(1)}$. When guessing this cut sequence recursively, we obtain an $(1+\epsilon)$-approximation algorithm with a running time of $n^{(\log n/\epsilon)^{O(1)}}$, i.e., quasi-polynomial. For our relaxed model, we construct a totally different cut sequence,

where any polygon arising in this sequence has *constant complexity*, and still only an ϵ-fraction of the overall weight is lost. Therefore, when embedding the search for this cut sequence into a dynamic program, we obtain a *polynomial* time approximation scheme.

Next, we study the rectangle coloring problem. Let us first give a formal statement of the problem. For any collection $\mathcal{R}$ of axis-parallel rectangles in the plane, one can define an intersection graph $G = (V, E)$ by introducing one vertex in V for each rectangle in $\mathcal{R}$ and connecting two vertices if and only if their corresponding rectangles overlap. We denote by $\omega(\mathcal{R})$ the clique number of the resulting intersection graph of $\mathcal{R}$ and by $\chi(\mathcal{R})$ its chromatic number. For rectangles, the clique number is identical to the minimum number q such that any point in the plane is contained in at most q rectangles. Clearly, $\chi(\mathcal{R}) \geq \omega(\mathcal{R})$. The main open question is whether $\chi(\mathcal{R}) = O(\omega(\mathcal{R}))$ for any collection of rectangles $\mathcal{R}$.

The relation between $\chi(\mathcal{R})$ and $\omega(\mathcal{R})$ is also interesting in our model. We now want to compute a minimum number of colors c for which there exists a c-coloring of the rectangles such that after the shrinking operation rectangles with the same color are pairwise non-overlapping. We prove the following result.

▶ **Theorem 2.** *For any $\delta > 0$, any collection of axis-parallel rectangles $\mathcal{R}$ in the plane can be colored with $O((\frac{1}{\delta})^2 \log^2(\frac{1}{\delta}))\omega(\mathcal{R})$ colors, such that after shrinking each rectangle by a multiplicative factor of $(1-\delta)$ the resulting rectangles with the same color are pairwise non-overlapping. Moreover, we can compute such a coloring in polynomial time.*

We prove this theorem by showing a rather general partitioning lemma that splits any collection of rectangles into $O((\frac{1}{\delta})^2 \log^2(\frac{1}{\delta}))$ sub-collections. Each of the resulting collections has the property that its rectangles can be shrunk by a factor of at most $(1-\delta)$ such that any two overlapping rectangles are either contained in one another or they do not overlap on a corner, i.e., they cross each other. It has been shown in [9] (building on the previous work [6, 13, 26]) that such collections of rectangles $\mathcal{R}'$ admit a coloring algorithm with at most $\omega(\mathcal{R}')$ colors. This gives us the desired result.

Due to a connection between coloring and the integrality gap of the natural LP-relaxation of MWISR (see, e.g, [9]), we obtain the following corollary (in fact, our partitioning lemma also yields this directly). We will define this relaxation formally in Section 3.

▶ **Corollary 3.** *The integrality gap for the natural LP relaxation for δ-MWISR is at most $O((\frac{1}{\delta})^2 \log^2(\frac{1}{\delta}))$ and there is a polynomial time $O((\frac{1}{\delta})^2 \log^2(\frac{1}{\delta}))$-approximation algorithm for δ-MWISR that rounds this LP.*

1.3 Other Related Work

The framework of Adamaszek and Wiese has been further extended in [2, 20] to give a QPTAS for the maximum independent set of polygons in general. In polynomial time, the best result is a n^ϵ-approximation by Fox and Pach [18] for independent set of arbitrary curves in the plane. For the rectangle coloring problem better bounds are known for some special cases of rectangles [26, 3]. Also, a small improvement over Asplund and Grünbaum was discussed in [23]. We refer the readers to a nice survey by Kostochka for a more complete literature on the coloring problem for other objects [25].

Finally, we remark that special cases of both MWISR and rectangle coloring when intersection patterns are restricted are much simpler than the general problem. When one rectangle is not allowed to contain any corner of another, the intersection graph is a perfect graph; therefore both problems are polynomial time solvable (see, e.g., [13, 26]).

1.4 Problem Definition and Notation

We are given a set of n axis-parallel rectangles $\mathcal{R} = \{R_1, \dots, R_n\}$ in the plane. Each input rectangle R_i is specified by an open set $R_i := \{(x,y) | x_i^{(1)} < x < x_i^{(2)} \wedge y_i^{(1)} < y < y_i^{(2)}\}$ together with its weight w_i[1]. For each rectangle R_i we denote its *width* and *height* by $g_i := |x_i^{(1)} - x_i^{(2)}|$ and $h_i = |y_i^{(1)} - y_i^{(2)}|$, respectively. We say that a subset of rectangles $\mathcal{S} \subseteq \mathcal{R}$ is an *independent set* if every pair of rectangles $R_i, R_j \in \mathcal{S}$ satisfies $R_i \cap R_j = \emptyset$.

Our model uses the following relaxed notion of an independent set. For $R_i \in \mathcal{R}$, a δ-shrunk rectangle $R_i^{-\delta}$ is defined by the x-coordinates $x_i^{(1)} + \frac{1}{2}\delta g_i$ and $x_i^{(2)} - \frac{1}{2}\delta g_i$, and the y-coordinates $y_i^{(1)} + \frac{1}{2}\delta h_i$ and $y_i^{(2)} - \frac{1}{2}\delta h_i$ respectively. Then, for any subset $\mathcal{S} \subseteq \mathcal{R}$ of rectangles, we denote by $\mathcal{S}^{-\delta}$ the collection of δ-shrunk rectangles of $\mathcal{S}$, i.e., $\mathcal{S}^{-\delta} = \{R_i^{-\delta} : R_i \in \mathcal{S}\}$. We say that a subset $\mathcal{S} \subseteq \mathcal{R}$ is a *δ-independent set* if $\mathcal{S}^{-\delta}$ is an independent set.

Now we define our problems formally. In δ-MWISR our goal is to find a maximum weight subset $\mathcal{S} \subseteq \mathcal{R}$ that is δ-independent. For the coloring problem, we define a δ-chromatic number, denoted by $\chi^{-\delta}(\mathcal{S})$, of a collection $\mathcal{S} \subseteq \mathcal{R}$ as the minimum integer c such that rectangles in $\mathcal{S}$ can be colored using c colors so that rectangles with the same color form a δ-independent set. Our goal is to bound $\chi^{-\delta}(\mathcal{R})$ in terms of $\omega(\mathcal{R})$[2].

2 Approximation Scheme for Independent Set

In this section, we present a polynomial time approximation scheme for δ-MWISR for any constant $\delta > 0$. More precisely, for any constants $\epsilon > 0$ and $\delta > 0$, we present a $(1+\epsilon)$-approximation algorithm for δ-MWISR with a running time of $n^{(\frac{1}{\epsilon\delta})^{O(1/\epsilon)}}$. Denote by $N := \max_i\{x_i^{(1)}, y_i^{(1)}, x_i^{(2)}, y_i^{(2)}\}$. Suppose for now that N is bounded by a polynomial in n. We will show later how to remove this assumption.

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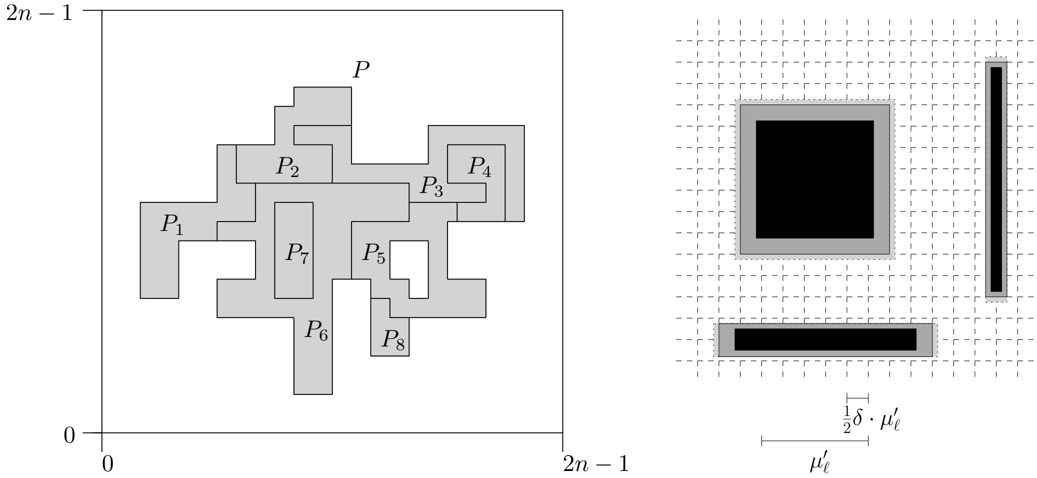
[1] By linear scaling we can assume the rectangle coordinates to be integers, even if in the actual input we are given rationals.

