Strictly Pareto Inefficient Nash Equilibria

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***Abstract*—We find 'strict' Pareto inefficiency of Nash equilibria (NE) in prisoners' dilemma and Braess paradox, wherein the utilities of 'all' players degrade in NE. The strict Pareto inefficiency is a narrower concept than (usually mentioned) Pareto inefficiency. In this paper, we present a measure that shows the magnitude of strict Pareto inefficiency, abbreviated as MoS. MoS distinguishes strict Pareto inefficiency of a state (an allocation/a strategy profile), say, a Nash equilibrium (NE). We present examples wherein the widely-used measure of ineffectiveness based on the social optimality, like price of anarchy (PoA), does not always distinguish strict Pareto inefficiency whereas MoS does.**

Furthermore, we show that, if there exists a Pareto optimum that is proportional to a state, MoS of the state is obtained as the constant of proportionality between the Pareto optimum and the state. Then, if the Pareto optimum is socially optimal, PoA of the state is identical to MoS of the state. We show that the magnitude of strict Pareto inefficiency, MoS, of NE can increase without bounds in a networking game, even though the network has a finite amount of resources and a small number of non-cooperative players.

Keywords: strict Pareto inefficiency, weak Pareto optimality, atomic and non-atomic Nash equilibrium, networking game, price of anarchy, prisoners' dilemma.

I. Introduction

Decision makers in many systems can be regarded as players in games. Non-cooperative decisions which lead to Nash equilibria (NE) have the advantage of independence and distribution of decision making. Non-cooperative decisions, however, may not always be beneficial. That is, NE may be Pareto inefficient (*e.g.,* Dubey [1]). Among the possible inefficiency of NE, we are strongly impressed by such examples as prisoners' dilemma and Braess paradox on transportation and communication networks [2], [3], [4], [5], [6]. In these examples, the utilities of **all** non-cooperative players degrade in the NE. In contrast, consider an example of the NE with two players, where one player's utility degrades whereas the other player's utility is not worse off. The situation does not seem to be so impressive. That is, what impresses us strongly is not mere Pareto inefficiency but **strict** Pareto inefficiency. In this article, we present a measure that shows the magnitude of strict Pareto inefficiency (abbreviated as MoS) of a system state (an allocation/a strategy profile) that distinguishes strict Pareto inefficiency.

To measure the ineffectiveness of an NE, it looks that many researchers (starting with *e.g.,* Roughgarden [7]) have been using the measure of social optimality of the NE, represented by the price of anarchy, PoA. In this article, we mean, by the price of anarchy (PoA) of a state, the ratio of the social (overall) optimum to the social utility of the state. We note that PoA (as used herein) does not seem to reflect the definition of strict Pareto inefficiency directly. In fact, we show in some examples that PoA does not always distinguish strict Pareto inefficiency from weak Pareto efficiency whereas MoS does.

The Braess paradox is strongly counter-intuitive in the sense that the utility of **every** non-cooperative player degrades after network resources are augmented. We note, however, that being strictly Pareto inefficient implies being not socially optimal, but the reverse does not necessarily hold true. We note, however, that there exist counter-intuitive phenomena observed only w.r.t. (with respect to) social optimality but not observed w.r.t strict Pareto inefficiency. We call such phenomena observed only w.r.t. social optimality **anomalous** [8], [9], instead of **paradoxical**.

Furthermore, in this article, we show that, if there exists a Pareto optimum that is proportional to a state, *i.e.,* if the ratio of the utility of a Pareto optimum to that of the state is identical for every player, then MoS of the state is obtained as the constant of proportionality between the Pareto optimum and the state. Moreover, we see that, if the Pareto optimum is socially optimal, PoA of the state is identical to MoS of the state. We then show an atomic networking game wherein the MoS of NE can increase without bounds, even though the network has a finite amount of resources and a small number of non-cooperative players.

In section II, we recall some definitions related to this article. In section III, we present the definition of a measure of the magnitude of strict Pareto inefficiency (MoS) of a system state. We show some properties of MoS in relation to PoA. In section IV, we show examples. We present discrete games like prisoners' dilemma in subsection IV-A and two kinds of networking games in subsection IV-B. In partcular, subsection IV-B2 shows the atomic networking game wherein the MoS of NE can increase without bounds. Section V concludes this article.

II. Definitions

We recall some definitions related to this article.

[**Strict Pareto superiority and inferiority**]: We consider a system that consists of a number of users or players $\boldsymbol{n}$, where $\boldsymbol{n}$ denotes the set $\{1, 2, \cdots, n\}$. Denote by S the system state [an allocation/a strategy profile] $(s_1, s_2, \cdots, s_n)$ where s_i denotes the decision/strategy chosen by player i, $i \in \boldsymbol{n}$. Denote by $\boldsymbol{S}$ the set of feasible system states each of which presents a realizable combination of player decisions. For each state S of the system, each player i has his/her own utility $U_i(S)$. Denote the combination of utilities of all players in a system state $S \in \boldsymbol{S}$ by $\vec{U}(S) = (U_1(S), U_2(S), \ldots, U_n(S))$. In general, we consider the cases where $U_i(S)$ has a positive real value, $U_i(S) > 0$, for all $i \in \boldsymbol{n}$, $S \in \boldsymbol{S}$. (When we need the value of the function that depends on $U_i(S)$ in the case where $U_i(S) = 0$ for some $i \in \boldsymbol{n}$, we may use the converged value of the function with $U_i(S) \to 0$, if applicable.)

Consider an arbitrary pair of two (achievable) states of the system, S^a and S^b, $\in \boldsymbol{S}$. If $U_i(S^a) \leq U_i(S^b)$ for all $i \in \boldsymbol{n}$ and $U_j(S^a) < U_j(S^b)$ for some $j \in \boldsymbol{n}$, then S^a is **Pareto inferior** to S^b and S^b is **Pareto superior** to S^a. Furthermore, if $U_i(S^a) < U_i(S^b)$ for all $i \in \boldsymbol{n}$, then S^a is **strictly Pareto inferior** to S^b and S^b is **strictly Pareto superior** (or strongly Pareto dominant) to S^a.

Consider the case where $U_i(S^a) = K_{S^aS^b} U_i(S^b)$ ($S^a, S^b \in \boldsymbol{S}$), for all $i \in \boldsymbol{n}$ and for some constant of proportionality $K_{S^aS^b} > 0$. We call state S^a **proportional** to state S^b. Consider, furthermore, the case where $U_i(S) = U(S)$ ($S \in \boldsymbol{S}$) for all $i \in \boldsymbol{n}$. We call such a state S **symmetric**.

