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A new third-order family of nonlinear solvers for multiple roots

Jafar Biazar, Behzad Ghanbari *

Department of Mathematics, Faculty of Sciences, University of Guilan, P.C. 41938, Rasht, Iran

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ABSTRACT

In this paper, a new family of third-order methods for finding multiple roots of nonlinear equations has been introduced. This family requires one-function and two-derivative evaluation per iteration. The family contains several known third-order methods, as special cases. Some examples are presented to show the performance of the presented family.

1. Introduction

The design of iterative formulae for solving such equations is very important and interesting tasks in applied mathematics and other disciplines. In this paper, iterative methods for finding a multiple root α of a nonlinear equation $f(x) = 0$ of multiplicity m, i.e. $f^{(j)}(\alpha) = 0,\ j = 0, 1, \ldots, m-1$ and $f^{(m)}(\alpha) \neq 0$, have been considered.

It is well known that Newton's method is the most widely used (second-order) method for solving such equations, giving by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

To improve the quadratic order of Newton's method, several methods including many multiple-root-finding methods of different orders are presented. For example, we refer the readers to [1–8] and the references therein.

Our new approach is based on third-order Euler–Chebyshev's method for finding multiple roots [1]

$$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{f(x_n)}{f'(x_n)} - \frac{m^2}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} \tag{2}$$

and the third-order Halley method [2]

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}. \tag{3}$$

This paper is organized as follows: In Section 2, we consider a general iterative scheme, analyze it to present a family of third-order methods. Section 3 is devoted to numerical comparisons between the results obtained in this work and some known iterative methods. Finally, conclusions are stated in the last section.

* Corresponding author.
E-mail addresses: biazar@guilan.ac.ir, jafar.biazar@gmail.com (J. Biazar), b.ghanbary@yahoo.com (B. Ghanbari).

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2. Development of methods and convergence analysis

To derive a third order method, the following iterative scheme, motivated by (2) and (3), is suggested

$$x_{n+1} = x_n - \frac{Af(x_n)f'^2(x_n)f''(x_n) + Bf'^4(x_n) + Cf^2(x_n)f''^2(x_n)}{f'(x_n)^3 f''(x_n) + Df(x_n)f'(x_n)f''^2(x_n)} \tag{4}$$

where A, B, C and D are parameters to be determined such that the iterative method defined by (4) has the order of convergence three. It can be easily seen that when $A = \frac{m(3-m)}{2}, B = 0, C = \frac{m^2}{2}$ and $D = 0$, Eq. (4) reduces to Euler–Chebyshev's third-order method defined by (2).

For the family of methods defined by (4), we have the following analysis of convergence.

Theorem 1. *Let $\alpha \in I$ be a multiple root of multiplicity m of a sufficiently differentiable function $f : I \to \Re$ on an open interval I which contains x_0 as a close initial approximation to α. In the case of*

$$A = \frac{m^2(Dm + m + 2D + 1) - 4Cm - 3Dm + 4C}{2m} \tag{5}$$

and

$$B = -\frac{(m-1)^2(m^2D + m^2 + Dm - 2C)}{2m^2} \tag{6}$$

the family of methods defined by (4), has third-order convergence.

Proof. Using Taylor expansion of $f(x)$ about α, we have

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!} e_n^m \left[1 + \overline{C}_1 e_n + \overline{C}_2 e_n^2 + \overline{C}_3 e_n^3 + O(e_n^4)\right], \tag{7}$$

$$f^2(x_n) = \frac{f^{(m)}(\alpha)^2}{(m!)^2} e_n^{2m} \left[1 + 2\overline{C}_1 e_n + \left[\overline{C}_1^{\,2} + 2\overline{C}_2\right] e_n^2 + \left[2\overline{C}_3 + 2\overline{C}_1\overline{C}_2\right] e_n^3 + O(e_n^4)\right], \tag{8}$$