[2] Notice that there is a collection $\mathcal{R}$ for which the lower bound of $\omega(\mathcal{R})$ still holds, e.g., consider a collection of identical rectangles.



(a) Partitioning of polygons **(b)** Shrinking of rectangles

Figure 1 Illustration of our algorithm. Figure 1a denotes the partition of polygon P (grey area) into at most k smaller polygons, each with at most k edges. In Figure 1b depicted in light grays are original rectangles R_i from the input, in dark gray the slightly shrunk rectangles R'_i, and in black the shrunk rectangles.

2.1 The Algorithm

Our algorithm, called GEO-DP, is exactly the same dynamic program which was used in [1] for obtaining a QPTAS for MWISR (without the shrinking assumption). GEO-DP is parametrized by a value $k \in \mathbb{N}$. For our purposes, we will later choose $k := (\frac{1}{\epsilon\delta})^{O(1/\epsilon)}$.

Fix a parameter $k \in \mathbb{N}$. Let $\mathcal{P}$ denote the set of all simple polygons within the $[0, N] \times [0, N]$ input square whose corners have only integer coordinates, and which have at most k axis-parallel edges each. We introduce a DP-cell for each polygon $P \in \mathcal{P}$, where a cell corresponding to P stores a near-optimal solution $sol(P) \subseteq \mathcal{R}_P^{-\delta}$, where $\mathcal{R}_P^{-\delta}$ denotes the set of all rectangles from $\mathcal{R}^{-\delta}$ contained in P. Here, near-optimal means with respect to the optimal solution using (original) rectangles from $\mathcal{R}$ contained in P.

▶ **Proposition 4.** *The number of DP-cells is at most $N^{O(k)}$.*

To compute the solution $sol(P)$ for some polygon $P \in \mathcal{P}$ we use the following procedure. If $\mathcal{R}_P^{-\delta} = \emptyset$ or $|\mathcal{R}_P^{-\delta}| = 1$ then we set $sol(P) := \mathcal{R}_P^{-\delta}$ and terminate. Otherwise, we enumerate all possibilities to partition P into k' polygons $P_1, \ldots, P_{k'} \in \mathcal{P}$ such that $k' \leq k$. See Figure 1a for an illustration. Since by Proposition 4 we have $|\mathcal{P}| \leq N^{O(k)}$, the number of potential partitions we need to consider is upper bounded by $\binom{N^{O(k)}}{k} = N^{O(k^2)}$. Let $P_1, \ldots, P_{k'}$, where $k' \leq k$, be a feasible partition, i.e., each P_j has at most k edges and they form a partition of $\mathcal{P}$. For any enumerated set $\{P_1, \ldots, P_{k'}\} \subseteq \mathcal{P}$, one can efficiently verify whether this is a feasible partition since all polygons have axis-parallel edges with integer coordinates in $\{0, \ldots, N\}$. For each polygon $P_i \in \{P_1, \ldots, P_{k'}\}$ we look up the DP-table value $sol(P_i)$ and compute $\sum_{i=1}^{k'} w(sol(P_i))$. We set $sol'(P) := \bigcup_{i=1}^{k'} sol(P_i)$ for the partition $\{P_1, \ldots, P_{k'}\}$ which yields the maximum profit. Now we define $sol(P) := sol'(P)$ if $w(sol'(P)) > \max_{R \in \mathcal{R}_P^{-\delta}} w(R)$, and otherwise $sol(P) := \{R_{\max}\}$ where $R_{\max} \in \mathcal{R}_P^{-\delta}$ is the rectangle with maximum weight in $\mathcal{R}_P^{-\delta}$. At the end, the algorithm outputs the value in the DP-cell which corresponds to the polygon containing the entire input region $[0, N] \times [0, N]$.

Thus, the entry in the DP-table for each polygon $\mathcal{P}$ can be computed in time $N^{O(k^2)}$, assuming that all entries for all polygons $\mathcal{P}' \subsetneq \mathcal{P}$ have been computed already. Since we have $|\mathcal{P}| \leq N^{O(k)}$, we get the following upper bound on the running time of GEO-DP.

▶ **Proposition 5.** *When parametrized by k the running time of GEO-DP is upper bounded by $N^{O(k^2)}$.*

For bounding the approximation ratio of GEO-DP for any parameter k, it is sufficient to consider only the special case that the input set $\mathcal{R}$ is already an optimal feasible solution. This can be proven formally by induction on the DP-cells. For $\mathcal{R}^* \subseteq \mathcal{R}$ being the optimal solution, we can prove that when GEO-DP is given $\mathcal{R}$ as input, the value for each DP-cell is at least as high as when given $\mathcal{R}^*$ as input. Therefore, we will assume from now on in our whole argumentation about GEO-DP that $\mathcal{R}$ is already the (optimal) independent set.

2.2 A Suitable Shrunk Solution

Consider $\epsilon, \delta > 0$ such that $\epsilon\delta < 1$. We define $k := (\frac{1}{\delta\epsilon})^{O(1/\epsilon)}$ and show that for this choice of the parameter, GEO-DP yields a $(1+\epsilon)$-approximate solution for δ-MWISR. Starting with an optimal solution $\mathcal{R}^* \subseteq \mathcal{R}$ for the (non-shrunk) input set $\mathcal{R}$, we first define a $(1+\epsilon)$-approximative set $\mathcal{R}'$ consisting of one rectangle R'_i for each rectangle $R_i \in \mathcal{R}^*$ such that $R_i^{-\delta} \subseteq R'_i \subseteq R_i$. Then, in the second step, we show that if the input consisted only of $\mathcal{R}'$, then GEO-DP would compute the whole set $\mathcal{R}'$ as a feasible solution. This implies that GEO-DP finds a $(1+\epsilon)$-approximate solution for δ-MWISR.

Now we start with the description of the first step. Let $\mathcal{R}^* \subseteq \mathcal{R}$ be the maximum weight set of pairwise non-overlapping rectangles, i.e., where $w(\mathcal{R}^*) = \mathsf{OPT}$. Assume for simplicity that $1/\epsilon$ and $1/\delta$ are integers. We partition the rectangles of $\mathcal{R}$ into $O_{\delta,\epsilon}(\log N)$ groups $\mathcal{R}_\ell$, according to the lengths of their respective longer edge (where $O_{\delta,\epsilon}$ hides constants that depend only on ϵ and δ). Using standard shifting techniques (see, e.g., Hochbaum and Maas [24]), by losing only a factor of $1+\epsilon$ in our objective function, we can assume that for any two rectangles in different groups, the lengths of their respective longer edge differ at least by a factor of $\frac{1}{\epsilon\delta}$, and for any two rectangles in the same group they differ at most by a factor of $(\frac{1}{\epsilon\delta})^{1/\epsilon}$.

▶ **Lemma 6.** *By losing a factor of $1+\epsilon$ in the value of the optimal solution, we can assume that there is a partition of the rectangles $\mathcal{R}$ into $O(\log N)$ groups $\mathcal{R}_\ell$ and values $\mu'_\ell, \mu_\ell \in \mathbb{N}$ for each group $\mathcal{R}_\ell$ such that*

- *$\mu'_\ell \leq \max\{g_i, h_i\} < \mu_\ell$ for each $R_i \in \mathcal{R}_\ell$ (recall that g_i and h_i are width and height of rectangle R_i respectively), and*
- *$\delta\epsilon \cdot \mu'_\ell = \mu_{\ell+1}$ and $\mu_\ell/\mu'_\ell = (1/\delta\epsilon)^{1/\epsilon}$ for each ℓ.*

Proof. We first group rectangles in $\mathcal{R}$ into $\mathcal{R}_1, \ldots, \mathcal{R}_m$ for $m = O(\log N)$ based on their values $v_i = \max\{h_i, g_i\}$, where $\mathcal{R}_j = \{R_i : v_i \in [(1/\delta\epsilon)^{j-1}, (1/\delta\epsilon)^j)\}$. Then, we again group every $1/\epsilon$ consecutive groups $\mathcal{R}_j$ together to obtain *supergroups*. We define supergroups with respect to different values of "shifts" as follows. For each shift $s \in \{1, \ldots, 1/\epsilon\}$, the supergroup $\mathcal{T}_{s,0} = \bigcup_{j=1}^{s-1} \mathcal{R}_j$ and for each $\alpha \geq 1$, we have $\mathcal{T}_{s,\alpha} = \bigcup_{j=s+(\alpha-1)/\epsilon+1}^{s+\alpha/\epsilon-1} \mathcal{R}_j$. Notice that for each fixed s, if we take the union of supergroups $\mathcal{T}_{s,\alpha}$, we would get $\mathcal{T}_s = \bigcup_\alpha \mathcal{T}_{s,\alpha} = \bigcup_{j:j\neq s \pmod{1/\epsilon}} \mathcal{R}_j$.