If S^a is proportional to S^b, the degree of Pareto inferiority of S^a to S^b can simply be defined to be, for example, $K_{S^aS^b}$. In general, however, system states may not always be proportional to each other, and the Pareto superiority/inferiority relations induce partial ordering among system states and are not subject to total ordering or single scalar measure straightforwardly.

[**Strict Pareto inefficiency**] If there exists some system state that is strictly Pareto superior to a system state, we call the latter state a **strictly Pareto inefficient** state. If there exists no system state that is strictly Pareto superior to a system state, the latter state is called a **weakly Pareto optimal** or **efficient** state[1]. Both atomic and non-atomic Nash equilibria can be strictly Pareto inefficient as we see in the Braess paradox and in prisoners' dilemma.

[**Price of Anarchy**] Denote the sum of the utilities of players in state $S \in \boldsymbol{S}$ (the social utility of S) by

$$O(\vec{U}(S)) = \sum_p U_p(S). \quad (1)$$

We may have one or more than one maximum $\bar{S}$ such that $O(\vec{U}(\bar{S})) = \max_{S \in \boldsymbol{S}} O(\vec{U}(S))$. We call $\bar{S}$ a **social optimum**. **It looks that many researchers use as the measure of ineffectiveness of a state (allocation), say S, the ratio of the social optimum to the social utility (the sum of the player-utilities) of the state**, $O(\vec{U}(\bar{S}))/O(\vec{U}(S))$. It also appears that the term 'price of anarchy (PoA)' is often used in this context.

[1]We note that there are coined the terms **strong** and **weak** Pareto optima. Commonly used 'Pareto optima' are the same as strong Pareto optima. The state to which no state is Pareto superior is a strong Pareto optimum. No state can be Pareto superior to a strongly Pareto optimal state. The state to which no state is strictly Pareto superior is a weak Pareto optimum. Being strictly Pareto superior implies being Pareto superior, but not vice versa. Thus, strong Pareto optima are also weak Pareto optima, but not vice versa.

In contrast, we note that there exist terms **strong** and **weak** Pareto inefficiency. It looks to be defined such that **strongly** Pareto inefficient states are the states that are not strongly Pareto optimal, and that **weakly** Pareto inefficient states are not weakly Pareto optimal. Then, since strong Pareto optima are also weak Pareto optima, weakly Pareto inefficient states must also be strongly Pareto inefficient, which sounds unnatural. We therefore use the term **strict Pareto inefficiency**, instead of weak Pareto inefficiency, in this article. We keep to use, however, the term **weak** Pareto optimum/efficiency.

Although the term 'anarchy' may imply the situation of non-cooperation that leads to NE, in this article, as **the price of anarchy, PoA, of a state (allocation)**, S, we refer to the following:

$$PoA(S) \triangleq O(\vec{U}(\bar{S}))/O(\vec{U}(S)) = \sum_p U_p(\bar{S}) / \sum_p U_p(S). \quad (2)$$

III. Magnitude of Strict Pareto Inefficiency

Given the definition of the degree of strict Pareto inferiority $Q(S^a, S)$ of system state S^a to S, naturally, we have the magnitude of strict Pareto inefficiency of a system state S^a ($S^a \in \boldsymbol{S}$), by

$$MoS(S^a) \triangleq \max_{S \in \boldsymbol{S}} Q(S^a, S) \triangleq Q(S^a, \ddot{S}^a) \quad (3)$$

(the larger magnitude for the greater inefficiency). It shows the maximum degree in which the strict Pareto inferiority can be improved by moving from state S^a to some other state $\ddot{S}^a$. As the base of MoS, we use such a measure of strict Pareto inferiority $Q(S^a, S^b)$ of system state S^a to S^b as follows:

$$Q(S^a, S^b) = Q_{min}(S^a, S^b) \triangleq \min_{p \in \boldsymbol{n}} U_p(S^b)/U_p(S^a). \quad (4)$$

If $Q_{min}(S^a, S^b) > 1$ and ≤ 1, respectively, then S^a is strictly Pareto inferior and not so to S^b. We note that $Q_{min}(S^a, S^b) = 1$ for $\vec{U}(S^a) = \vec{U}(S^b)$.

Proposition 1: If $MoS(S) > 1$ then S is strictly Pareto inefficient ($S \in \boldsymbol{S}$). If $MoS(S) = 1$ then S is weakly Pareto optimal.

[Proof] Clearly, from the definitions (3) and (4), $MoS(S) \geq 1$ since $Q(S, S) = 1$. If $MoS(S) > 1$, then $Q(S, S') > 1$ for some S' ($S' \in \boldsymbol{S}$), and thus S is strictly Pareto inefficient. If $MoS(S) = 1$, then $Q(S, S') \leq 1$ for all S' ($S' \in \boldsymbol{S}$), and thus S is not strictly Pareto inefficient. □

We thus have the following definition in this article:

$$MoS(S^a) = \max_{S \in \boldsymbol{S}} \min_{p \in \boldsymbol{n}} U_p(S)/U_p(S^a) \text{ for the utility base and}$$
$$\max_{S \in \boldsymbol{S}} \min_{p \in \boldsymbol{n}} C_p(S^a)/C_p(S) \text{ for the cost base,} \quad (5)$$

where $C_i(S)$ denotes the cost for player i in system state S.

If S is a Nash equilibrium, $MoS(S)$ is closely related to 'Selfishness Degradation Factor (SDF)' [10]. We note that the current measure MoS has clear emphasis on the **strictness** of Pareto inefficiency distinguished from usual Pareto inefficiency.

[**Proportionate Cases**] Consider a system state $\grave{S}$ with players' utility $\vec{U}(\grave{S}) = (U_1(\grave{S}), U_2(\grave{S}), \ldots, U_n(\grave{S}))$.

Condition 1: There exists such a Pareto optimum, $\breve{S}$, that is proportional to $\grave{S}$, $\vec{U}(\grave{S})$.

Theorem 1: Assume that condition 1 is satisfied. That is, $U_i(\breve{S})/U_i(\grave{S}) = K_{\breve{S},\grave{S}}$, $i \in \boldsymbol{n}$, for some constant $K_{\breve{S},\grave{S}}$, with a Pareto optimum, $\vec{U}(\breve{S})$. (I) $MoS(\grave{S})$ is obtained such that $MoS(\grave{S}) = K_{\breve{S},\grave{S}}$. In this case, $MoS(\grave{S}) > 1$ and $= 1$, respectively, if $\grave{S}$ is strictly Pareto inefficient and Pareto optimal. (II) Thus, $MoS(\grave{S})$ distinguishes the strict Pareto inefficiency / Pareto optimality of state $\grave{S}$.