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} e_n^{m-1} \left[1 + \overline{D}_1 e_n + \overline{D}_2 e_n^2 + \overline{D}_3 e_n^3 + O(e_n^4)\right], \tag{9}$$

$$f''(x_n) = \frac{f^{(m)}(\alpha)}{(m-2)!} e_n^{m-2} \left[1 + \overline{S}_1 e_n^2 + \overline{S}_2 e_n^2 + \overline{S}_3 e_n^3 + O(e_n^4)\right], \tag{10}$$

$$f'(x_n)^2 = \frac{f^{(m)}(\alpha)^2}{(m-1)!^2} e_n^{2m-2} \left[1 + 2\overline{D}_1 e_n + \left(2\overline{D}_2 + \overline{D}_1^2\right) e_n^2 + (2\overline{D}_3 + 2\overline{D}_1\overline{D}_2) e_n^3 + O(e_n^4)\right], \tag{11}$$

$$f'(x_n)^4 = \frac{f^{(m)}(\alpha)^4}{(m-1)!^4} e_n^{4m-4} \left[1 + 4\overline{D}_1 e_n + \left(4\overline{D}_2 + 6\overline{D}_1^2\right) e_n^2 + \left(12\overline{D}_1\overline{D}_2 + 4\overline{D}_3 + 4\overline{D}_1^3\right) e_n^3 + O(e_n^4)\right], \tag{12}$$

$$f''(x_n)^2 = \frac{f^{(m)}(\alpha)^2}{(m-2)!^2} e_n^{2m-4} \left[1 + 2\overline{S}_1 e_n^2 + \left(2\overline{S}_2 + \overline{S}_1^2\right) e_n^2 + (2\overline{S}_3 + 2\overline{S}_1\overline{S}_2) e_n^3 + O(e_n^4)\right], \tag{13}$$

where $e_n = x_n - \alpha$ and

$$\overline{C}_j = \frac{(m)!}{(m+j)!}\frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}, \qquad \overline{D}_j = \frac{(m-1)!}{(m+j-1)!}\frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)} \quad \text{and} \quad \overline{S}_j = \frac{(m-2)!}{(m+j-2)!}\frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}.$$

Using Eqs. (7)–(13):

$$\begin{aligned} f(x_n)^2 f'(x_n)^2 \;=\; & \frac{f^{(m)}(\alpha)^4}{m!^2(m-1)!^2} e_n^{4m-2}\Big[1 + (2\overline{D}_1 + 2\overline{C}_1)e_n + (2\overline{D}_2 + \overline{D}_1^2 + 4\overline{C}_1\overline{D}_1 + 2\overline{C}_2 + \overline{C}_1^2)e_n^2 \\ & \times (4\overline{C}_2\overline{D}_1 + 2\overline{D}_1\overline{C}_1^2 + 2\overline{D}_3 + 2\overline{D}_1\overline{D}_2 + 2\overline{C}_3 + 2\overline{C}_1\overline{C}_2 + 4\overline{C}_1\overline{D}_2 + 2\overline{C}_1\overline{D}_1^2)e_n^3 + O(e_n^4)\Big] \end{aligned}$$

$$\begin{aligned} f(x_n)f'(x_n)^2 f''(x_n) \;=\; & \frac{f^{(m)}(\alpha)^4}{m!(m-2)!(m-1)!^2} e_n^{4m-2}\Big[1 + (2\overline{D}_1 + \overline{C}_1 + \overline{S}_1)e_n + (2\overline{D}_2 + \overline{D}_1^2 + 2\overline{S}_1\overline{D}_1 + 2\overline{C}_1\overline{D}_2 \\ & + \overline{S}_2 + \overline{S}_1\overline{C}_1 + \overline{C}_2)e_n^2 + (2\overline{S}_2\overline{D}_1 + 2\overline{D}_1\overline{C}_1\overline{S}_1 + 2\overline{C}_2\overline{D}_1 + 2\overline{D}_3 \\ & + 2\overline{D}_1\overline{D}_2 + \overline{S}_3 + \overline{C}_1\overline{S}_2 + \overline{C}_2\overline{S}_1 + \overline{C}_3 + 2\overline{S}_1\overline{D}_2 + \overline{S}_1\overline{D}_1^2 + 2\overline{C}_1\overline{D}_2 + \overline{C}_1\overline{D}_1^2)e_n^3 + O(e_n^4)\Big] \end{aligned}$$

