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A Simplification of Stenger's Topological Degree Formula

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Summary. A formula due to F. Stenger expresses the topological degree of a continuous mapping defined on a polyhedron in R^n as a constant times a sum of determinants of $n \times n$ matrices. We replace these determinant evaluations by a scanning procedure which examines their associated matrices and at each of $n-1$ steps discards at least half of the matrices remaining from the previous step. Finally we obtain a lower bound for the number of matrices present originally, thus giving an estimate for the minimum amount of labour needed in many cases to compute the degree using this method.

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1. Introduction

Let P^n, $n \geqq 1$, be a connected n-dimensional polyhedron (definition below) in R^n and let Φ^n: $P^n \to R^n$ be continuous with $\Phi^n(p) \neq \theta^n = (0, 0, \ldots, 0)$ $\forall p \in b(P^n)$, the boundary of P^n. Then the topological degree of Φ^n on P^n relative to θ^n, an integer denoted by $d(\Phi^n, P^n, \theta^n)$, can be defined [1, 7]. A new formula for $d(\Phi^n, P^n, \theta^n)$, which rendered computation of the degree far more feasible than was previously thought to be the general case, was given in [7, equation (4.26)].

At present there is much interest in using this formula to compute the degree [4, 5, 8, 9]. The formula involves a sum of determinants of $n \times n$ matrices, whose evaluation is time-consuming in practice. In this paper we replace these determinant evaluations by a procedure which scans their associated matrices for entries of ± 1 in certain combinations. This procedure is easy to implement and at each of $n-1$ steps can be arranged so as to discard at least half of the matrices remaining from the previous step. A less general version of this procedure was recently given by R.B. Kearfott [4, 5]; however the proof given in [4, 5] is quite different from the one described here.

We conclude by giving a lower bound for the number of $n \times n$ matrices in a sufficient refinement (definition below) of $b(P^n)$ in the case where $d(\Phi^n, P^n, \theta^n) \neq 0$

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(for most applications this is the interesting case [1, 6]). This gives an estimate for the minimum amount of labour needed to compute $d(\Phi^n, P^n, \theta^n)$ using our procedure.

2. Background Material

We shall usually use superscripts to denote dimension and subscripts for indexing.

Definition 2.1. The points $a_0, a_1, \ldots, a_q$ in R^n are said to be linearly independent if the vectors $a_0-a_1, a_0-a_2, \ldots, a_0-a_q$ are linearly independent. A q-simplex, $q \geqq 0$, is the closed convex hull of $q+1$ linearly independent points called its vertices.

An n-simplex in R^n together with a list of its vertices can be assigned an orientation number which is equal to $+1$ or -1. This number is invariant (changes sign) under an even (odd) permutation of the vertices. The assignment is equivalent to orienting R^n itself (see [1]). The orientation of R^n induces an orientation on the vector subspaces of R^n and their translations in a natural way. Thus a q-simplex lying in an oriented R^n (so $q \leqq n$) must lie in a translation of a q-dimensional subspace of R^n and consequently has an assigned orientation number. We shall denote oriented q-simplexes by

$$S^q = \langle y_0\, y_1 \cdots y_q \rangle,$$

where the y_i, $0 \leqq i \leqq q$, are points in R^n. We write

$$\langle y_0\, y_1\, y_2 \cdots y_q \rangle = -\langle y_1\, y_0\, y_2 \cdots y_q \rangle = \langle y_2\, y_0\, y_1 \cdots y_q \rangle,$$

etc.

Definition 2.2. A q-dimensional polyhedron P^q is a union of a finite number of oriented q-simplexes S_i^q, $i=1,2,\ldots,m$, such that for every pair S_i^q, S_j^q of these simplexes either $S_i^q \cap S_j^q$ is the empty set or $S_i^q \cap S_j^q$ is a common face, i.e., an r-simplex $(0 \leqq r \leqq q)$ whose vertices are vertices of both S_i^q and S_j^q. We write

$$P^q = \sum_{i=1}^{m} S_i^q.$$

Definition 2.3. A q-region is a connected q-dimensional polyhedron.

