Inapproximability of Rainbow Colouring

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Abstract

A *rainbow colouring* of a connected graph G is a colouring of the edges of G such that every pair of vertices in G is connected by at least one path in which no two edges are coloured the same. The minimum number of colours required to rainbow colour G is called its *rainbow connection number*. Chakraborty, Fischer, Matsliah and Yuster have shown that it is NP-hard to compute the rainbow connection number of graphs [J. Comb. Optim., 2011]. Basavaraju, Chandran, Rajendraprasad and Ramaswamy have reported an $(r + 3)$-factor approximation algorithm to rainbow colour any graph of radius r [Graphs and Combinatorics, 2012]. In this article, we use a result of Guruswami, Håstad and Sudan on the NP-hardness of colouring a 2-colourable 4-uniform hypergraph using constantly many colours [SIAM J. Comput., 2002] to show that for every positive integer k, it is NP-hard to distinguish between graphs with rainbow connection number $2k + 2$ and $4k + 2$. This, in turn, implies that there cannot exist a polynomial time algorithm to rainbow colour graphs with less than twice the optimum number of colours, unless P = NP.

The authors have earlier shown that the rainbow connection number problem remains NP-hard even when restricted to the class of chordal graphs, though in this case a 4-factor approximation algorithm is available [COCOON, 2012]. In this article, we improve upon the 4-factor approximation algorithm to design a linear-time algorithm that can rainbow colour a chordal graph G using at most $3/2$ times the minimum number of colours if G is bridgeless and at most $5/2$ times the minimum number of colours otherwise. Finally we show that the rainbow connection number of bridgeless chordal graphs cannot be polynomial-time approximated to a factor less than $5/4$, unless P = NP.

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1 Introduction

When a network (transport, communication, social, etc) is modelled as a graph, connectivity gives a way of quantifying its robustness. Due to the diverse application scenarios and manifold theoretical interests, many variants of the connectivity problem have been studied. One typical case is when there are different possible types of connections (edges) between nodes and additional restrictions on connectivity based on the types of edges that can be used in a path. In this case we can model the network as an edge-coloured graph. One natural restriction to impose on connectivity is that any two nodes should be connected by a path in which no edge of the same type (colour) occurs more than once. This is precisely the property

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called *rainbow connectivity.* Such a restriction for the paths can arise, for instance, in routing packets in a cellular network with transceivers that can operate in multiple frequency bands or in routing secret messages between security agencies using different handshaking passwords in different links [18, 5]. The problem was formalised in graph theoretic terms by Chartrand, Johns, McKeon, and Zhang [10].

An *edge colouring* of a graph is a function from its edge set to the set of natural numbers. A path in an edge coloured graph with no two edges sharing the same colour is called a *rainbow path.* An edge coloured graph is said to be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. Such a colouring is called a *rainbow colouring* of the graph. A rainbow colouring using minimum possible number of colours is called *optimal.* The minimum number of colours required to rainbow colour a connected graph G is called its *rainbow connection number*, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, that of an even cycle is half its length, and that of a tree is its number of edges. Note that disconnected graphs cannot be rainbow coloured and hence their rainbow connection number is left undefined. Any connected graph can be rainbow coloured by giving distinct colours to the edges of a spanning tree of the graph. Hence the rainbow connection number of any connected graph is less than its number of vertices. It is trivial to see that that $rc(G)$ is at least the diameter of G. It is easy to see that no two bridges in a graph can get the same colour under a rainbow colouring and hence $rc(G)$ is lower bounded by the number of bridges in the G.

While formalising the concept of rainbow colouring, Chartrand et al. also determined the rainbow connection number for some special graphs [10]. Subsequently, there have been various investigations towards finding good upper bounds for rainbow connection number in terms of other graph parameters [4, 20, 6, 3] and for many special graph classes [19, 6, 2]. Behaviour of rainbow connection number in random graphs is also well studied [4, 15, 21, 13]. A basic introduction to the topic can be found in Chapter 11 of the book *Chromatic Graph Theory* by Chartrand and Zhang [9] and a survey of most of the recent results in the area can be found in the article by Li and Sun [18] and also in their monograph *Rainbow Connection of Graphs* [17].

The first result showing the computational difficulty of the rainbow colouring problem was due to Chakraborty, Fischer, Matsliah, and Yuster [5]. They showed that it is NP-hard to compute the rainbow connection number of an arbitrary graph. In particular, it was shown that the problem of deciding whether a graph can be rainbow coloured using 2 colours is NP-complete. Later, Ananth, Nasre, and Sarpatwar [1] complemented the above result and now we know that for every integer $k \geq 2$, it is NP-complete to decide whether a given graph can be rainbow coloured using k colours. To the best of our knowledge, hitherto no results are reported on the hardness of approximation of rainbow connection number.