▶ **Observation 7.** $\sum_{s=1}^{1/\epsilon} \mathsf{OPT}(\mathcal{T}_s) \geq (1-\epsilon)\mathsf{OPT}/\epsilon$.

Proof. Let $\mathcal{R}^*$ be an optimal solution. We argue that

$$\sum_{s=1}^{1/\epsilon} w(\mathcal{T}_s \cap \mathcal{R}^*) \geq (1-\epsilon) w(\mathcal{R}^*)/\epsilon$$

Notice that each rectangle $R_i \in \mathcal{R}^*$ appears in $(1/\epsilon) - 1$ terms on the left-hand-side (more precisely, if $R_i \in \mathcal{R}_j$ where $j = s \pmod{1/\epsilon}$, then the contribution from rectangle R_i does not appear). The claim then follows. ◀

Then there must be a shift $s \in \{0, \ldots, 1/\epsilon - 1\}$ such that $w(\mathcal{T}_s \cap \mathcal{R}^*) \geq (1-\epsilon)w(\mathcal{R}^*)$. We complete the proof of this lemma by observing that for each s, the collection $\mathcal{T}_s$ has the following properties:

- For any α, for any two rectangles $R_i, R_{i'} \in \mathcal{T}_{s,\alpha}$, we have $v_i/v_{i'} \leq (1/\delta\epsilon)^{(1/\epsilon)}$.
- For two integers $\alpha < \alpha'$, for rectangles $R_i \in \mathcal{T}_{s,\alpha}, R_{i'} \in \mathcal{T}_{s,\alpha'}$, we have $h_{i'}/h_i \geq \frac{1}{\delta\epsilon}$.

◀

The readers may think of the values $\mu_0, \mu'_0, \mu_1, \ldots, \mu'_q$ as being the values $N, N(\delta\epsilon)^{1/\epsilon}$, $N(\delta\epsilon)^{1+1/\epsilon}, N(\delta\epsilon)^{1+2/\epsilon}, \ldots$. Next, we place a grid with a random offset in the plane. Let $a \in \{0, \ldots, \mu_0 - 1\}$ be a random offset. We draw the grid cells of various granularities, and we use the notion of *levels* to indicate the granularities of the cells. Denote by G_ℓ the grid of level ℓ. Each grid cell of G_ℓ has a width and height of $w_\ell = 2\delta \cdot \mu'_\ell$ and there is one grid cell whose top left corner has the coordinates (a, a). More formally, the horizontal (resp. vertical) grid lines at level ℓ are those with y-coordinates (resp. x-coordinates) $a, a + w_\ell, a + 2w_\ell, \ldots$. Observe that each grid line in G_ℓ is a also a grid line in $G_{\ell'}$ whenever $\ell' > \ell$.

For each set $\mathcal{R}_\ell$ we remove all rectangles which are intersected by a grid $G_{\ell'}$ with $\ell' < \ell$. The next lemma shows that this comes at a negligible cost, by exploiting the fact that the grid granularity $w_{\ell'}$ of each grid $G_{\ell'}$ is at least by a factor of $1/\epsilon$ larger than $\max\{g_i, h_i\}$ for any rectangle R_i in a set $\mathcal{R}_\ell$ with $\ell' < \ell$, and the fact that a was a random offset.

▶ **Lemma 8.** *Let $\epsilon > 0$ be any constant. There is a randomized algorithm that, given a collection $\mathcal{R}$ of rectangles, produces a new collection $\mathcal{R}' \subseteq \mathcal{R}$ together with grid lines $\{G_\ell\}$ such that no rectangle in group $\mathcal{R}_\ell \cap \mathcal{R}'$ is intersected by grid lines $G_{\ell'}$ for $\ell' < \ell$. Moreover,* $\mathsf{OPT}(\mathcal{R}') \geq (1-\epsilon)\mathsf{OPT}(\mathcal{R})$ *in expectation.*

Proof. We first argue that, for any $\ell' < \ell$, the probability that a rectangle $R_i \in \mathcal{R}_\ell$ is intersected by a grid line of $G_{\ell'}$ is at most $\epsilon^{\ell-\ell'}$: Consider a rectangle $R_i \in \mathcal{R}_\ell$. Two consecutive parallel grid lines of the grid $G_{\ell'}$ have a distance of $w_{\ell'} = 2\delta\mu'_{\ell'} > \frac{2}{\epsilon^{\ell-\ell'}}\mu_\ell > \frac{2}{\epsilon^{\ell-\ell'}}\max\{g_i, h_i\}$. Therefore, the probability that R_i is intersected by a horizontal grid line of $G_{\ell'}$ is at most $\epsilon^{\ell-\ell'}/2$; similarly, the probability that R_i is intersected by a vertical grid line of $G_{\ell'}$ is at most $\epsilon^{\ell-\ell'}/2$. By the union bound the probability that R_i is intersected by some grid line of $G_{\ell'}$ is bounded by $\epsilon^{\ell-\ell'}$.

Now let $\mathcal{R}^*$ be an optimal solution. Observe that any rectangle $R \in \mathcal{R}_\ell \cap \mathcal{R}^*$ is removed if it intersects some a grid line of $G_{\ell'}$ with $\ell' < \ell$. So the probability that R is removed from the instance is, by the union bound, at most $\sum_{\ell':\ell'>\ell} \epsilon^{\ell'-\ell} \leq 2\epsilon$. Therefore, in expectation, the total weight of the remaining rectangles in $\mathcal{R}^*$ is at least $(1-2\epsilon)w(\mathcal{R}^*) \geq \frac{1}{(1+3\epsilon)}w(\mathcal{R}^*)$. ◀

We remark that if N is polynomially bounded in the number of input rectangles, our algorithm does not need to execute this lemma; only the existential statement is sufficient for the DP to find a good solution. The lemma is only needed when N is superpolynomial.

Denote by $\tilde{\mathcal{R}}$ the set of rectangles from the optimal solution in the set obtained by Lemma 8. We will now shrink these rectangles for the purpose of proving that GEO-DP finds a good solution. We remark that our algorithm does not need to compute this shrinking. For each rectangle $R_i \in \tilde{\mathcal{R}}$ we define a new rectangle R_i' such that $R_i^{-\delta} \subseteq R_i' \subseteq R_i$. Consider a rectangle $R_i \in \tilde{\mathcal{R}} \cap \mathcal{R}_\ell$. If $\mu_\ell' \leq h_i < \mu_\ell$ then we move the top and bottom boundaries of R_i towards each other so that they align with the closest horizontal grid lines of G_ℓ. If $\mu_\ell' \leq g_i < \mu_\ell$, then we move the left and right boundaries of R_i towards each other so that they align with the closest vertical grid lines of G_ℓ. Note that $R_i^{-\delta} \subseteq R_i' \subseteq R_i$ since we apply the above procedure only to the edges that are at least $\mu_\ell' = w_\ell/2\delta$ units long and in their corresponding dimension R_i crosses at least $1/2\delta$ grid lines of G_ℓ. See Figure 1b for an illustration. Note, the actual shrinking for R_i is always $R_i^{-\delta}$ (R_i' is defined only for analysis.) Denote by $\mathcal{R}'$ the solution consisting of all rectangles R_i' for $R_i \in \tilde{\mathcal{R}}$.

