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Advances in Mathematics 211 (2007) 152–203

ADVANCES IN Mathematics

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Regular submanifolds, graphs and area formula in Heisenberg groups

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Received 9 December 2005; accepted 28 July 2006

Available online 11 September 2006

Communicated by Luis Caffarelli

Abstract

We describe intrinsically regular submanifolds in Heisenberg groups $\mathbb{H}^n$. Low dimensional and low codimensional submanifolds turn out to be of a very different nature. The first ones are Legendrian surfaces, while low codimensional ones are more general objects, possibly non-Euclidean rectifiable. Nevertheless we prove that they are graphs in a natural group way, as well as that an area formula holds for the intrinsic Hausdorff measure. Finally, they can be seen as Federer–Fleming currents given a natural complex of differential forms on $\mathbb{H}^n$.

MSC: 28A78; 28A75; 22E25

Keywords: Heisenberg groups; Intrinsic Hausdorff measure; Regular submanifolds; Intrinsic graphs; Area formula; Federer–Fleming currents

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[1] The authors are supported by GNAMPA of INdAM, project “Analysis in metric spaces and subelliptic equations.”
[2] The author is supported by MURST, Italy and by University of Bologna, Italy, funds for selected research topics.
[3] The authors are supported by MURST, Italy and University of Trento, Italy.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2006.07.015

Contents

1. Introduction

Our aim is studying *intrinsically regular submanifolds* of the Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$. By that we mean submanifolds which, in the geometry of the Heisenberg group, have the same role as $\mathcal{C}^1$ submanifolds have inside Euclidean spaces. Here and in what follows, 'intrinsic' will denote properties defined only in terms of the group structure of $\mathbb{H}^n$ or, equivalently, of its Lie algebra $\mathfrak{h}$.

We postpone complete definitions of $\mathbb{H}^n$ to the next section. Here we remind that $\mathbb{H}^n$, with group operation $\cdot$, is a (connected and simply connected) Lie group identified through exponential coordinates with $\mathbb{R}^{2n+1}$. If $\mathfrak{h}$ denotes the Lie algebra of all left invariant vector fields on $\mathbb{H}^n$, then $\mathfrak{h}$ admits the stratification $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$; $\mathfrak{h}_1$ is called horizontal layer. The horizontal layer defines, by left translation, the horizontal fiber bundle $H\mathbb{H}^n$. Since $H\mathbb{H}^n$ depends only on the stratification of $\mathfrak{h}$, we call 'intrinsic' any notion depending only on $H\mathbb{H}^n$. The stratification of $\mathfrak{h}$ induces, through the exponential map, a family of anisotropic dilations δ_λ for $\lambda > 0$. We refer to δ_λ as intrinsic dilations. A privileged role in the geometry of $\mathbb{H}^n$ is played by *horizontal curves*, i.e. curves tangent at any point to the fiber of $H\mathbb{H}^n$ at that point (if we think $\mathbb{H}^n$ as the configuration space of a non-holonomic mechanical system, horizontal curves describe admissible trajectories of the system).

We recall the notions of Carnot–Carathéodory distance and Hausdorff measures in $\mathbb{H}^n$. Once a scalar product is defined in $\mathfrak{h}$, each fiber of the horizontal bundle over a generic point p is consequently endowed with a scalar product $\langle\cdot,\cdot\rangle_p$. We denote also by $|\cdot|_p$ the associated norm. Thus, we can define the (sub-Riemannian) length of a horizontal curve $\gamma : [0, T] \to \mathbb{H}^n$ as $\int_0^T |\gamma'(t)|_{\gamma(t)}\, dt$. Given $p, q \in \mathbb{H}^n$, their *Carnot–Carathéodory distance* $d_c(p, q)$ is the minimal length of horizontal curves connecting p and q.

Intrinsic s-dimensional Hausdorff measures $\mathcal{H}_c^s$ and $\mathcal{S}_c^s$, $s \geqslant 0$, are obtained from d_c, following classical Carathéodory construction as in Federer's book [9, Section 2.10.2]. The intrinsic metric (or Hausdorff) dimension $\dim_{\mathbb{H}}(S)$ of a set S is the number $\dim_{\mathbb{H}}(S) \stackrel{\text{def}}{=} \inf\{s \geqslant 0\colon \mathcal{H}_c^s(S) = 0\}$.

Heisenberg groups provide the simplest non-trivial examples of nilpotent stratified, connected and simply connected Lie groups (Carnot groups in most of the recent literature).

Let us start with some comments about possible notions of regular submanifolds of a group.

It is barely worth to say that Euclidean regular submanifolds of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ are not a satisfactory choice for many reasons. Indeed, Euclidean regular submanifolds need not to be group regular; this is clear for low dimensional submanifolds: the 1-dimensional, group regular, objects are horizontal curves that are a small subclass of $\mathcal{C}^1$ lines, but, also a low codimensional Euclidean submanifold need not to be group regular due to the presence of the so-called *characteristic points* where no intrinsic notion of tangent space to the surface exists (see [6,18]). On the contrary, in Carnot groups exist 1-codimensional surfaces, sometimes called $\mathbb{H}$-regular or $\mathbb{G}$-regular surfaces, that can be highly irregular as Euclidean objects but that enjoy intrinsic regularity properties, so that it is natural to think of them as 1-codimensional regular submanifolds of the group (see [11,12,16]).

What do we mean by 'intrinsic regularity properties'? We have already stated how *intrinsic* should be meant here. We believe that the most natural requirements to be made on a subset $S \subset \mathbb{H}^n$ to be considered as an intrinsic regular submanifold are

(i) S has, at each point, a tangent 'plane' and a normal 'plane' (or better a 'transversal plane');
(ii) tangent 'planes' depend continuously on the point;

the notion of 'plane' has to depend only on the group structure and on the differential structure as given by the horizontal bundle. Since subgroups are the natural counterpart of Euclidean subspaces, it seems accordingly natural to ask that

(iii) both the tangent 'plane' and the transversal 'plane' are subgroups (or better cosets of subgroups) of $\mathbb{H}^n$; $\mathbb{H}^n$ is the direct product of them (see Section 3.2);
(iv) the tangent 'plane' to S in a point is the limit of group dilations of S centered in that point (see Definition 3.4).