[Proof] Note that we can find such a Pareto optimum $\vec{U}(\breve{S})$ that satisfies $U_i(\breve{S})/U_i(\grave{S}) = K_{\breve{S},\grave{S}}$, $i \in \boldsymbol{n}$. Consider another state $S' \in \boldsymbol{S}$. Since $\breve{S}$ is a Pareto optimum, then there must exist some i ($i \in \boldsymbol{n}$) such that $U_i(S') \leq U_i(\breve{S})$ and, thus, such that $U_i(S')/U_i(\grave{S}) \leq U_i(\breve{S})/U_i(\grave{S}) = K_{\breve{S},\grave{S}}$. Then, $Q_{min}(\grave{S}, S') = \min_p U_p(S')/U_p(\grave{S}) \leq K_{\breve{S},\grave{S}}$. Then

$MoS(\grave{S}) = \max[\{\max_{S \in \boldsymbol{S}, S \neq \breve{S}} Q_{min}(\grave{S}, S)\}, Q_{min}(\grave{S}, \breve{S})] = K_{\breve{S},\grave{S}}$ (by noting that $Q_{min}(\grave{S}, \breve{S}) = K_{\breve{S},\grave{S}}$). Thus, $MoS(\grave{S})$ is given by $K_{\breve{S},\grave{S}}$.

Naturally, $MoS(\grave{S}) = K_{\breve{S},\grave{S}} > 1$ means that $\grave{S}$ is strictly Pareto inefficient, and that $MoS(\grave{S}) = K_{\breve{S},\grave{S}} = 1$ means that $\grave{S}$ is Pareto optimal. □

Note that, if condition 1 holds true for state $\grave{S}$, $\grave{S}$ is Pareto inefficient iff $\grave{S}$ is strictly Pareto inefficient and $\grave{S}$ is weakly Pareto optimal iff $\grave{S}$ is (strongly) Pareto optimal.

If $\grave{S}$ is an NE, we call $\breve{S}$ Nash-proportionate fair [11].

[**MoS and PoA**]

Condition 2: We can find such a social optimum $\bar{S}$ that is proportional to a system state $\grave{S}$ that satisfies $U_i(\bar{S})/U_i(\grave{S}) = K_{\bar{S},\grave{S}}$, $i \in \boldsymbol{n}$, for certain $K_{\bar{S},\grave{S}}$.

Since a social optimum is Pareto optimal, we have the following:

Corollary 1: PoA of a state (denoted by $\grave{S}$), say an NE, is identical to MoS of the state under the condition 2, although other feasible states may not be proportional to the NE. In that case, $PoA(\grave{S}) = MoS(\grave{S}) > 1$ and $= 1$, respectively, if $\grave{S}$ is strictly Pareto inefficient and Pareto optimal. Thus, in this proportional case, PoA of the state, distinguishes the strict Pareto inefficiency of the state.

Note also that, among the states $\grave{S}$ for which condition 2 holds true, there exists no state that is Pareto inefficient but not strictly Pareto inefficient and no state that is weakly Pareto optimal but not (strongly) Pareto optimal.

We then have:

Proposition 2: For an arbitrary state S, $PoA(S) \geq MoS(S)$ where the equality holds true if condition 2 is satisfied.

[Proof] We note that $\sum_{p \in \boldsymbol{n}} U_p(S)/\sum_{p \in \boldsymbol{n}} U_p(S^a) \geq \min_{p \in \boldsymbol{n}} U_p(S)/U_p(S^a)$. Then, from (2) and (5), we have $PoA(S) \geq MoS(S)$. It is clear from the corollary 1 that the equalities hold true if condition 2 is satisfied. □

We have the following:

$PoA(S) = 1$ iff S is socially optimal. $PoA(S) > 1$ iff S is not socially optimal.

$MoS(S) = 1$ iff S is weakly Pareto optimal (not strictly Pareto inefficient). $MoS(S) > 1$ iff S is strictly Pareto inefficient.

IV. Examples

We present case studies on one kind of discrete games and on two kinds of networking games.

A. Discrete Games: Distinction of strict Pareto inefficiency from weak Pareto efficiency by MoS and PoA

We present below examples of games with two players and two strategies for each player. These examples show some relation between the price of anarchy (PoA) and the magnitude of strict Pareto inefficiency (MoS). Denote by S^{kl} ($\in \boldsymbol{S}$) the state wherein players 1 and 2, respectively, choose strategies k and l, where $k, l \in \{1, 2\}$. In the following frameworks of the two-player games, "(U_1, U_2) for S^{kl}" means that the utilities of players 1 and 2 are, respectively, U_1 and U_2 in state S^{kl}, $k, l \in \{1, 2\}$. In all of the following examples, S^{11} is an NE, $\tilde{S}$.

Note that $MoS(S^{11}) =$

$$\max\{\min\left\{\frac{U_1(S^{11})}{U_1(S^{11})}, \frac{U_2(S^{11})}{U_2(S^{11})}\right\}, \min\left\{\frac{U_1(S^{12})}{U_1(S^{11})}, \frac{U_2(S^{12})}{U_2(S^{11})}\right\}, \min\left\{\frac{U_1(S^{21})}{U_1(S^{11})}, \frac{U_2(S^{21})}{U_2(S^{11})}\right\}, \min\left\{\frac{U_1(S^{22})}{U_1(S^{11})}, \frac{U_2(S^{22})}{U_2(S^{11})}\right\}\}.$$

$$MoA(S^{11}) = \frac{\max\left\{\begin{matrix} U_1(S^{11}) + U_2(S^{11}), U_1(S^{12}) + U_2(S^{12}), \\ U_1(S^{21}) + U_2(S^{21}), U_1(S^{22}) + U_2(S^{22}) \end{matrix}\right\}}{U_1(S^{11}) + U_2(S^{11})}.$$

1) The case of a Pareto optimal Nash equilibrium:

PoA>MoS=1		
player 1 ＼ player 2	strategy 1	strategy 2
strategy 1	(4,5) for S^{11}	(5,1) for S^{12}
strategy 2	(1,8) for S^{21}	(3,7) for S^{22}

S^{11} is the Nash equilibrium $\tilde{S}$. S^{22} is the social optimum $\bar{S}$. Clearly, all states (including the Nash equilibrium $\tilde{S} = S^{11}$) are Pareto efficient. As we anticipate,

$$MoS(S^{11}) = \max\{\min\{4/4, 5/5\}, \min\{5/4, 1/5\},$$

$$\min\{1/4, 8/5\}, \min\{3/4, 7/5\}\} = \max\{1, 1/5, 1/4, 3/4\} = 1.$$

In contrast,

$$PoA(S^{11}) = (3 + 7)/(4 + 5) = 10/9 > 1,$$

although S^{11} is Pareto optimal. For this Pareto optimal Nash equilibrium, PoA shows the value greater than 1 whereas MoS shows the value 1. **We thus see that PoA does not always distinguish strict Pareto inefficiency from efficiency whereas MoS does**.