2.3 Analysis of the Dynamic Program

In this section we show that, when given the set $\mathcal{R}$ as an input, GEO-DP will find the solution $\mathcal{R}'$ when parametrized by $k := (\frac{1}{\epsilon\delta})^{10/\epsilon}$. Using the fact that $w(\mathcal{R}') \geq (1-O(\epsilon))w(\mathcal{R}^*)$ (from Lemmas 6 and 8), this implies that GEO-DP is a $(1+\epsilon)$-approximation algorithm for δ-MWISR.

In its recursion, GEO-DP tries all possibilities to partition the input square $[0,N]\times[0,N]$ into at most k smaller polygons and then selects the most profitable partition. For each polygon in the latter partition, it again computes an optimal partition into at most k smaller polygons and so on. The sequence of cuts produced by GEO-DP can be described by a tree T where each node v is associated with a region P_v in the plane. We say that a tree T is a *good* $(k,\mathcal{R}')$*-region decomposition* if the following holds:

- For each node v in T and each rectangle $R \in \mathcal{R}'$, we have that if R does not coincide with P_v, i.e., $R \neq P_v$, then either R is contained in P_v, or R is disjoint from P_v.
- For tree nodes u and v such that u is a parent of v, we have $P_v \subseteq P_u$. Each node $v \in T$ has at most $k' \leq k$ children $u_1, \ldots, u_{k'}$ in T, and $\bigcup_{i=1}^{k'} P_{u_i} = P_v$.
- For each leaf node v of T, the polygon P_v coincides with a rectangle in $\mathcal{R}'$ or P_v has empty intersection with every rectangle in $\mathcal{R}'$.

▶ **Lemma 9.** *If a good $(k,\mathcal{R}')$-region decomposition exists, then the algorithm GEO-DP parametrized by k is a $(1+\epsilon)$-approximation algorithm for δ-MWISR.*

Proof. We assume that there is a non-overlapping set of rectangles $\mathcal{R}'$ with $w(\mathcal{R}') \geq (1-O(\epsilon))\mathsf{OPT}$ for which a $(k,\mathcal{R}')$-region decomposition exists. For each $R_i' \in \mathcal{R}'$, we denote by R_i the original, non-shrunk counterpart of R_i'. Let T be the tree that represents the region decomposition for $\mathcal{R}'$. We now prove the following statement by induction on the structure of T from its leaves to the root:

> For any node $u \in T$, when GEO-DP processes the instance given by the input rectangles that are contained in P_u, it outputs a set of rectangles $\bar{\mathcal{R}}_u$ whose weight $w(\bar{\mathcal{R}}_u)$ is at least the total weight of the rectangles in $\mathcal{R}'$ that are contained in P_u.

In particular, this statement implies that for the root node r with $P_r = [0,N]\times[0,N]$ GEO-DP computes a set of rectangles $\bar{\mathcal{R}}_r$ with weight $w(\bar{\mathcal{R}}_r) \geq w(\mathcal{R}') \geq (1-O(\epsilon))\mathsf{OPT}$ as desired.

The base case is obvious: For each leaf node v its polygon P_v coincides with a rectangle $R_i' \in \mathcal{R}'$ and thus $R_i^{-\delta}$ is in P_v; so GEO-DP returns a solution whose weight is at least $w(R_i')$. Now for the inductive step, consider a node v for which the induction hypothesis holds for all children of v. Let $\mathcal{R}_v'$ denote all rectangles from $\mathcal{R}'$ that are contained in

P_v. Denote the children of v by $v_1, \ldots, v_{k'}$ for some $k' \leq k$. We have that $P_v = \bigcup_{j=1}^{k'} P_{v_j}$ and that the polygons $P_{v_1}, \ldots, P_{v_{k'}}$ are pairwise disjoint. For each $j \in \{1, \ldots, k'\}$ let $\mathcal{R}'_{v_j}$ denote the rectangles from $\mathcal{R}'$ that are contained in P_{v_j}. Since each rectangle in $\mathcal{R}'_v$ is contained in some polygon P_{v_j} the sets $\mathcal{R}'_{v_j}$ form a partition. In particular, this implies that $w(\mathcal{R}'_v) = \sum_{j=1}^{k'} w(\mathcal{R}'_{v_j})$. Moreover, GEO-DP considers the cut which partitions P_v into $P_{v_1}, \ldots, P_{v_{k'}}$ and returns, by the induction hypothesis, a solution $\bar{\mathcal{R}}_v$ consisting of one solution $\bar{\mathcal{R}}_{v_j}$ for each polygon P_{v_j} such that $w(\bar{\mathcal{R}}_v) = \sum_{j=1}^{k'} w(\bar{\mathcal{R}}_{v_j}) \geq \sum_{j=1}^{k'} w(\mathcal{R}'_{v_j}) = w(\mathcal{R}'_v)$. This completes the proof. ◀

We prove the existence of a $(k, \mathcal{R}')$-region decomposition by iteratively cutting the polygons. Initially, before the first iteration, we have the tree T which contains only the root r with corresponding region $P_r = [0, N] \times [0, N]$ (the whole input square). Denote the grid lines we have by $\{G_\ell\}_{\ell=0}^{q}$. In each iteration ℓ, we use grid G_ℓ as a template to further cut the polygons into sub-polygons (updating the tree T accordingly). We will ensure that the following invariant holds at the beginning of iteration ℓ: For each leaf node $v \in T$, the polygon P_v has only four edges (i.e., it is a rectangular region[3]), and P_v is either contained in a grid cell of $G_{\ell-1}$ or P_v coincides with some rectangle in $\mathcal{R}'$; each region P_v has empty intersection with every rectangle in $\mathcal{R}' \cap (\bigcup_{\ell' < \ell} \mathcal{R}_{\ell'})$. Finally, every internal node has degree at most k. It is not hard to see that if we have maintained the invariant until the last iteration q, the tree T would satisfy all properties of good $(k, \mathcal{R}')$-region decomposition.

Partition into groups of cells

Now assume that we have so far maintained the invariant up to iteration ℓ, and we will provide a sequence of cuts extending the so far constructed tree such that the invariant holds for $\ell + 1$. Consider a leaf node v of T. If P_v coincides with a rectangle in $\mathcal{R}'$, no further partition is necessary (it satisfies the invariant until the end). Otherwise, we consider the grid G_ℓ restricted to P_v. Denote by $\mathcal{R}^{cor}_{v,\ell} \subseteq \mathcal{R}' \cap \mathcal{R}_\ell$ all rectangles of $\mathcal{R}' \cap \mathcal{R}_\ell$ that overlap corners of G_ℓ inside P_v. We add each such rectangle as a child node of v. Notice that these nodes satisfy the invariant for level $\ell + 1$. Let $M = (\mu'_\ell / \mu'_{\ell+1})^2$ (i.e., M equals the maximum number of grid cells of G_ℓ within P_v). Since $|\mathcal{R}^{cor}_{v,\ell}| \leq M$, the polygon P_v after removing such rectangles has at most $4M + 4$ edges. We then focus on the other rectangles. The way we shrunk rectangles guarantees the following.

▶ **Observation 10.** *Consider a grid cell C in G_ℓ. Either the cell C is not touched by any rectangle $R'_i \in \mathcal{R}_\ell \cap \mathcal{R}'$, i.e., $C \cap R'_i = \emptyset$ for all $R'_i \in \mathcal{R}_\ell \cap \mathcal{R}'$, or C is crossed by a rectangle $R'_i \in \mathcal{R}_\ell \cap \mathcal{R}'$, i.e., C without the relative interior of R'_i has two connected components.*

Since their longer edges start and end at grid coordinates, the rectangles in $\mathcal{R}^{cor}_{v,\ell}$ partition the grid cells into three disjoint groups: cells which are not crossed by any rectangle in $\mathcal{R}_\ell \cap \mathcal{R}'$, cells which are horizontally crossed, and cells which are vertically crossed (see Figure 2). The cells of the first group already satisfy the invariant for $\ell + 1$ because no rectangle in $\mathcal{R}_\ell \cap \mathcal{R}'$ intersects it (but we remark that there may be rectangles in $\mathcal{R}_{\ell+1}, \ldots,$ that may still be in such cells). For each of them we create a child node v' of v. We partition the remaining grid cells into at most M *groups* $\mathcal{C}_1, \mathcal{C}_2, \ldots$ such that two adjacent grid cells are in the same group if and only if there is a rectangle $R'_i \in \mathcal{R}_\ell$ crossing both of them. For each

[3] A rectangular region refers to a region in the plane which may not coincide with any input rectangle.

group $\mathcal{C}_j$ we add a child node v_j to v and we define the region $Q_j = (\bigcup_{C \in \mathcal{C}_j} C) \setminus \bigcup_{R_i \in \mathcal{R}^{cor}_{v,\ell}} R_i$ corresponding to node v_j.