Notice that the requirement that the limit of a blow up procedure is a subgroup comes out naturally even in much more general settings than $\mathbb{H}^n$, see [22]. Moreover, the explicit requirement of existence of both a tangent space and a transversal space is not pointless, because there are subgroups in $\mathbb{H}^n$, as the T axis for example, without a complementary subgroup, i.e. a subgroup $\mathbb{G} \subset \mathbb{H}^n$ such that $\mathbb{G} \cap T = \{0\}$ and $\mathbb{H}^n = \mathbb{G} \cdot T$. Finally, the distinction between normal and transversal planes is natural, because not necessarily at each point a normal subgroup exists, even if a (possibly not normal but) transversal subgroup exists.

Condition (iv) entails that the tangent plane has the natural geometric meaning of 'surface seen at infinite scale,' the scale however being meant with respect to intrinsic dilations. Notice that—if S is both an Euclidean smooth manifold and a group regular manifold—the intrinsic tangent plane is usually different from the Euclidean one. On the other hand, there are sets, 'bad' from the Euclidean point of view, that behave as regular sets with respect to group dilations.

It is natural to check if requirements (i)–(iv) are met by the classes of regular submanifolds of $\mathbb{H}^n$ considered in the literature.

$\mathcal{C}^1$ *horizontal curves*: they are Euclidean $\mathcal{C}^1$ curves; their (Euclidean) tangent space in a point is a 1-dimensional affine subspace contained in the horizontal fiber through the point, hence it is also a coset of a 1-dimensional subgroup of $\mathbb{H}^n$. The normal space is the complementary subspace of the tangent space, and it is again a subgroup. Clearly both of them depend continuously on the point. It can also be shown (see Theorem 3.5) that the Euclidean tangent lines are also limits of group dilation of the curve, so that they are also tangent in the group sense.

Legendre submanifolds: they are n-dimensional, hence maximal dimensional, integral manifolds of the horizontal distribution (see [4]). The tangent spaces are n-dimensional affine subspaces of the horizontal fiber that are also cosets of subgroups of $\mathbb{H}^n$. The complementary affine subspaces are the normal subgroups. As before the tangent spaces are limit of intrinsic dilations of the surface (see Theorem 3.5).

1-*codimensional* $\mathbb{H}$-*regular surfaces*: (see [10,11]) we recall that, locally, they are given as level sets of $\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ functions from $\mathbb{H}^n$ to $\mathbb{R}$ (see Definition 2.12), with P-differential of maximal rank (the notion of P-differential for maps between Carnot groups, introduced by Pansu in [24], provides the intrinsic notion we use systematically to be coherent with our purpose). It has been proved in [11] that $\mathbb{H}$-regular surfaces have a natural normal space (i.e. the span of the *horizontal normal vector*) at each point, hence it is a coset of a 1-dimensional subgroup contained in the horizontal fibre; that the natural tangent space is a subgroup obtained as limit of intrinsic dilations of the surface; and finally, notwithstanding that these surfaces can be highly irregular as Euclidean surfaces, the intrinsic normal subgroup and the intrinsic tangent subgroup depend continuously on the point.

In conclusion, all the surfaces in these examples are *intrinsically regular submanifolds* in the sense that they satisfy requirements (i)–(iv). Notice that $\mathcal{C}^1$ horizontal curves have topological dimension 1, Legendre submanifolds have topological dimension n, and 1-codimensional $\mathbb{H}$-regular surfaces have topological dimension $2n$ (the systematic specification 'topological' is not pointless, because, as already noticed in [10,11], other different dimensions play a role in the geometry of Carnot groups). Our aim is now to fill the picture, finding other classes of intrinsically regular submanifolds of arbitrary topological dimension.

Notice that, from the analytical point of view, horizontal curves and Legendre surfaces are given locally as images in $\mathbb{H}^n$ respectively of intervals $I \subset \mathbb{R}$ or of open sets in $\mathbb{R}^n$ through P-differentiable maps with injective differentials. On the contrary, 1-codimensional $\mathbb{H}$-regular surfaces are given locally as level sets of P-differentiable functions with surjective differentials.

The first idea coming to the mind, and the one we take here, is to generalize both these approaches. Notice that, even if in the Euclidean setting they are fully equivalent, this is no longer true in Heisenberg groups. Thus, if $1 \leqslant k \leqslant n$, we define

- *k-dimensional regular surfaces of $\mathbb{H}^n$ as images of continuously Pansu differentiable functions $\mathcal{V} \to \mathbb{H}^n$, $\mathcal{V}$ open in $\mathbb{R}^k$, with differentials of maximal rank, hence injective* (*see Definition* 3.1);
- *k-codimensional regular surfaces of $\mathbb{H}^n$ as level sets of continuously Pansu differentiable functions $\mathcal{U} \to \mathbb{R}^k$, $\mathcal{U}$ open in $\mathbb{H}^n$, with P-differential of maximal rank, hence surjective* (*see Definition* 3.2).

These two approaches are naturally different ones: indeed no non-trivial geometric object falls under the scope of both definitions. The reason is that, for $k > n$, there is no k-dimensional subgroup of the horizontal fibre; hence surfaces having as a tangent space a subgroup of the horizontal fibre are limited to have dimension $\leqslant n$ and, dually, the ones with an horizontal normal space are limited to have codimension $\leqslant n$ (both phenomena depend on the fact that we can find at most n linearly independent and commuting elements of $\mathfrak{h}_1$).

We will call the first ones *low dimensional* (or *k-dimensional*) *$\mathbb{H}$-regular surfaces* and the second ones *low codimensional* (or *k-codimensional*) *$\mathbb{H}$-regular surfaces*. It is the object of part of this paper to prove that these $\mathbb{H}$-regular surfaces enjoy properties (i)–(iv).

The two families of low dimensional and low codimensional $\mathbb{H}$-regular surfaces contain very different objects. We give here a first brief sketch of their basic properties; some of them are well known while other ones are proved in this paper.

Proposition. *k-dimensional $\mathbb{H}$-regular surfaces are Euclidean submanifolds. For $k = 1$, they are horizontal curves. For $k = n$, they are Legendrian manifolds and for $k < n$ they are submanifolds of Legendrian manifolds. They have equal topological dimension, metric dimension and Euclidean dimension. Their intrinsic tangent k-planes coincide with their Euclidean tangent k-planes* (*both are cosets of subgroups of $\mathbb{H}^n$ contained in the horizontal fibre*).