For the rest of this paper let P^n be an n-region and suppose $\Phi^n = (\varphi_1, \varphi_2, \ldots, \varphi_n): P^n \to R^n$ is continuous with $\Phi^n(p) \neq \theta^n = (0, 0, \ldots, 0)$ $\forall p \in b(P^n)$, the boundary of P^n. Then the topological degree of Φ^n on P^n relative to θ^n, denoted by $d(\Phi^n, P^n, \theta^n)$, is defined [1, 7]. We always assume that the $(n-1)$-simplexes in any representation of $b(P^n)$ are oriented in such a way that we can write

$$b(P^n) = \sum_{j=1}^{k} t_j \langle y_1^{(j)} \ldots y_n^{(j)} \rangle, \qquad t_j = \pm 1,$$

where all possible cancellations have been carried out (see [1]) (for a proof that $b(P^n)$ is an $(n-1)$-dimensional polyhedron see [8]).

For t any real number, set

$$\operatorname{sgn} t = \begin{cases} -1 & \text{if } t<0 \\ 0 & \text{if } t=0 \\ 1 & \text{if } t>0. \end{cases}$$

Definition 2.4. If $n=1$, $b(P^1)=\langle x_m\rangle - \langle x_0\rangle$ (say) is said to be sufficiently refined relative to $\operatorname{sgn}\Phi^1 = \operatorname{sgn}\varphi_1$ if $\varphi_1(x_0)\varphi_1(x_m)\neq 0$. If $n>1$, $b(P^n)$ is said to be sufficiently refined relative to $\operatorname{sgn}\Phi^n$ if $b(P^n)$ has been subdivided so that it may be written as a union of a finite number of $(n-1)$-regions $\beta_1^{n-1}, \beta_2^{n-1}, \ldots, \beta_m^{n-1}$ in such a way that

(i) the $(n-1)$-dimensional interiors of the β_i^{n-1} are pairwise disjoint;

(ii) at least one of the functions $\varphi_1, \ldots, \varphi_n$, say φ_{s_i}, does not vanish on each region β_i^{n-1};

(iii) if $\varphi_{s_i}\neq 0$ on β_i^{n-1}, then $b(\beta_i^{n-1})$ is sufficiently refined relative to $\operatorname{sgn}\Phi_{s_i}^{n-1}$, where $\Phi_{s_i}^{n-1}=(\varphi_1, \ldots, \hat{\varphi}_{s_i}, \ldots, \varphi_n)$ ($\hat{}$ denotes omission).

Remark. Our Definition 2.4 differs from Definition 4.4 of [7], where an extra condition is included. However, this extra condition is unnecessary for the proof of the results we borrow from [7]; for a full explanation see [8].

Notation. For any q-simplex S^q and any function

$$\Gamma^{q+1}=(\gamma_1, \gamma_2, \ldots, \gamma_{q+1})\colon S^q \to R^{q+1}$$

where $S^q=\langle y_1\, y_2 \ldots y_{q+1}\rangle$, we will denote by $\operatorname{sgn}\Gamma^{q+1}(S^q)$ the $(q+1)\times(q+1)$ matrix whose (i,j) entry is $\operatorname{sgn}\gamma_j(y_i)$, $1\leqq i, j\leqq q+1$. We shall denote the determinant of this matrix by $\det(\operatorname{sgn}\Gamma^{q+1}(S^q))$.

Theorem 2.5 ([7, equation (4.26)]). *Suppose that $b(P^n)$ has been subdivided into $(n-1)$-regions β_k^{n-1}, $1\leqq k\leqq m$, so that it is sufficiently refined relative to* $\operatorname{sgn}\Phi^n$; *with this subdivision suppose*

$$b(P^n)=\sum_{k=1}^{l} t_k\langle y_1^{(k)} \ldots y_n^{(k)}\rangle, \qquad t_k=\pm 1.$$

Then

$$d(\Phi^n, P^n, \theta^n)=\frac{1}{2^n n!}\sum_{k=1}^{l} t_k \det(\operatorname{sgn}\Phi^n(\langle y_1^{(k)} \ldots y_n^{(k)}\rangle)) \tag{2.1}$$

Proof. See [7], or for a proof via the theory developed in [1] see [8].

It is formula (2.1), with its many determinants of $n\times n$ matrices, that we seek to simplify.