In this article we show that, for any positive integer k, it is NP-hard to distinguish between graphs with rainbow connection number $2k + 2$ and $4k + 2$ (Corollary 3). This precludes the possibility of having a polynomial time algorithm to rainbow colour graphs using less than twice the optimum number of colours, unless P = NP (Corollary 4). The proof is by a reduction from a hypergraph colouring problem. It was shown by Guruswami, Håstad, and Sudan that, for any constant c, it is NP-hard to colour a 2-colourable 4-uniform hypergraph with c colours [14]. It follows almost directly from their arguments that, for any constant $c \geq 2$, given a 4-uniform hypergraph H with chromatic number either 2 or more than c, it is NP-complete to decide whether the chromatic number of H is 2. We reduce this problem to a problem of determining whether a given bridgeless bipartite graph G with $rc(G) \in \{2k + 2, 4k + 2\}$ has $rc(G) = 2k + 2$ for any constant $k \leq (c - 1)/4$ (Theorem 2).

Currently, the best approximation guarantee for this problem on general graphs is given by an $O(nm)$-time $(r + 3)$-factor approximation algorithm for rainbow colouring a graph with radius r by Basavaraju et al. [3]. The gap between the algorithmic guarantee and the hardness of approximation shown here invites a deeper investigation into the problem. We are inclined towards believing that there might exist a polynomial-time constant-factor approximation algorithm for this problem.

One large subclass of graphs for which a constant-factor approximation algorithm is known to exist for the rainbow connection number problem is the class of chordal graphs. Chandran et al. have shown that any bridgeless chordal graph can be rainbow coloured using at most $3r$ colours, where r is the radius of the graph [6]. The proof given there is constructive and can be easily extended to a polynomial-time algorithm which will colour any chordal graph G with b bridges and radius r using at most $3r + b$ colours. Since $\max\{r, b\}$ is a lower bound for $rc(G)$, this immediately gives us a 4-factor approximation algorithm. We modify this algorithm slightly using a technique used by Li and Dong in [12] to reuse bridge colours and design a linear-time algorithm to rainbow colour chordal graphs (Algorithm 1). We then do a careful analysis using distance properties of chordal graphs to show that for any chordal graph G with diameter at least 3, the above algorithm rainbow colours G using at most $\frac{5}{2}rc(G)$ colours. Further, it follows that if G is bridgeless, then this algorithm uses at most $\frac{3}{2}rc(G) + 3$ colours only (Corollary 6). This brings to table the question of approximation hardness of the problem when restricted to chordal graphs.

We have shown, in an earlier work, that for every $k \geq 3$, the problem of deciding whether a given graph can be rainbow coloured using k colours remains NP-complete even when restricted to the class of chordal graphs [7]. In this article we go further and show an inapproximability result for the case. From the same hypergraph colouring problem that we used for the previous reduction, we give a different and more involved reduction to show that, for any positive integer k, given a bridgeless chordal graph G with $rc(G) \in \{4k, 5k\}$, it is NP-complete to decide whether $rc(G) = 4k$ (Corollary 8). As before, this precludes the possibility of having a polynomial-time algorithm to rainbow colour bridgeless chordal graphs with less than 5/4 times the optimal number of colours (Corollary 9). This should be contrasted with the case of split graphs, which are a proper subclass of chordal graphs. We have shown in an earlier work that it is NP-hard to determine the rainbow connection number of split graphs, but nevertheless designed a linear-time algorithm which will rainbow colour any split graph G using at most $rc(G) + 1$ colours [7].

2 Preliminaries

First we recall some standard graph theoretic terminology that we will use in this article. All graphs considered here are finite, simple and undirected. For a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set respectively. Let G be a connected graph. The *distance* between two vertices u and v in G, denoted by $d_G(u, v)$ is the length of a shortest path between them in G, where the *length* of a path is the number of edges in that path. The *eccentricity* of a vertex v is $ecc_G(v) := \max_{x \in V(G)} d_G(v, x)$. The *diameter* of G is $diam(G) := \max_{x \in V(G)} ecc_G(x)$ and *radius* of G is $radius(G) := \min_{x \in V(G)} ecc_G(x)$. A vertex is called *central* in G if its eccentricity is equal to the radius of G. The *boundary of G with respect to a vertex x* is $\{v \in V(G) : d_G(x, v) = ecc_G(x)\}$. If G has a unique central vertex x, then the *boundary* of G with respect to x will be referred to as just the *boundary* of G. The distance of a vertex v from a subset S of $V(G)$ is $\min_{s \in S} d_G(v, s)$. The *neighbourhood* $N(v)$ of a vertex v is the set of vertices adjacent to v but not including v. A *bridge* in a connected graph G is an edge of G whose deletion disconnects G. A connected graph is called *bridgeless* if it has no bridges. A graph is called *chordal*, if it has no induced cycle of length greater than 3. A graph is called a *split graph*, if its vertex set can be partitioned into a clique and an independent set. It is easy to see that split graphs form a subclass of chordal graphs.

