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The Jacobi elliptic function method and its application for two component BKP hierarchy equations

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ABSTRACT

The periodic wave solutions for the two component BKP hierarchy are obtained by using of Jacobi elliptic function method, in the limit cases, the multiple soliton solutions are also obtained. The properties of some periodic and soliton solution for this system are shown by some figures.

1. Introduction

Many phenomena in physics and engineering are described by nonlinear partial differential equations (PDEs). When we want to understand the physical mechanism of phenomena in nature, described by nonlinear PDEs, exact solution for the nonlinear PDEs have been explored. Thus the methods for finding exact solutions for the governing equations have to be developed. To study exact solutions of nonlinear PDEs has become one of the most important topics in mathematical physics. For instances the nonlinear wave phenomena observed in fluid dynamics, plasma, and optical fiber are often modeled by the bell shaped sech solutions and the kink shaped tanh solutions. The availability of these exact solutions for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solutions.

Nonlinear differential equations have many wide array of application of many fields, which describe the motion of the isolated waves, localized in a small part of space, such as in physics, in which applications extend over magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas, biology, chemistry, and several other fields.

Looking for exact solitary wave solutions to nonlinear evolution equations has long been a major concern for both mathematicians and physicists. These solutions may describe various phenomena in physics and other fields, such as solitons and propagation with a finite speed, and thus they may give more insight into the physical aspects of the problems.

In order to obtain the periodic wave and soliton solutions of nonlinear evolution equations, a number of methods have been proposed, such as the homogenous balance method [1–9], the hyperbolic function expansion method [10,11], the sine-cosine method [12], the nonlinear transformation method [13–15] and the trial function method [16,17]. These methods, however, can only lead to the shock or solitary wave solutions, or the periodic wave solutions in terms of the elementary functions can not be used to derive the generalized periodic solutions.

In this paper we used the Jacobi elliptic functions to obtain the solitary wave solutions that were found by the previous methods.

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2. Summary of the method

This method can be summarized as follows: for a given system of nonlinear evolution equation, say, in three variables

$$\begin{aligned} &F(u, v, w, u_t, v_t, w_t, u_x, v_x, w_x, u_{xt}, v_{xt}, w_{xt}, \ldots) = 0, \\ &G(u, v, w, u_t, v_t, w_t, u_x, v_x, w_x, u_{xt}, v_{xt}, w_{xt}, \ldots) = 0, \\ &K(u, v, w, u_t, v_t, w_t, u_x, v_x, w_x, u_{xt}, v_{xt}, w_{xt}, \ldots) = 0. \end{aligned} \tag{1}$$

We seek the following wave traveling solutions:

$$u(x, t) = u(\zeta), \qquad v(x, t) = v(\zeta), \qquad w(x, t) = w(\zeta), \quad \zeta = kx + \lambda y + \nu t + d, \tag{2}$$

which are important physical significance, and k and λ are constants to be determined. Then system (1) reduce to a system of nonlinear ordinary equations.

$$\begin{aligned} &F_0(u, v, w, u_\zeta, v_\zeta, w_\zeta, u_{\zeta\zeta}, v_{\zeta\zeta}, w_{\zeta\zeta}, \ldots) = 0, \\ &G_0(u, v, w, u_\zeta, v_\zeta, w_\zeta, u_{\zeta\zeta}, v_{\zeta\zeta}, w_{\zeta\zeta}, \ldots) = 0, \\ &K_0(u, v, w, u_\zeta, v_\zeta, w_\zeta, u_{\zeta\zeta}, v_{\zeta\zeta}, w_{\zeta\zeta}, \ldots) = 0. \end{aligned} \tag{3}$$

Taking the following transformation

$$\begin{aligned} u(\zeta) &= \sum_{i=0}^{n} a_i f^i(\zeta), \\ v(\zeta) &= \sum_{i=0}^{n} b_i f^i(\zeta), \\ w(\zeta) &= \sum_{i=0}^{n} c_i f^i(\zeta) \end{aligned} \tag{4}$$

in which a_i $(i = 0, 1, 2, \ldots, n)$, b_i $(i = 0, 1, 2, \ldots, n)$ and c_i $(i = 0, 1, 2, \ldots, n)$ are real constants to be determined. The balancing number n is a positive integer which can be determined by balancing the highest order derivative terms with the highest power nonlinear terms in Eq. (3) and $f(\zeta)$ expresses the solutions of the following new anzata [18]

$$f'(\zeta) = \sqrt{r + af^2(\zeta) + \frac{b}{2}f^4(\zeta) + \frac{c}{3}f^6(\zeta)}. \tag{5}$$

Where r, a, b and c are real parameters and the prime means the derivative with respect to ζ.

We substitute anzatz Eqs. (5) and (4) into Eq. (3) and with computerized symbolic computation, we obtain a set of algebraic equations for r, a, b, , c, k, λ, a_i and b_i.

Inserting each solutions of this set of algebraic equations into (4) and the solutions of Eq. (5) and setting $\zeta = kx \pm \lambda t$, then we obtain the exact traveling wave solutions of Eq. (1).

In Eq. (4), if we assume $f = \tanh\zeta$, this is called the tanh-function method [19], $f = \operatorname{sech}\zeta$, this is called sech-function method [20], $f = sn\zeta, cn\zeta, cs\zeta$, this is called Jacobi elliptic function method [21], we choose Eq. (5) because the solitary wave $f = \operatorname{sech}\zeta$, the shock wave $f = \tanh\zeta$ and the periodic waves in terms of Jacobi elliptic functions $f = sn\zeta, cn\zeta, cs\zeta$ etc. are all the solutions of it for appropriate values of a, b, c and r.