3. Simplification of the Degree Formula

Consider the matrices

$$t_k \operatorname{sgn}\Phi^n(\langle y_1^{(k)} \ldots y_n^{(k)}\rangle), \qquad k=1,2,\ldots,l \tag{3.1}$$

whose determinants were used to compute $d(\Phi^n, P^n, \theta^n)$ in Theorem 2.5. We now give a procedure which to each matrix $t_k \operatorname{sgn} \Phi^n$ assigns a number called its signature and denoted by $\operatorname{sig}(t_k \operatorname{sgn} \Phi^n)$. It is shown in Theorem 3.2 below that when these signatures are added we get $d(\Phi^n, P^n, \theta^n)$.

Procedure. Choose integers r_i and Δ_i, $i=1,2,\dots,n$, with $1 \leqq r_i \leqq i$ and $\Delta_i = +1$ or -1 $\forall i$.

Choose from (3.1) all $t_k \operatorname{sgn} \Phi^n$ whose r_nth column consists entirely of Δ_n's. Assign a temporary signature $t_k \Delta_n (-1)^{r_n+1}$ to each such matrix; assign the signature zero to every other $t_k \operatorname{sgn} \Phi^n$.

Delete the r_nth column from each chosen matrix to form an $n \times (n-1)$ array. In the r_{n-1}th column of this array pick all combinations of $n-1$ rows having $n-1$ Δ_{n-1}'s as entries, if any such combination exists (if not, assign $t_k \operatorname{sgn} \Phi^n$ the signature zero, i.e., discard the matrix) (if the matrix is not discarded there will be either one or n such combinations).

If there is one such combination, suppose that the q_{n-1}th row is the unique row with entry 0 or $-\Delta_{n-1}$. Delete this row to give an $(n-1) \times (n-1)$ matrix and assign a temporary signature

$$t_k \Delta_n \Delta_{n-1} (-1)^{r_n+1+r_{n-1}+q_{n-1}}$$

to the matrix.

If there are n such combinations (i.e., r_{n-1}th column contains n Δ_{n-1}'s) then deleting each row in turn gives $n-1$ Δ_{n-1}'s in the r_{n-1}th column. Do this, obtaining n $(n-1) \times (n-1)$ matrices with associated temporary signatures

$$t_k \Delta_n \Delta_{n-1} (-1)^{r_n+1+r_{n-1}+q_{n-1}},$$

where the q_{n-1}th row was the one deleted (so q_{n-1} runs through the values $1,2,\dots,n$).

Now deal with each $(n-1) \times (n-1)$ matrix just as each original $n \times n$ matrix was dealt with after assigning the first temporary signature, replacing n by $n-1$ throughout. Continue reducing until left with 1×1 matrices. At this stage the sum of the temporary signatures of those 1×1 matrices whose ancestor was a particular $t_k \operatorname{sgn} \Phi^n$ is taken as the signature of that $t_k \operatorname{sgn} \Phi^n$. Finally add all the signatures of the $t_k \operatorname{sgn} \Phi^n$.

Observation 3.1. For $n=1$, suppose $b(P^1) = \langle x_m \rangle - \langle x_0 \rangle$. Then $d(\Phi^1, P^1, \theta^1)$ is defined iff $\varphi_1(x_m)\varphi_1(x_0) \neq 0$ (so we immediately have a sufficient refinement by definition 2.4) and in fact

$$d(\Phi^1, P^1, \theta) = \tfrac{1}{2}\{\operatorname{sgn} \Phi^1(x_m) - \operatorname{sgn} \Phi^1(x_0)\} \quad [1, 7].$$

To apply the procedure here, of necessity $r_1 = 1$. As usual $\Delta_1 \in \{-1, 1\}$. If $\operatorname{sgn} \Phi^1(x_m) = \Delta_1$, then

$$\operatorname{sig}(\operatorname{sgn} \Phi^1(x_m)) = \Delta_1 (-1)^{r_1+1} = \Delta_1;$$

if not, the signature is zero. If $\operatorname{sgn} \Phi^1(x_0) = \Delta_1$, then

$$\operatorname{sig}(\operatorname{sgn} \Phi^1(x_0)) = -\Delta_1;$$

if not, the signature is zero. Finally the signatures are added. By examining the various cases we see that the procedure calculates $d(\Phi^1, P^1, \theta^1)$.

