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Association schemes based on maximal totally isotropic subspaces in singular pseudo-symplectic spaces

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ABSTRACT

This paper provides some new families of symmetric association schemes based on maximal totally isotropic subspaces in (singular) pseudo-symplectic spaces. All intersection numbers of these schemes are computed.

1. Introduction

Dual polar schemes are well-known association schemes [1,2]. Applying the matrix method, Wan et al. [5] computed all parameters of dual polar schemes; Wang et al. [4,8,9] determined all subconstituents of dual polar graphs. As a generalization of bilinear forms schemes, Wang et al. [7] constructed association schemes in attenuated spaces, and computed their parameters. As a generalization of dual polar schemes, Guo et al. [3] constructed association schemes in singular classical spaces. In this paper, we continue this research, and consider the similar problems in singular pseudo-symplectic spaces.

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Let $\mathbb{F}_q$ be a finite field with q elements, where q is a power of 2. Let E denote the subspace of $\mathbb{F}_q^{2\nu+\delta+l}$ generated by $e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \ldots, e_{2\nu+\delta+l}$, where e_i is the row vector in $\mathbb{F}_q^{2\nu+\delta+l}$ whose ith coordinate is 1 and all other coordinates are 0.

Let

$$S_{s,\tau;\,l} = \begin{pmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & \Delta & \\ & & & 0^{(l)} \end{pmatrix}, \quad \text{where } \Delta = \begin{cases} \emptyset, & \text{if } \tau = 0, \\ (1), & \text{if } \tau = 1, \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \tau = 2. \end{cases}$$

For $\delta = 1$ or 2, the *singular pseudo-symplectic group* of degree $2\nu+\delta+l$ over $\mathbb{F}_q$, denoted by $Ps_{2\nu+\delta+l,\,2\nu+\delta}(\mathbb{F}_q)$, consists of all nonsingular matrices T over $\mathbb{F}_q$ satisfying $TS_{\nu,\delta;\,l}T^t = S_{\nu,\delta;\,l}$. The vector space $\mathbb{F}_q^{2\nu+\delta+l}$ together with the right multiplication action of $Ps_{2\nu+\delta+l,\,2\nu+\delta}(\mathbb{F}_q)$ is called the $(2\nu+\delta+l)$-dimensional *singular pseudo-symplectic space* over $\mathbb{F}_q$. If $l = 0$, $Ps_{2\nu+\delta+l,\,2\nu+\delta}(\mathbb{F}_q)$ is the pseudo-symplectic group, and $\mathbb{F}_q^{2\nu+\delta+l}$ is the pseudo-symplectic space.

For an m-dimensional subspace P of $\mathbb{F}_q^{2\nu+\delta+l}$, we use the same letter P to denote the $m \times (2\nu+\delta+l)$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P. An m-dimensional subspace P in the $(2\nu+\delta+l)$-dimensional singular pseudo-symplectic space is a subspace of *type* $(m, 2s+\tau, s, \varepsilon, k)$, where $\tau = 0, 1$ or 2, $\varepsilon = 0$ or 1, if $\dim(P \cap E) = k$ and $PS_{\nu,\delta;\,l}P^t$ is cogredient to $S_{s,\tau;\,m-2s-\tau}$ and P does not or does contain a vector of the form

$$\begin{cases} (0,0,\ldots,0,1,x_{2\nu+2},\ldots,x_{2\nu+1+l}), & \text{if } \delta = 1, \\ (0,0,\ldots,0,1,0,x_{2\nu+3},\ldots,x_{2\nu+2+l}), & \text{if } \delta = 2, \end{cases}$$

corresponding to the cases $\varepsilon = 0$ or 1, respectively. In particular, subspaces of type $(\nu+\varepsilon, 0, 0, \varepsilon, 0)$ are called *maximal totally isotropic subspaces*.

Let $X = \mathcal{M}(m, 2s+\tau, s, \varepsilon, k;\ 2\nu+\delta+l, 2\nu+\delta)$ denote the set of all subspaces of type $(m, 2s+\tau, s, \varepsilon, k)$ of $\mathbb{F}_q^{2\nu+\delta+l}$. By [6, Theorem 4.18], X is an orbit under the action of $Ps_{2\nu+\delta+l,\,2\nu+\delta}(\mathbb{F}_q)$. Hence X forms an association scheme according to all the orbits of the action of $Ps_{2\nu+\delta+l,\,2\nu+\delta}(\mathbb{F}_q)$ on $X \times X$ (see [1]). In this paper, we determine all these orbits, and compute all the parameters of these association schemes.

This paper is organized as follows. In Section 2, we construct association schemes based on maximal isotropic subspaces in pseudo-symplectic spaces, and compute their parameters. In Section 3, we discuss the corresponding problem in singular pseudo-symplectic spaces.

2. The case $l = 0$

In this section we study the association schemes based on maximal isotropic subspaces in pseudo-symplectic spaces, and compute their parameters.

Let $X = \mathcal{M}(\nu+\varepsilon, 0, 0, \varepsilon;\ 2\nu+\delta)$, where $\delta = 1$, or $\delta = 2$ and $\varepsilon = 1$. Define

$$R_i = \{(P', Q') \in X \times X \mid \dim(P' \cap Q') = \nu + \delta - 1 - i\}.$$

Then the configuration $(X, \{R_i\}_{0 \leqslant i \leqslant \nu})$ forms a symmetric association scheme isomorphic to the association scheme in [5, Chapter 7, Theorem 8].

Now we consider the case $\delta = 2$ and $\varepsilon = 0$.

Theorem 2.1. *Let $X = \mathcal{M}(\nu, 0, 0, 0;\ 2\nu+2)$. For any two elements of X*

$$P' = \begin{matrix} {\scriptstyle 2\nu} & {\scriptstyle 1} & {\scriptstyle 1} \\ (P'_1 & z^t & 0) \end{matrix}, \quad Q' = \begin{matrix} {\scriptstyle 2\nu} & {\scriptstyle 1} & {\scriptstyle 1} \\ (Q'_1 & w^t & 0) \end{matrix},$$

define $(P', Q') \in R_{(i,a)}$ if and only if $\dim(P' \cap Q') + a = \dim(P'_1 \cap Q'_1) = \nu - i$. Then the configuration $\mathcal{X} = (X, \{R_{(i,a)}\}_{0 \leqslant i \leqslant \nu - a,\ 0 \leqslant a \leqslant 1})$ forms a symmetric association scheme of class 2ν, and with parameters v and $n_{(i,a)}$ given by (1) and (2); intersection numbers $p^{(i,a)}_{(s,b)\,(u,c)}$'s given by (5) and (6).