▶ **Lemma 11.** *All cells in each group $\mathcal{C}_j$ are contained in either a grid row or a grid column of G_ℓ. Moreover, the region Q_j has at most $9M$ edges, and no rectangle in $\mathcal{R}' \cap (\bigcup_{\ell' \geq \ell} \mathcal{R}_{\ell'})$ touches its boundary.*

Proof. Assume for contradiction that there is a group $\mathcal{C}_j$ that is not horizontally or vertically contained in a grid row or column. Then $\mathcal{C}_j$ contains more than one cell and thus each cell in $\mathcal{C}_j$ is crossed horizontally or vertically but no both. If there is no cell in $\mathcal{C}_j$ that is crossed vertically then no two cells from $\mathcal{C}_j$ in different rows can be in the same group which is a contradiction since we assumed $\mathcal{C}_j$ not to be contained in one grid row. The same reasoning applies if no cell in $\mathcal{C}_j$ is crossed horizontally. Thus, there must be a grid cell C in $\mathcal{C}_j$ that is crossed horizontally and another grid cell C' that is crossed vertically. However, then the cells in $\mathcal{C}_j$ that are crossed horizontally and the ones that are crossed vertically must be in different groups.

Moreover, the edges of Q_j consist of the grid cell boundaries of G_ℓ (at most $4M$ edges as there are M such cells with 4 edges each), the boundaries of rectangles in $\mathcal{R}^{cor}_\ell$ (at most $4M$ edges as there are at most M such rectangles), and the boundaries of the polygon P_v (at most 4 edges by the induction hypothesis). So Q_j has at most $8M + 4 \leq 9M$ edges. Also, no rectangle in a set $\mathcal{R}_{\ell'}$ with $\ell' \geq \ell$ touches the boundary of Q_j because no rectangle in $\mathcal{R}' \cap \mathcal{R}_{\ell'}$ can cross a grid line of G_ℓ (by Lemma 8), the boundary of other rectangles in $\mathcal{R}'$, or the boundary of the polygon P_v (by the induction hypothesis). ◀

So the "correct" partition of P_v has one polygon for each cell that is not crossed by a rectangle in $\mathcal{R}_\ell$, one polygon for each group $\mathcal{C}_j$, and one polygon for each rectangle in $\mathcal{R}^{cor}_{v,\ell}$. Note that in total those are at most $5M$ many. Notice that these tree nodes for a group $\mathcal{C}_j$ do not necessarily satisfy the invariant since Q_j might not be contained in a grid cell of G_ℓ.

While this partition has similarities to quad-tree approaches like in Arora's algorithm for Euclidean TSP [5] we note that in such classical approaches the pieces arising in the recursive partition (typically squares) do not depend on the instance and are predetermined. In constrast, in our case this partition depends on the structure of the optimal solution $\mathcal{R}'$ and the algorithm has to guess the correct one. Furthermore, before proceeding to the next level we must further refine the partitions that correspond to groups $\mathcal{C}_j$ step-by-step as we explain in the sequel.

Further partitioning of each group

Next, we show that there is a sequence of cuts that further partition each group $\mathcal{C}_j$ into a family of smaller polygons such that at each intermediate step each polygon has at most k edges. Consider group $\mathcal{C}_j$ that is horizontally crossed (the other case is symmetric). We construct a (planar) graph $H_j = (V_j, E_j)$ within Q_j, see Figure 3 for a sketch. The set V_j has a node for each vertex of the polygon Q_j and for each intersection of the top or bottom edge of a rectangle $R'_i \in \mathcal{R}' \cap \mathcal{R}_\ell$ with a vertical grid line in G_ℓ (including the corners of R'_i). Denote by $V_j^{(0)}, V_j^{(1)}, V_j^{(2)}, \ldots$ the vertices in V_j ordered by the vertical grid lines they appear on, i.e. $V_j^{(p)}$ contains the vertices in V_j on the p^{th} vertical grid line in G_ℓ inside Q_j. For each p, we introduce a horizontal edge in E_j between two vertices $v \in V_j^{(p)}$, $v' \in V_j^{(p+1)}$ if and only if v and v' lie on the same edge of a rectangle in $\mathcal{R}' \cap \mathcal{R}_\ell$; also we add a vertical edge in E_j between two vertices $v \in V_j^{(p)}$, $v' \in V_j^{(p)}$ if the line segment L between v and v'

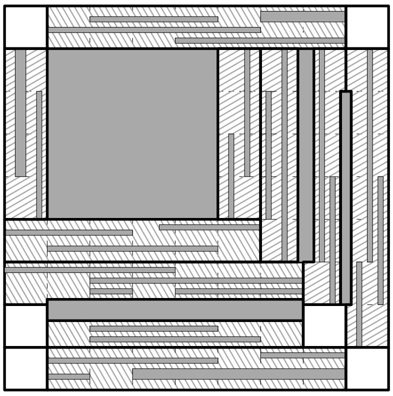


Figure 2 The pieces P'_j for the groups $\mathcal{C}_j$ and the grid cells that are not touched by any rectangle. The shading indicates whether the group is a horizontal or a vertical group.

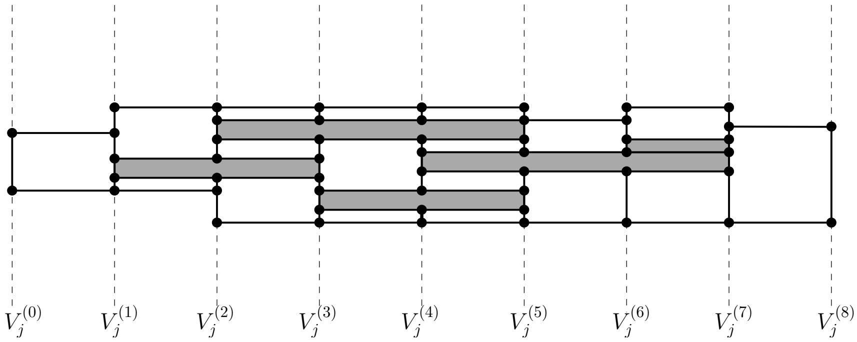


Figure 3 The graph H_j for one piece P'_j. The thick lines represent the edges E_j of H_j.

does not cross any rectangle $\mathcal{R}' \cap \mathcal{R}_\ell$ and also no other vertex $v'' \in V_j^{(p)}$ with $v'' \notin \{v, v'\}$. By construction, no edge in E_j crosses through any rectangle in $\mathcal{R}'$.

Now we cut the region Q_j step-by-step along simple paths in H_j which go from left to right, visiting a vertex in $V_j^{(p)}$ after having visited a vertex in $V_j^{(p-1)}$, for each p. We call such paths *cutting paths*. Each polygon arising in this partition sequence can be described as the polygon $P(\sigma, \sigma')$ between two cutting paths σ and σ' that start at some common point s and end at t; also they are disjoint except at the two endpoints. Observe that such polygons have at most $O(M)$ edges each and that Q_j itself equals $P(\sigma_T, \sigma_B)$ where σ_T and σ_B denote the paths describing the top and bottom boundary of Q_j, respectively. Now the idea is that if a polygon $P(\sigma, \sigma')$ for two cutting paths σ, σ' does not satisfy the invariant, then it can be further partitioned along another cutting path σ'', as the following lemma shows (we will prove it later in Section 2.4).