Next we define hypergraphs and their colouring. A *hypergraph* H is a tuple (V, E), where V is a finite set and E is a collection of subsets of V. Elements of V and E are called vertices and (hyper)edges respectively. A vertex of H is called *isolated* if its not part of any edge of H. The hypergraph H is called r*-uniform* if $|e| = r$ for every $e \in E$. Given a hypergraph $H(V, E)$ and a colouring $c : V \to \mathbb{N}$, an edge is called k*-coloured* if it contains vertices of k different colours. An edge is called *monochromatic* if it is 1-coloured. The colouring c is called *proper* if no edge in E is monochromatic under c. The minimum number of colours required to properly colour H is called its *chromatic number* and is denoted by $\chi(H)$.

In the two approximation hardness results that we prove in this article, we make use of the following deep result of Guruswami, Håstad, and Sudan [14].

▶ **Theorem 1** (Guruswami, Håstad, Sudan)**.** *For every constant $c \geq 2$, given a 4-uniform hypergraph H with either $\chi(H) = 2$ or $\chi(H) > c$, it is* NP*-complete to decide whether $\chi(H) = 2$.*

Theorem 1 is not explicitly stated as above in their work, but it will follow easily from Corollary 4.3 in [14]. Corollary 4.3 in [14] is a result about a problem called 4-set splitting which easily translates into the 4-uniform hypergraph colouring problem as noted in the proof of Theorem 4.4 there.

Given a minimisation problem P, an (α, β)-approximation algorithm for P is an algorithm whose output on every instance I to P is a solution of P for I with cost at most $\alpha x + \beta$, where x denotes the cost of an optimum solution of P for I. If α and β are independent of the instance I, then the (α, β)-approximation algorithm is called a *constant factor approximation algorithm* for P. An $(\alpha, 0)$-approximation algorithm will also be referred to as an α-factor algorithm.

Throughout this article, the shorthand $[n]$ denotes the set $\{1, \ldots, n\}$. The cardinality of a set S is denoted by $|S|$.

3 General graphs: Hardness of approximation

▶ **Theorem 2.** *For every positive integer k, the first problem below* (P1) *is polynomial-time reducible to the second* (P2)*.*

P1. *Given a 4-uniform hypergraph H with either $\chi(H) = 2$ or $\chi(H) \geq 4k+2$, decide whether $\chi(H) = 2$.*

P2. *Given a bipartite bridgeless graph G with $rc(G) \in \{2k+2, 4k+2\}$, decide whether $rc(G) = 2k+2$.*

Proof. Let k be arbitrary and $H = (V_H, E_H)$ be the 4-uniform hypergraph given as an instance of P1. Since isolated vertices do not affect the chromatic number of a hypergraph, we can safely assume that H does not contain an isolated vertex. We construct a bridgeless graph G from H as follows.

First we construct a graph G'_0 with $V(G'_0) = \{x\} \cup V_H \cup E_H$ and $E(G'_0) = E_1 \cup E_2$ where $E_1 = \{\{x, v\} : v \in V_H\}$ and $E_2 = \{\{v, e\} : v \in V_H, e \in E_H, v \in e \text{ in } H\}$. From G'_0 we construct G_0 by replacing each edge $\{v, e\}$ in E_2 with a new k-length path between v and e. Let $G_1, \ldots, G_{4k+2}$ be copies of G_0. We obtain our desired graph G by identifying the vertex

x as common in every G_i, $i \in [4k+2]$. It is easy to see that G is bipartite, bridgeless and has radius $k+1$ with x as the unique central vertex. The $(4k+2)|V_H|$ neighbours of x in G are the vertices corresponding to the vertices of H and their collection is denoted by V_{V_H}. Similarly the $(4k+2)|E_H|$ vertices in G at distance $k+1$ from x (the boundary vertices of G) are those corresponding to the hyperedges of H and their collection is denoted by V_{E_H}. It is evident that the construction of G from H takes time at most polynomial in size of H.

We prove the theorem by establishing the following two claims. The converses of both the claims are also true since the converse of one is the contrapositive of the other.

Claim 2.1. *If* $\chi(H) = 2$ *then* $rc(G) = 2k+2$.

Claim 2.2. *If* $\chi(H) \geq 4k+2$ *then* $rc(G) = 4k+2$.