Proof. Similar to [3, Lemma 2.1], each $R_{(i,a)}$ is an orbit of $Ps_{2\nu+2}(\mathbb{F}_q)$ on $X \times X$, and $\mathcal{X}$ is a symmetric association scheme. Now we compute its parameters. Pick

$$P_1' = (I^{(\nu)} \quad 0^{(\nu,\nu)}), \quad Q_1' = \begin{pmatrix} 0^{(\nu-i,i)} & I^{(\nu-i)} & 0 & 0 \\ 0 & 0 & I^{(i)} & 0 \end{pmatrix}.$$

By [6, Theorem 4.14]

$$v = |\mathcal{M}(\nu,0,0,0;2\nu+2)| = q^{\nu}\prod_{t=1}^{\nu}(q^t+1). \tag{1}$$

For $P' = (P_1', 0^{(\nu,2)}) \in X$, $n_{(i,a)}$ is the number of subspaces

$$U' = \begin{matrix} \scriptstyle 2\nu & \scriptstyle 1 & \scriptstyle 1 \\ (U_1' & u^t & 0) \end{matrix} \in X$$

satisfying $(P',U') \in R_{(i,a)}$. Then U_1' is a maximal totally isotropic subspace in $\mathbb{F}_q^{2\nu}$ intersecting P_1' at a $(\nu-i)$-dimensional subspace. By [5, Chapter 7, Theorem 8], there are $q^{i(i+1)/2}\begin{bmatrix}\nu\\ i\end{bmatrix}$ choices for U_1'. By the transitivity of $Ps_{2\nu+2}(\mathbb{F}_q)$, the number of u is independent of the choice of U_1'. Pick $U_1' = Q_1'$. Then U' has the unique matrix representation of the form

$$\begin{matrix} \scriptstyle i & \scriptstyle \nu-i & \scriptstyle i & \scriptstyle \nu-i & \scriptstyle 1 & \scriptstyle 1 \end{matrix}$$
$$\begin{pmatrix} 0 & I & 0 & 0 & u_1^t & 0 \\ 0 & 0 & I & 0 & u_2^t & 0 \end{pmatrix} \begin{matrix} \nu-i \\ i \end{matrix},$$

which implies that

$$n_{(i,1)} = (q^{\nu-i}-1)q^{i(i+3)/2}\begin{bmatrix}\nu\\ i\end{bmatrix} \quad \text{and} \quad n_{(i,0)} = q^{i(i+3)/2}\begin{bmatrix}\nu\\ i\end{bmatrix}. \tag{2}$$

Next, we compute the intersection numbers $p^{(i,0)}_{(s,0)\,(u,0)}$. By the transitivity of $Ps_{2\nu+2}(\mathbb{F}_q)$ on $R_{(i,0)}$, we pick $P' = (P_1', 0^{(\nu,2)})$ and $Q' = (Q_1', 0^{(\nu,2)})$. Then $(P',Q') \in R_{(i,0)}$, and $p^{(i,0)}_{(s,0)\,(u,0)}$ is the number of subspaces

$$S' = \begin{matrix} \scriptstyle 2\nu & \scriptstyle 1 & \scriptstyle 1 \\ (S_1' & w^t & 0) \end{matrix} \in X$$

satisfying $(P',S') \in R_{(s,0)}$ and $(S',Q') \in R_{(u,0)}$. By [5, Chapter 7, Theorem 8], there are

$$p^i_{su} = \sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i \\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} p(\alpha)$$

choices for S_1', where

$$p(\alpha) = q^{\lfloor\frac{\mu}{2}\rfloor(\lfloor\frac{\mu}{2}\rfloor+1)+\rho i+\frac{\rho(\rho+1)}{2}}\,\frac{\begin{bmatrix}\nu-i\\ \beta\end{bmatrix}\prod_{t=\gamma+1}^{i}(q^t-1)}{\prod_{t=1}^{\lfloor\frac{\mu}{2}\rfloor}(q^{2t}-1)\prod_{t=1}^{\alpha-\mu}(q^t-1)}.$$

By the transitivity of $Ps_{2\nu+2}(\mathbb{F}_q)$, the number of w is independent of the choice of S_1'. By [5, Chapter 7, Lemma 7] we may pick

$$S_1' = \begin{pmatrix} I & 0^{(\alpha,\gamma)} & 0 & 0 & \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & I & 0^{(\beta,\rho)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(\gamma)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(\rho)} \end{pmatrix}, \tag{3}$$

where

$$B=\begin{cases}\begin{pmatrix}0 & I^{(p)} & \\ I^{(p)} & 0 & \\ & & 0\end{pmatrix}, & \text{if } \mu=2p+1,\\[2ex] \begin{pmatrix}0 & I^{(p)}\\ I^{(p)} & \end{pmatrix} \text{ or } \begin{pmatrix}0 & I^{(p-1)} & & \\ I^{(p-1)} & 0 & & \\ & & 0 & 1\\ & & 1 & 1\end{pmatrix}, & \text{if } \mu=2p.\end{cases} \tag{4}$$

Since $\dim(P_1'\cap Q_1'\cap S_1')=\beta$, we obtain

$$\dim((P_1'\cap S_1')+(Q_1'\cap S_1'))=\dim(P_1'\cap S_1')+\dim(Q_1'\cap S_1')-\beta=\nu+\rho-s-u+i,$$

and the number of vectors w is $q^{\nu-(\nu+\rho-s-u+i)}=q^{s+u-i-\rho}$. It follows that

$$p^{(i,0)}_{(s,0)\,(u,0)}=\sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i\\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} q^{s+u-i-\rho}p(\alpha). \tag{5}$$