Before going on to the general case $n>1$ we introduce some notation.

Let J_n denote the subset of $\{1, 2, \ldots, m\}$ such that $j \in J_n$ implies that $\varphi_{r_n} \neq 0$ on β_j^{n-1} with $\operatorname{sgn} \varphi_{r_n}|_{\beta_j^{n-1}} = \Delta_n$. Then equations (4.15) and (4.16) of [7] may be combined in our notation as

$$d(\Phi^n, P^n, \theta^n) = (-1)^{r_n+1} \Delta_n \sum_{j \in J_n} d(\Phi_{r_n}^{n-1}, \beta_j^{n-1}, \theta^{n-1}). \tag{3.2}$$

For each $j \in J_n$ take

$$\beta_j^{n-1} = \sum_{k \in K_j} \langle y_1^{(k)} \ldots y_n^{(k)} \rangle$$

so

$$b(\beta_j^{n-1}) = \sum_{k \in K_j} \sum_{i=1}^{n} (-1)^{i+1} \langle y_1^{(k)} \ldots \hat{y}_i^{(k)} \ldots y_n^{(k)} \rangle$$

($\hat{}$ denotes omission) (see [1]) with associated matrices

$$(-1)^{i+1} \operatorname{sgn} \Phi_{r_n}^{n-1} (\langle y_1^{(k)} \ldots \hat{y}_i^{(k)} \ldots y_n^{(k)} \rangle),$$
$$i = 1, 2, \ldots, n, \quad k \in K_j, \ j \in J_n. \tag{3.3}$$

Theorem 3.2. *Suppose $b(P^n)$ is sufficiently refined relative to* $\operatorname{sgn} \Phi^n$ *with notation as above. Fix the choice of r_i and $\Delta_i (1 \leqq i \leqq n)$ in the procedure.*

Then

(i) the procedure computes $d(\Phi^n, P^n, \theta^n)$ for $n \geqq 1$

(ii) if $\operatorname{sgn} \Phi^n(S^{n-1})$ is an $n \times n$ matrix, $n \geqq 2$, whose r_j^{th} column consists entirely of Δ_j's for $j = n-1$, n, then $\operatorname{sig}(\operatorname{sgn} \Phi^n(S^{n-1})) = 0$.

Proof. By observation 3.1 above (i) holds for $n=1$. We use the following argument for the rest of the proof: fix $\nu > 1$. Show that (i) true for $n = \nu - 1 \Rightarrow$ (ii) true for $n = \nu \Rightarrow$ (i) true for $n = \nu$.

Fix $\nu > 1$ and assume that (i) holds for $n = \nu - 1$. Choose a ν-simplex $S^\nu = \langle x_0 x_1 \ldots x_\nu \rangle$ and a continuous function $\Gamma^\nu = (\gamma_1, \gamma_2, \ldots, \gamma_\nu): S^\nu \to R^\nu$ such that

$$\operatorname{sgn} \gamma_j(x_i) = (i,j) \text{ entry of } \operatorname{sgn} \Phi^\nu(S^{\nu-1}) \quad \text{for } 1 \leqq i, j \leqq \nu$$

and moreover

$$\operatorname{sgn} \gamma_{r_j} = \Delta_j \quad \text{on all of } S^\nu \quad \text{for } j = \nu - 1 \text{ and } \nu$$

($\operatorname{sgn} \gamma_j(x_0)$ is arbitrary for $j \neq r_{\nu-1}, r_\nu$). Then $d(\Phi^\nu, S^\nu, \theta^\nu)$ is defined and equal to zero because $\Gamma^\nu(x) \neq \theta^\nu$ for any $x \in S^\nu$ [1, p. 32]. We get a sufficient refinement of $b(S^\nu)$ relative to $\operatorname{sgn} \Gamma^\nu$ by taking $\beta_1^{\nu-1} = \langle x_1 x_2 \ldots x_\nu \rangle$ with associated coordinate function $\gamma_{r_\nu} \neq 0$ on $\beta_1^{\nu-1}$, and

$$\beta_2^{\nu-1} = \sum_{i=1}^{\nu} (-1)^i \langle x_0 x_1 \ldots \hat{x}_i \ldots x_\nu \rangle$$