2) The case of a Pareto optimal Nash equilibrium:

PoA≫MoS=1		
player 1＼player 2	strategy 1	strategy 2
strategy 1	(4,5)for S^{11}(NE)	(5,1) for S^{12}
strategy 2	(1,898)for S^{21}	(3,897)forS^{22}(SO)

S^{11} is the Nash equilibrium $\tilde{S}$. S^{22} is the social optimum $\bar{S}$. Clearly, all states are Pareto optimal. As we anticipate,

$$MoS(S^{11}) = \max\{\min\{4/4, 5/5\}, \min\{5/4, 1/5\},$$

$$\min\{1/4, 898/5\}, \min\{3/4, 897/5\}\} = \max\{1, 1/5, 1/4, 3/4\} = 1.$$

In contrast,

$$PoA(S^{11}) = (3 + 897)/(4 + 5) = 100 > 1,$$

although S^{11} is Pareto optimal. For this Pareto optimal Nash equilibrium, PoA shows a big value, 100, greater than 1, whereas MoS shows the value 1. **We thus see that PoA does not always distinguish strict Pareto inefficiency from efficiency whereas MoS does**. Note, furthermore, that **PoA of Pareto-optimal NE can have great values much bigger than 1**. We also see that PoA does not always distinguish Pareto inefficiency.

3) The case of a strictly Pareto inefficient Nash equilibrium (like a prisoners' dilemma):

PoA>MoS>1		
player 1\player 2	strategy 1	strategy 2
strategy 1	(2,3)for S^{11} (NE)	(9,2)forS^{12}(SO)
strategy 2	(1,8) for S^{21}	(4,6) for S^{22}

S^{11} is the Nash equilibrium $\tilde{S}$. S^{12} is a social optimum $\bar{S}$, which is Pareto indifferent to NE $\tilde{S}$. Thus, **we cannot apply corollary 1**. In contrast, S^{22} is a Pareto optimum $\breve{S}$, which is proportional to NE $\tilde{S}$. NE $\tilde{S} = S^{11}$ is strictly Pareto inefficient. **Since Pareto optimum $\breve{S}$ is proportional to $\tilde{S}$, we can apply theorem 1**, which leads to $MoS(S^{11}) = 4/2 = 6/3 = 2$. In fact, by definition,

$$MoS(S^{11}) = \max\{\min\{2/2, 3/3\}, \min\{9/2, 2/3\},$$

$$\min\{1/2, 8/3\}, \min\{4/2, 6/3\}\} = \max\{1, 2/3, 1/2, 2\} = 2 > 1.$$

$$PoA(S^{11}) = (9 + 2)/(2 + 3) = 11/5 = 2.2 > 1.$$

MoS shows the value different from the value of PoA, both greater than 1 for this case of strictly Pareto inefficient Nash equilibrium.

4) The case of a strictly Pareto inefficient Nash equilibrium (like a prisoners' dilemma):

PoA=MoS>1		
player 1\player 2	strategy 1	strategy 2
strategy 1	(2,3)for S^{11}(NE)	(5,2) for S^{12}
strategy 2	(1,8) for S^{21}	(4,6)forS^{22}(SO)

S^{11} is the Nash equilibrium $\tilde{S}$. S^{22} is the social optimum (SO) $\bar{S}$. Clearly, the Nash equilibrium $\tilde{S} = S^{11}$ is strictly Pareto inefficient. **Since $\bar{S}$ is proportional to $\tilde{S}$, we can apply corollary 1**. Thus, MoS is equal to PoA, and

$$MoS(S^{11}) = PoA(S^{11}) = (4 + 6)/(2 + 3) = 2 > 1.$$

In fact, from the definition of MoS,

$$MoS(S^{11}) = \max\{\min\{2/2, 3/3\}, \min\{5/2, 2/3\}, \min\{1/2, 8/3\},$$

$$\min\{4/2, 6/3\}\} = \max\{1, 2/3, 1/2, 2\} = 2 > 1.$$

Two measures show the same value greater than 1 for this case of strictly Pareto inefficient Nash equilibrium.

5) The case of a strictly Pareto inefficient Nash equilibrium (like a prisoners' dilemma):

PoA>MoS >1		
player 1\player 2	strategy 1	strategy 2
strategy 1	(2,3)for S^{11}(NE)	(5,2) for S^{12}
strategy 2	(1,8) for S^{21}	(3,7)for S^{22}(SO)

S^{11} is the Nash equilibrium $\tilde{S}$. S^{22} is the social optimum $\bar{S}$. Clearly, the Nash equilibrium $\tilde{S} = S^{11}$ is strictly Pareto inefficient. **$\bar{S}$ is Pareto superior but not proportional to $\tilde{S}$. We thus cannot apply corollary 1**.

$$MoS(S^{11}) = \max\{\min\{2/2, 3/3\}, \min\{5/2, 2/3\},$$

$$\min\{1/2, 8/3\}, \min\{3/2, 7/3\}\} = \max\{1, 2/3, 1/2, 3/2\} = 3/2.$$

$$PoA(S^{11}) = (3 + 7)/(2 + 3) = 2 > 3/2 = MoS(S^{11}).$$

In this asymmetric game, MoS and PoA show mutually different values both greater than 1 for this case of strictly Pareto inefficient Nash equilibrium.

6) The case of a weakly Pareto efficient (not strictly Pareto inefficient) Nash equilibrium:

PoA>MoS=1		
player 1\player 2	strategy 1	strategy 2
strategy 1	(3,5)for S^{11}(NE)	(5,1) for S^{12}
strategy 2	(1,7) for S^{21}	(3,6)for S^{22}(SO)

S^{11} is the Nash equilibrium $\tilde{S}$. S^{22} is the social optimum $\bar{S}$ that is **Pareto superior but not strictly Pareto superior** to NE $\tilde{S}$. Then, we cannot apply corollary 1. Clearly, NE $\tilde{S} = S^{11}$ is **Pareto inefficient but not strictly Pareto inefficient** (is weakly Pareto optimal), but all other states are Pareto optimal. Note that MoS of NE $\tilde{S} = S^{11}$ is

$$MoS(S^{11}) = \max\{\min\{3/3, 5/5\}, \min\{5/3, 1/5\},$$

$$\min\{1/3, 7/5\}, \min\{3/3, 6/5\}\} = \max\{1, 1/5, 1/3, 1\} = 1.$$

$$PoA(S^{11}) = (3 + 6)/(3 + 5) = 9/8 > 1.$$

PoA is greater than 1 for the weakly Pareto optimal NE. **We thus see that PoA does not always distinguish strict Pareto inefficiency from weak Pareto efficiency whereas MoS does**.