For $a=0$ or 1, we claim that

$$q^sp^i_{su}=p^{(i,a)}_{(s,0)\,(u,0)}+p^{(i,a)}_{(s,0)\,(u,1)},\quad (q^\nu-q^s)p^i_{su}=p^{(i,a)}_{(s,1)\,(u,0)}+p^{(i,a)}_{(s,1)\,(u,1)}. \tag{6}$$

Pick $P'=(P_1',\,0^{(\nu,2)})$, $Q'=(Q_1',\,0^{(\nu,2)})$, and let $S'=(S_1',\,w^t,\,0)\in X$ such that

$$\dim(P_1'+S_1')=\dim(P'+S')=\nu+s \text{ and } \dim(Q_1'+S_1')=\nu+u.$$

By the transitivity of $Ps_{2\nu+2}(\mathbb{F}_q)$, the number of vectors w is independent of the choice of S_1'. Pick S_1' as (3). Then the number of vectors w is q^s, which implies that

$$q^sp^i_{s,u}=p^{(i,0)}_{(s,0)\,(u,0)}+p^{(i,0)}_{(s,0)\,(u,1)}.$$

Since

$$p^{(i,0)}_{(s,0)\,(u,0)}+p^{(i,0)}_{(s,0)\,(u,1)}+p^{(i,0)}_{(s,1)\,(u,0)}+p^{(i,0)}_{(s,1)\,(u,1)}=q^\nu,$$

we obtain $(q^\nu-q^s)p^i_{s,u}=p^{(i,0)}_{(s,1)\,(u,0)}+p^{(i,0)}_{(s,1)\,(u,1)}$. Similarly, (6) also holds for $a=1$. □

For brevity, by (6) we may assume that

$$p^{(i,a)}_{(s,b)\,(u,c)}=\sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i\\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} p(\alpha;a,b,c).$$

3. The case $l>0$

In this section we study the association schemes based on maximal totally isotropic subspaces in singular pseudo-symplectic spaces, and compute their parameters.

Let $X=\mathcal{M}(\nu+\varepsilon,0,0,\varepsilon,0;\ 2\nu+\delta+l,2\nu+\delta)$. If $\delta=1$ and $\varepsilon=0$, then for any two elements of X

$$P=\overset{\begin{matrix}2\nu & 1 & l\end{matrix}}{\begin{pmatrix}P_1' & 0 & P''\end{pmatrix}},\quad Q=\overset{\begin{matrix}2\nu & 1 & l\end{matrix}}{\begin{pmatrix}Q_1' & 0 & Q''\end{pmatrix}},$$

define $(P,Q)\in R_{(i,j-i)}$ if and only if $\dim(P\cap Q)=\nu-j$, $\dim(P_1'\cap Q_1')=\nu-i$. Then the configuration $(X,\{R_{(i,j-i)}\}_{0\leqslant i\leqslant\nu,\,0\leqslant j-i\leqslant\min\{\nu-i,l\}})$ forms a symmetric association scheme isomorphic to the association scheme in [3, Theorem 1.1].

Now we consider the case $\delta=2$ and $\varepsilon=1$.

Theorem 3.1. *Let* $X = \mathcal{M}(\nu+1, 0, 0, 1, 0; 2\nu+2+l, 2\nu+2)$. *For any two elements of X*

$$P = \begin{pmatrix} \overset{2\nu}{P_1'} & \overset{1}{0} & \overset{1}{0} & \overset{l}{P''} \\ 0 & 1 & 0 & P_2 \end{pmatrix}, \quad Q = \begin{pmatrix} \overset{2\nu}{Q_1'} & \overset{1}{0} & \overset{1}{0} & \overset{l}{Q''} \\ 0 & 1 & 0 & Q_2 \end{pmatrix},$$

define $(P, Q) \in R_{(i,j-i)}$ *if and only if* $\dim(P \cap Q) = \nu + 1 - j, \dim(P_1' \cap Q_1') = \nu - i$. *Then the configuration* $\mathcal{X} = (X, \{R_{(i,j-i)}\}_{0 \leqslant i \leqslant \nu,\, 0 \leqslant j-i \leqslant \min\{\nu+1-i,l\}})$ *forms a symmetric association scheme with parameters d, v and* $n_{(i,j-i)}$ *given by* (7), (8) *and* (9); *intersection numbers* $p^{(i,j-i)}_{(s,t-s)\,(u,v-u)}$*'s given by* (14) *and* (15).

Proof. Similar to [3, Lemma 2.1], each $R_{(i,j-i)}$ is an orbit of $Ps_{2\nu+2+l,\,2\nu+2}(\mathbb{F}_q)$ on $X \times X$, and $\mathcal{X}$ is a symmetric association scheme. Now we compute its parameters.

By the definition of $R_{(i,j-i)}$ we have that $0 \leqslant i \leqslant \nu, 0 \leqslant \nu+1-i-(\nu+1-j) \leqslant \nu+1-i$ and $0 \leqslant \nu+1+j-(\nu+1+i) \leqslant l$. It follows that this scheme has

$$d = -1 + \sum_{i=0}^{\nu}(1 + \min\{\nu+1-i, l\}) \tag{7}$$

classes. By [6, Theorem 4.20],

$$v = q^{(\nu+1)l}\prod_{i=1}^{\nu}(q^i+1). \tag{8}$$

Pick

$$P' = \begin{pmatrix} I^{(\nu)} & 0^{(\nu)} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Q' = \begin{pmatrix} 0^{(\nu-i,\,i)} & I^{(\nu-i)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0^{(1,\,\nu-i)} & 1 & 0 \\ 0 & 0 & I^{(i)} & 0 & 0 & 0 \end{pmatrix}.$$