with associated coordinate function $\gamma_{r_{\nu-1}} \neq 0$ on $\beta_2^{\nu-1}$. To check (iii) of definition 2.4 for say $\beta_2^{\nu-1}$, we observe that $b(\beta_2^{\nu-1})$ is sufficiently refined relative to $\operatorname{sgn} \Gamma_{r_{\nu-1}}^{\nu-1}$ because all of $b(\beta_2^{\nu-1})$ may be taken as a $(\nu-2)$-region on which $\gamma_{r_\nu} \neq 0$ (this is assuming $\nu>2$; the $\nu=2$ case is even easier). The other conditions of definition 2.4 are easily verified.

Now (3.2) above gives

$$(-1)^{r_\nu+1}\Delta_\nu d(\Gamma_{r_\nu}^{\nu-1}, \beta_1^{\nu-1}, \theta^{\nu-1}) = d(\Gamma^\nu, S^\nu, \theta^\nu) = 0 \tag{3.4}$$

However since we are assuming that (i) of the theorem holds for $n=\nu-1$ the procedure yields

$$d(\Gamma_{r_\nu}^{\nu-1}, \beta_1^{\nu-1}, \theta^{\nu-1}) = \sum_{i=1}^{\nu} (-1)^{i+1} \operatorname{sig}(\operatorname{sgn} \Gamma_{r_\nu}^{\nu-1}(\langle x_1 \dots \hat{x}_i \dots x_\nu \rangle)).$$

Thus from (3.4)

$$0 = (-1)^{r_\nu+1}\Delta_\nu \sum_{i=1}^{\nu} (-1)^{i+1} \operatorname{sig}(\operatorname{sgn} \Gamma_{r_\nu}^{\nu-1}(\langle x_1 \dots \hat{x}_i \dots x_\nu \rangle))$$
$$= \operatorname{sig}(\operatorname{sgn} \Gamma^\nu(\langle x_1 \dots x_\nu \rangle)),$$

by inspection of the procedure. Thus (ii) holds for $n=\nu$, since $\operatorname{sgn} \Gamma^\nu(\langle x_1 \dots x_\nu \rangle) = \operatorname{sgn} \Phi^\nu(S^{\nu-1})$.

We now show that (i) holds for $n=\nu$. We are assuming that (i) is true for $n=\nu-1$, and since $b(\beta_j^{\nu-1})$ is sufficiently refined relative to $\operatorname{sgn} \Phi_{r_\nu}^{\nu-1}\ \forall j \in J_\nu$, the procedure may be used to calculate $\sum_{j\in J_\nu} d(\Phi_{r_\nu}^{\nu-1}, \beta_j^{\nu-1}, \theta^{\nu-1})$. Thus, letting j vary over J_ν, choose from (3.3) all those matrices $(-1)^{i+1}$ $\operatorname{sgn} \Phi_{r_\nu}^{\nu-1}$ whose $r_{\nu-1}{}^{\text{th}}$ column consists entirely of $\Delta_{\nu-1}$'s and assign a temporary signature

$$(-1)^{i+1}\Delta_{\nu-1}(-1)^{r_{\nu-1}+1} = \Delta_{\nu-1}(-1)^{r_{\nu-1}+i}$$

to each. Apply the procedure from this point to compute the signature of each chosen $(-1)^{i+1} \operatorname{sgn} \Phi_{r_\nu}^{\nu-1}$, then add these signatures. To now calculate $d(\Phi^\nu, P^\nu, \theta^\nu)$ use (3.2): multiply the signature of each chosen $(-1)^{i+1} \operatorname{sgn} \Phi_{r_\nu}^{\nu-1}$ by $(-1)^{r_\nu+1}\Delta_\nu$ and then add these values.