▶ **Lemma 12.** *Let σ, σ' be two cutting paths in Q_j. Then either*

- *$P(\sigma, \sigma')$ has rectangular shape, is contained in a grid cell of G_ℓ, and has empty intersection with each rectangle in $\mathcal{R}_\ell \cap \mathcal{R}'$, or*
- *$P(\sigma, \sigma')$ has rectangular shape and it coincides with a rectangle in $\mathcal{R}' \cap \mathcal{R}_\ell$, or*
- *there is a cutting path σ'' with $\sigma \neq \sigma'' \neq \sigma'$ such that $P(\sigma, \sigma') = P(\sigma, \sigma'')\dot{\cup}P(\sigma'', \sigma')$.*

We invoke Lemma 12 on each region Q_j until the invariant is satisfied: If invoking the lemma on Q_j holds with the first or second cases, then we are done; otherwise, Q_j can be further partitioned into Q' and Q'' based on the cutting path. In such case, we add two nodes corresponding to regions Q' and Q'' into the tree T as children of Q_j, and then invoke the lemma on Q' and Q''. Since these polygons are always defined by two cutting paths, their complexities are bounded by $O(M)$. Now each leaf node that does not coincide with a rectangle in $\mathcal{R}' \cap \mathcal{R}_\ell$ satisfied the invariant. The above shows that there is a $(k, \mathcal{R}')$-region decomposition for $k = O(M) = (1/\delta\epsilon)^{1/\epsilon}$.

Note that already in the last part—the partitioning of each group—one single group might be partitioned into up to $\Omega(n)$ pieces. Thus, we cannot use an approach which guesses this partition in a single step only. In particular, to ensure polynomial running time we crucially need our DP and cannot replace it by a brute-force recursive algorithm since the depth of T can be up to $\Omega(\log n)$. This is a key difference to the QPTAS in [1] where instead of the DP one could alternatively use such a recursion and obtain the same result.

Superpolynomial input data

To remove the assumption that N is bounded by a polynomial, observe that there are only $O(\log N)$ recursion levels, which is polynomial in the length of the input encoding. Each coordinate used in our cut sequence coincides with a coordinate of a rectangle in $\mathcal{R}'$ or with a horizontal or vertical grid line (these coordinates can be computed efficiently in a randomized fashion by Lemma 8). While the last recursion level can give rise to up to $\Omega(N)$ of those, it suffices to consider only grid lines belonging to grid cells C such that there exists an input rectangle R with $R \subseteq C$. In each of the $O(\log N)$ levels, there can be only n such grid cells which bounds the total number of needed coordinates by $O(n \log N)$. This completes the proof of Theorem 1.

2.4 Proof of Lemma 12

In this section we prove Lemma 12. Assume w.l.o.g. that the polygon Q_j is completely contained in a grid row. Consider the polygon $P(\sigma, \sigma')$ defined by two cutting paths where σ is above σ', i.e. paths σ and σ' contain the upper and lower boundaries of polygon $P(\sigma, \sigma')$ respectively. Let H'_j be the subgraph of H_j induced by all vertices that are used by σ or σ' or which lie in the relative interior of $P(\sigma, \sigma')$. We assume w.l.o.g. that paths σ and σ' do not intersect except at the endpoints, i.e. they both start at some node $s \in V(H'_j)$ and end at some node $t \in V(H'_j)$. We will argue that one of the three cases of Lemma 12 applies.

We say that a path τ in H'_j is *monotone* if τ is empty or can be written as $\tau = (v_0, v_1, \ldots, v_z)$ such that for each i, vertex v_i is either on the left of v_{i+1} or on the top (i.e. the monotone path only goes right or down.) First, we need the following lemma.

▶ **Lemma 13.** *Let $u \in V(H'_j)$ be a vertex that corresponds to the bottom-right corner of a rectangle R in $\mathcal{R}_\ell$. Then there is a monotone path τ from vertex u to some vertex v' on path σ'; symmetrically, any top-left corner of a rectangle is reachable from a vertex in σ by a monotone path.*

Proof. We only prove this statement when $u \notin \sigma'$; otherwise, it is trivial (notice that u cannot be on σ.) To prove this statement, it is sufficient to show that there is a monotone path τ' that either connects vertex u to the bottom-right corner of another rectangle R' or to some vertex on σ': Applying this claim iteratively gives us the lemma.

Now notice that vertex u is on the right boundary of rectangle R, so $u \in V_j^{(p)}$ for some p. From the way we construct graph H_j, there must be a downward edge from u to either a vertex on the top boundary of some other rectangle R' or on the path σ'. In the latter case, we are immediately done. In the former case, let u' be a vertex on the top boundary of R'

that is connected to u via an edge (u, u'). We define path τ' that first takes an edge (u, u') and then from u' there is always a monotone path to the bottom-right corner of R' using edges on the boundary of R'. ◀

Using this lemma, we now prove Lemma 12. We have the following cases:

- First, if there is a vertex $u \in V(H_j)$ that is a corner of some rectangle $R \in \mathcal{R}_\ell \cap \mathcal{R}'$, we show that we can find a cutting path σ'' implying the third case of the lemma. Define σ''_t to be the monotone path that connects the top-left corner u_{top} of R to u (this path could be empty). Also σ''_b is the monotone path that connects u to the bottom-right corner u_{bot} of R. Observe that σ''_t is disjoint from σ''_b and that at least one edge in $\sigma''_b \cup \sigma''_t$ is in the interior of $P(\sigma, \sigma')$.
  We now apply Lemma 13 to find a path τ_t that connects a vertex v_{top} on σ to u_{top}, and similarly we can find a path τ_b that connects vertex u_{bot} to some vertex v_{bot} on σ'. It is easy to see that all paths $\sigma''_b, \sigma''_t, \tau_b, \tau_t$ are disjoint. Now the cutting path σ'' is easily defined: Start from s, follow path σ until it reaches v_{top}, then follow the paths $\tau_t, \sigma''_t, \sigma''_b, \tau_b$ in this order until v_{bot} is reached, and finally from v_{bot} we use the path σ' towards vertex t. This is a cutting path because we always go from left to right and the path cuts through the interior.
- Now assume that there is no such corner in the interior. There are two possibilities. First if there is no rectangle in $\mathcal{R}_\ell \cap \mathcal{R}'$ that lies in polygon $P(\sigma, \sigma')$, then either $P(\sigma, \sigma')$ is contained in one cell (in which case we are done with the first case of Lemma 12 applied), or there is a vertical edge that connects two vertices in $V_j^{(p)}$ for some p where we can cut. Otherwise, there is a rectangle $R \in \mathcal{R}_\ell \cap \mathcal{R}'$ that lies in $P(\sigma, \sigma')$ where all four corners lie on the border of polygon $P(\sigma, \sigma')$, i.e. on $\sigma \cup \sigma'$. If the upper boundary of R does not lie on σ, we could cut the polygon $P(\sigma, \sigma')$ using this upper boundary as our σ'' (in which case, the third case of Lemma 12 applies.) Similar arguments hold for the bottom boundary of R. Hence, the only case left to analyze is when the top and bottom boundaries of R lie on σ and σ' respectively. In such case, polygon $P(\sigma, \sigma')$ coincides with rectangle R, and the second case of Lemma 12 applies.

3 Coloring and Integrality Gap

In this section, we consider the rectangle coloring problem and bound the integrality gap of the LP for MWISR in our model. Both results rely on a partitioning lemma that divides rectangles into sub-collections with “nice” properties. We will first define these properties precisely and state the partitioning lemma. Then we will describe how it can be used to prove Theorem 2 and Corollary 3.

For pairs of intersecting rectangles we distinguish three types of intersections: crossing, containment, and corner intersections. We say that two rectangles R, R' have a *crossing* intersection if no rectangle contains a corner of the other, a *containment* intersection if one rectangle completely contains the other, and otherwise they have *corner* intersection. We call a collection of rectangles *nice* if no two rectangles in $\mathcal{R}$ have corner intersections (but may still have containment).

It is known that if a collection of rectangles $\mathcal{R}$ is nice then we have $\chi(\mathcal{R}) = \omega(\mathcal{R})$, see e.g., [9, Theorem 4] (which implies that then the intersection graph is perfect). Note that this statement is slightly more general than the classical result in [6] that the latter equality holds if the rectangles in $\mathcal{R}$ have only crossing intersections (and thus no containment intersections). Our partitioning scheme is formally summarized in the following lemma that we will prove later in Section 3.1.

▶ **Lemma 14** (Partitioning lemma). *Let $\mathcal{R}$ be a set of rectangles. For any $\delta > 0$, there is a value $M = O((\frac{1}{\delta})^2 \log^2(1/\delta))$ and a polynomial time algorithm computing a partition of $\mathcal{R}$ into groups $\mathcal{R}_1, \ldots, \mathcal{R}_M$ and a rectangle S_i for each rectangle $R_i \in \mathcal{R}$ such that $R_i^{-\delta} \subseteq S_i \subseteq R_i$. The computed partition and the rectangles S_i have the property that each collection $\mathcal{S}_j = \{S_i : R_i \in \mathcal{R}_j\}$ is nice.*

We explain now how to use Lemma 14 in order to prove Theorem 2 and Corollary 3.