For $P = (P', 0^{(\nu,l)})$, $n_{(i,j-i)}$ is the number of subspaces

$$U = \begin{pmatrix} \overset{2\nu+2}{U'} & \overset{l}{U''} \end{pmatrix} \in X$$

satisfying $(P, U) \in R_{(i,j-i)}$. Then U' is a subspace of type $(\nu+1, 0, 0, 1)$ in $\mathbb{F}_q^{2\nu+2}$ intersecting P' at a subspace of type $(\nu+1-i, 0, 0, 1)$ in $\mathbb{F}_q^{2\nu+2}$. By [5, Chapter 7, Theorem 8] there are $q^{i(i+1)/2}\begin{bmatrix}\nu\\i\end{bmatrix}$ choices for U'. By the transitivity of $Ps_{2\nu+2+l,\,2\nu+2}(\mathbb{F}_q)$, we may take $U' = Q'$. Then U has the unique matrix representation of the form

$$\begin{pmatrix} \overset{i}{0} & \overset{\nu-i}{I} & \overset{i}{0} & \overset{\nu-i}{0} & \overset{1}{0} & \overset{1}{0} & \overset{l}{A_1} \\ 0 & 0 & 0 & 0 & 1 & 0 & A_2 \\ 0 & 0 & I & 0 & 0 & 0 & A_3 \end{pmatrix} \begin{matrix} \nu-i \\ 1 \\ i \end{matrix}$$

where $\operatorname{rank}\begin{pmatrix}A_1\\A_2\end{pmatrix} = j - i$. By [5, Chapter 1, Theorem 5], there are

$$N(j-i; (\nu+1-i)\times l) = q^{(j-i)(j-i-1)/2}\begin{bmatrix}\nu+1-i\\j-i\end{bmatrix}\prod_{t=l-(j-i)+1}^{l}(q^t-1)$$

choices for A_1, A_2, which implies that

$$n_{(i,j-i)} = q^{il+i(i+1)/2+(j-i)(j-i-1)/2}\begin{bmatrix}\nu\\i\end{bmatrix}\begin{bmatrix}\nu+1-i\\j-i\end{bmatrix}\prod_{t=l-(j-i)+1}^{l}(q^t-1). \tag{9}$$

Now we compute the intersection numbers. By the transitivity of $Ps_{2\nu+2+l,\,2\nu+2}(\mathbb{F}_q)$ on $R_{(i,j-i)}$, we may choose two fixed maximal totally isotropic subspaces $P=(P',\,0^{(\nu,l)})$ and $Q=(Q',\,Q'')$, where

$$Q''=\begin{pmatrix} I^{(j-i)} & 0^{(j-i,l+i-j)} \\ 0 & 0 \end{pmatrix}.$$

Then $(P,Q)\in R_{(i,j-i)}$, and $p^{(i,j-i)}_{(s,t-s)\,(u,\nu-u)}$ is the number of subspaces

$$S=\begin{matrix} & 2\nu+2 & l & \\ (& S' & S'' &)\end{matrix}\in X$$

satisfying $(P,S)\in R_{(s,t-s)}$ and $(S,Q)\in R_{(u,\nu-u)}$. By [5, Chapter 7, Theorem 8] there are $p^i_{s,u}$ choices for S'. By the transitivity of $Ps_{2\nu+2+l,\,2\nu+2}(\mathbb{F}_q)$, the number of matrices S'' is independent of the choice of S'. Pick

$$S'=\begin{pmatrix} I & 0^{(\alpha,\gamma)} & 0 & 0 & \begin{pmatrix} B & 0\\ 0 & 0\end{pmatrix} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0^{(\beta,\rho)} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & I^{(\gamma)} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(\rho)} & 0 & 0 \end{pmatrix}, \tag{10}$$

where B is given by (4). Write

$$S''=\begin{pmatrix} A_1\\ A_2\\ A_3\\ A_4\\ A_5\\ A_6 \end{pmatrix}\begin{matrix} s-\rho-\gamma\\ \nu-\beta-s\\ \beta\\ 1\\ \gamma\\ \rho \end{matrix}$$

Since $\dim(P'\cap Q'\cap S')=\beta+1$, there are the following two cases to be considered.

Case 1: $j-i\leqslant\beta+1$. Then $(P,S)\in R_{(s,t-s)}$ and $(S,Q)\in R_{(u,\nu-u)}$ if and only if A_2, A_3, A_4 and A_5 satisfy

$$\operatorname{rank}\begin{pmatrix} A_2\\ A_3\\ A_4\end{pmatrix}=t-s \tag{11}$$

and

$$\operatorname{rank}\begin{pmatrix} \begin{pmatrix} A_3\\ A_4\end{pmatrix}-A_7\\ A_5\end{pmatrix}=\nu-u, \tag{12}$$

where

$$A_7=\begin{pmatrix} I^{(j-i)} & 0^{(j-i,l+i-j)}\\ 0^{(\beta+1+i-j,j-i)} & 0\end{pmatrix}.$$

Since $\dim(P'\cap Q'\cap S')=\beta+1$ and A_2 is a $(\nu-\beta-s)\times l$ matrix, we have

$$\max\{t-s-(\nu-\beta-s),0\}\leqslant\theta=\operatorname{rank}\begin{pmatrix} A_3\\ A_4\end{pmatrix}\leqslant\min\{t-s,\beta+1\}.$$

Similarly, since A_5 is a $\gamma\times l$ matrix, $\max\{\nu-u-\gamma,0\}\leqslant\xi\leqslant\min\{\nu-u,\beta+1\}$, where

$$\xi=\operatorname{rank}\left(\begin{pmatrix} A_3\\ A_4\end{pmatrix}-A_7\right). \tag{13}$$

By [3, Proposition 2.4] there are $N^{j-i}_{\theta,\xi}((\beta+1)\times l)$ choices of $\binom{A_3}{A_4}$ with rank θ satisfying (13). Once the $\binom{A_3}{A_4}$ is chosen, by [7, Lemma 2.4] there are $q^{(\nu-s-\beta)\theta}N(t-s-\theta;\,(\nu-s-\beta)\times(l-\theta))$ choices of