Low codimensional $\mathbb{H}$-regular surfaces, on the contrary, can be very irregular objects from an Euclidean point of view. In general, these surfaces are not Euclidean $\mathcal{C}^1$ submanifolds, not even locally (see [16] where it is constructed a 1-codimensional $\mathbb{H}$-regular surface in $\mathbb{H}^1 \equiv \mathbb{R}^3$ that is a fractal set with Euclidean dimension 2.5). Nevertheless we prove that

Proposition. *k-codimensional $\mathbb{H}$-regular surfaces have metric dimension* $(2n+2-k)$, *and topological dimension* $(2n+1-k)$. *At each point there is an intrinsic tangent* $(2n+1-k)$*-plane that is a coset of a subgroup of $\mathbb{H}^n$ and depends continuously on the point.*

Besides (i)–(iv), $\mathbb{H}$-regular surfaces also enjoy the following properties (see Theorems 3.5, 3.27, and 4.1):

Theorem 1. *$\mathbb{H}$-regular surfaces locally are graphs, provided we define intrinsically the notion of graph in $\mathbb{H}^n$.*

Theorem 2. *$\mathbb{H}$-regular surfaces locally have finite intrinsic Hausdorff measure*: *k-dimensional ones have finite $\mathcal{S}_c^k$ measure*; *k-codimensional ones have finite $\mathcal{S}_c^{2n+2-k}$ measure. The measures can be explicitly computed.*

About the notion of graph in $\mathbb{H}^n$, observe that $\mathbb{H}^n$ is (in many different ways) a direct product of subgroups; that is there are couples of subgroups, let us call them $\mathbb{G}_{\mathfrak{w}}$ and $\mathbb{G}_{\mathfrak{v}}$, such that any $p \in \mathbb{H}^n$ can be written in a unique way as $p = p_{\mathfrak{w}} \cdot p_{\mathfrak{v}}$, with $p_{\mathfrak{w}} \in \mathbb{G}_{\mathfrak{w}}$ and $p_{\mathfrak{v}} \in \mathbb{G}_{\mathfrak{v}}$. Simply split the algebra $\mathfrak{h}$ as the direct sum, $\mathfrak{h} = \mathfrak{w} \oplus \mathfrak{v}$, of two subalgebras $\mathfrak{w}$ and $\mathfrak{v}$ and set $\mathbb{G}_{\mathfrak{w}} := \exp \mathfrak{w}$, $\mathbb{G}_{\mathfrak{v}} := \exp \mathfrak{v}$.

Hence $\mathbb{H}^n$ is foliated by the family $\mathcal{L}_{\mathfrak{v}}(\xi)$ of cosets of (say) $\mathbb{G}_{\mathfrak{v}}$, where $\mathcal{L}_{\mathfrak{v}}(\xi) := \xi \cdot \mathbb{G}_{\mathfrak{v}}$ for each $\xi \in \mathbb{G}_{\mathfrak{w}}$; the subgroup $\mathbb{G}_{\mathfrak{w}}$ is the 'space of parameters' of the foliation. Then, it is natural to say:

Definition. A set $S \subset \mathbb{H}^n$ is a *graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$ if, for each $\xi \in \mathbb{G}_{\mathfrak{w}}$, $S \cap \mathcal{L}_{\mathfrak{v}}(\xi)$ contains at most one point. Equivalently if there is a function $\varphi : E \subset \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ such that

$$S = \{\xi \cdot \varphi(\xi) \colon \ \xi \in E\}$$

we say that *S is the graph of* φ.

Going back to Theorem 1, it is easy to check that low dimensional $\mathbb{H}$-regular surfaces are graphs because they are Euclidean $\mathcal{C}^1$ submanifolds and because low dimensional intrinsic graphs in $\mathbb{H}^n$ turn out to be Euclidean graphs.

On the contrary, low codimensional $\mathbb{H}$-regular surfaces need not to be graphs in the Euclidean sense. An easy example is shown in Example 3.8. One of the main result proved here (Theorem 3.27) states that any low codimensional $\mathbb{H}$-regular surface is, locally, the graph of a continuous function φ.

The proof follows from two results of independent interest. The first one (Proposition 3.13) is an Implicit Function Theorem that essentially states that if $f : \mathbb{H}^n \to \mathbb{R}^k$, $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$ is locally a bijective map from each leaf of a foliation as described above, then locally the level sets of f are intrinsic graphs with respect to that foliation.

The second result (Propositions 3.25) states that if S is a low codimensional $\mathbb{H}$-regular surface, then a foliation of $\mathbb{H}^n$ as required in the hypotheses of the Implicit Function Theorem, in fact, exists. Notice that this result is an algebraic one and that it has no counterpart in the Euclidean theory.

A more precise statement of Theorem 2 is

Theorem. *Let S be a k-codimensional* $\mathbb{H}$*-regular surface,* $1 \leqslant k \leqslant n$*. Then, by definition, there are an open* $\mathcal{U} \subset \mathbb{H}^n$ *and* $f = (f_1, \ldots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$ *such that* $S \cap \mathcal{U} = \{x \in \mathcal{U} \colon \ f(x) = 0\}$*. We know that S is locally a regular graph, that is it is possible to choose two subalgebras* $\mathfrak{v}$, $\mathfrak{w}$ *with* $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{w}$*, a relatively open subset* $\mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}$ *and a continuous function* $\varphi : \mathcal{V} \to \mathbb{G}_{\mathfrak{v}}$ *such that*

$$S \cap \mathcal{U} = \{x \in \mathcal{U} \colon \ f(x) = 0\} = \{\Phi(\xi) \stackrel{\text{def}}{=} \xi \cdot \varphi(\xi), \ \xi \in \mathcal{V}\}.$$

Let $v_1, \ldots, v_k$ *be such that* $\mathfrak{v} = \operatorname{span}\{v_1, \ldots, v_k\}$, $[v_i, v_j] = 0$ *for* $1 \leqslant i < j \leqslant k$, $|v_1 \wedge \cdots \wedge v_k| = 1$ *and*

$$\Delta(p) \stackrel{\text{def}}{=} \left|\det\left[v_i f_j(p)\right]_{1 \leqslant i, j \leqslant k}\right| \neq 0 \quad \textit{for } p \in \mathcal{U}.$$

Then

$$\mathcal{S}^{2n+2-k}_{\infty} \llcorner (S \cap \mathcal{U}) = \Phi_{\sharp}\left(\left(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi\right) \mathcal{H}^{2n+1-k}_{E} \llcorner \mathbb{G}_{\mathfrak{w}}\right).$$

Here, for $s \geqslant 0$, $\mathcal{H}^s_E$ denotes the usual Euclidean s-dimensional Hausdorff measure, while $\mathcal{S}^s_\infty$ is the s-dimensional spherical Hausdorff measures, associated with the distance d_∞ and appropriately normalized as in (4). The left invariant distance d_∞ is defined by $d_\infty(p, q) = d_\infty(q^{-1} \cdot p, 0)$, where, if $p = (p', p_{2n+1}) \in \mathbb{R}^{2n} \times \mathbb{R}^1 \equiv \mathbb{H}^n$, then $d_\infty(p, 0) = \max\{|p'|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\}$. Notice that $\mathcal{S}^s_\infty$ is equivalent with $\mathcal{S}^s_c$. For a measure μ, $\Phi_\sharp \mu$ is the image measure of μ [21, Definition 1.17].