The Jacobi elliptic functions $sn\zeta = sn(\zeta \mid m)$, $cn\zeta = cn(\zeta \mid m)$ and $dn\zeta = dn(\zeta \mid m)$, m $(0 < m < 1)$ is the modulus of the elliptic function, are double periodic posses properties of triangular functions, namely

$$\begin{aligned} &sn^2(\zeta) + cn^2(\zeta) = 1, \qquad dn^2(\zeta) + m^2 sn^2(\zeta) = 1, \qquad (cn(\zeta))' = -sn(\zeta)dn(\zeta), \\ &(sn(\zeta))' = cn(\zeta)dn(\zeta), \qquad (dn(\zeta))' = -m^2 sn(\zeta)cn(\zeta). \end{aligned} \tag{6}$$

When $m \longrightarrow 0$, the Jacobi elliptic function degenerate to the triangular functions,

$$\begin{aligned} &sn(\zeta) \to \sin(\zeta), \qquad cn(\zeta) \to \cos(\zeta), \qquad dn(\zeta) \to 1, \\ &cs(\zeta) \to \cot(\zeta), \qquad ds(\zeta) \to csc(\zeta). \end{aligned} \tag{7}$$

When $m \longrightarrow 1$, the Jacobi elliptic function degenerate to the hyperbolic functions,

$$\begin{aligned} &sn(\zeta) \to \tanh(\zeta), \qquad cn(\zeta) \to \operatorname{sech}(\zeta), \qquad dn(\zeta) \to \operatorname{sech}(\zeta), \\ &cs(\zeta) \to csch(\zeta), \qquad ds(\zeta) \to csch(\zeta). \end{aligned} \tag{8}$$

3. The periodic wave and solitary wave solutions of the two component BKP hierarchy

We consider a system of the two component BKP hierarchy [22]

$$u_t - u_{xxx} - u_{yyy} - 6(u_x v + u v_x + u_y w + u w_y) = 0, \qquad v_y = u_x, \qquad w_x = u_y. \tag{9}$$

which $u = u(x, y, t)$, $v = v(x, y, t)$ and $w = w(x, y, t)$ are real and functions.

A good understanding of the traveling wave solutions of Eq. (9) which describe water waves, very helpful for coastal and civil engineers to apply the nonlinear water model in harbor and coastal design. Therefore, the present work is motivated by the desire to find periodic wave solutions with the use of the Jacobi elliptic function. This means that the method will led to a deeper and more comprehensive understanding of the structure of the nonlinear PDEs. On the other hand, the periodic solutions of nonlinear PDEs are useful for physicists in studying more complicated physical phenomena.

Long wave in shallow water is a subject of broad interests and has a long and colorful history. Physically, it has a rich variety of phenomenological manifestation, especially the existence of wave permanent in form and robust in maintaining their entities through mutual interaction and collision, as well as the remarkable property of exhibiting recurrences of initial data when circumstances should prevail. These characteristics are due to the intimate interplay between the roles of nonlinearity and dispersion.

Seeking for the traveling wave solutions of Eq. (9), we let

$$\begin{aligned} &u(x, y, t) = u(\zeta), \qquad v(x, y, t) = v(\zeta), \qquad w(x, y, t) = w(\zeta) \\ &\zeta = kx + \lambda y + \nu t + d, \end{aligned} \tag{10}$$

where k, l, m and d are constants.

Substituting (10) into (9), then (9) is reduced to the following nonlinear ordinary differential equation

$$\nu u' - (k^3 + \lambda^3) u''' - 6(k(uv)' + \lambda(uw)') = 0, \qquad \lambda v' - k u' = 0, \qquad k w' - \lambda u' = 0. \tag{11}$$

Balancing the highest order derivative terms with nonlinear terms u''' with uv' gives leading order $n = 4$, so, Eq. (4) can be simplified as follows

$$u(\zeta) = \sum_{i=0}^{4} a_i f^i(\zeta), \tag{12}$$

$$v(\zeta) = \sum_{i=0}^{4} b_i f^i(\zeta), \tag{13}$$

$$w(\zeta) = \sum_{i=0}^{4} c_i f^i(\zeta). \tag{14}$$

After the substitution of Eq. (12) with (5) into (11) and setting coefficients of $f^i(\zeta)$, $f^i\sqrt{r + af^2 + \frac{b}{2}f^4 + \frac{c}{3}f^6}$ to zero, we can deduce the following set of equations with respect to unknowns a_0, a_1, a_2, $a3$, $a4$, b_0, b_1, b_2, b_3, b_4, c_0, c_1, c_2, c_3, c_4, k, λ, ν, a, b, c, r