On the other hand applying the procedure to the matrices in (3.1) means that we shall immediately discard all those $t_k \operatorname{sgn} \Phi^\nu$ which do not have an $r_\nu{}^{\text{th}}$ column consisting entirely of Δ_ν's. We thus retain all $t_k \operatorname{sgn} \Phi^\nu$ corresponding to simplexes lying in $\beta_j^{\nu-1}$'s for which $j \in J_\nu$ and possibly as well some $t_k \operatorname{sgn} \Phi^\nu$ associated with other $\beta_j^{\nu-1}$; however these spurious $t_k \operatorname{sgn} \Phi^\nu$ are assigned signature zero by part (ii) of the theorem since they must have another column besides the $r_\nu{}^{\text{th}}$ which consists entirely of $+1$'s or of -1's. Consequently by inspection the procedure is seen to coincide with the method given above for computing $d(\Phi^\nu, P^\nu, \theta^\nu)$. Thus (ii) holds for $n=\nu$ and the proof is complete.

Part (ii) of the theorem indicates that there is superfluous computation in the original procedure. We now give a modified procedure designed to avoid this, and which also discards matrices arising in the procedure which have a column consisting entirely of zeroes, since such matrices are clearly discarded at some stage.

Modified procedure. Chosen integers r_i and Δ_i, $i=1,2,\ldots,n$, with $1 \leqq r_i \leqq i$ and $\Delta_i = +1$ or $-1 \; \forall i$.

Choose from (3.1) all $t_k \operatorname{sgn} \Phi^n$ whose $r_n{}^{\text{th}}$ column consists entirely of Δ_n's. Assign a temporary signature $t_k \Delta_n (-1)^{r_n+1}$ to each such matrix; assign the signature zero to every other $t_k \operatorname{sgn} \Phi^n$.

If any other column of this matrix is constant (i.e., all its entries have the same value) assign the signature zero to the matrix, i.e., discard it. Otherwise delete the $r_n{}^{\text{th}}$ column to form an $n \times (n-1)$ array. In the $r_{n-1}{}^{\text{th}}$ column of this array pick the combination of $n-1$ rows having $n-1$ Δ_{n-1}'s as entries, if such a combination exists; if not, discard the matrix. Suppose the $q_{n-1}{}^{\text{th}}$ row is the unique row with entry 0 or $-\Delta_{n-1}$. Delete this row to give an $(n-1) \times (n-1)$ matrix and assign a temporary signature

$$t_k \Delta_n \Delta_{n-1} (-1)^{r_n+1+r_{n-1}+q_{n-1}} \quad \text{to} \quad t_k \operatorname{sgn} \Phi^n.$$

Deal with this $(n-1) \times (n-1)$ matrix just as we dealt with the $n \times n$ situation after assigning the first temporary signature, replacing n by $n-1$. Continue reducing until left with a 1×1 matrix. The temporary signature at this stage is the signature of $t_k \operatorname{sgn} \Phi^n$, i.e.,

$$\operatorname{sig}(t_k \operatorname{sgn} \Phi^n) = t_k \Delta_n \Delta_{n-1} \cdots \Delta_1 (-1)^{r_n+1+r_n+1+q_{n-1}+\cdots+r_1+q_1}.$$

Finally add the signatures of the chosen $t_k \operatorname{sgn} \Phi^n$.

Remark 3.3. It can be shown that with a judicious choice of r_n and Δ_n the number of matrices retained after the first step of the modified procedure is less than the original number by a factor of at worst $\dfrac{1}{2n}$, and by choosing r_i and Δ_i carefully for $n-1 \geqq i \geqq 2$ we can discard at least half of the remaining matrices at each of these $n-2$ steps. It is easy to see how this may be accomplished by examining the proof of Corollary 3.5 below.

Remark 3.4. In the modified procedure it is clear that the signature of any matrix is -1, 0, or $+1$.

Corollary 3.5 (to Theorem 3.2). *Suppose $b(P^n)$ is sufficiently refined relative to* $\operatorname{sgn} \Phi^n$. *Let $d' = |d(\Phi^n, P^n, \theta^n)|$. Then $b(P^n)$ is subdivided into at least $d' n 2^{n-1} (n-1)$-simplexes.*

Proof. By Theorem 3.2 the modified procedure yields $d(\Phi^n, P^n, \theta^n)$ for any choice of r_i, Δ_i $(1 \leqq i \leqq n)$ satisfying $1 \leqq r_i \leqq i$, $\Delta_i \in \{-1, 1\} \; \forall i$.