Rectangle Coloring

It is straightforward to see that Lemma 14 implies the coloring algorithm. Partition the input collection $\mathcal{R}$ into $M = O((\frac{1}{\delta})^2 \log^2(1/\delta))$ collections $\mathcal{R}_1, \ldots, \mathcal{R}_M$. Now we know that each set $\mathcal{S}_j$ is nice, so we can color its rectangles with $\omega(\mathcal{S}_j) \leq \omega(\mathcal{R})$ colors while using a different set of colors for each set $\mathcal{S}_j$. In total, the number of used colors is at most $M \cdot \omega(\mathcal{R})$. This proves that $\chi^{-\delta}(\mathcal{R}) \leq O((\frac{1}{\delta})^2 \log^2(1/\delta))\omega(\mathcal{R})$ and thus Theorem 2.

Integrality Gap

We use Lemma 14 in order to bound the integrality gap of the natural LP-formulation of MWISR in our shrinking model. To this end, we first define this LP and the meaning of an integrality gap in our model and subsequently prove Corollary 3.

First recall the following standard LP relaxation for MWISR. For each rectangle R_i, we have a variable x_i which indicates whether rectangle R_i is included in the solution.

$$\begin{aligned} \text{(LP-IS)} \quad \max \quad & \sum_{R_i \in \mathcal{R}} w_i x_i \\ \text{s.t.} \quad & \sum_{R_i : p \in R_i} x_i \leq 1 \text{ for all } p \in \mathcal{P} \\ & x_i \geq 0 \ \text{ for all } R_i \in \mathcal{R} \end{aligned}$$

Here $\mathcal{P}$ denotes the set of "interesting points" defined as follows: define a non-uniform grid by drawing a horizontal and a vertical line through each corner of an input rectangle. Note that each point in the interior of a grid cell is overlapped by exactly the same set of rectangles. For each grid cell add an arbitrary point from its interior to $\mathcal{P}$. Note that $|\mathcal{P}| \leq O(|\mathcal{R}|^2)$. In the MWISR problem, the integrality gap is the maximum possible ratio $\sup_{\mathcal{R}} \frac{\mathsf{LP}(\mathcal{R})}{\mathsf{OPT}(\mathcal{R})}$ where $\mathsf{LP}(\mathcal{R})$ denotes the optimal value of (LP-IS) on the instance $\mathcal{R}$. For the model of shrinking the rectangles, we use the following natural modification of the integrality gap definition. For each collection $\mathcal{R}$, let $\mathsf{OPT}_\delta(\mathcal{R})$ be the weight of a maximum-weight δ-feasible independent set $\mathcal{R}' \subseteq \mathcal{R}$. Notice that for any $\delta > 0$ we have that $\mathsf{OPT}_\delta(\mathcal{R}) \geq \mathsf{OPT}(\mathcal{R})$. Then the *$\delta$-shrunk integrality gap* is defined as $\sup_{\mathcal{R}} \frac{\mathsf{LP}(\mathcal{R})}{\mathsf{OPT}_\delta(\mathcal{R})}$. We need the following lemma.

▶ **Lemma 15** (Implied by Theorem 4 in [9]). *Let $\mathcal{R}$ be a nice collection of rectangles and let x be a solution to (LP-IS) for $\mathcal{R}$. Then there is a set of independent rectangles $\mathcal{R}' \subseteq \mathcal{R}$ with $w(\mathcal{R}') \geq \sum_{R_i \in \mathcal{R}} w_i x_i$.*

Now we prove Corollary 3. Let x^* be an optimal LP solution to an input collection $\mathcal{R}$ of rectangles, so we have $\sum_{R_i \in \mathcal{R}} w_i x_i^* = \mathsf{LP}(\mathcal{R})$. Use Lemma 14 to partition $\mathcal{R}$ into $\mathcal{R}_1, \ldots, \mathcal{R}_M$. By the pigeon hole principle there must be a group $\mathcal{R}_j$ with $\sum_{R_i \in \mathcal{R}_j} w_i x_i^* \geq \mathsf{LP}(\mathcal{R})/M$. Together with Lemma 15, applied on a nice set $\mathcal{R}_j$, this yields the proof of Corollary 3.

3.1 Proof of the Partitioning Lemma

We prove Lemma 14 now. Our algorithm deals with the x and y coordinates of the input rectangles separately in the following way: We compute two collections of intervals $\mathcal{I}^x, \mathcal{I}^y$ obtained by projecting the rectangles in $\mathcal{R}$ onto the x and y-axes, respectively. Then for each such collection we invoke the following lemma where for any interval $I = (a, a+x)$ we define $I^{-\delta} := (a + \frac{\delta}{2}x, a + (1 - \frac{\delta}{2})x)$. For simplicity, we prove the following lemma only for open intervals, as also our rectangles are defined as open sets. However, it holds also for general intervals.

▶ **Lemma 16.** *Let $\mathcal{I} = \{I_1, \dots, I_n\}$ be a set of open intervals with integral start and end points. There is a value $M = O((1/\delta)\log(1/\delta))$ and a polynomial time algorithm computing a partition of $\mathcal{I}$ into groups $\mathcal{I}_1, \dots, \mathcal{I}_M$ and an open interval I'_i with $I_i^{-\delta} \subseteq I'_i \subseteq I_i$ for each interval $I_i \in \mathcal{I}$ such that each collection $\mathcal{I}'_j = \{I'_i : I_i \in \mathcal{I}_j\}$ is* nested *(i.e. any two intervals in it are either disjoint or one is contained in another.)*

It follows straightforwardly that invoking this lemma for $\mathcal{I}^x$ and $\mathcal{I}^y$ gives the desired result: Let $\{\mathcal{I}_j^x\}_{j=1}^M$ and $\{\mathcal{I}_j^y\}_{j=1}^M$ be the partitioning obtained by the lemma. We can define a partition $\{\mathcal{R}_{j,k}\}_{j,k=1}^M$ where $\mathcal{R}_{j,k} = \{R_i : I_i^x \in \mathcal{I}_j^x \text{ and } I_i^y \in \mathcal{I}_k^y\}$. Notice that any two overlapping rectangles in the same set $\mathcal{R}_{j,k}$ must be nested in both x and y coordinates, so either they are crossing or one is contained in the other.

The proof of the above lemma has two main steps. In the first step, we group intervals into many groups by their lengths, where intervals in the same groups have roughly the same length, and the ratio of lengths of two intervals in different groups is sufficiently large. We pay a factor of $O(\log(1/\delta))$ in this step. In the second step, we partition the intervals into at most $O(1/\delta)$ groups and shrink intervals in each group to obtain the claimed properties.

Step 1: Preprocessing

We first group the intervals geometrically by their lengths into $\mathcal{I} = \bigcup_j \mathcal{I}_j$ such that each set $\mathcal{I}_j$ contains all intervals whose lengths are within $[2^j, 2^{j+1})$. Let $L := \lceil \log \frac{8}{\delta} \rceil$. For each $r \in \{0, \dots, L-1\}$ we define a collection $\Gamma_r = \{\mathcal{I}_j : j \equiv r \mod L\}$. Notice that, for any collection Γ_r, if we take two intervals from different sets $\mathcal{I}_j$, their lengths differ by at least a factor of $4/\delta$. This property will be crucial in our algorithm. In the next step, we further partitioning each collection Γ_r into $O(1/\delta)$ sub-collections.

Step 2: Shrinking

Recall that our intervals have integral start and end points and assume w.l.o.g. that they are all contained in $[0, N]$ for some large integer N. Consider a collection Γ_r. By the first step, we know that $\Gamma_r = \{\mathcal{I}_r, \mathcal{I}_{L+r}, \mathcal{I}_{2L+r}, \dots, \mathcal{I}_{\ell_{\max}L+r}\}$ with $\ell_{\max}$ being the largest integer such that $\mathcal{I}_{\ell_{\max}L+r} \neq \emptyset$. We say that an interval is at level-ℓ if it belongs to $\mathcal{I}_{\ell L+r}$, i.e., its length is in the interval $[2^{\ell L+r}, 2 \cdot 2^{\ell L+r})$. For later convenience, we define $\mu'_\ell = 2^{\ell L+r}$ and $\mu_\ell = 2^{\ell L+r+1}$. Note that $\mu'_{\ell+1}/\mu'_\ell = 2^L \geq 8/\delta$ for each ℓ. Moreover, for each ℓ we define a collection of level-ℓ points $\mathcal{P}_\ell = \{k \cdot \delta\mu'_\ell | k \in \mathbb{Z}\}$.