Since the diameter of G is $2k+2$ we see that $rc(G) \geq 2k+2$ always. Hence to prove Claim 2.1 it suffices to show that $rc(G) \leq 2k+2$ whenever $\chi(H) = 2$. Let $c_H : V_H \to \{a, b\}$ be a proper 2-colouring of H. For each vertex $v \in V_{V_H}$ consider the subtree T_v of G formed by all the $(k+1)$-length paths starting with the edge $\{x, v\}$ and reaching some vertex $e \in V_{E_H}$. We remark that, in this case, e will correspond to a hyperedge of H which contains v. If $c_H(v) = a$ (b) then colour every edge of T_v at a distance j from x in T_v with colour a_j (b_j) for each $j \in \{0, \ldots, k\}$. Note that all the subtrees considered above are pairwise edge-disjoint and hence every edge gets a unique colour. We argue that the above edge-colouring obtained for G is a rainbow colouring. First observe that since c_H is a proper colouring of H, every vertex u in G is part of a $(2k+2)$-cycle C_u of G containing x and with edge colours $a_0, \ldots, a_k, b_k, \ldots, b_0$ in that order starting from an edge incident on x. Given any two vertices $u, v \in V(G)$, it is not difficult to see that at least one of the four possible walks from u to v along C_u, x and C_v contains a rainbow path. Hence Claim 2.1.

In order to prove Claim 2.2, we first show the easy fact that G can always be rainbow coloured using $4k+2$ colours. Let H' be a hyperedge-maximal sub-hypergraph of H which can be properly 2-coloured. Let $V_{E_{H'}}$ be the set of $(4k+2)|E(H')|$ vertices in $V_{E_H} \subset V(G)$ which correspond to the hyperedges in H'. Let G' be the induced subgraph of G consisting of the vertices on all the $(k+1)$-length paths from x to $V_{E_{H'}}$. The maximality of H' ensures every vertex v of H is part of some hyperedge in H'. Hence $V_{V_H} \subset V(G')$ and we can rainbow colour G' using the $2k+2$ colours from $\{a_0, \ldots, a_k, b_0, \ldots, b_k\}$ as we did while showing Claim 2.1. Now contract entire G' in G to a single vertex y to obtain a minor G'' of G. The graph G'' consists of the vertex y and separate sets of 4 edge-disjoint k-length paths from y to each vertex in $V_{E_H} \setminus V_{E_{H'}}$. It is easy to edge-colour G'' using the $2k$ colours from $\{c_1, \ldots, c_k, d_1, \ldots, d_k\}$ so that every vertex u of G'' is part of a $2k$-cycle of G'' containing y and with edge colours $c_1, \ldots, c_k, d_k, \ldots, d_1$ in that order starting from an edge incident on y. This makes G'' rainbow connected as earlier. Now we can combine the above rainbow colourings of G' and G'' in the obvious manner to obtain a rainbow colouring of G using $4k+2$ colours.

Finally we show that if $\chi(H) > 4k+1$ then $rc(G) > 4k+1$. For the sake of contradiction, assume that $\chi(H) > 4k+1$ and $rc(G) \leq 4k+1$. Let c_G be a rainbow colouring of G using $rc(G)$ colours. From c_G we obtain $(4k+2)$ different vertex colourings $\{c_H^1, \ldots, c_H^{4k+2}\}$ of H as follows. For a vertex v of H and $i \in [4k+2]$, let v_i be the vertex in G_i corresponding to v. Then set $c_H^i(v) = c_G(\{x, v_i\})$. Since every one of the colourings thus obtained uses at most $4k+1$ colours and $\chi(H) > 4k+1$, none of these colourings is proper. That is, for each $i \in [4k+2]$, there exists a hyperedge $e_i \in E(H)$ which is monochromatic under c_H^i. Among them, by pigeonhole principle, there exist two hyperedges e_s and e_t, $s, t \in [4k+2]$ and $s \neq t$, such that $c_H^s(v_s) = c_H^t(v_t), \forall (v_s, v_t) \in e_s \times e_t$. Let V_s (V_t) be the set of four vertices in G_s (G_t) corresponding to the vertices of H belonging to the edge e_s (e_t) and let f_s (f_t) be the vertex in G_s (G_t) corresponding to the hyperedge e_s (e_t). We see that every edge in $\{\{x, v\} : v \in V_s \cup V_t\}$ gets the same colour under c_G. Hence none of the 16 shortest paths between f_s and f_t in G is rainbow coloured under c_G. Any other path between f_s and f_t is of length at least $4k+2$ and since c_G uses at most $4k+1$ colours it is not rainbow coloured. This contradicts the fact that c_G was a rainbow colouring of G. ◀

Since Problem P1 is known to be NP-hard (Theorem 1), so is Problem P2. Further, it is easy to see that problem P2 is in NP. Hence the following corollary and hardness of approximation.

▶ **Corollary 3.** *For every positive integer k, given a bipartite bridgeless graph G with $rc(G) \in \{2k+2, 4k+2\}$, it is* NP*-complete to decide whether $rc(G) = 2k+2$.*

▶ **Corollary 4.** *For any $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ there does not exist a polynomial time $(2-\alpha, \beta)$ approximation algorithm for determining the rainbow connection number of graphs unless* P = NP.