A_2 satisfying (11), and there are $q^{\gamma\xi}N(\nu-u-\xi;\gamma\times(l-\xi))$ choices of A_5 satisfying (12). Therefore there are

$$\alpha_{j-i}=\sum_{\theta=\max\{t-s-(\nu-\beta-s),0\}}^{\min\{t-s,\beta+1\}}\sum_{\xi=\max\{\nu-u-\gamma,0\}}^{\min\{\nu-u,\beta+1\}}q^{(\nu-s-\beta)\theta+\gamma\xi}N_{\theta,\xi}^{j-i}((\beta+1)\times l)\,\Omega$$

choices for A_2, A_3, A_4, A_5, where $\Omega=N(t-s-\theta;(\nu-s-\beta)\times(l-\theta))\,N(\nu-u-\xi;\gamma\times(l-\xi))$. Since A_1 and A_6 may be any $(s-\rho-\gamma)\times l$ and $\rho\times l$ matrices over $\mathbb{F}_q$, respectively, there are $q^{(s-\gamma)l}$ choices for A_1, A_6. It follows that there are $\alpha_{j-i}q^{(s-\gamma)l}$ choices of S'' for a fixed S'; and so

$$p_{(s,t-s)\,(u,\nu-u)}^{(i,j-i)}=\sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i\\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}}\sum_{\theta=\max\{t-s-(\nu-\beta-s),0\}}^{\min\{t-s,\beta+1\}}\sum_{\xi=\max\{\nu-u-\gamma,0\}}^{\min\{\nu-u,\beta+1\}}q^{\Lambda}p(\alpha)N_{\theta,\xi}^{j-i}((\beta+1)\times l)\,\Omega,\quad(14)$$

where $\Lambda=(s-\gamma)l+(\nu-s-\beta)\theta+\gamma\xi$.

Case 2: $j-i>\beta+1$. Similar to Case 1, there are $\alpha_{\beta+1}q^{(s-\gamma)l}$ choices of S'' for a fixed S'. Hence

$$p_{(s,t-s)\,(u,\nu-u)}^{(i,j-i)}=\sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i\\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}}\sum_{\theta=\max\{t-s-(\nu-\beta-s),0\}}^{\min\{t-s,\beta+1\}}\sum_{\xi=\max\{\nu-u-\gamma,0\}}^{\min\{\nu-u,\beta+1\}}q^{\Lambda}p(\alpha)N_{\theta,\xi}^{\beta+1}((\beta+1)\times l)\,\Omega.\qquad\square\quad(15)$$

Next we consider the case $\delta=2$ and $\varepsilon=0$.

Theorem 3.2. *Let $X=\mathcal{M}(\nu,0,0,0,0;2\nu+2+l,2\nu+2)$. For any two elements of X*

$$P=\begin{pmatrix}\overset{2\nu}{P_1'} & \overset{1}{z^t} & \overset{1}{0} & \overset{l}{P''}\end{pmatrix},\quad Q=\begin{pmatrix}\overset{2\nu}{Q_1'} & \overset{1}{w^t} & \overset{1}{0} & \overset{l}{Q''}\end{pmatrix},$$

define $(P,Q)\in R_{(i,a,j-i-a)}$ if and only if

$$\dim(P_1'\cap Q_1')=\dim((P_1'\ \ z^t)\cap(Q_1'\ \ w^t))+a=\nu-i,\quad \dim(P\cap Q)=\nu-j.$$

Then the configuration $\mathcal{X}=(X,\{R_{(i,a,j-i-a)}\}_{0\leqslant i\leqslant\nu-a,\,0\leqslant a\leqslant1,\,0\leqslant j-i-a\leqslant\min\{\nu-i-a,l\}})$ forms a symmetric association scheme with parameters d, v and $n_{(i,a,j-i-a)}$ given by (16), (17) and (18); intersection numbers $p_{(s,b,t-s-b)\,(u,c,\nu-u-c)}^{(i,a,j-i-a)}$'s given by (22), (23), (28), (29), (34), (35).

Proof. Similar to [3, Lemma 2.1], each $R_{(i,a,j-i-a)}$ is an orbit of $Ps_{2\nu+2+l,2\nu+2}(\mathbb{F}_q)$ on $X\times X$, and $\mathcal{X}$ is a symmetric association scheme. Now we compute its parameters.

By the definition of $R_{(i,a,j-i-a)}$ we have that $0\leqslant i\leqslant\nu-a,\ a=0$ or $1, 0\leqslant\nu-i-a-(\nu-j)\leqslant\nu-i-a$ and $0\leqslant\nu+j-(\nu+i+a)\leqslant l$. It follows that this scheme has

$$d=\min\{\nu,l\}+\sum_{i=1}^{\nu}2(1+\min\{\nu-i,l\})\tag{16}$$

classes. By [6, Theorem 4.20],

$$v=q^{\nu(l+1)}\prod_{i=1}^{\nu}(q^i+1).\tag{17}$$

Pick

$$P' = \begin{pmatrix} I^{(\nu)} & 0^{(\nu,\nu+2)} \end{pmatrix},$$

$$Q' = \begin{pmatrix} 0 & I^{(a)} & 0 & 0 & 0^{(a)} & 0 & x & 0 \\ 0^{(\nu-i-a,i)} & 0 & I & 0 & 0 & 0^{(\nu-i-a)} & 0 & 0 \\ 0 & 0 & 0 & I^{(i)} & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $x = 1$ or empty according to $a = 1$ or 0, respectively. For $P = (P', 0^{(\nu,l)})$, $n_{(i,a,j-i-a)}$ is the number of subspaces

$$U = \begin{pmatrix} \overset{2\nu+2}{U'} & \overset{l}{U''} \end{pmatrix} \in X$$

satisfying $(P, U) \in R_{(i,a,j-i-a)}$. By Theorem 2.1, there are $n_{(i,a)}$ (see (2)) choices for U'. By the transitivity of $Ps_{2\nu+2+l,\,2\nu+2}(\mathbb{F}_q)$, we may take $U' = Q'$. Then U has the unique matrix representation of the form

$$\begin{matrix} i & a & \nu-i-a & i & a & \nu-i-a & 1 & 1 & l & \\ \end{matrix}$$
$$\begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & x & 0 & A_1 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & A_3 \end{pmatrix} \begin{matrix} a \\ \nu-i-a, \\ i \end{matrix}$$

where rank $A_2 = j - i - a$. By [5, Chapter 1, Theorem 5], there are

$$N(j-i-a; (\nu-i-a) \times l) = q^{(j-i-a)(j-i-a-1)/2} \begin{bmatrix} \nu-i-a \\ j-i-a \end{bmatrix} \prod_{t=l-(j-i-a)+1}^{l} (q^t - 1)$$

choices for A_2, which implies that

$$n_{(i,a,j-i-a)} = n_{(i,a)} q^{(i+a)l+(j-i-a)(j-i-a-1)/2} \begin{bmatrix} \nu-i-a \\ j-i-a \end{bmatrix} \prod_{t=l-(j-i-a)+1}^{l} (q^t - 1), \tag{18}$$

where $n_{(i,a)}$ is given by (2).