$$\begin{aligned}
&-6\,k^3a_3r^2 - k^3a_1ar + \nu\,a_1r - 6\,\lambda^3a_3r^2 - 6\,\lambda\,a_1c_0r - 6\,\lambda\,a_0c_1r - \lambda^3a_1ar - 6\,ka_0b_1r - 6\,ka_1b_0r = 0,\\
&-3\,k^3a_1br - 33\,k^3a_3ar - 6\,ka_1b_0a - 18\,\lambda\,a_2c_1r - 6\,ka_0b_1a - 33\,\lambda^3a_3ar\\
&\quad + \nu\,a_1a - 18\,ka_0b_3r - 3\,\lambda^3a_1br - \lambda^3a_1a^2 - 18\,\lambda\,a_1c_2r - 6\,\lambda\,a_0c_1a - k^3a_1a^2\\
&\quad - 18\,ka_3b_0r - 18\,ka_2b_1r - 18\,\lambda\,a_0c_3r - 18\,\lambda\,a_3c_0r + 3\,\nu\,a_3r - 6\,\lambda\,a_1c_0a - 18\,ka_1b_2r = 0,\\
&-12\,\lambda\,a_1c_1r - 8\,k^3a_2ar - 12\,ka_0b_2r - 8\,\lambda^3a_2ar - 12\,ka_2b_0r - 12\,ka_1b_1r - 24\,k^3a_4r^2\\
&\quad - 24\,\lambda^3a_4r^2 - 12\,\lambda\,a_2c_0r + 2\,\nu\,a_2r - 12\,\lambda\,a_0c_2r = 0,\\
&-16\,ka_4b_4c - \frac{64}{3}\,k^3a_4c^2 - 16\,\lambda\,a_4c_4c - \frac{64}{3}\,\lambda^3a_4c^2 = 0,\\
&-8\,\lambda\,a_2c_2c - 8\,\lambda\,a_0c_4c - 8\,ka_3b_1c - 48\,\lambda\,a_4c_4a - 18\,\lambda\,a_4c_2b - 18\,\lambda\,a_3c_3b\\
&\quad - 8\,\lambda\,a_4c_0c - 8\,\lambda\,a_3c_1c - 18\,\lambda\,a_2c_4b + \frac{4}{3}\,\nu\,a_4c - 30\,k^3a_4b^2 - 12\,\lambda^3a_2bc\\
&\quad - 30\,\lambda^3a_4b^2 - 12\,k^3a_2bc - \frac{256}{3}\,k^3a_4ac - \frac{256}{3}\,\lambda^3a_4ac - 18\,ka_3b_3b - 8\,ka_1b_3c\\
&\quad - 8\,\lambda\,a_1c_3c - 8\,ka_0b_4c - 8\,ka_2b_2c - 18\,ka_2b_4b - 8\,ka_4b_0c - 48\,ka_4b_4a - 18\,ka_4b_2b = 0,\\
&-\frac{35}{3}\,\lambda^3a_3c^2 - 14\,\lambda\,a_4c_3c - \frac{35}{3}\,k^3a_3c^2 - 14\,\lambda\,a_3c_4c - 14\,ka_4b_3c - 14\,ka_3b_4c = 0,\\
&\nu\,a_2b - 24\,\lambda\,a_2c_2a - 6\,ka_2b_0b - 24\,ka_3b_1a - 24\,\lambda\,a_0c_4a - 6\,\lambda\,a_0c_2b - 36\,\lambda\,a_4c_2r
\end{aligned}$$