Proof. Suppose for some $\alpha > 0$, there exists a polynomial time $(2-\alpha, \beta)$ approximation algorithm $\mathcal{A}$ to determine the rainbow connection number of graphs. Choose a positive integer k large enough so that $(2+\beta)/(2k+2) < \alpha$. If the input graph G to $\mathcal{A}$ has $rc(G)$ at most $2k+2$, then $\mathcal{A}$ will certify that $rc(G) \leq (2-\alpha)(2k+2)+\beta < (2-\frac{2+\beta}{2k+2})(2k+2)+\beta = 4k+2$. This contradicts Corollary 3. ◀

4 Chordal graphs: An approximation algorithm and inapproximability

▶ **Theorem 5.** *For every connected chordal graph G with b bridges and radius r, Algorithm 1* (ColourChordalGraph) *returns a rainbow colouring of G using at most $3r+\max\{0, b-3\}$ colours in linear time.*

Proof. Let B_i be the number of edges between V_{i-1} and V_i which are bridges of G. First we argue that $\forall i \in [r]$, and $\forall v \in V_i$, if $N(v) \cap V_{i-1} = \{u\}$ and $N(v) \cap V_i = \emptyset$, then the edge $e = \{v, u\}$ is a bridge in G. Otherwise, a smallest cycle C containing the edge e has vertices from V_{i-1} and V_{i+1} and hence should have a length at least 4. The cycle C does not have any chords since it is a smallest cycle containing e. This contradicts the fact that G is chordal. Hence for each i, b_i counts the number of bridges of G between V_{i-1} and V_i and thus the number of colours used in round i is at most $\max\{3, B_i\}$. Hence in total, Algorithm 1 uses at most $\sum_{i=1}^{r} \max\{3, B_i\} \leq 3r + \max\{0, b-3\}$ colours.

Next we show that c_G makes G rainbow connected. For each $i \in [r]$, let $k_i = \max\{3, B_i\}$ and $K_i = \{c_{i,1}, \ldots, c_{i,k_i}\}$. In c_G, every bridge in G between V_i and V_{i-1} gets a distinct colour from K_i. Every vertex $v \in V_i$ which is not an end point of a bridge from V_{i-1} to V_i is contained in a 2- or 3-length path starting from V_{i-1} touching v and going back to V_{i-1} which is coloured $(c_{i,1}, c_{i,2})$ or $(c_{i,1}, c_{i,3}, c_{i,2})$, respectively. Hence for any two vertices $u, v \in V_i$, there exist rainbow paths P_u from u to V_{i-1} and P_v from v to V_{i-1} such that the colours assigned by c_G to P_u and P_v form disjoint subsets of K_i. Since this is true for each $i \in [r]$, for any two distinct vertices $u, v \in V(G)$, there exist two rainbow paths P_u from u to x and P_v from v to x such that the set of colours assigned to P_u and P_v by c_G are disjoint till the first common vertex in these two paths. Hence $P_u \cup P_v$ contains a u–v rainbow path.

Finally we show that Algorithm 1 runs in $O(m)$ time where m is the number of edges in the input graph. Chepoi and Dragan have designed an algorithm which finds a central vertex of a chordal graph in $O(m)$ time [11]. Hence the initialisation steps in Algorithm 1 runs in

Algorithm 1: ColourChordalGraph

Data: $G(V, E)$, a connected chordal graph
Result: A rainbow colouring c_G of G

Initialisation:
$x \leftarrow$ a central vertex of G
$r \leftarrow$ radius of G
$V_i \leftarrow \{v \in V(G) : d_G(v, x) = i\}$ for each $i \in [r]$
$b_i \leftarrow 0$ for each $i \in [r]$ `//` b_i `counts the number of` V_{i-1} `to` V_i `bridges in` G.

for $i = 1$ **to** r **do**
  $c_G(\{v, v'\}) = c_{i,3}$, $\forall v, v' \in V_i$ and $\{v, v'\} \in E(G)$
  foreach $v \in V_i$ *such that* $|N(v) \cap V_{i-1}| \geq 2$ **do**
    Let $\{u_1, \ldots, u_t\} = N(v) \cap V_{i-1}$
    $c_G(\{v, u_1\}) = c_{i,1}$
    $c_G(\{v, u_s\}) = c_{i,2}, \forall s \in \{2, \ldots, t\}$
  end
  foreach $v \in V_i$ *such that* $|N(v) \cap V_{i-1}| = 1$ **do**
    Let u be the single vertex in $N(v) \cap V_{i-1}$
    if $N(v) \cap V_i = \emptyset$ **then**
      $b_i = b_i + 1$ `//` $\{v, u\}$ `is a bridge in` G
      $c_G(\{v, u\}) = c_{i,b_i}$
    else if $\exists v' \in N(v) \cap V_i$ *and* $\exists u' \in N(v') \cap V_{i-1}$ *such that* $c_G(v', u') = c_{i,1}$ **then**
      $c_G(\{v, u\}) = c_{i,2}$
    else
      $c_G(\{v, u\}) = c_{i,1}$
    end
  end
end

return c_G

linear time. Each for-loop visits each vertex in G at most once. If we flag every vertex $v \in V_i$ when it gets a $c_{i,1}$-coloured edge to V_{i-1}, the algorithm, when it visits a vertex, needs to examine only its neighbours and incident edges. Hence the total running time in $O(m)$. ◀

Chang and Nemhauser have shown that the radius r and diameter d of any chordal graph are constrained by the inequality $r \leq d/2 + 1$ [8]. Hence we get the following corollary to Theorem 5.