Now we compute the intersection numbers. By the transitivity of $Ps_{2\nu+2+l,\,2\nu+2}(\mathbb{F}_q)$ on $R_{(i,a,j-i-a)}$, we may choose two fixed maximal totally isotropic subspaces $P = (P', 0^{(\nu,l)})$ and $Q = (Q', Q'')$, where

$$Q'' = \begin{pmatrix} 0^{(a)} & 0 & 0 \\ 0 & I^{(j-i-a)} & 0^{(j-i-a,l+i-j)} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $(P, Q) \in R_{(i,a,j-i-a)}$ and $p^{(i,a,j-i-a)}_{(s,b,t-s-b)\,(u,c,\nu-u-c)}$ is the number of subspaces

$$S = \begin{pmatrix} \overset{2\nu+2}{S'} & \overset{l}{S''} \end{pmatrix} \in X$$

satisfying $(P, S) \in R_{(s,b,t-s-b)}$ and $(S, Q) \in R_{(u,c,\nu-u-c)}$. By Theorem 2.1 there are $p^{(i,a)}_{(s,b)\,(u,c)}$ choices for S'. By the transitivity of $Ps_{2\nu+2+l,\,2\nu+2}(\mathbb{F}_q)$, the number of matrices S'' is independent of the choice of S'. Then there are the following three cases to be considered:

Case 1: $b = 0$ and $c = a$. Without loss of generality, we may pick S' as (10). Write

$$S'' = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{pmatrix} \begin{matrix} s-\rho-\gamma \\ \nu-\beta-s \\ a \\ \beta-a \\ \gamma \\ \rho \end{matrix}.$$

Since $\dim(P' \cap Q' \cap S') = \beta - a$, there are the following two cases to be considered:

Case 1.1: $j - i - a \leqslant \beta - a$. Then $(P, S) \in R_{(s,0,t-s)}$ and $(S, Q) \in R_{(u,a,v-u-a)}$ if and only if A_2, A_3, A_4 and A_5 satisfy

$$\operatorname{rank}\begin{pmatrix} A_2 \\ A_3 \\ A_4 \end{pmatrix} = t - s \tag{19}$$

and

$$\operatorname{rank}\begin{pmatrix} A_4 - A_7 \\ A_5 \end{pmatrix} = v - u - a, \tag{20}$$

where

$$A_7 = \begin{pmatrix} I^{(j-i-a)} & 0^{(j-i-a,l+i+a-j)} \\ 0^{(\beta+i-j,j-i-a)} & 0 \end{pmatrix}.$$

Since $\dim(P' \cap Q' \cap S') = \beta - a$, A_2 and A_3 are $(v - \beta - s) \times l$ and $a \times l$ matrices, respectively, we have

$$\max\{t - s - (v - \beta - s + a), 0\} \leqslant \theta = \operatorname{rank} A_4 \leqslant \min\{t - s, \beta - a\}.$$

Similarly, since A_5 is a $\gamma \times l$ matrix, $\max\{v - u - a - \gamma, 0\} \leqslant \xi \leqslant \min\{v - u - a, \beta - a\}$, where

$$\xi = \operatorname{rank}\ (A_4 - A_7). \tag{21}$$

By [3, Proposition 2.4] there are $N_{\theta,\xi}^{j-i-a}((\beta - a) \times l)$ choices of A_4 with rank θ satisfying (21). Once the A_4 is chosen, by [7, Lemma 2.4] there are $q^{(v-s-\beta+a)\theta} N(t - s - \theta;\ (v - s - \beta + a) \times (l - \theta))$ choices of A_2, A_3 satisfying (19), and there are $q^{\gamma\xi} N(v - u - a - \xi;\ \gamma \times (l - \xi))$ choices of A_5 satisfying (20). Therefore there are

$$\tilde{\alpha}_{j-i-a} = \sum_{\theta=\max\{t-s-(v-\beta-s+a),\,0\}}^{\min\{t-s,\beta-a\}} \sum_{\xi=\max\{v-u-a-\gamma,\,0\}}^{\min\{v-u-a,\beta-a\}} q^{(v-s-\beta+a)\theta+\gamma\xi} N_{\theta,\xi}^{j-i-a}((\beta - a) \times l)\, \tilde{\Omega}$$

choices for A_2, A_3, A_4, A_5, where $\tilde{\Omega} = N(t - s - \theta;\ (v - s - \beta + a) \times (l - \theta))\, N(v - u - a - \xi;\ \gamma \times (l - \xi))$. Since A_1 and A_6 may be any $(s - \rho - \gamma) \times l$ and $\rho \times l$ matrices over $\mathbb{F}_q$, respectively, there are $q^{(s-\gamma)l}$ choices for A_1, A_6. It follows that there are $\tilde{\alpha}_{j-i-a} q^{(s-\gamma)l}$ choices of S'' for a fixed S'; and so

$$p_{(s,0,t-s)\ (u,a,v-u-a)}^{(i,a,j-i-a)} = \sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=v-i \\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} \sum_{\theta=\max\{t-s-(v-\beta-s+a),\,0\}}^{\min\{t-s,\beta-a\}} \sum_{\xi=\max\{v-u-a-\gamma,\,0\}}^{\min\{v-u-a,\beta-a\}} p(\alpha; a, 0, a) q^{\tilde{\Lambda}} N_{\theta,\xi}^{j-i-a}((\beta - a) \times l)\, \tilde{\Omega}, \tag{22}$$

where $\tilde{\Lambda} = (v - s - \beta + a)\theta + \gamma\xi + (s - \gamma)l$.