Given the notion of $\mathbb{H}$-regular surfaces, it is natural to define rectifiable sets in $\mathbb{H}^n$. For $1 \leqslant k \leqslant n$, we say that M is a k-dimensional $\mathbb{H}$-rectifiable set if $\mathcal{S}^k_\infty(M) < \infty$ and $\mathcal{S}^k_\infty$ almost all of M is contained in the countable union of k-dimensional $\mathbb{H}$-regular surfaces. Analogously, for $n+1 \leqslant k \leqslant 2n+1$, we say that M is a k-dimensional $\mathbb{H}$-rectifiable set if $\mathcal{S}^{k+1}_\infty(M) < \infty$ and $\mathcal{S}^{k+1}_\infty$ almost all of M is contained in the countable union of $(2n+1-k)$-dimensional $\mathbb{H}$-regular surfaces.

As a consequence of the theory of $\mathbb{H}$-regular surfaces developed before, in Theorem 5.6 we prove that k-dimensional $\mathbb{H}$-rectifiable sets have at almost every point (with respect to, respectively, $\mathcal{S}^k_\infty$ or $\mathcal{S}^{k+1}_\infty$) an intrinsic approximate tangent group.

A further insight on the fact that low dimensional $\mathbb{H}$-regular surfaces are particular Euclidean $\mathcal{C}^1$ surfaces, whereas low codimensional $\mathbb{H}$-regular surfaces are ‘more general’ objects than Euclidean submanifolds, is provided by Rumin’s construction (see [28]) of a complex of differential forms $\mathcal{D}^k_{\mathbb{H}}$ (*Heisenberg k-differential forms*) playing in $\mathbb{H}^n$ the same role of the De Rham complex in Euclidean spaces.

Precise construction of Rumin’s complex is given later. We only say that Rumin proves that there is a locally exact sequence of forms in $\mathbb{H}^n$

$$0 \to \mathcal{C}^0_\infty(\mathcal{U}) \xrightarrow{d} \mathcal{D}^1_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}^n_{\mathbb{H}}(\mathcal{U}) \xrightarrow{D} \mathcal{D}^{n+1}_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}^{2n+1}_{\mathbb{H}}(\mathcal{U}) \to 0$$

where d are first order differential operators and D is a second order operator.

Since surfaces are duals of forms, the picture is perfectly coherent: indeed the objects of Rumin’s complex in dimension $k \leqslant n$ are quotient of Euclidean k-forms, so that their duals are ‘smaller’ than duals of Euclidean k-forms, coherently with the fact that low dimensional surfaces are particular Euclidean $\mathcal{C}^1$ surfaces. On the other hand, Rumin’s forms in dimension $k \geqslant n$ are subspaces of Euclidean of k-forms, so that their duals are ‘larger’ than the duals of usual k-forms, coherently with the fact that low codimensional surfaces can be very singular sets from the Euclidean point of view. Finally observe that the fact that D is a second order operator is related with the jump of the metric dimension passing from low dimensional to low codimensional $\mathbb{H}$-regular surfaces.

The existence of Rumin’s complex suggests to define, by duality, (Federer–Fleming) currents in $\mathbb{H}^n$, together with boundaries and masses.

Precisely, for $1 \leqslant k \leqslant 2n+1$, we call *Heisenberg current of dimension k* in $\mathcal{U}$, any continuous linear functional on $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$. If T is a k-dimensional Heisenberg current, its *Heisenberg boundary* is the $(k-1)$-dimensional Heisenberg current $\partial_{\mathbb{H}} T$, defined by the identities $\partial_{\mathbb{H}} T(\alpha) = T(d\alpha)$ if $k \neq n+1$ and $\partial_{\mathbb{H}} T(\alpha) = T(D\alpha)$ if $k = n+1$. The mass $\mathbf{M}_{\mathcal{V}}(T)$, of T in an open $\mathcal{V}$, is given as one can imagine. Though, its definition requires a few algebraic preliminaries so that it will be given in full detail in Section 5.

As in the Euclidean setting, oriented $\mathbb{H}$-regular surfaces induce naturally, by integration, Heisenberg currents. The following proposition sketches the interplay among $\mathbb{H}$-regular surfaces, intrinsic Hausdorff measures, Rumin’s complex and Heisenberg currents.

Proposition. *Assume $S \subset \mathcal{U}$ is a k-dimensional $\mathbb{H}$-regular surface oriented by a* (*horizontal*) *tangent k-vector field $t_{\mathbb{H}}$. Then the map*

$$\alpha \mapsto [\![S]\!](\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | t_{\mathbb{H}} \rangle \, d\mathcal{S}^k_\infty$$

from $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$ to $\mathbb{R}$ is a k-dimensional Heisenberg current with locally finite mass. Precisely, if $\mathcal{V} \Subset \mathcal{U}$,

$$\mathbf{M}_{\mathcal{V}}\big([\![S]\!]\big) = \mathcal{S}^k_\infty(S \cap \mathcal{V}).$$

Assume S is a k-codimensional $\mathbb{H}$-regular surface oriented by a tangent $(2n+1-k)$-vector field $t_{\mathbb{H}}$, then the map

$$\alpha \mapsto [\![S]\!](\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | t_{\mathbb{H}} \rangle \, d\mathcal{S}^{2n+2-k}_\infty$$

from $\mathcal{D}^{2n+1-k}_{\mathbb{H}}(\mathcal{U})$ to $\mathbb{R}$ is a $(2n+1-k)$-dimensional Heisenberg current with locally finite mass and there exists a geometric constant $c_{n,k} \in (0,1)$ such that, for any open $\mathcal{V} \Subset \mathcal{U}$

$$c_{n,k}\mathcal{S}^{2n+2-k}_\infty(S \cap \mathcal{V}) \leqslant \mathbf{M}_{\mathcal{V}}\big([\![S]\!]\big) \leqslant \mathcal{S}^{2n+2-k}_\infty(S \cap \mathcal{V}).$$

In Proposition 5.15 the last statement is made more precise, providing an explicit form of the mass of the current carried by a low codimensional $\mathbb{H}$-regular surface.