$$
\begin{aligned}
& - 36\,\lambda\, a_3c_3r - 24\,\lambda\, a_4c_0a - 24\,\lambda\, a_3c_1a - 36\,\lambda\, a_2c_4r + 4\,\nu\, a_4a - 64\,k^3a_4a^2 - 64\,\lambda^3a_4a^2 \\
& - 16\,\lambda^3a_2ab - 16\,k^3a_2ab - 6\,ka_1b_1b - 72\,k^3a_4br - 16\,k^3a_2cr - 36\,ka_3b_3r - 72\,\lambda^3a_4br \\
& - 16\,\lambda^3a_2cr - 6\,\lambda\, a_2c_0b - 24\,ka_1b_3a - 24\,\lambda\, a_1c_3a - 6\,\lambda\, a_1c_1b - 24\,ka_0b_4a - 24\,ka_2b_2a \\
& - 24\,ka_4b_0a - 6\,ka_0b_2b - 36\,ka_4b_2r - 36\,ka_2b_4r = 0, \\
& -15\,\lambda\, a_2c_3b + \nu\, a_3c - 6\,\lambda\, a_2c_1c - 15\,\lambda\, a_1c_4b - 6\,\lambda\, a_0c_3c - 42\,\lambda\, a_4c_3a - 42\,\lambda\, a_3c_4a \\
& - 15\,\lambda\, a_4c_1b - 6\,\lambda\, a_3c_0c - 15\,\lambda\, a_3c_2b - 6\,ka_1b_2c - 15\,k^3a_3b^2 - 15\,\lambda^3a_3b^2 - \frac{7}{2}\,k^3a_1bc \\
& - 6\,ka_3b_0c - 44\,k^3a_3ac - \frac{7}{2}\,\lambda^3a_1bc - 42\,ka_3b_4a - 44\,\lambda^3a_3ac - 15\,ka_1b_4b - 15\,ka_3b_2b \\
& - 6\,\lambda\, a_1c_2c - 6\,ka_0b_3c - 15\,ka_2b_3b - 6\,ka_2b_1c - 15\,ka_4b_1b - 42\,ka_4b_3a = 0, \\
& -24\,\lambda\, a_4c_4b - 52\,\lambda^3a_4bc - 12\,ka_2b_4c - 24\,ka_4b_4b - \frac{16}{3}\,k^3a_2c^2 - 52\,k^3a_4bc - \frac{16}{3}\,\lambda^3a_2c^2 \\
& - 12\,\lambda\, a_2c_4c - 12\,\lambda\, a_4c_2c - 12\,ka_4b_2c - 12\,ka_3b_3c - 12\,\lambda\, a_3c_3c = 0, \\
& -21\,\lambda\, a_4c_3b - 10\,ka_3b_2c - 10\,\lambda\, a_1c_4c - 21\,ka_4b_3b - 21\,\lambda\, a_3c_4b - 10\,\lambda\, a_4c_1c - 10\,ka_4b_1c \\
& - \frac{55}{2}\,k^3a_3bc - 21\,ka_3b_4b - \frac{5}{3}\,k^3a_1c^2 - \frac{55}{2}\,\lambda^3a_3bc - 10\,ka_1b_4c - 10\,ka_2b_3c - 10\,\lambda\, a_2c_3c \\
& - \frac{5}{3}\,\lambda^3a_1c^2 - 10\,\lambda\, a_3c_2c = 0, \\
& -88\,\lambda^3a_4ar - 12\,ka_2b_0a - 12\,\lambda\, a_1c_1a - 12\,ka_0b_2a - 24\,ka_4b_0r - 12\,\lambda\, a_2c_0a - 24\,\lambda\, a_4c_0r \\
& - 24\,ka_0b_4r - 24\,\lambda\, a_3c_1r - 24\,\lambda\, a_2c_2r - 12\,k^3a_2br - 8\,k^3a_2a^2 - 8\,\lambda^3a_2a^2 - 88\,k^3a_4ar \\
& - 12\,ka_1b_1a - 24\,\lambda\, a_0c_4r + 4\,\nu\, a_4r - 12\,\lambda^3a_2br - 24\,ka_2b_2r - 24\,ka_3b_1r + 2\,\nu\, a_2a \\
& - 24\,\lambda\, a_1c_3r - 12\,\lambda\, a_0c_2a - 24\,ka_1b_3r = 0, \\
& -30\,\lambda\, a_2c_3a + \frac{3}{2}\,\nu\, a_3b - 30\,\lambda\, a_1c_4a - 9\,\lambda\, a_2c_1b - 9\,\lambda\, a_0c_3b - 42\,\lambda\, a_4c_3r - 42\,\lambda\, a_3c_4r \\
& - 30\,\lambda\, a_4c_1a - 9\,\lambda\, a_3c_0b - 30\,\lambda\, a_3c_2a - 9\,ka_1b_2b + \frac{1}{3}\,\nu\, a_1c - 3/2\,k^3a_1b^2 - \frac{87}{2}\,k^3a_3ab \\
& - 37\,k^3a_3cr - 16/3\,\lambda^3a_1ac - \frac{87}{2}\,\lambda^3a_3ab - 37\,\lambda^3a_3cr - \frac{16}{3}\,k^3a_1ac - 30\,ka_1b_4a \\
& - 30\,ka_3b_2a - 2\,ka_0b_1c - 2\,ka_1b_0c - 9\,\lambda\,{}_1c_2b - 9\,ka_0b_3b - 2\,\lambda\, a_0c_1c - 2\,\lambda\, a_1c_0c \\
& - 30\,ka_2b_3a - 9\,ka_2b_1b - 42\,ka_3b_4r - 30\,ka_4b_1a - 3/2\,\lambda^3a_1b^2 - 42\,ka_4b_3r - 9\,ka_3b_0b = 0, \\
& \frac{2}{3}\,\nu\, a_2c - 12\,\lambda\, a_2c_2b - 4\,ka_2b_0c - 12\,\lambda\, a_0c_4b - 12\,ka_3b_1b - 48\,\lambda\, a_4c_4r - 4\,\lambda\, a_0c_2c \\
& - 36\,\lambda\, a_4c_2a - 36\,\lambda\, a_3c_3a - 12\,\lambda\, a_4c_0b - 12\,\lambda\, a_3c_1b - 36\,\lambda\, a_2c_4a + 2\,\nu\, a_4b - 6\,k^3a_2b^2 \\
& - 6\,\lambda^3a_2b^2 - \frac{56}{3}\,\lambda^3a_2ac - \frac{56}{3}\,k^3a_2ac - 4\,ka_1b_1c - 72\,k^3a_4cr - 92\,k^3a_4ab - 92\,\lambda^3a_4ab \\
& - 72\,\lambda^3a_4cr - 4\,\lambda\, a_2c_0c - 12\,ka_1b_3b - 12\,\lambda\, a_1c_3b - 4\,\lambda\, a_1c_1c - 12\,ka_0b_4b - 4\,ka_0b_2c \\
& - 36\,ka_3b_3a - 12\,ka_2b_2b - 36\,ka_2b_4a - 12\,ka_4b_0b - 48\,ka_4b_4r - 36\,ka_4b_2a = 0, \\
& \lambda\, b_1 - ka_1 = 0, \\
& 3\,\lambda\, b_3 - 3\,ka_3 = 0, \\
& 2\,\lambda\, b_2 - 2\,ka_2 = 0, \\
& 4\,\lambda\, b_4 - 4\,ka_4 = 0, \\
& kc_1 - \lambda\, a_1 = 0, \\
& 3\,kc_3 - 3\,\lambda\, a_3 = 0, \\
& 2\,kc_2 - 2\lambda\, a_2 = 0, \\
& 4\,kc_4 - 4\,\lambda\, a_4 = 0.
\end{aligned}
\tag{15}
$$