▶ **Corollary 6.** *If G is a connected chordal graph with diameter at least 3, then Algorithm 1 returns a rainbow colouring of G using at most $\frac{5}{2}rc(G)$ colours. If G is bridgeless then Algorithm 1 uses only $\frac{3}{2}rc(G) + 3$ colours.*

Proof. Let G be a chordal graph with b bridges, diameter $d \geq 3$ and radius r. Let $a(G)$ be the number of colours used by Algorithm 1 in rainbow colouring G. Then by Theorem 5 and the bound by Chang and Nemhauser, we have

$$a(G) \leq \frac{3}{2}d + 3 + \max\{0, b - 3\} = \frac{3}{2}d + \max\{3, b\} \tag{1}$$

Since $d, b \leq rc(G)$ (easy observations) and $d \geq 3$ (by assumption), we get $a(G) \leq \frac{3}{2}rc(G) + rc(G) = \frac{5}{2}rc(G)$. Further if G is bridgeless then $a(G) \leq \frac{3}{2}rc(G) + 3$. ◀

▶ Remark. The requirement that $diam(G) \geq 3$ in Corollary 6 is a consequence of the generality of Algorithm 1 and not due to any inherent difficulty in the problem. If $diam(G) = 1$, then G is a clique and hence a colouring which gives every edge of G the same colour is a rainbow colouring. Li, Li, and Liu have shown that any graph G with diameter 2 has $rc(G) \leq 5$ if it is bridgeless, and $rc(G) \leq b + 2$ if it has b bridges [16].

In the context of the above approximation algorithm, we now state and prove the inapproximability result on chordal graphs.

▶ **Theorem 7.** *For every positive integer k, the first problem below* (P1) *is polynomial-time reducible to the second* (P2).

P1. *Given a* 4*-uniform hypergraph H with either $\chi(H) = 2$ or $\chi(H) \geq 5k$, decide whether $\chi(H) = 2$.*

P2. *Given a bridgeless chordal graph G with $rc(G) \in \{4k, 5k\}$, decide whether $rc(G) = 4k$.*

Proof. Let k be arbitrary and $H = (V_H, E_H)$ be the 4-uniform hypergraph given as an instance of P1. Since isolated vertices do not affect the chromatic number of a hypergraph, we can safely assume that H does not contain an isolated vertex. We construct a bridgeless chordal graph G from H as follows.

First we construct a graph G_H with $V(G_H) = \{x\} \cup V_H \cup E_H$ and $E(G_H) = E_1 \cup E_2 \cup E_3$ where $E_1 = \{\{x, v\} : v \in V_H\}$, $E_2 = \{\{v, e\} : v \in V_H, e \in E_H, v \in e \text{ in } H\}$, and $E_3 = \{\{v, v'\} : v, v' \in V_H\}$. The graph G_H is easily verified to be bridgeless and chordal. In fact, it is a split graph with V_H as the clique and $\{x\} \cup E_H$ as the independent set. Now we construct a graph G_1 by taking $(5k)^k + 1$ copies of G_H and identifying x as common in every copy. The common vertex x is relabelled as x_0. The vertex x_0 will serve as the unique central vertex for G_1 and all the $G_i, i \in \{2, \ldots, k\}$, to be constructed next. Once we have G_i for some $i < k$, we construct G_{i+1} by joining to every vertex e in the boundary of G_i, another copy of G_H identifying $x \in V(G_H)$ with e. Note that the boundary of G_i is the set of vertices at distance of $radius(G_i) = 2i$ from x_0, which in our case, turns out to be all the vertices corresponding to some hyperedge of H in a copy of G_H added in the i-th step. The graph $G = G_k$ constructed this way is our desired graph. Since a graph obtained from two bridgeless chordal graphs by identifying a single vertex as common to both is bridgeless and chordal, we see that G is a bridgeless chordal graph as desired. G has diameter $4k$ and radius $2k$ with x_0 as the unique central vertex. If $h = |E_H|$, then G has $\left((5k)^k + 1\right)\left(1 + h + \cdots + h^{k-1}\right)$ copies of G_H and hence the construction of G takes only a time polynomial in the size of H.