Finally, let us mention a few open problems that should be attacked starting from the results of the present paper.

- Is there a unifying approach to low dimensional and low codimensional $\mathbb{H}$-regular surfaces? Likely intrinsic graphs will play a role here and one should understand how to characterize functions whose intrinsic graphs are $\mathbb{H}$-regular surfaces. For hypersurfaces, see [3].
- Can we extend the theory of intrinsically regular surfaces to arbitrary Carnot groups? Observe that, though Rumin's theory has a counterpart in general Carnot groups, the extension of our theory of regular submanifolds, as well as the associated area formula, is not even at an embryonal state due to the increasing complexity of the group structure. In general, the theory is well understood only for codimension 1 surfaces (see [11]).
- The theory of rectifiable sets in any dimension and codimension is only hinted in the last section and it should be developed. Probably the notion of intrinsic Lipschitz graph (see [13]) will play a role here. Moreover, our approach should be compared with the ones already existing in the literature (see the approach of Scott Pauls in [25]).
- Appropriate versions of closure and compactness theorems for Federer–Fleming currents should be investigated.

2. Multilinear algebra and miscellanea

2.1. Notations

For a general review on Heisenberg groups and their properties we refer to [15,30] and to [31]. We limit ourselves to fix some notations.

$\mathbb{H}^n$ is the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p', p_{2n+1})$, with $p' \in \mathbb{R}^{2n}$ and $p_{2n+1} \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot q = \big(p' + q', p_{2n+1} + q_{2n+1} + 2\langle Jp', q'\rangle_{\mathbb{R}^{2n}}\big)$$

where $J = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$ is the $2n \times 2n$ symplectic matrix. We denote as $p^{-1} := (-p', -p_{2n+1})$ the inverse of p and as 0 the identity of $\mathbb{H}^n$.

For fixed $q \in \mathbb{H}^n$ and for $r > 0$ left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ and non-isotropic dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are automorphisms of the group defined as

$$\tau_q(p) := q \cdot p \quad \text{and as} \quad \delta_r p := \big(rp', r^2 p_{2n+1}\big).$$

We denote as $\mathfrak{h}^n$ or, more frequently, as $\mathfrak{h}$ when the dimension n is intended, the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_i + 2(Jp')_i \partial_{2n+1}, \qquad Y_i := \partial_{i+n} + 2(Jp')_{i+n} \partial_{2n+1}, \qquad T := \partial_{2n+1}.$$

The only non-trivial commutation relations among them are $[X_j, Y_j] = -4T$, for $j = 1, \dots, n$. Sometimes we will shift notations putting

$$W_i := X_i, \quad W_{i+n} := Y_i, \quad W_{2n+1} := T, \quad \text{for } i = 1, \dots, n.$$

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

Hence Heisenberg groups are a special instance of Carnot groups of step 2. A *Carnot group* $\mathbb{G}$ *of step* k is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a *step* k *stratification*, i.e. there exist linear subspaces $V_1, \dots, V_k$ such that

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_k, \qquad [V_1, V_j] = V_{j+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \quad \text{if } i > k.$$

An intrinsic distance on $\mathbb{H}^n$ is the *Carnot–Carathéodory* distance $d_c(\cdot,\cdot)$. To define it recall that an absolutely continuous curve $\gamma : [0, T] \to \mathbb{H}^n$ is a *subunit curve* with respect to $X_1, \dots, X_n, Y_1, \dots, Y_n$ if there are real measurable functions $c_1, \dots, c_{2n}$, defined in $[0, T]$, such that $\sum_j c_j^2(s) \leqslant 1$ and $\dot\gamma(s) = \sum_{j=1}^n c_j(s) X_j(\gamma(s)) + c_{j+n}(s) Y_j(\gamma(s))$, for a.e. $s \in [0, T]$. Then, if $p, q \in \mathbb{H}^n$, the cc-distance (Carnot–Carathéodory distance) $d_c(p, q)$ is

$$d_c(p, q) \stackrel{\text{def}}{=} \inf\big\{T > 0\colon \gamma \text{ is subunit, } \gamma(0) = p, \ \gamma(T) = q\big\}.$$

The set of subunit curves joining p and q is not empty, by Chow's theorem, since the rank of the Lie algebra generated by $X_1, \dots, X_n, Y_1, \dots, Y_n$ is $2n + 1$; hence d_c is a distance on $\mathbb{H}^n$ inducing the same topology as the standard Euclidean distance.

Several distances *equivalent* to d_c have been used in the literature. We use the following one, that can also be computed explicitly

$$d_\infty(p, q) = d_\infty\big(q^{-1} \cdot p, 0\big),$$

where, if $p = (p', p_{2n+1}) \in \mathbb{H}^n$, $d_\infty(p, 0) := \max\{|p'|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\}$.

$U_c(p, r)$ and $B_c(p, r)$ are the open and the closed ball associated with d_c, $U_\infty(p, r)$ and $B_\infty(p, r)$ are the open and closed balls associated with d_∞.

Both the cc-metric d_c and the metric d_∞ are well behaved with respect to left translations and dilations, that is

$$\begin{aligned} &d_c(z\cdot x, z\cdot y) = d_c(x,y), && d_c(\delta_\lambda x, \delta_\lambda y) = \lambda d_c(x,y),\\ &d_\infty(z\cdot x, z\cdot y) = d_\infty(x,y), && d_\infty(\delta_\lambda x, \delta_\lambda y) = \lambda d_\infty(x,y) \end{aligned} \tag{1}$$

for $x, y, z \in \mathbb{H}^n$ and $\lambda > 0$.

We recall that, because the topologies induced by d_c, d_∞ and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^n$ is $2n+1$. On the contrary, the Hausdorff dimension of $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$, with respect to the cc-distance d_c or with respect to any other equivalent distance, is the integer $Q := 2n+2$ usually called the *homogeneous dimension* of $\mathbb{H}^n$ (see [23]).