B. Networking Games

We present examples with two aspects of network management: flow control (Subsection IV-B1) and routing/load-balancing (Subsection IV-B2). In subsection IV-B1, we show an application of theorem 1 to network flow control. In subsection IV-B2, we show an example of network routing/load-balancing to which we can apply corollary 1 and wherein the magnitude of strict Pareto inefficiency, MoS, of the NE can increase without bounds, in the network that has only a finite amount of resources and a small number of non-cooperative players.

1) Flow Control in Networks:

a) Assumptions on Networks: Consider a communication network modeled by an open product-form network of m state-independent queues, $k = 1, 2, \ldots, m$ that model communication links, or, simply, links [12]. Denote the set of the links $\{1, 2, \ldots, m\}$ by $\mathcal{M}$. The vertices or nodes connected by links model the routers of the communication network. There are n independent users/players, $1, 2, \ldots, n$. User-i decides the feasible rate λ_i of packets that she/he will send through the communication network. Denote the set of the users $\{1, 2, \ldots, n\}$ by $\boldsymbol{n}$. Denote $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $\mathcal{L}$ be the product of the strategy spaces, that is, $\mathcal{L} = \{\boldsymbol{\lambda} \mid \lambda_i \geq 0, i \in \boldsymbol{n}\}$. Denote by C ($\subset \mathcal{L}$) the set of feasible values of $\boldsymbol{\lambda}$.

T_i is the average end-to-end delay of the packets in control of user i. μ_{ik} is the state-independent service rate of user-i packets at link k. In this article, it is assumed that each router (node) has a sufficient capacity of storing packets, and, thus, losses of packets may not occur. q_{ik} is the resulting visit rate of user-i packets to link k. That is, q_{ik}, for all i, k, is the solution of the following system of equations:

$$q_{ik} = p^i_{0k} + \sum_{l \in \mathcal{M}} q_{il} p^i_{lk} \text{ for all } i \in \boldsymbol{n},\ k \in \mathcal{M}, \tag{6}$$

where p^i_{lk} and p^i_{0k}, respectively, are the probabilities that a user-i packet goes to link k when it leaves link l and when it enters the network; they are fixed and not subject to optimal control. Define $p^i_{k0} = 1 - \sum_l p^i_{kl},\ i \in \boldsymbol{n}, k \in \mathcal{M}$. In this subsection, we are concerned only with optimal flow control and not with optimal routing in this subsection. Then, if user i injects the rate λ_i of packets into the network, user-i packets visit link k at the rate of $q_{ik}\lambda_i$, where q_{ik} is given by eq. (6). User i injects the rate, $p^i_{0k}\lambda_i$, of packets into link k from the outside of the network. User-i packets departing link k leave the network at the frequency (or, probability) q^i_{k0}. That is, the network has multiple ports of entry and of exit. Consider the case where the mean response time, $T^{(k)}_i$, for a user-i packet to pass through link k, is

$$T^{(k)}_i = \mu^{-1}_{ik} T^{(k)} \quad \text{and} \quad T^{(k)} = \frac{1}{1 - s_k \sum_{p \in \boldsymbol{n}} q_{pk}\lambda_p/\mu_{pk}}, \tag{7}$$
$$\text{if } 1 - s_k \sum_{p \in \boldsymbol{n}} q_{pk}\lambda_p/\mu_{pk} > 0, \quad \text{otherwise infinite,}$$

where s_k is 1 for a link modeled by a single-server, $1/h$ for a link consisting of h parallel channels each of which is chosen with probability $1/h$ and is modeled by a single server, and 0 for a link modeled by an infinite server, for $1 - s_k \sum_{p \in \boldsymbol{n}} q_{pk}\lambda_p/\mu_{pk} > 0$ [12]. Denote $\mathcal{K} = \{l | s_l \neq 0\}$. Then, using the Little's result on the average number of user-i packets that stay in the network ($= \lambda_i T_i(\boldsymbol{\lambda})$),

$$T_i(\boldsymbol{\lambda}) = \sum_{l \in \mathcal{K}} \frac{Q_{il}}{1 - s_l \sum_{p \in \boldsymbol{n}} Q_{pl}\lambda_p} + \sum_{l \in \mathcal{M} - \mathcal{K}} Q_{il}, \tag{8}$$
$$\text{if } 1 - s_k \sum_{p \in \boldsymbol{n}} Q_{pl}\lambda_p > 0 \text{ for all } l, \quad \text{otherwise infinite,}$$
$$\text{where} \quad Q_{il} = \frac{q_{il}}{\mu_{il}}.$$

Clearly, $T_i(\boldsymbol{\lambda})$ is increasing in $\boldsymbol{\lambda}$ as long as $1 - s_l \sum_{p \in \boldsymbol{n}} Q_{pl}\lambda_p > 0, l \in \mathcal{K}$. We note that $\sum_{l \in \mathcal{M} - \mathcal{K}} Q_{il}$ is constant and independent of the strategy. In order that the statistical equilibrium of this network be attained, it must hold true that $\boldsymbol{\lambda} \in C$, where the feasible region C is

$$C = (\boldsymbol{\lambda} \mid \lambda_i \geq 0, i \in \boldsymbol{n}, \text{ and } 1 - s_l \sum_{p \in \boldsymbol{n}} Q_{pl}\lambda_p > 0, l \in \mathcal{K}). \tag{9}$$

b) Flow Control with the Power Criterion: Denote the set of the players $\{1, 2, \ldots, n\}$ by $\boldsymbol{n}$. Thus, the strategy profile is presented by a vector, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Consider a non-cooperative game that has n players each of whom decides the value of $\lambda_i \geq 0$. Each of network users (user-i) has two important major concerns in choosing the protocol to use: one is the amount of packets user-i can send per unit time (throughput), denoted by λ_i, and the other is the expected time of each packet taken from its origin to its destination (mean response time), denoted by T_i. In the following, we denote by $\tilde{\boldsymbol{\lambda}}$ ($\in C$) a strategy profile that presents a Nash equilibrium (with finite utilities).