We prove the theorem by establishing the following two claims. The converses of both the claims are also true since the converse of one is the contrapositive of the other.

Claim 7.1. *If $\chi(H) = 2$ then $rc(G) = 4k$.*

Claim 7.2. *If $\chi(H) \geq 5k$ then $rc(G) = 5k$.*

Since the diameter of G is $4k$, it follows that $rc(G) \geq 4k$ always. Hence to prove Claim 7.1 it suffices to show that $rc(G) \leq 4k$ whenever $\chi(H) = 2$. We define an edge-colouring c_G of G based on a red-blue colouring c_H of H by describing the colours assigned to the edges of each copy of G_H in G. Let G_H^i be a copy of G_H added at the i-the level, that is, $x \in G_H^i$ is at a distance $2i-2$ from x_0 in G. Recall that $E(G_H^i) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{\{x, v\} : v \in V_H\}$, $E_2 = \{\{v, e\} : v \in V_H, e \in E_H, v \in e \text{ in } H\}$, and $E_3 = \{\{v, v'\} : v, v' \in V_H\}$. An edge

$\{x, v\} \in E_1$ is given colour a_i (c_i) if the vertex corresponding to v in H is coloured red (blue) in c_H. An edge $\{v, e\} \in E_2, v \in V_H$ is given colour b_i (d_i) if the vertex corresponding to v in H is coloured red (blue) in c_H. This ensures that every vertex in G^i_H is part of a 4-cycle of G^i_H containing $x \in V(G^i_H)$ and with edge colours a_i, b_i, d_i, c_i in that order starting from an edge incident on x. Colours on the edges in E_3 do not matter to us and hence we can give them any colour that is already used. This colouring c_G of G thus uses $4k$ colours and has the property that if u and v are two distinct vertices of G with $\{d_G(u, x_0), d_G(v, x_0)\} \subset \{2i-1, 2i\}, i \in [k]$, then there exist two vertices u' and v' (not necessarily distinct) with $d_G(u', x_0) = d_G(v', x_0) = 2i-2$ and rainbow paths P_u from u to u' and P_v from v to v' such that the colours used in P_u and P_v form two disjoint subsets of $\{a_i, b_i, c_i, d_i\}$. It is easy to see that this property ensures that c_G is a rainbow colouring of G.

If we give G as an input graph to Algorithm 1, then the colouring returned by it will use 3 colours at all odd levels and 2 colours at all even levels. Hence the total number of colours used is $5k$ which means $rc(G) \leq 5k$. Hence to prove Claim 7.2 we only need to show that if $\chi(H) \geq 5k$, then $rc(G) \geq 5k$. Assume, for the sake of contradiction, that c_G is a rainbow colouring of G using less than $5k$ colours. Every copy G'_H of G_H in G then induces a vertex colouring of c'_H of H as $c'_H(v) = c_G(\{x, v\}), \{x, v\} \in E(G'_H)$ for every $v \in V_H$. Since c'_H uses less than $5k$ colours and $\chi(H) \geq 5k$, there exists a hyperedge $e \in E_H$ that is monochromatic under c'_H, which means that the second edge in all the 2-length paths from e to x in G'_H is of the same colour. Let us call such a vertex e a *trapped vertex* of G and the common colour on the second edge of all the 2-length paths from e to x the *blocking colour* of e. Since every copy of G_H in G has at least one trapped vertex we get $(5k)^k + 1$ disjoint sequences of the form $(t_1, \ldots, t_k)$ such that t_1 is a trapped vertex in G_1, and $t_i, i \geq 2$ is a trapped vertex in a copy of G_H attached to t_{i-1}. Since we have $(5k)^k + 1$ such disjoint sequences there exists at least 2 sequences $(t_1, \ldots, t_k)$ and $(s_1, \ldots, s_k)$ which induce the same sequence of blocking colours. Hence in any rainbow path P between t_k and s_k, if $d_P(t_i, t_{i-1}) = 2$, then $d_P(s_i, s_{i-1}) \geq 3$ and vice versa. Hence the length of P is at least $5k$ and so P cannot be a rainbow path in a colouring which uses less than $5k$ colours. This contradiction proves Claim 7.2. ◀

Since Problem P1 is known to be NP-hard (Theorem 1), so is Problem P2. Further, it is easy to see that problem P2 is in NP. Hence the following corollary and hardness of approximation.

▶ **Corollary 8.** *For every positive integer k, given a bridgeless chordal graph G with $rc(G) \in \{4k, 5k\}$, it is* NP*-complete to decide whether $rc(G) = 4k$.*

▶ **Corollary 9.** *For any $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, there does not exist a polynomial time $(5/4-\alpha, \beta)$ approximation algorithm for determining the rainbow connection number of chordal graphs unless* P = NP.