For a non-negative integer k, $\mathcal{L}^k$ denotes the k-dimensional Lebesgue measure. $\mathcal{L}^{2n+1}$ is the bi-invariant Haar measure of $\mathbb{H}^n$, hence, if $E \subset \mathbb{R}^{2n+1}$ is measurable, then $\mathcal{L}^{2n+1}(\tau_p(E)) = \mathcal{L}^{2n+1}(E)$ for all $p \in \mathbb{H}^n$. Moreover, if $\lambda > 0$ then $\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E)$. We explicitly observe that, $\forall p \in \mathbb{H}^n$ and $\forall r > 0$,

$$\mathcal{L}^{2n+1}\big(B_c(p,r)\big) = r^{2n+2}\mathcal{L}^{2n+1}\big(B_c(p,1)\big) = r^{2n+2}\mathcal{L}^{2n+1}\big(B_c(0,1)\big) \tag{2}$$

and as a consequence

$$\mathcal{L}^{2n+1}\big(\partial B_c(p,r)\big) = 0 \quad \text{and} \quad \mathcal{L}^{2n+1}\big(B_c(p,r)\big) = \mathcal{L}^{2n+1}\big(U_c(p,r)\big). \tag{3}$$

Analogously for the d_∞ distance.

Related with the previously defined distances d_c and d_∞, different Hausdorff measures, obtained following Carathéodory's construction as in [9, Section 2.10.2], are used in this paper. For $m \geqslant 0$, we denote by $\mathcal{H}_E^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$ and by $\mathcal{H}_c^m$ and $\mathcal{H}_\infty^m$ the m-dimensional Hausdorff measures in $\mathbb{H}^n$, obtained, respectively, from the distances d_c and d_∞. Analogously, $\mathcal{S}_E^m$, $\mathcal{S}_c^m$, and $\mathcal{S}_\infty^m$ denote the corresponding spherical Hausdorff measures. We have to be more precise about the constants appearing in the various definitions. Since explicit computations will be carried out only for the measures $\mathcal{S}_\infty^m$, with m a positive integer, we limit ourselves to this case. For each $A \subset \mathbb{H}^n$ and $\delta > 0$, $\mathcal{S}_\infty^m(A) := \lim_{\delta\to 0}\mathcal{S}_{\infty,\delta}^m(A)$, where

$$\mathcal{S}_{\infty,\delta}^m(A) = \inf\left\{\sum_i \zeta\big(B_\infty(p_i,r_i)\big)\colon\ A \subset \bigcup_i B_\infty(p_i,r_i) \text{ and } r_i \leqslant \delta\right\}$$

and the evaluation function ζ is

$$\zeta\big(B_\infty(p,r)\big) := \begin{cases} \omega_m r^m & \text{if } 1 < m \leqslant n,\\ 2\omega_{m-1} r^m & \text{if } m = n+1,\\ 2\omega_{m-2} r^m & \text{if } n+2 \leqslant m \end{cases} \tag{4}$$

where ω_m is m-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^m$. The motivation for this choice appears in the proof of Theorem 4.1.

Since d_c and d_∞ are equivalent distances, for each fixed $m > 0$, all the measures $\mathcal{H}_c^m$, $\mathcal{S}_c^m$, $\mathcal{H}_\infty^m$, and $\mathcal{S}_\infty^m$ are equivalent measures. We notice however that, due to the lack of an optimal isodiametric inequality in $\mathbb{H}^n$, it is not known if, in general, $\mathcal{H}^m_\infty(E) = \mathcal{S}^m_\infty(E)$ even for 'nice' subsets of $\mathbb{H}^n$ and for $m = Q$. Related to this point see the recent paper [27] by Severine Rigot. This is why we state some of the theorems in this paper in terms of the measures $\mathcal{S}^m_\infty$ that are more explicitly computable by blow-up analysis.

Translation invariance and homogeneity under dilations of Hausdorff measures follow as usual from (1). More precisely we have

Proposition 2.1. *Let $A \subseteq \mathbb{H}^n$, $p \in \mathbb{H}^n$ and $m, r \in [0, \infty)$. Then*

$$\mathcal{S}^m_\infty(\tau_p A) = \mathcal{S}^m_\infty(A) \quad \textit{and} \quad \mathcal{S}^m_\infty(\delta_r A) = r^m \mathcal{S}^m_\infty(A).$$

The same holds for $\mathcal{S}^m_c$, $\mathcal{H}^m_\infty$ and $\mathcal{H}^m_c$.

Finally we recall the following geometric property of spheres, whose easy proof can be found in [12].

Proposition 2.2. *Let d be a translation invariant and 1-homogeneous distance in $\mathbb{H}^n$, that is d is such that $d(z \cdot x, z \cdot y) = d(x, y)$ and $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$ for $x, y, z \in \mathbb{H}^n$ and $\lambda > 0$, and denote by U_d or B_d the open or closed d-balls. Then*

$$\operatorname{diam}_d\big(B_d(x, r)\big) = \operatorname{diam}_d\big(U_d(x, r)\big) = 2r, \quad \textit{for } r > 0.$$

2.2. Horizontal and integrable k-vectors and k-covectors

We consider the vector spaces $\mathfrak{h} := \operatorname{span}\{X_1, \dots, Y_n, T\}$ and $\mathfrak{h}_1 := \operatorname{span}\{X_1, \dots, Y_n\}$, endowed with an inner product, indicated as $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and T orthonormal.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $X_1, \dots, Y_n, T$, is the family of covectors $\{dx_1, \dots, dx_{2n}, \theta\}$ where $\theta := dx_{2n+1} - 2\langle (Jx'), dx' \rangle_{\mathbb{R}^{2n}}$ is the *contact form* in $\mathbb{H}^n$. We indicate as $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $dx_1, \dots, dx_{2n}, \theta$ an orthonormal basis. Sometimes it will be notationally convenient to put $\theta_1 := dx_1, \dots, \theta_{2n} := dx_{2n}, \theta_{2n+1} := \theta$.