The power is defined as $P_i = \lambda_i/T_i$ for user-i. In this subsection, we consider the case where the utility, U_i, of user i is its power, P_i, i.e., $U_i = P_i$ for all $i \in \boldsymbol{n}$. Denote the vector $(P_1, P_2, \ldots, P_n)$ by $\boldsymbol{P}$. From (8), $P_i(\boldsymbol{\lambda})$ is defined for all $\boldsymbol{\lambda} \in C$. The existence of a Nash equilibrium flow control, which is Pareto inefficient, has been shown [13]. Furthermore, for this network, a stronger result, *i.e.,* the existence of a Pareto-optimal flow control that is proportional to a feasible state, say NE, will be shown by Theorem 2.

Denote by $\mathcal{G}$ the graph $(\mathcal{V}, \mathcal{E})$ such that $\mathcal{V} = \boldsymbol{n} \cup \mathcal{M}$ and $\mathcal{E} = \{(i, k) \mid i \in \boldsymbol{n}, k \in \mathcal{M} \text{ and } q_{ik} > 0\}$.

Assumption 1: $\mathcal{G}$ is connected.

Note that the graph $\mathcal{G}$ (as will be described in Example 1 below) is undirected. and different from the graph that shows the network link connection.

Theorem 2: If Assumption 1 holds true, for any feasible flow-control allocation $\boldsymbol{\lambda}$, say an NE, of this network, there exists a Pareto-optimal flow control allocation that is proportional to it. Thus, condition 1 is satisfied in the flow control networks for any feasible $\boldsymbol{\lambda}$.

[Proof] We can prove this by following the Appendix A of Kameda *et al.* [11] with replacing specific term 'Nash equilibrium' therein by less specific term 'flow-control allocation.' In fact, the proof given in that appendix of [11]. □

Thus, **condition 1 (*i.e.,* the assumption of theorem 1) is satisfied for any feasible $\boldsymbol{\lambda}$ in the flow control networks. We therefore obtain MoS of $\grave{\boldsymbol{\lambda}}$, on the basis of theorem 1**, as follows:

$MoS(\grave{\boldsymbol{\lambda}}) = \max_{\boldsymbol{\lambda} \in C} K(\boldsymbol{\lambda}, \grave{\boldsymbol{\lambda}})$, where $P_i(\boldsymbol{\lambda})/P_i(\grave{\boldsymbol{\lambda}}) = K(\boldsymbol{\lambda}, \grave{\boldsymbol{\lambda}}), i \in \boldsymbol{n}$ ($K(\boldsymbol{\lambda}, \grave{\boldsymbol{\lambda}})$ is a constant of proportionality).

In comparison, note that, from the definition, we obtain MoS of $\grave{\boldsymbol{\lambda}}$ as follows:

$MoS(\grave{\boldsymbol{\lambda}}) = \max_{\boldsymbol{\lambda} \in C}\{\min_j[P_j(\boldsymbol{\lambda})/P_j(\grave{\boldsymbol{\lambda}})]\}$.

Example 1 Consider a simple network consisting of three users $\boldsymbol{n} = \{1, 2, 3\}$ and two links $\mathcal{K} = \{1, 2\}$,

where $p^1_{01} = p^2_{02} = 1$, $p^3_{01} = p^3_{02} = 0.5$, $p^1_{10} = p^2_{20} = p^3_{10} = p^3_{20} = 1$, $\mu^i_1 = \mu_1 = 3, \mu^i_2 = \mu_2 = 6, (i = 1, 2, 3)$ for case A, (as to the structure of the network for case A parameters see Fig. 1) and where $p^1_{01} = p^2_{02} = 1$, $p^3_{01} = 0.4, p^3_{02} = 0.6$, $p^1_{10} = p^2_{20} = p^3_{10} = p^3_{20} = 1$, $\mu^i_1 = \mu_1 = 10$ $\mu^i_2 = \mu_2 = 12, (i = 1, 2, 3)$ for case B.

Then, as to its graph $\mathcal{G}$ we have $q_{11} = q_{22} = 1$, and $q_{31} = q_{32} = 0.5$ for case A (Fig. 2) and $q_{31} = 0.4, q_{32} = 0.6$ for case B. Each case of the network satisfies Assumption 1.

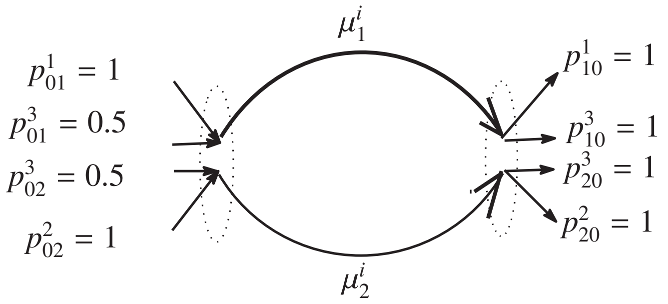

Fig. 1. A simple network for $i = 1, 2, 3$ (with Case A parameters).

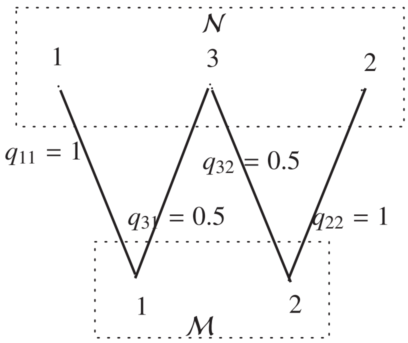

Fig. 2. Graph $\mathcal{G}$ for the simple network for $i = 1, 2, 3$ (with Case A parameters).

We can obtain numerically the following:
Case A: The Nash equilibrium flow control $\tilde{\lambda}_A$: The powers of users, 1, 2, and 3, are
($\tilde{\lambda}_A$: $P_1 = 0.8755\ldots, P_2 = 5.93263\ldots,$ $\boldsymbol{P_3}$= **3.05175**...).
Based on the definition, $MoS(\lambda_A) \triangleq Q(\lambda_A, \ddot{\lambda}_A) = 1.1034\ldots$.
($\ddot{\lambda}_A$: $P_1 = 0.96607\ldots, P_2 = 6.54611\ldots, P_3 = 3.36733\ldots$).
Based on theorems 2 and 1), $MoS(\lambda_A) = Q(\lambda_A, \ddot{\lambda}_A) = 1.1034\ldots$.
($\ddot{\lambda}_A$: $P_1 = 0.96607\ldots, P_2 = 6.54611\ldots, P_3 = 3.36733\ldots$).
In contrast, $PoA(\lambda_A) = 1.14098\cdots > 1$, although, in the social optimum $\bar{\lambda}_A$,
($\bar{\lambda}_A$: $P_1 = 2.25, P_2 = 9$, and $\boldsymbol{P_3}$= **0**), which implies that
this social optimum is not Pareto superior to the NE λ_A.