$$f'(x_n)^3 f''(x_n) = \frac{f^{(m)}(\alpha)^4}{(m-2)!(m-1)!^3} e_n^{4m-2}\Big[1 + (3\overline{D}_1 + \overline{S}_1)e_n + (3\overline{D}_2 + 3\overline{D}_1^2 + 3\overline{C}_1\overline{D}_1 + \overline{S}_2)e_n^2 + (3\overline{D}_1\overline{S}_2 + 6\overline{D}_1\overline{D}_2 + 3\overline{D}_3 + \overline{D}_1^3 + \overline{S}_3 + 3\overline{D}_2\overline{S}_1 + 3\overline{D}_1^2\overline{S}_1)e_n^3 + O(e_n^4)\Big]$$

$$f(x_n)f'(x_n)f''(x_n)^2 = \frac{f^{(m)}(\alpha)^4}{m!(m-1)!(m-2)!^2} e_n^{4m-2}\Big[1 + (2\overline{S}_1 + \overline{C}_1 + \overline{D}_1)e_n + (2\overline{S}_2 + \overline{S}_1^2 + 2\overline{S}_1\overline{D}_1 + 2\overline{C}_1\overline{S}_1 + \overline{D}_2 + \overline{D}_1\overline{C}_1 + \overline{C}_2)e_n^2 + (2\overline{D}_2\overline{S}_1 + 2\overline{D}_1\overline{C}_1\overline{S}_1 + 2\overline{C}_2\overline{S}_1 + 2\overline{S}_3 + 2\overline{S}_1\overline{S}_2 + \overline{D}_3 + \overline{C}_1\overline{D}_2 + \overline{C}_2\overline{D}_1 + \overline{C}_3 + 2\overline{D}_1\overline{S}_2 + \overline{D}_1\overline{S}_1^2 + 2\overline{C}_1\overline{S}_2 + \overline{C}_1\overline{S}_1^2)e_n^3 + O(e_n^4)\Big].$$

From (4) and some symbolic computational in Maple, we have

$$e_{n+1} = K_1 e_n^+ K_2 e_n^2 + K_3 e_n^3 + O(e_n^4), \tag{14}$$

where

$$K_1 = 1 - \frac{Bm^2 + Cm^2 + Am^2 + C - 2Cm - Am}{m(m-1)(m-D+Dm)}.$$

It can be easily shown that if, whenever

$$A = -\frac{-m^2(m-1) + C(m-1)^2 + Bm^2 - Dm(m-1)^2}{m(m-1)} \tag{15}$$

then $K_1 = 0$. Substituting of (15) into K_2, leads to

$$K_2 = \frac{-2Bm^2 + 2C(m-1)^2 - m^2(m-1)^2 + D(-m^4 + m^3 + m^2 - m)}{m(m^2-1)} \frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}.$$

This can be vanished, for the following value of

$$B = -\frac{(m-1)^2(m^2D + m^2 + Dm - 2C)}{2m^2} \tag{16}$$

setting (16) in (15), yields to

$$A = \frac{m^2(Dm + m + 2D + 1) - 4Cm - 3Dm + 4C}{2m}.$$

This completes the proof. □

In our knowledge, the family (4), includes following known third-order methods as its particular cases.

Case 1: For $C = D = 0$, family (4) leads to the well-known Osada's third-order method [6]

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)}. \tag{17}$$

Case 2: For $C = \frac{m^2}{2}$, $D = 0$, family (4) leads to Euler–Chebyshev's method (2).

Case 3: For $C = 0$, $D = -\frac{m}{m+1}$, family (4) leads to the well-known Halley method (3).

By setting $D = 0$ in (4) for those parameters satisfying conditions (5) and (6), the following one-parameter family of third order methods will be obtained

$$x_{n+1} = x_n - \left[\frac{m^2(m+1) - 4C(m-1)}{2m}\frac{f(x_n)}{f'(x_n)} + \frac{(m-1)^2(-m^2+2C)}{2m^2}\frac{f'(x_n)}{f''(x_n)} + C\frac{f^2(x_n)f''(x_n)}{f'(x_n)^3}\right],$$

which were obtained by Chun et al. [8].