Following Federer (see [9, 1.3]), the exterior algebras of $\mathfrak{h}$ and of $\bigwedge^1 \mathfrak{h}$ are the graded algebras indicated as

$$\bigwedge\nolimits_* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge\nolimits_k \mathfrak{h} \quad \text{and} \quad \bigwedge\nolimits^* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge\nolimits^k \mathfrak{h}$$

where $\bigwedge_0 \mathfrak{h} = \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \leqslant k \leqslant 2n + 1$,

$$\bigwedge\nolimits_k \mathfrak{h} := \operatorname{span}\{W_{i_1} \wedge \dots \wedge W_{i_k} \colon 1 \leqslant i_1 < \dots < i_k \leqslant 2n + 1\},$$
$$\bigwedge\nolimits^k \mathfrak{h} := \operatorname{span}\{\theta_{i_1} \wedge \dots \wedge \theta_{i_k} \colon 1 \leqslant i_1 < \dots < i_k \leqslant 2n + 1\}.$$

The elements of $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$ are called *k-vectors* and *k-covectors*.

The dual space $\bigwedge^1(\bigwedge_k \mathfrak{h})$ of $\bigwedge_k \mathfrak{h}$ can be naturally identified with $\bigwedge^k \mathfrak{h}$. The action of a k-covector φ on a k-vector v is denoted as $\langle \varphi | v \rangle$.

The *symplectic two form* $d\theta \in \bigwedge^2 \mathfrak{h}_1$ is $d\theta = 4 \sum_{i=1}^n dx_i \wedge dx_{i+n}$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{h}$ and to $\bigwedge^k \mathfrak{h}$ making the bases $W_{i_1} \wedge \cdots \wedge W_{i_k}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ orthonormal.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$. This way we obtain the algebras

$$\bigwedge\nolimits_* \mathfrak{h}_1 = \bigoplus_{k=1}^{2n} \bigwedge\nolimits_k \mathfrak{h}_1 \quad \text{and} \quad \bigwedge\nolimits^* \mathfrak{h}_1 = \bigoplus_{k=1}^{2n} \bigwedge\nolimits^k \mathfrak{h}_1$$

whose elements are the *horizontal k-vectors* and *horizontal k-covectors*; here

$$\bigwedge\nolimits_k \mathfrak{h}_1 := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k}\colon\ 1 \leqslant i_1 < \cdots < i_k \leqslant 2n\},$$
$$\bigwedge\nolimits^k \mathfrak{h}_1 := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}\colon\ 1 \leqslant i_1 < \cdots < i_k \leqslant 2n\}$$

and clearly $\bigwedge_k \mathfrak{h}_1 \subset \bigwedge_k \mathfrak{h}$ for $1 \leqslant k \leqslant 2n$.

Definition 2.3. We define linear isomorphisms (see [9, 1.7.8])

$$*\colon \bigwedge\nolimits_k \mathfrak{h} \leftrightarrow \bigwedge\nolimits_{2n+1-k} \mathfrak{h} \quad \text{and} \quad *\colon \bigwedge\nolimits^k \mathfrak{h} \leftrightarrow \bigwedge\nolimits^{2n+1-k} \mathfrak{h},$$

for $1 \leqslant k \leqslant 2n$, putting, for $v = \sum_I v_I W_I$ and $\varphi = \sum_I \varphi_I \theta_I$,

$$*v := \sum_I v_I (*W_I) \quad \text{and} \quad *\varphi := \sum_I \varphi_I (*\theta_I)$$

where

$$*W_I := (-1)^{\sigma(I)} W_{I^*} \quad \text{and} \quad *\theta_I := (-1)^{\sigma(I)} \theta_{I^*}$$

with $I = \{i_1, \ldots, i_k\}$, $1 \leqslant i_1 < \cdots < i_k \leqslant 2n+1$, $W_I = W_{i_1} \wedge \cdots \wedge W_{i_k}$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$, $I^* = \{i_1^* < \cdots < i_{2n+1-k}^*\} = \{1, \ldots, 2n+1\} \setminus I$ and $\sigma(I)$ is the number of couples (i_h, i_ℓ^*) with $i_h > i_\ell^*$.

The following properties of the $*$ operator follow readily from the definition: $\forall v, w \in \bigwedge_k \mathfrak{h}$ and $\forall \varphi, \psi \in \bigwedge^k \mathfrak{h}$

$$**v = (-1)^{k(2n+1-k)} v = v, \qquad **\varphi = (-1)^{k(2n+1-k)} \varphi = \varphi,$$
$$v \wedge *w = \langle v, w \rangle W_{\{1,\ldots,2n+1\}}, \qquad \varphi \wedge *\psi = \langle \varphi, \psi \rangle \theta_{\{1,\ldots,2n+1\}},$$
$$\langle *\varphi | *v \rangle = \langle \varphi | v \rangle. \tag{5}$$

Notice that, if $v = v_1 \wedge \cdots \wedge v_k$ is a simple k-vector, then $*v$ is a simple $(2n+1-k)$-vector. Moreover, notice that

$$\text{if} \quad v \in \bigwedge\nolimits_k \mathfrak{h}_1, \quad \text{then} \quad *v = \xi \wedge T, \quad \text{with } \xi \in \bigwedge\nolimits_{2n-k} \mathfrak{h}_1. \tag{6}$$

If $v \in \bigwedge_k \mathfrak{h}$ we define $v^* \in \bigwedge^k \mathfrak{h}$ by the identity $\langle v^* | w \rangle := \langle v, w \rangle$, and analogously we define $\varphi^* \in \bigwedge_k \mathfrak{h}$ for $\varphi \in \bigwedge^k \mathfrak{h}$.

Remark 2.4. A simple non-zero k-vector $v = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k \mathfrak{h}$ is naturally associated with a left invariant distribution of k-dimensional planes in $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$. In general, if $k > 1$, this distribution is not integrable—by Frobenius Theorem—because not necessarily $[v_i, v_j] \in \operatorname{span}\{v_1, \dots, v_k\}$. An example is provided by the 2-vector $X_1 \wedge Y_1 \in \bigwedge_2 \mathfrak{h}_1$. Horizontal k-vectors that are also integrable (more precisely: k-vectors such that the associated distribution is integrable) play an important role in the following. Notice that if $T \in \{v_1, \dots, v_k\}$ then certainly (the distribution associated with) v is integrable. On the other hand, $v \in \bigwedge_k \mathfrak{h}_1$ can be integrable only if $k \leqslant n$. Explicit algebraic characterizations of k-vectors associated with integrable distributions are proved in Theorem 2.8.

We define the vector spaces ${}_H\bigwedge_k$ and ${}_H\bigwedge^k$ of *integrable k-vectors and k-covectors* as follows.