Case B: The Nash equilibrium flow control $\tilde{\lambda}_B$: The powers of users, 1, 2, and 3, are
($\tilde{\lambda}_B$: $P_1 = 12.625\ldots, P_2 = 14.6669\ldots,$ $\boldsymbol{P_3}$= **26.8685**...).
Based on the definition, $MoS(\lambda_B) \triangleq Q(\lambda_B, \ddot{\lambda}_B) = 1.12173\ldots$.
($\ddot{\lambda}_B$: $P_1 = 14.1618\ldots, P_2 = 16.452\ldots$, and $P_3 = 30.139\ldots$).
Based on theorems 2 and 1, $MoS(\lambda_B) = Q(\lambda_B, \ddot{\lambda}_B) = 1.12173\ldots$.
($\ddot{\lambda}_B$: $P_1 = 14.1618\ldots, P_2 = 16.452\ldots$, and $P_3 = 30.139\ldots$).
In contrast, $PoA(\lambda_A) = 1.12628\cdots > 1$ although, in the social optimum, $\bar{\lambda}_B$,
($\bar{\lambda}_B$: $P_1 = 25, P_2 = 36$, and $\boldsymbol{P_3}$= **0**), which implies that **this social optimum is not Pareto superior to the NE λ_B.**

2) Routing/Load Balancing in a Two-Node Network Model: We present an example of network routing/load-balancing. Since the optima are symmetric as we show below, we can apply corollary 1, and we have the results on the magnitude of strict Pareto inefficiency, MoS, of the NE in the model. Therein the magnitude of strict Pareto inefficiency, MoS, of the NE can increase without bounds, in the network that has only a finite amount of resources and a small number of non-cooperative players.

a) The Model and Assumptions: We consider a model that consists of two identical servers (nodes) and a communication means that connects both servers. Servers are numbered 1 and 2. Jobs (or customers) are classified into $2n$ classes $R_{ik}, i = 1, 2, k = 1, 2, \cdots, n$. Jobs of class R_{ik} arrive only at server i with identical rate $1/n$. Out of each class arrival, the rate x_{ik}, $i = 1, 2$, $k = 1, 2, \cdots, n$, of jobs are forwarded upon arrival through the communication means to the other server j $(i \neq j)$ to be processed there. Therefore, the remaining rate $1/n - x_{ik}$ of class R_{ik} jobs are processed at server i. We have $0 \leq x_{ik} \leq 1/n$. We denote the vector $(x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n})$ by $\boldsymbol{x}$. We denote the set of $\boldsymbol{x}$'s that satisfy the above constraint by $\boldsymbol{C}$. Within the constraint, a set of values of x_{ik} $(i = 1, 2, k = 1, 2, \cdots, n)$ are chosen to achieve optimization. Thus, the load β_i on server i is given by $\beta_i = 1 - \sum_l x_{il} + \sum_l x_{jl}, (i \neq j)$. Then, the expected processing (including queueing) time $D_i(\beta_i)$ of a job that is processed at server i (*i.e.,* the cost function at server i) is

$$D_i(\beta_i) = 1/(\mu - \beta_i) \text{ for } \beta_i < \mu \text{ (otherwise it is infinite).}$$

As to the communication means, we consider two communication lines 1 and 2 separately for each server. One line i is used for forwarding of a job that arrives at server i, $i = 1, 2$. The communication time of a job arriving at server i and being processed at server j $(\neq i)$ is simply t, *i.e.,* independent of the traffic and the job class and with no queueing delay.

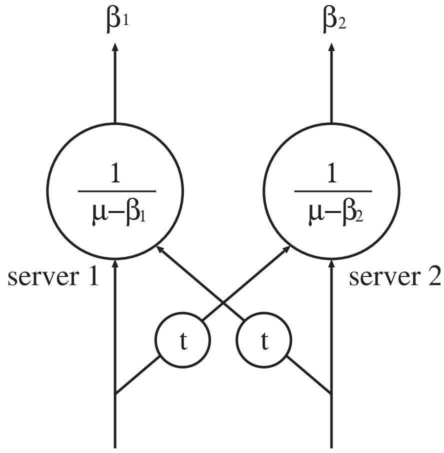

Fig. 3. The system model.

The expected sojourn time of a class R_{ik} job that arrives at server i is

$$T_{ik}(\boldsymbol{x}) = n[(\frac{1}{n} - x_{ik})T_{iik}(\boldsymbol{x}) + x_{ik}T_{ijk}(\boldsymbol{x})], \quad (10)$$

where $T_{iik}(\boldsymbol{x}) = D_i(\beta_i)$ and $T_{ijk}(\boldsymbol{x}) = D_j(\beta_j)+t$, for $j \neq i$. (The above expression holds true, again, only for positive values of denominators, and are otherwise infinite.) Then, the overall expected sojourn time of a job that arrives at the system is

$$T(\boldsymbol{x}) = \frac{1}{2n}\sum_{i,k} T_{ik}(\boldsymbol{x}). \quad (11)$$

b) The Optima: We have three optima as follows: , the social optimum, the non-atomic Nash equilibrium, and the atomic Nash equilibrium, as in the following.

(1) [Social optimum — Completely centralized optimization]: There exists only one decision maker over all of $2n$ classes. The social optimum is given by such $\bar{\boldsymbol{x}}$ that satisfies the following: $T(\bar{\boldsymbol{x}}) = \min_{\boldsymbol{x}\in \boldsymbol{C}} T(\boldsymbol{x})$.
The solution $\bar{\boldsymbol{x}}$ is unique and simply given as follows: $\bar{\boldsymbol{x}} = \boldsymbol{0}$, *i.e.*, $x_{1k} = x_{2k} = 0$ for all k
and

$$T(\bar{\boldsymbol{x}}) = T_{ik}(\bar{\boldsymbol{x}}) = 1/(\mu - 1),\ i = 1, 2,\ k = 1, 2, \cdots, n. \quad (12)$$

(2) [Non-atomic Nash equilibrium — Completely distributed optimization]: Each class has infinitely many infinitesimal decision makers (players). Thus, infinitely many non-cooperative decision makers exist in total. The non-atomic Nash equilibrium (or Wardrop equilibrium) is given by such $\hat{\boldsymbol{x}}$ that satisfies the following for all i, k,

$$T_{ik}(\hat{\boldsymbol{x}}) = \min\{T_{iik}(\hat{\boldsymbol{x}}), T_{ijk}(\hat{\boldsymbol{x}})\}\ (i \neq j) \quad \text{s.t.} \quad \hat{\boldsymbol{x}} \in \boldsymbol{C}. \quad (13)$$