Solving above algebraic equations, we obtain

$$
a_0 = \frac{4\,k\lambda^4a - k\lambda\,\nu + 6\,k\lambda^2c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3}, \qquad a_2 = -\frac{1}{2}\,\lambda\,kb, \qquad a_1 = a_3 = a_4 = 0,
$$

$$
b_0 = b_0, \qquad b_2 = -\frac{1}{2}\,k^2b, \qquad b_1 = b_3 = b_4 = 0
$$

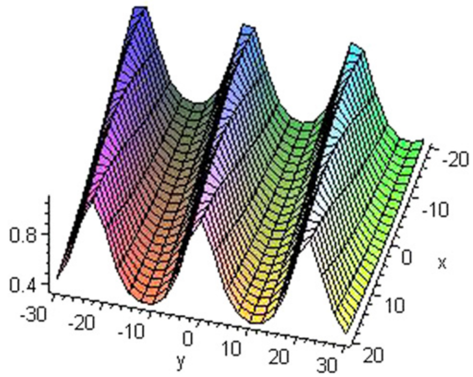

Fig. 1a. The periodic solution u of Eq. (17).

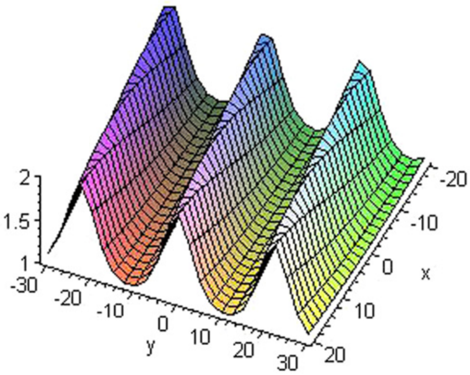

Fig. 1b. The periodic solution v of Eq. (17).

$$c_0 = c_0, \quad c_2 = -\frac{1}{2}\, b\lambda^2, \quad c_1 = c_3 = c_4 = 0, \quad r \neq 0, \quad c = 0.$$

Hence the solution of Eq. (11) reads

$$\begin{aligned} u(\zeta) &= \frac{4\, k\lambda^4 a - k\lambda\, \nu + 6\, k\lambda^2 c_0 + 4\, k^4\lambda\, a + 6\, k^2\lambda\, b_0}{-6\, k^3 - 6\, \lambda^3} - \frac{1}{2}\, \lambda\, kb\, f^2(\zeta), \\ v(\zeta) &= b_0 - \frac{1}{2}\, k^2 b\, f^2(\zeta), \\ w(\zeta) &= c_0 - \frac{1}{2}\, b\lambda^2\, f^2(\zeta). \end{aligned} \tag{16}$$

Depending on a, b, c and r in Eq. (5), we obtain multiple traveling wave solutions of Eq. (11).

Case 1. $a = -(1 + m^2),\ b = 2m^2,\ r = 1,\ c = 0.$

The solution of Eq. (5) reads $f = sn(\zeta, m)$, so we get the periodic wave solution to Eq. (11)

$$\begin{aligned} u(\zeta) &= \frac{4\, k\lambda^4 a - k\lambda\, \nu + 6\, k\lambda^2 c_0 + 4\, k^4\lambda\, a + 6\, k^2\lambda\, b_0}{-6\, k^3 - 6\, \lambda^3} - \frac{1}{2}\, \lambda\, kb\, sn^2(\zeta, m), \\ v(\zeta) &= b_0 - \frac{1}{2}\, k^2 b\, sn^2(\zeta, m), \\ w(\zeta) &= c_0 - \frac{1}{2}\, b\lambda^2\, sn^2(\zeta, m). \end{aligned} \tag{17}$$

Whose typical structure is shown in Fig. 1

As $m \to 1$, Eq. (17) degenerates to shock wave solution

$$\begin{aligned} u(\zeta) &= \frac{4\, k\lambda^4 a - k\lambda\, \nu + 6\, k\lambda^2 c_0 + 4\, k^4\lambda\, a + 6\, k^2\lambda\, b_0}{-6\, k^3 - 6\, \lambda^3} - \frac{1}{2}\, \lambda\, kb\, \tanh^2(\zeta), \\ v(\zeta) &= b_0 - \frac{1}{2}\, k^2 b\, \tanh^2(\zeta), \end{aligned}$$

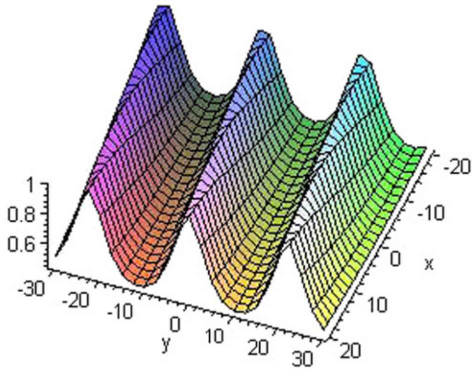

Fig. 1c. The periodic solution w of Eq. (17).

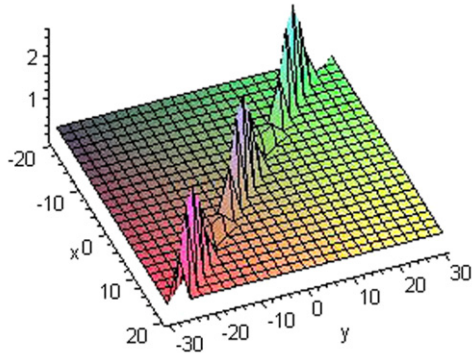

Fig. 2a. The shock wave solution u of Eq. (18).