▶ **Observation 17.** *Each level-ℓ interval contains at least $1/\delta$ points in $\mathcal{P}_\ell$. Moreover, for any two consecutive points $p, p' \in \mathcal{P}_{\ell+1}$ there are $\mu'_{\ell+1}/\mu'_\ell - 1$ points in $\mathcal{P}_\ell \cap (p, p')$.*

We now describe our shrinking process. For each interval $I_i = (x_i, y_i)$ at level ℓ, we shrink the left-endpoint of I_i towards its centroid to the closest point in $\mathcal{P}_\ell$; similarly for the right endpoint. Formally, we define $I_i' := \left(\left\lceil \frac{x_i}{\delta\mu_\ell'} \right\rceil \cdot \delta\mu_\ell', \left\lfloor \frac{y_i}{\delta\mu_\ell'} \right\rfloor \cdot \delta\mu_\ell'\right)$ to be the shrunk interval corresponding to I_i. From the above observation, each interval gets shrunk by a factor of at most $(1-2\delta)$. Let $\mathcal{I}^{good}$ be the set of intervals that do not contain points of levels higher than the interval itself, i.e., each interval I_i is contained in $\mathcal{I}^{good}$ if and only if I_i does not contain a point in $\mathcal{P}_{\ell+1}$. Note that the latter condition implies that I_i does not contain a point in $\mathcal{P}_{\ell+2}, \mathcal{P}_{\ell+3}, \ldots$ since the values $\mu_\ell', \mu_{\ell+1}', \ldots$ pairwise divide each other.

▶ **Lemma 18.** *The collection of intervals $\mathcal{I}^{good}$ can be partitioned into $M' = O(1/\delta)$ sub-collections such that each shrunk sub-collection is nested.*

Proof. We define $M' := 2^L = O(1/\delta)$. Note that $\mu_{\ell+1}'/\mu_\ell' = M'$ for each ℓ. We partition $\mathcal{I}^{good}$ into $\{\mathcal{J}_a\}_{a=0}^{M'-1}$ as follows. Since each good level-ℓ interval I_i does not contain a point in $\mathcal{P}_{\ell+1}$, its shrunk counterpart I_i' is of the form $(K_i\delta\mu_{\ell+1}' + a_i(\delta\mu_\ell'), K_i\delta\mu_{\ell+1}' + b_i(\delta\mu_\ell'))$ for some integers K_i, a_i, b_i, where $a_i, b_i \in \{0, \ldots, M'-1\}$; that is for an interval $I_i = (x_i, y_i)$ with $I_i' = (x_i', y_i')$, we have that

$$K_i = \left\lfloor \frac{x_i'}{\delta\mu_{\ell+1}'} \right\rfloor, a_i = \frac{x_i' - K_i\delta\mu_{\ell+1}'}{\delta\mu_\ell'} \text{ and } b_i = \frac{y_i' - K_i\delta\mu_{\ell+1}'}{\delta\mu_\ell'}$$

We include each such interval I_i in the set $\mathcal{J}_{a_i}$. There can be at most $O(1/\delta)$ such sets.

Now we argue that each set $\mathcal{J}_a$ is nested. Consider a set $\mathcal{J}_a$ for some a and two intervals $I_i, I_j \in \mathcal{J}_a$ that are in levels ℓ_i and ℓ_j respectively. If I_i and I_j are disjoint, we are done, so assume that they are overlapping. If $\ell_i \neq \ell_j$ then one interval must contain the other. Here we use that for each ℓ no level-ℓ interval contains a point in $\mathcal{P}_{\ell+1}, \mathcal{P}_{\ell+2}, \ldots$. If $\ell_i = \ell_j$ we have $K_i = K_j$, and therefore one interval must contain the other. ◀

Finally, we need to deal with intervals in $\mathcal{I}^{bad} = \mathcal{I} \setminus \mathcal{I}^{good}$. The intuition is that, if we define point sets similar to $\mathcal{P}_\ell$ but with respect to some shift s, then the bad intervals are behaving like the good intervals above. Formally, we define $s = \delta \sum_\ell \frac{\mu_\ell'}{4}$ be the shift and for each ℓ we define $\mathcal{P}_\ell' = \{s + k \cdot \delta\mu_\ell' | k \in \mathbb{Z}\}$. The intervals in $\mathcal{I}^{bad}$ are shrunk with respect to these new points in a way similar to intervals in $\mathcal{I}^{good}$ but instead we use the points in $\{\mathcal{P}_\ell'\}$ rather than $\{\mathcal{P}_\ell\}$. Formally, for each interval $I_i = (x_i, y_i) \in \mathcal{I}^{bad}$ we define a new shrunk counterpart $I_i'' := \left(\left\lceil \frac{x_i - s}{\delta\mu_\ell'} \right\rceil \cdot \delta\mu_\ell' + s, \left\lfloor \frac{y_i - s}{\delta\mu_\ell'} \right\rfloor \cdot \delta\mu_\ell' + s\right)$.

▶ **Lemma 19.** *Any level-ℓ interval $I_i \in \mathcal{I}^{bad}$ does not contain any point in $\mathcal{P}_{\ell+1}'$.*

Proof. Assume otherwise that some level-ℓ interval $I_i \in \mathcal{I}^{bad}$ intersects some new point $q' \in \mathcal{P}_{\ell+1}'$. Since $I_i \in \mathcal{I}^{bad}$, the interval intersects some old point q in $\mathcal{P}_{\ell+1}$ as well. Recall that the length of the interval I_i is strictly smaller than $2\mu_\ell' \leq \frac{\delta}{4}\mu_{\ell+1}'$.

It must be the case that the coordinate of q is a multiple of $\delta\mu_{\ell+1}'$, while the coordinate of q' is equal to $s + k'\delta\mu_{\ell+1}'$ for some $k' \in \mathbb{Z}$. The shift s can also be written as $k''\delta\mu_{\ell+1}' + \delta\sum_{\ell' \leq \ell+1} \frac{\mu_{\ell'}'}{4}$ for some $k'' \in \mathbb{Z}$ (because the terms $\delta\mu_{\ell+2}', \delta\mu_{\ell+3}', \ldots,$ are multiples of $\delta\mu_{\ell+1}'$.) Observe that the term $\delta\sum_{\ell' \leq \ell+1} \frac{\mu_{\ell'}'}{4}$ is at least $\delta\mu_{\ell+1}'/4$ and at most $3\delta\mu_{\ell+1}'/4$ as the values of μ_ℓ' are geometrically increasing in ℓ. This implies that the distance between q and q' is at least $\delta\mu_{\ell+1}'/4$, and since the interval I_i contains both points, its length must be at least that much. This is a contradiction. ◀

With similar arguments as in Lemma 18 we can partition $\mathcal{I}^{bad}$ into $O(1/\delta)$ sub-collections whose respective shrunk counterparts I_i'' are nested.

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A NP-Hardness Proof

We give a sketch for the proof that δ-MWISR is NP-hard. We note that (ordinary) MWISR is NP-hard even for unit squares [17, Theorem 2]. Let $\mathcal{R}$ be an instance produced by this reduction. By analysing the proof in [17] one can easily show that the intersection area between any pair of intersecting squares in $\mathcal{R}$ is at least a constant ϵ. Notice that shrinking a (unit) square $R \in \mathcal{R}$ by a factor of $(1-\delta)$ reduces its area by at most $1-(1-\delta)(1-\delta) = 2\delta - \delta^2 \leq 2\delta$. This implies that if δ is chosen such that $4\delta < \epsilon$ then any collection of rectangles $\mathcal{S} \subseteq \mathcal{R}$ is non-overlapping if and only if $\mathcal{S}^{-\delta}$ is non-overlapping. Thus, for the instance $\mathcal{R}$, any subset $\mathcal{S} \subseteq \mathcal{R}$ is δ-independent if and only if $\mathcal{S}$ is independent. So if one were able to compute a δ-independent set of value $\mathsf{OPT}(\mathcal{R})$ in polynomial time, it would also imply that such an algorithm can compute an optimal independent set of $\mathcal{R}$.