There are $2n$ possibilities for the ordered pair (r_n, Δ_n). Any matrix chosen for two distinct pairs will have signature zero (two columns will be constant). Consequently we can assume that there is no overlap in choice; we count the minimum number of matrices needed to give $|d(\Phi^n, P^n, \theta^n)| = d'$ for a fixed pair (r_n, Δ_n) then multiply this number by $2n$.

Fix (r_n, Δ_n). Consider the matrices $t_k \operatorname{sgn} \Phi^n$ chosen for this pair. Delete the $r_n{}^{\text{th}}$ column from each $t_k \operatorname{sgn} \Phi^n$ to leave an $n \times (n-1)$ array. Choose r_{n-1} satisfying $1 \leqq r_{n-1} \leqq n-1$. Then the matrices chosen at this stage for the pair $(r_{n-1}, 1)$ cannot overlap with those chosen for the pair $(r_{n-1}, -1)$ if $n > 2$, because ovarlapping implies the existence of an $n \times (n-1)$ array with a column containing $(n-1)$ $+1$'s and $(n-1)$ -1's. Thus the minimum number of matrices needed for the pair (r_n, Δ_n) is at least twice the minimum number needed for the pair (r_{n-1}, Δ_{n-1}).

We can apply the least argument repeatedly going from the i^{th} to the $(i-1)^{\text{th}}$ stage for $i > 2$. Each stage yields a factor of 2, so with the original factor of $2n$ this gives a factor $n 2^{n-1}$.

When $i = 2$ is reached all that can be said is that by Remark 3.4 at least d' matrices are needed. This gives the final lower bound of $d' n 2^{n-1}$.

Corollary 3.6 *Suppose* $b(P^n)$ *is sufficiently refined relative to* $\operatorname{sgn} \Phi^n$. *If* $d(\Phi^n, P^n, \theta^n) \neq 0$, *then* $b(P^n)$ *is subdivided into at least* $n 2^{n-1} (n-1)$*-simplexes.*

Proof. Immediate from Corollary 3.5.

Remark 3.7. The result of Corollary 3.5 can be improved for $n \geqq 6$ if we recall formula (2.1):

$$d(\Phi^n, P^n, \theta^n) = \frac{1}{2^n n!} \sum_{k=1}^{l} t_k \det(\operatorname{sgn} \Phi^n(\langle y_1^{(k)} \dots y_n^{(k)} \rangle)).$$

By Hadamard's determinant theorem [2], $|t_k \det(\operatorname{sgn} \Phi^n)| \leqq n^{n/2}$ for all k, and it follows under the hypotheses of Corollary 3.5 that the above sum must contain at least $d' 2^n n! / n^{n/2}$ terms; for $n \geqq 6$ this is greater than $d' n 2^{n-1}$ (if $d' \neq 0$).

A further improvement, valid for all $n \geqq 1$, may be obtained as follows. For any matrix $\operatorname{sgn} \Phi^n(S^{n-1})$, if there exists a choice of (r_i, Δ_i), $1 \leqq i \leqq n$, for which $\operatorname{sig}(\operatorname{sgn} \Phi^n(S^{n-1})) \neq 0$, it can be then shown that $|\det(\operatorname{sgn} \Phi^n(S^{n-1}))| \leqq 2^{n-1}$. Thus by Corollary 3.5 there are at least $d' n 2^{n-1}$ determinants in (2.1), each having absolute value at most 2^{n-1}. Now, using the same argument as in the previous paragraph, it is straightforward to show that the sum in (2.1) contains at least

$$d' n 2^{n-1} + \frac{d' 2^n n}{n^{n/2}} ((n-1)! - 2^{n-2}) \qquad \text{terms.}$$

We conjecture that in fact the lower bound can be increased to $2 d' n!$ Then Corollary 3.6 would have the lower bound $2n!$ If so, this is certainly the best possible estimate: let P^n be the cube of side 2 in R^n with vertices all of whose coordinates are ± 1. We give R^n the standard 'counterclockwise' orientation. If $\Phi^n(x) = x$ for all x in P^n, then it is easy to show that $d(\Phi^n, P^n, \theta^n) = 1$. Now [3] gives a simplicial decomposition of P^n which, as can be checked, readily yields a sufficient refinement of $b(P^n)$ relative to $\operatorname{sgn} \Phi^n$, and this sufficient refinement contains $2n!$ $(n-1)$-simplexes.

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