Proof. Suppose for some $\alpha > 0$, there exists a polynomial time $(5/4-\alpha, \beta)$ approximation algorithm $\mathcal{A}$ to determine the rainbow connection number of chordal graphs. Choose a positive integer $k > \beta/4\alpha$. If the input chordal graph G to $\mathcal{A}$ has $rc(G)$ at most $4k$, then $\mathcal{A}$ will certify that $rc(G) \leq (5/4-\alpha)4k + \beta = 5k - (4\alpha k - \beta) < 5k$. This contradicts Corollary 8. ◀

References

1 Prabhanjan Ananth, Meghana Nasre, and Kanthi K. Sarpatwar. Rainbow Connectivity: Hardness and Tractability. In *FSTTCS 2011*, LIPIcs, volume 13, pages 241–251, Schloss Dagstuhl, 2011. doi: 10.4230/LIPIcs.FSTTCS.2011.241

2 Manu Basavaraju, L. Sunil Chandran, Deepak Rajendraprasad, and Arunselvan Ramaswamy. Rainbow connection number of graph power and graph products. *Accepted for publication in Graphs and Combinatorics. Preprint arXiv:1104.4190v2 [math.CO]*, 2011.

3 Manu Basavaraju, L. Sunil Chandran, Deepak Rajendraprasad, and Arunselvan Ramaswamy. Rainbow connection number and radius. *Graphs and Combinatorics*, pages 1–11, 2012.

4 Yair Caro, Arie Lev, Yehuda Roditty, Zsolt Tuza, and Raphael Yuster. On rainbow connection. *Electron. J. Combin.*, 15(1):Research paper 57, 13, 2008.

5 Sourav Chakraborty, Eldar Fischer, Arie Matsliah, and Raphael Yuster. Hardness and algorithms for rainbow connection. *J. Comb. Optim.*, 21(3):330–347, 2011.

6 L. Sunil Chandran, Anita Das, Deepak Rajendraprasad, and Nithin M. Varma. Rainbow connection number and connected dominating sets. *Journal of Graph Theory*, 71(2):206–218, 2012.

7 L. Sunil Chandran and Deepak Rajendraprasad. Rainbow colouring of split and threshold graphs. In Joachim Gudmundsson, Julián Mestre, and Taso Viglas, editors, *Computing and Combinatorics*, volume 7434 of *LNCS*, pages 181–192. Springer, 2012. doi: 10.1007/978-3-642-32241-9_16.

8 Gerard J Chang and George L Nemhauser. The k-domination and k-stability problems on sun-free chordal graphs. *SIAM Journal on Algebraic Discrete Methods*, 5(3):332–345, 1984.

9 G. Chartrand and P. Zhang. *Chromatic Graph Theory*. Chapman & Hall, 2008.

10 Gary Chartrand, Garry L. Johns, Kathleen A. McKeon, and Ping Zhang. Rainbow connection in graphs. *Math. Bohem.*, 133(1):85–98, 2008.

11 Victor Chepoi and Feodor Dragan. Linear-time algorithm for finding a central vertex of a chordal graph. In *ESA'94, Second Annual European Symposium*, pages 159–170. Springer, 1994.

12 Jiuying Dong and Xueliang Li. Rainbow connection number, bridges and radius. *Graphs and Combinatorics*, pages 1–7, 2012.

13 A. Frieze and C.E. Tsourakakis. Rainbow connectivity of $G(n,p)$ at the connectivity threshold. *Preprint arXiv:1201.4603*, 2012.

14 Venkatesan Guruswami, Johan Håstad, and Madhu Sudan. Hardness of approximate hypergraph coloring. *SIAM Journal on Computing*, 31(6):1663–1686, 2002.

15 Jing He and Hongyu Liang. On rainbow-k-connectivity of random graphs. *Information Processing Letters*, 112(10):406–410, 2012.

16 Hengzhe Li, Xueliang Li, and Sujuan Liu. Rainbow connection of graphs with diameter 2. *Discrete Mathematics*, 312(8):1453–1457, 2012.

17 X. Li and Y. Sun. *Rainbow Connections of Graphs*. Springerbriefs in Mathematics. Springer, 2012.

18 Xueliang Li, Yongtang Shi, and Yuefang Sun. Rainbow connections of graphs: A survey. *Graphs and Combinatorics*, 29(1):1–38, 2013.

19 Xueliang Li and Yuefang Sun. Upper bounds for the rainbow connection numbers of line graphs. *Graphs and Combinatorics*, pages 1–13, 2011. 10.1007/s00373-011-1034-1.

20 Ingo Schiermeyer. Rainbow connection in graphs with minimum degree three. In *Combinatorial Algorithms*, volume 5874 of *LNCS*, pages 432–437. Springer, Berlin, 2009.

21 Yilun Shang. A sharp threshold for rainbow connection of random bipartite graphs. *Int. J. Appl. Math.*, 24(1):149–153, 2011.