Case 4: For $C = \frac{m^2}{4}$, $D = 0$, family (4) leads to a third-order method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} + \frac{(m-1)^2}{4}\frac{f'(x_n)}{f''(x_n)} - \frac{m^2}{4}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} \tag{18}$$

which were obtained by Chun et al. [8].

Case 5: For $C = m^2$, $D = 0$, family (4) leads to a third-order method

$$x_{n+1} = x_n - \frac{1}{2}m(5-3m)\frac{f(x_n)}{f'(x_n)} - \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)} - m^2\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} \tag{19}$$

which was obtained by Chun et al. [8].

Table 1
Comparison of various third-order multiple-root-finding methods and Newton's method.

$f(x)$	x_0	NM	ECM	HM	OM	CM1	CM2	BGM
$f_1(x)$	2	6	4	4	4	4	4	3
	1	6	4	4	5	4	4	3
$f_2(x)$	2.3	7	5	5	5	5	5	4
	2	7	5	5	5	5	5	4
$f_3(x)$	0	4	3	3	3	3	3	2
	1	4	4	4	4	4	4	3
$f_4(x)$	1.7	5	4	4	4	4	4	3
	1	5	4	4	4	4	4	3
$f_5(x)$	3	6	5	4	5	5	4	4
	−1	10	23	11	24	26	32	4
$f_6(x)$	−2	8	6	5	6	6	6	5
	−1	6	4	3	5	4	4	3
$f_7(x)$	1.7	6	4	4	5	5	4	3
	2	5	4	4	4	4	4	3

For $C = 0$, $D = -\frac{m(m+1)}{m^2+2m-3}$, we introduce a new third-order method for multiple

$$x_{n+1} = x_n - \frac{f'(x_n)}{\frac{(m+3)}{2(m-1)}f''(x_n) - \frac{m(m+1)}{2(m-1)^2}\frac{f(x_n)f''^2(x_n)}{f'(x_n)}}. \tag{20}$$

Obviously, the number of function evaluations per iteration required in the methods defined by (20) is three. We consider the definition of efficiency index [9] as $\sqrt[r]{p}$, where p is the order of the method and r is the number of function evaluations per iteration required by the method. We have that the family of methods defined by (4) has the efficiency index equal to $\sqrt[3]{3} \approx 1.442$, which is much better than the $\sqrt{2} \approx 1.4241$ of Newton's method.

3. Numerical examples

In this section, some numerical test of some various multiple-root-finding methods as well as our new methods and Newton's method are presented. Compared methods were Newton's method (1) (NM), Euler–Chebyshev's method (2) (ECM), Halley-like method (3) (HM), Osada's method (17) (OM), the Chun methods (18) (CM1) and (19) (CM2), and the method (20) (BGM) introduced in this contribution. All computations were done using MAPLE with 128 digit floating point arithmetics (Digits := 128). Displayed in Table 1 are the number of iterations required such that $|f(x)| < 10^{-32}$. The following functions are used for the comparison and we display the approximate zeros x_* found, up to the 28th decimal place.

$f(x)$	m	x_*
$f_1(x) = (x^3 + 4x^2 - 10)^3$	3	1.3652300134140968457608068290
$f_2(x) = (\sin^2(x) - x^2 + 1)^2$	2	1.4044916482153412260350868178
$f_3(x) = (x^2 - e^x - 3x + 2)^5$	5	0.2575302854398607604553673049
$f_4(x) = (\cos(x) - x)^3$	3	0.7390851332151606416553120876
$f_5(x) = ((x-1)^3 - 1)^6$	6	2.0
$f_6(x) = (xe^{x^2} - \sin^2(x) + 3\cos(x) + 5)^4$	4	1.2076478271309189270094167584
$f_7(x) = (\sin(x) - x/2)^2$	2	1.8954942670339809471440357381

The results presented in Table 1 show that for the functions we tested, the new method introduced in this contribution can be competitive to the known third-order methods and Newton's method and converges faster than the other multiple-root-finding methods.

4. Conclusion

In this paper a new third order method for finding multiple root of nonlinear equations was obtained. This family contains some known methods and a recently proposed family as its particular cases. Efficiently of this family were tested via some numerical examples.

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