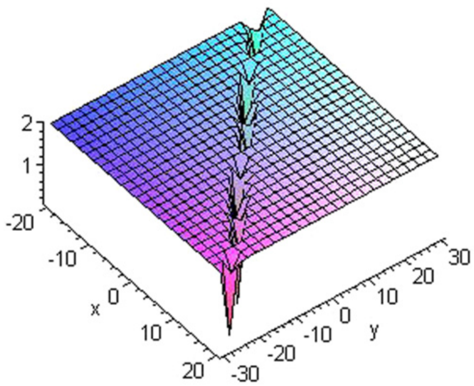

Fig. 2b. The shock wave solution v of Eq. (18).

$$w(\zeta) = c_0 - \frac{1}{2}\, b\lambda^2 \tanh^2(\zeta). \tag{18}$$

Which is illustrated in Fig. 2.

Case 2. $a = 2m^2 - 1,\ b = 2,\ r = -m^2(1 - m^2),\ c = 0.$

The solution of Eq. (5) reads $f = ds(\zeta, m)$, so we get the periodic wave solution to Eq. (11)

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,ds^2(\zeta, m),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,ds^2(\zeta, m),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,ds^2(\zeta, m). \tag{19}$$

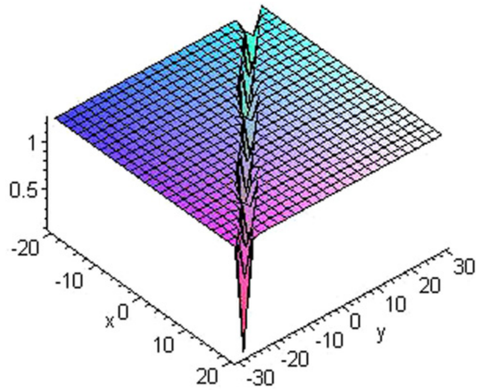

Fig. 2c. The shock wave solution w of Eq. (18).

As $m \to 1$, Eq. (20) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,csch^2(\zeta),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,csch^2(\zeta),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,csch^2(\zeta). \tag{20}$$

Case 3. $a = 2 - m^2,\ b = 2,\ r = 1 - m^2,\ c = 0.$

The solution of Eq. (5) reads $f = cs(\zeta, m)$, so we get the periodic wave solution to Eq. (11)

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,cs^2(\zeta, m),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,cs^2(\zeta, m),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,cs^2(\zeta, m). \tag{21}$$

As $m \to 0$, Eq. (21) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\cot^2(\zeta),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,cot^2(\zeta),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,cot^2(\zeta). \tag{22}$$

As $m \to 1$, Eq. (21) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,csch^2(\zeta),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,csch^2(\zeta),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,csch^2(\zeta). \tag{23}$$

Case 4. $a = 2m^2 - 1,\ b = -2m^2,\ r = 1 - m^2,\ c = 0.$

The solution of Eq. (5) reads $f = cn(\zeta, m)$, so we get the periodic wave solution to Eq. (11)

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,cn^2(\zeta, m),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,cn^2(\zeta, m),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,cn^2(\zeta, m). \tag{24}$$

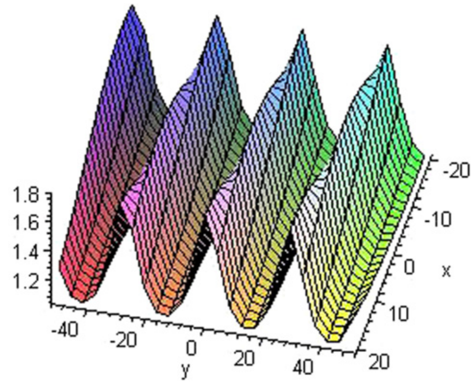

Fig. 3a. The periodic wave solution u of Eq. (24).

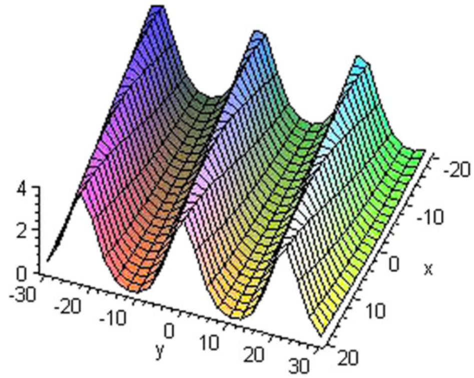

Fig. 3b. The periodic wave solution v of Eq. (24).

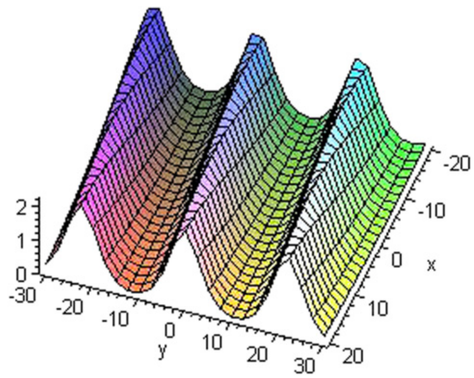

Fig. 3c. The periodic wave solution w of Eq. (24).

As $m \to 1$, Eq. (25) degenerates to the following solitary wave solution

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,{b_0}^3}{-6\,k^3 - 6\,\lambda} - \frac{1}{2}\,\lambda\,kb\,\mathrm{sech}^2(\zeta),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\mathrm{sech}^2(\zeta),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\mathrm{sech}^2(\zeta). \qquad (25)$$

The properties of these periodic and solitary solutions are shown in Figs. 3 and 4.