The solution $\hat{\boldsymbol{x}}$ is unique and given as follows: $\hat{\boldsymbol{x}} = \boldsymbol{0}$, *i.e.*, $\hat{x}_{1k} = \hat{x}_{2k} = 0$, for all k. Aagain,

$$T(\hat{\boldsymbol{x}}) = T_{ik}(\hat{\boldsymbol{x}}) = 1/(\mu - 1)\ i = 1, 2,\ k = 1, 2, \cdots, n.$$

(3) [Atomic Nash equilibrium — Intermediately distributed optimization]: Each class has one decision maker (an atomic player). Thus, $2n$ non-cooperative decision makers exist in total. The atomic Nash equilibrium is given by such $\tilde{\boldsymbol{x}}$ that, for all i, k,

$$T_{ik}(\tilde{\boldsymbol{x}}) = \min_{x_{ik}} T_{ik}(\tilde{\boldsymbol{x}}_{-(ik)}; x_{ik}), \quad \text{s.t. that } (\tilde{\boldsymbol{x}}_{-(ik)}; x_{ik}) \in \boldsymbol{C}.$$

where $(\tilde{\boldsymbol{x}}_{-(ik)}; x_{ik})$ denotes the $2n$ vector wherein the element corresponding to $\tilde{x}_{ik}$ has been replaced by x_{ik}.

(A) The case where $t > 1/[n(\mu - 1)^2]$: The solution $\tilde{\boldsymbol{x}}$ is unique is given as follows: $\tilde{\boldsymbol{x}} = \boldsymbol{0}$. Again,

$$T(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = 1/(\mu - 1),\ i = 1, 2,\ k = 1, 2, \cdots, n.$$

(B) The case where $t \leq 1/[n(\mu - 1)^2]$: The solution $\tilde{\boldsymbol{x}}$ is unique is given as follows:

$$\tilde{x}_{1k} = \tilde{x}_{2k} = \frac{1}{2}[\frac{1}{n} - t(\mu - 1)^2], \quad \text{for all } k. \quad (14)$$

In that case, we have

$$T(\tilde{\boldsymbol{x}}) = T_{1k}(\tilde{\boldsymbol{x}}) = T_{2k}(\tilde{\boldsymbol{x}}) = \frac{1}{\mu - 1} + \frac{t}{2}[1 - nt(\mu - 1)^2], \quad \text{for all } k. \quad (15)$$

The solutions for the models that are more general, in the number of nodes and in the forms of cost functions, than the above have been obtained [6].

Consider the case (B) in the atomic Nash equilibrium. In this case (B), each player mutually forwards a part of his/her jobs through the communication means to the other server for remote processing, and thereby his/her mean sojourn time degrades. In this case, the atomic Nash equilibrium is strictly Pareto inefficient. In contrast, the solutions of the social optimum, the non-atomic Nash equilibrium, and the atomic Nash equilibrium in case (A) are identical and Pareto optimal (also weakly Pareto optimal). As n increases in the atomic Nash equilibrium with case (B) (see eq. (15)), $T(\tilde{\boldsymbol{x}})$ decreases as far as $t \leq 1/[n(\mu - 1)^2]$ holds true. Then, as n increases further, $t > 1/[n(\mu - 1)^2]$ (case (A)) holds true and $T(\tilde{\boldsymbol{x}})$ becomes the same as those of the social optimum and the non-atomic Nash equilibrium.

Since the solutions of the social optimum and of all the Nash equilibria are symmetric (thus mutually proportionate), we can use the corollary 1. Then, from (12) and (15), in the case (B)

$$MoS = \frac{T(\tilde{\boldsymbol{x}})}{T(\bar{\boldsymbol{x}})}$$
$$= 1 + \frac{t}{2}[1 - nt(\mu - 1)^2](\mu - 1) \text{ for } t \leq 1/[n(\mu - 1)^2] \text{ and}$$
$$= 1 \text{ for } t > 1/[n(\mu - 1)^2].$$

Thus, **the magnitude of strict Pareto inefficiency MoS of the atomic Nash equilibrium decreases as the number of players** $2n$ **increases** and finally it reaches that of the non-atomic Nash equilibrium that is Pareto optimal, 1. On the other hand, we cannot let the atomic Nash equilibrium be down to the social optimum as we cannot reduce the number of atomic players $2n$ down to 1.

[**The worst-case performance**]: Furthermore, in the case (B), we can easily see that $T_{ik}(\tilde{\boldsymbol{x}})$ (= $T(\tilde{\boldsymbol{x}})$), for every i, k, has its maximum $\tilde{T}(\mu, n)$ w.r.t. t (*i.e.,* the worst-case performance) for given μ, n.

$$\tilde{T}(\mu, n) = \frac{1}{\mu - 1}[1 + \frac{1}{8n(\mu - 1)}], \text{ when } t = \frac{1}{2n(\mu - 1)^2}. \quad (16)$$

The magnitude of strict Pareto inefficiency MoS of the atomic Nash equilibrium for given μ, n is to be

$$MoS = \Delta(\mu, n) = \frac{\tilde{T}(\mu, n)}{T_0(\mu)}, \quad (17)$$

where $T_0(\mu) = 1/(\mu - 1)$ is the mean sojourn time of the social optimum for given μ. Then, we have

$$MoS = \Delta(\mu, n) = 1 + \frac{1}{8n(\mu - 1)}. \quad (18)$$

As $\mu \to 1$ with n fixed, $\Delta(\mu, n)$ increases without bounds.

Theorem 3: There exist networking games wherein, with a finite amount of network resources and with a fixed number of non-cooperative players, the magnitudes of strict Pareto inefficiency, MoS, of atomic Nash equilibria can increase without bounds.

This is in contrast to the studies that seek the bounds of the degrees of ineffectiveness of non-atomic Nash equilibria for congestion games [14].

V. Concluding Remarks

We have tried to make clear a definition of the magnitude of strict Pareto inefficiency, MoS, of a system state, *e.g.,* Nash equilibrium (NE). We have presented some examples wherein the measure of social optimality like price of anarchy, PoA, of a state, does not always distinguish 'strict' Pareto inefficiency from others whereas MoS of the state does. Furthermore, we have shown that, if there exists a Pareto optimum that is proportional to a state, MoS of the state is obtained as the constant of proportionality between the Pareto optimum and the state. We have seen, moreover, that, if the Pareto optimum is socially optimal, PoA of the state is identical to MoS of the state. We have examined networking games. We have shown that the magnitude of strict Pareto inefficiency, MoS, of NE can increase without bounds, even though the system has a finite amount of resources and a small number of non-cooperative players.

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