Case 1.2: $j - i - a > \beta - a$. Similar to Case 1.1, there are $\tilde{\alpha}_{\beta-a} q^{(s-\gamma)l}$ choices of S'' for a fixed S'. Hence

$$p_{(s,0,t-s)\ (u,a,v-u-a)}^{(i,a,j-i-a)} = \sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=v-i \\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} \sum_{\theta=\max\{t-s-(v-\beta-s+a),\,0\}}^{\min\{t-s,\beta-a\}} \sum_{\xi=\max\{v-u-a-\gamma,\,0\}}^{\min\{v-u-a,\beta-a\}} p(\alpha; a, 0, a) q^{\tilde{\Lambda}} N_{\theta,\xi}^{\beta-a}((\beta - a) \times l)\, \tilde{\Omega}. \tag{23}$$

Case 2: $a = b = 0$ and $c = 1$. Without loss of generality, we may pick

$$S' = \begin{pmatrix} I & 0^{(\alpha,\gamma)} & 0 & 0 & \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0^{(\beta,\rho)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(\gamma)} & 0 & 0 & \tilde{e}_1^t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(\rho)} & 0 & 0 \end{pmatrix}, \tag{24}$$

where B is given by (4), and $\tilde{e}_1 = (1, 0, \dots, 0)$. Write

$$S'' = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{pmatrix} \begin{matrix} s-\rho-\gamma \\ \nu-\beta-s \\ \beta \\ 1 \\ \gamma-1 \\ \rho \end{matrix}$$

Since $\dim(P' \cap Q' \cap S') = \beta$, there are the following two cases to be considered:

Case 2.1: $j - i \leqslant \beta$. Then $(P, S) \in R_{(s,0,t-s)}$ and $(S, Q) \in R_{(u,1,\nu-u-1)}$ if and only if A_2, A_3 and A_5 satisfy

$$\operatorname{rank} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = t - s \tag{25}$$

and

$$\operatorname{rank} \begin{pmatrix} A_3 - A_7 \\ A_5 \end{pmatrix} = \nu - u - 1, \tag{26}$$

where

$$A_7 = \begin{pmatrix} I^{(j-i)} & 0^{(j-i,l+i-j)} \\ 0^{(\beta+i-j,j-i)} & 0 \end{pmatrix}.$$

Since $\dim(P' \cap Q' \cap S') = \beta, A_2$ is a $(\nu - \beta - s) \times l$ matrix, we have

$$\max\{t - s - (\nu - \beta - s), 0\} \leqslant \theta = \operatorname{rank} A_3 \leqslant \min\{t - s, \beta\}.$$

Similarly, since A_5 is a $(\gamma - 1) \times l$ matrix, $\max\{\nu - u - \gamma, 0\} \leqslant \xi \leqslant \min\{\nu - u - 1, \beta\}$, where

$$\xi = \operatorname{rank}(A_3 - A_7). \tag{27}$$

By [3, Proposition 2.4], there are $N_{\theta,\xi}^{j-i}(\beta \times l)$ choices of A_3 with rank θ satisfying (27). Once the A_3 is chosen, by [7, Lemma 2.4], there are $q^{(\nu-s-\beta)\theta} N(t - s - \theta; (\nu - s - \beta) \times (l - \theta))$ choices of A_2 satisfying (25), and there are $q^{(\gamma-1)\xi} N(\nu - u - 1 - \xi; (\gamma - 1) \times (l - \xi))$ choices of A_5 satisfying (26). Therefore there are

$$\bar{\alpha}_{j-i} = \sum_{\theta=\max\{t-s-(\nu-\beta-s),\,0\}}^{\min\{t-s,\beta\}} \sum_{\xi=\max\{\nu-u-\gamma,\,0\}}^{\min\{\nu-u-1,\beta\}} q^{(\nu-s-\beta)\theta+(\gamma-1)\xi} N_{\theta,\xi}^{j-i}(\beta \times l)\, \bar{\Omega}$$

choices for A_2, A_3, A_5, where $\bar{\Omega} = N(t - s - \theta; (\nu - s - \beta) \times (l - \theta))\, N(\nu - u - 1 - \xi; (\gamma - 1) \times (l - \xi))$. Since $A_1,\ A_4$ and A_6 may be any $(s - \rho - \gamma) \times l,\ 1 \times l$ and $\rho \times l$ matrices over $\mathbb{F}_q$, respectively, there are $q^{(s-\gamma+1)l}$ choices for A_1, A_4, A_6. It follows that there are $\bar{\alpha}_{j-i} q^{(s-\gamma+1)l}$ choices of S'' for a fixed S'; and so

$$p_{(s,0,t-s)\,(u,1,\nu-u-1)}^{(i,0,j-i)} = \sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i \\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} \sum_{\theta=\max\{t-s-(\nu-\beta-s),\,0\}}^{\min\{t-s,\beta\}} \sum_{\xi=\max\{\nu-u-\gamma,\,0\}}^{\min\{\nu-u-1,\beta\}} p(\alpha; 0, 0, 1) q^{\bar{\Lambda}} N_{\theta,\xi}^{j-i}(\beta \times l)\, \bar{\Omega}, \tag{28}$$

where $\bar{\Lambda} = (\nu - s - \beta)\theta + (\gamma - 1)\xi + (s - \gamma + 1)l$.