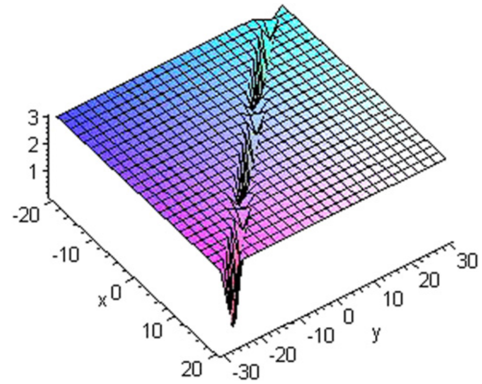

Fig. 4a. The solitary wave solution u of Eq. (25).

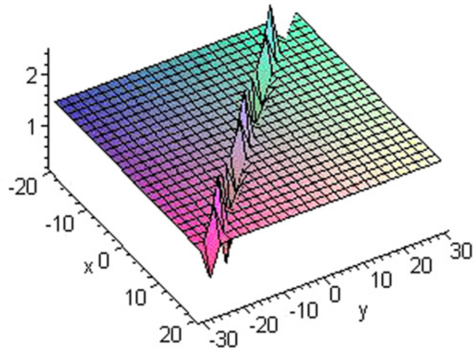

Fig. 4b. The solitary wave solution v of Eq. (25).

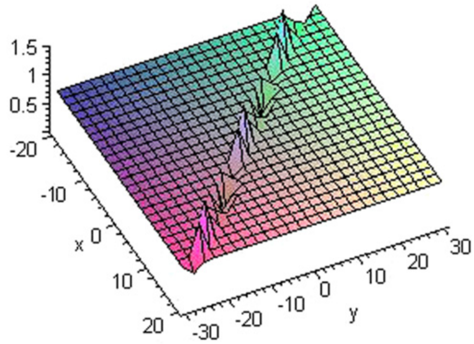

Fig. 4c. The solitary wave solution w of Eq. (25).

Case 5. $a = 2 - m^2,\ b = -2,\ r = m^2 - 1,\ c = 0.$

Now the solution of Eq. (5) reads $f = dn(\zeta, m)$. Thus we have another periodic wave solution to Eq. (11)

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,dn^2(\zeta, m),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,dn^2(\zeta, m),$$

$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,dn^2(\zeta, m). \qquad (26)$$

As $m \to 1$, Eq. (26) degenerates to the following solitary wave solution as follows

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\mathrm{sech}^2(\zeta),$$

$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\mathrm{sech}^2(\zeta),$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\mathrm{sech}^2(\zeta). \tag{27}$$

Case 6. $a = \frac{m^2-2}{2},\ b = \frac{m^2}{2},\ r = \frac{1}{4},\ c = 0.$

The solution of Eq. (5) reads $f = \frac{sn(\zeta,m)}{1\pm dn(\zeta,m)}$. Thus the double periodic wave solution to Eq. (11)

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{sn^2(\zeta,m)}{(1\pm dn(\zeta,m))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{sn^2(\zeta,m)}{(1\pm dn(\zeta,m))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{sn^2(\zeta,m)}{(1\pm dn(\zeta,m))^2}. \tag{28}$$

As $m \to 1$, Eq. (28) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{\tanh^2(\zeta)}{(1\pm \mathrm{sech}(\zeta))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{\tanh^2(\zeta)}{(1\pm \mathrm{sech}(\zeta))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{\tanh^2(\zeta)}{(1\pm \mathrm{sech}(\zeta))^2}. \tag{29}$$

Case 7. $a = \frac{m^2-2}{2},\ b = \frac{m^2}{2},\ r = \frac{m^2}{4},\ c = 0.$

Eq. (5) has solution $f = \frac{dn(\zeta,m)}{(m^2+1)(sn(\zeta,m)1\pm dn(\zeta,m))}$, from which we get the following double periodic wave solutions of Eq. (11)

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{dn^2(\zeta,m)}{(m^2+1)(sn(\zeta,m)1\pm dn(\zeta,m))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{dn^2(\zeta,m)}{(m^2+1)^2(sn(\zeta,m)1\pm dn(\zeta,m))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{dn^2(\zeta,m)}{(m^2+1)^2(sn(\zeta,m)1\pm dn(\zeta,m))^2}. \tag{30}$$

As $m \to 1$, Eq. (30) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{\mathrm{sech}^2(\zeta)}{(\tanh(\zeta)\pm \mathrm{sech}(\zeta))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{\mathrm{sech}^2(\zeta)}{(\tanh(\zeta)\pm \mathrm{sech}(\zeta))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{\mathrm{sech}^2(\zeta)}{(\tanh(\zeta)\pm \mathrm{sech}(\zeta))^2}. \tag{31}$$

Case 8. $a = \frac{m^2+1}{2},\ b = -\frac{1}{2},\ r = -\frac{(1-m^2)^2}{4},\ c = 0.$

Eq. (5) has the solution $f = mcn(\zeta,m) \pm dn(\zeta,m)$, from which we get the following double periodic wave solutions of Eq. (11)

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,(mcn(\zeta,m)\pm dn(\zeta,m))^2,$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,(mcn(\zeta,m)\pm dn(\zeta,m))^2,$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,(mcn(\zeta,m)\pm dn(\zeta,m))^2. \tag{32}$$

As $m \to 1$, Eq. (32) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,(\text{sech}(\zeta) \pm \text{sech}(\zeta))^2,$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,(\text{sech}(\zeta) \pm \text{sech}(\zeta))^2,$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,(\text{sech}(\zeta) \pm \text{sech}(\zeta))^2. \tag{33}$$

Case 9. $a = \frac{m^2+1}{2},\ b = \frac{m^2-1}{2},\ r = \frac{m^2-1}{4},\ c = 0.$

The solution of Eq. (5) reads $f = \frac{dn(\zeta,m)}{1 \pm msn(\zeta,m)}$. Thus we have another double periodic wave solutions of Eq. (11) in the form

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{dn^2(\zeta,m)}{(1 \pm msn(\zeta,m))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{dn(\zeta,m)^2}{(1 \pm msn(\zeta,m))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{dn(\zeta,m)^2}{(1 \pm msn(\zeta,m))^2}. \tag{34}$$

As $m \to 1$, Eq. (34) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{\text{sech}^2(\zeta)}{(1 \pm \tanh(\zeta))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{\text{sech}(\zeta)^2}{(1 \pm \tanh(\zeta))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{\text{sech}(\zeta)^2}{(1 \pm \tanh(\zeta))^2}. \tag{35}$$

Case 10. $a = \frac{m^2+1}{2},\ b = \frac{1-m^2}{2},\ r = \frac{1-m^2}{4},\ c = 0.$

The solution of Eq. (5) reads $f = \frac{cn(\zeta,m)}{1 \pm sn(\zeta,m)}$. Thus we have another double periodic wave solutions of Eq. (11) in the form

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{cn^2(\zeta,m)}{(1 \pm sn(\zeta,m))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{cn(\zeta,m)^2}{(1 \pm sn(\zeta,m))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{cn(\zeta,m)^2}{(1 \pm sn(\zeta,m))^2}. \tag{36}$$

As $m \to 1$, Eq. (36) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{\text{sech}^2(\zeta)}{(1 \pm \tanh(\zeta))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{\text{sech}(\zeta)^2}{(1 \pm \tanh(\zeta))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{\text{sech}(\zeta)^2}{(1 \pm \tanh(\zeta))^2}. \tag{37}$$

Case 11. $a = \frac{m^2+1}{2},\ b = \frac{(1-m^2)^2}{2},\ r = \frac{1}{4},\ c = 0.$

The solution of Eq. (5) reads $f = \frac{sn(\zeta,m)}{dn(\zeta,m) \pm cn(\zeta,m)}$. Then we get another periodic wave solutions of Eq. (11) in the form

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{sn^2(\zeta,m)}{(dn(\zeta,m) \pm msn(\zeta,m))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{sn(\zeta,m)^2}{(dn(\zeta,m) \pm msn(\zeta,m))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{sn(\zeta,m)^2}{(dn(\zeta,m) \pm msn(\zeta,m))^2}. \tag{38}$$

As $m \to 1$, Eq. (38) degenerates to

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\frac{\tanh^2(\zeta)}{(\operatorname{sech}(\zeta) \pm \operatorname{sech}(\zeta))^2},$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\frac{\tanh(\zeta)^2}{(\operatorname{sech}(\zeta) \pm \operatorname{sech}(\zeta))^2},$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\frac{\tanh(\zeta)^2}{(\operatorname{sech}(\zeta) \pm \operatorname{sech}(\zeta))^2}. \quad (39)$$

Case 12. $a = 0,\ b = 2,\ r = 0,\ c = 0.$

In this case, the solution of Eq. (5) reads $f = \frac{G}{\zeta}$, where G is a constant. Therefore, we get the rational solutions of Eq. (11) in the form

$$u(\zeta) = \frac{4\,k\lambda^4 a - k\lambda\,\nu + 6\,k\lambda^2 c_0 + 4\,k^4\lambda\,a + 6\,k^2\lambda\,b_0}{-6\,k^3 - 6\,\lambda^3} - \frac{1}{2}\,\lambda\,kb\,\left(\frac{G}{\zeta}\right)^2,$$
$$v(\zeta) = b_0 - \frac{1}{2}\,k^2 b\,\left(\frac{G}{\zeta}\right)^2,$$
$$w(\zeta) = c_0 - \frac{1}{2}\,b\lambda^2\,\left(\frac{G}{\zeta}\right)^2. \quad (40)$$

4. Conclusion

There is no systemic way for solving Eq. (5). Nevertheless, this ansatz with four arbitrary parameters r, a, b and c is reasonable since its solution can be expressed in terms of functions, such as the Jacobi elliptic function, that appear only in the nonlinear problems. In addition, these functions go back, in some limiting cases, to sech ζ, tanh ζ that describe the solitary and shock wave propagation. The values of the constants a_i ($i = 0, 1, 2, \ldots, n$) and b_i ($i = 0, 1, 2, \ldots, m$) in (4) depend crucially on the nature of differential equations whereas different types of their solutions can be classified in terms of r, a, b and c as shown in cases 1–12.

In this work, making use of Jacobi elliptic functions, the periodic wave solutions and multiple soliton solutions for the two component BKP hierarchy are obtained. Many different new forms of traveling wave solutions such as the periodic wave solution, solitary wave solution or bell-shaped soliton solutions and shock wave solution or kink-shaped soliton solutions are obtained. A kink is a solution with boundary values 0 and 2π at the left infinity and the right infinity respectively. Some of the properties of them are shown graphically. This method can be applied to solve other systems of nonlinear differential equations, we can obtain for more new solutions for Eq. (9) by using a transformed rational function method [23].

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