Case 2.2: $j-i>\beta$. Similar to Case 2.1, there are $\bar{\alpha}_\beta q^{(s-\gamma+1)l}$ choices of S'' for a fixed S'. Hence

$$p^{(i,0,j-i)}_{(s,0,t-s)\,(u,1,\nu-u-1)}=\sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i\\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}}\ \sum_{\theta=\max\{t-s-(\nu-\beta-s),\,0\}}^{\min\{t-s,\beta\}}\ \sum_{\xi=\max\{\nu-u-\gamma,\,0\}}^{\min\{\nu-u-1,\beta\}} p(\alpha;0,0,1)q^{\bar{\Lambda}}N^{\beta}_{\theta,\xi}(\beta\times l)\,\bar{\Omega}. \tag{29}$$

Case 3: $a=b=c=1$. Without loss of generality, we may pick

$$S'=\begin{pmatrix} I & 0^{(\alpha,\gamma)} & 0 & 0 & \begin{pmatrix}B&0\\0&0\end{pmatrix} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0^{(\beta,\rho)} & 0 & 0 & 0 & 0 & h\bar{e}_1^t & 0\\ 0&0&0&0&0&I^{(\gamma)}&0&0&0&0\\ 0&0&0&0&0&0&0&I^{(\rho)}&0&0\end{pmatrix}, \tag{30}$$

where B is given by (4), $\bar{e}_1=(1,0,\ldots,0)$ and $1\neq h\in\mathbb{F}_q\backslash\{0\}$. Write

$$S''=\begin{pmatrix}A_1\\A_2\\A_3\\A_4\\A_5\\A_6\end{pmatrix}\begin{matrix}s-\rho-\gamma\\ \nu-\beta-s\\ 1\\ \beta-1\\ \gamma\\ \rho\end{matrix}$$

Since $\dim(P'\cap Q'\cap S')=\beta-1$, there are the following two cases to be considered:

Case 3.1: $j-i-1\leqslant\beta-1$. Then $(P,S)\in R_{(s,1,t-s-1)}$ and $(S,Q)\in R_{(u,1,\nu-u-1)}$ if and only if A_2,A_4 and A_5 satisfy

$$\operatorname{rank}\begin{pmatrix}A_2\\A_4\end{pmatrix}=t-s-1 \tag{31}$$

and

$$\operatorname{rank}\begin{pmatrix}A_4-A_7\\A_5\end{pmatrix}=\nu-u-1, \tag{32}$$

where

$$A_7=\begin{pmatrix}I^{(j-i-1)} & 0^{(j-i-1,l+i+1-j)}\\ 0^{(\beta+i-j,j-i-1)} & 0\end{pmatrix}.$$

Since $\dim(P'\cap Q'\cap S')=\beta-1$ and A_2 is a $(\nu-\beta-s)\times l$ matrix, we have

$$\max\{t-s-1-(\nu-\beta-s),0\}\leqslant\theta=\operatorname{rank}A_4\leqslant\min\{t-s-1,\beta-1\}.$$

Similarly, since A_5 is a $\gamma\times l$ matrix, $\max\{\nu-u-\gamma-1,0\}\leqslant\xi\leqslant\min\{\nu-u-1,\beta-1\}$, where

$$\xi=\operatorname{rank}(A_4-A_7). \tag{33}$$

By [3, Proposition 2.4], there are $N^{j-i-1}_{\theta,\xi}((\beta-1)\times l)$ choices of A_4 with rank θ satisfying (33). Once the A_4 is chosen, by [7, Lemma 2.4], there are $q^{(\nu-s-\beta)\theta}N(t-s-1-\theta;\,(\nu-s-\beta)\times(l-\theta))$ choices of A_2 satisfying (31), and there are $q^{\gamma\xi}N(\nu-u-1-\xi;\,\gamma\times(l-\xi))$ choices of A_5 satisfying (32). Therefore there are

$$\acute{\alpha}_{j-i-1}=\sum_{\theta=\max\{t-s-1-(\nu-\beta-s),\,0\}}^{\min\{t-s-1,\beta-1\}}\ \sum_{\xi=\max\{\nu-u-1-\gamma,\,0\}}^{\min\{\nu-u-1,\beta-1\}} q^{(\nu-s-\beta)\theta+\gamma\xi}N^{j-i-1}_{\theta,\xi}((\beta-1)\times l)\,\acute{\Omega}$$

choices for A_2, A_4, A_5, where $\acute{\Omega} = N(t-s-1-\theta; (\nu-s-\beta)\times(l-\theta))\, N(\nu-u-1-\xi; \gamma\times(l-\xi))$. Since A_1, A_3 and A_6 may be any $(s-\rho-\gamma)\times l$, $1\times l$ and $\rho\times l$ matrices over $\mathbb{F}_q$, respectively, there are $q^{(s-\gamma+1)l}$ choices for A_1, A_3, A_6. It follows that there are $\acute{\alpha}_{j-i-1}q^{(s-\gamma+1)l}$ choices of S'' for a fixed S'; and so

$$p^{(i,1,j-i-1)}_{(s,1,t-s-1)\,(u,1,\nu-u-1)} = \sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i\\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} \sum_{\theta=\max\{t-s-1-(\nu-\beta-s),0\}}^{\min\{t-s-1,\beta-1\}} \sum_{\xi=\max\{\nu-u-1-\gamma,0\}}^{\min\{\nu-u-1,\beta-1\}} p(\alpha;1,1,1)q^{\acute{\Lambda}}N^{j-i-1}_{\theta,\xi}((\beta-1)\times l)\,\acute{\Omega}, \tag{34}$$

where $\acute{\Lambda} = (\nu-s-\beta)\theta + \gamma\xi + (s-\gamma+1)l$.

Case 3.2: $j-i-1 > \beta-1$. Similar to Case 3.1, there are $\acute{\alpha}_{\beta-1}q^{(s-\gamma+1)l}$ choices of S'' for a fixed S'. Hence

$$p^{(i,1,j-i-1)}_{(s,1,t-s-1)\,(u,1,\nu-u-1)} = \sum_{\substack{\alpha+\gamma=i,\,\beta+\rho=\nu-i\\ \alpha+\rho=u,\,\mu+\rho+\gamma=s}} \sum_{\theta=\max\{t-s-1-(\nu-\beta-s),0\}}^{\min\{t-s-1,\beta-1\}} \sum_{\xi=\max\{\nu-u-1-\gamma,0\}}^{\min\{\nu-u-1,\beta-1\}} p(\alpha;1,1,1)q^{\acute{\Lambda}}N^{\beta-1}_{\theta,\xi}((\beta-1)\times l)\,\acute{\Omega}. \tag{35}$$

By above discussion, the proof of the theorem is completed. □

By the basic equalities of parameters of an association scheme, all the parameters of the scheme may be derived.

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