Definition 2.5. We set ${}_H\bigwedge_0 = \mathbb{R}$ and, for $1 \leqslant k \leqslant n$,

$$\begin{aligned} {}_H\bigwedge\nolimits_k &\stackrel{\text{def}}{=} \operatorname{span}\Big\{ v \in \bigwedge\nolimits_k \mathfrak{h}_1 \colon\ v \text{ is simple and integrable} \Big\}, \\ {}_H\bigwedge\nolimits_{2n+1-k} &\stackrel{\text{def}}{=} *\Big({}_H\bigwedge\nolimits_k \Big). \end{aligned}$$

Integrable covectors are defined by duality: for $0 \leqslant k \leqslant 2n+1$ we set

$${}_H\bigwedge\nolimits^k \stackrel{\text{def}}{=} \bigwedge\nolimits^1 \Big({}_H\bigwedge\nolimits_k \Big) \simeq \Big\{ \varphi \in \bigwedge\nolimits^k \mathfrak{h} \colon\ \varphi^* \in {}_H\bigwedge\nolimits_k \Big\}.$$

Notice that ${}_H\bigwedge_1 = \bigwedge_1 \mathfrak{h}_1 = \mathfrak{h}_1$. On the contrary, for $1 < k \leqslant n$, $0 \neq {}_H\bigwedge_k \subsetneq \bigwedge_k \mathfrak{h}_1$.

If $1 \leqslant k \leqslant n$ and if $w \in {}_H\bigwedge_{2n+1-k}$ is a simple $(2n+1-k)$-vector, then one can choose $w_1, \dots, w_{2n+1-k}$ so that:

$$w = w_1 \wedge \cdots \wedge w_{2n+1-k},\ w_1 \wedge \cdots \wedge w_{2n-k} \in \bigwedge\nolimits_{2n-k} \mathfrak{h}_1 \quad \text{and} \quad w_{2n+1-k} = T.$$

Recall now the definition of H-linear map (*horizontal linear map*) between Carnot groups (see [24] and also Chapter 3 of [19]). H-linear maps play the same central role as linear maps between vector spaces. Here we deal only with H-linear maps from $\mathbb{R}^k \to \mathbb{H}^n$ and, vice versa, from $\mathbb{H}^n \to \mathbb{R}^k$. Nevertheless we provide the general definition.

Definition 2.6. Let $\mathbb{G}^1$ and $\mathbb{G}^2$ be Carnot groups with dilation automorphisms δ^1_λ and δ^2_λ. We say that $L : \mathbb{G}^1 \to \mathbb{G}^2$ is a H-linear map if L is a homogeneous Lie groups homomorphism, where homogeneous means that $\delta^2_\lambda(Lx) = L(\delta^1_\lambda x)$, for all $\lambda > 0$ and $x \in \mathbb{G}^1$.

H-linear maps are closely related with integrable k-vectors. In Theorem 2.8 we show that there is a one-to-one correspondence between injective H-linear maps $\mathbb{R}^k \to \mathbb{H}^n$ and integrable simple k-vectors.

The following proposition, characterizing H-linear maps $\mathbb{R}^k \to \mathbb{H}^n$, is a special instance of a more general statement proved in [19].

Proposition 2.7. *Let $k \geqslant 1$ and $L : \mathbb{R}^k \to \mathbb{H}^n$ be H-linear. Then there is a $2n \times k$ matrix A with $A^T J A = 0$, such that*

$$Lx = (Ax, 0), \quad \forall x \in \mathbb{R}^k.$$

Moreover, L can be injective only if $1 \leqslant k \leqslant n$.

Proof. First notice that $L(\mathbb{R}^k) \subset \{p \in \mathbb{H}^n\colon\ p_{2n+1} = 0\}$. Indeed, for all $x \in \mathbb{R}^k$: $2(Lx)_{2n+1} = (Lx \cdot Lx)_{2n+1} = (L(2x))_{2n+1} = (\delta_2 Lx)_{2n+1} = 4(Lx)_{2n+1}$. Here we used the notations $\lambda p = (\lambda p', \lambda p_{2n+1})$ for $\lambda \in \mathbb{R}$ while $\delta_\lambda p = (\lambda p', \lambda^2 p_{2n+1})$. Moreover L is linear as a map $\mathbb{R}^k \to \mathbb{R}^{2n}$, hence $Lx = (Ax, 0)$ for some matrix A. Finally, for all $x, y \in \mathbb{R}^k$,

$$0 = \big(L(x+y)\big)_{2n+1} = (Lx \cdot Ly)_{2n+1} = 2\langle JAx, Ay\rangle_{\mathbb{R}^{2n}}$$

that yields $A^T J A = 0$. $\square$

Theorem 2.8. *Assume $2 \leqslant k \leqslant n$ and $v = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k \mathfrak{h}_1$, $v \neq 0$. Then the following four statements are equivalent*:

(1) $v \in {}_H\bigwedge_k$;
(2) $[v_i, v_j] = 0$ *for* $1 \leqslant i, j, \leqslant k$;
(3) $\langle \gamma \wedge d\theta | v\rangle = 0$ *for all* $\gamma \in \bigwedge^{k-2} \mathfrak{h}$;
(4) *there is an injective H-linear map $L : \mathbb{R}^k \to \mathbb{H}^n$ such that $Le_1 \wedge \cdots \wedge Le_k = v$; L can be explicitly defined as $Lx = \delta_{x_1} v_1 \cdot \delta_{x_2} v_2 \cdot \cdots \cdot \delta_{x_k} v_k$.*

Notice that for $k = 1$ statements (1)–(4) are either meaningless or trivially equivalent.

Proof. (1) $\Rightarrow$ (2). Because $[v_i, v_j]$ is always a multiple of T and $v_i, v_j \in \mathfrak{h}_1$, the necessity of (2) for the integrability of the distribution associated with v is just Frobenius Theorem.

(2) $\Rightarrow$ (1) follows from Frobenius Theorem.

(2) $\Leftrightarrow$ (3). A direct computation yields $[v_i, v_j] = \langle d\theta | v_i \wedge v_j\rangle = 4\langle J v_i, v_j\rangle_{\mathbb{R}^{2n}}$. If $v = v_1, \ldots, v_k \in \mathfrak{h}_1$ and if $\gamma \in \bigwedge^{k-2} \mathfrak{h}_1$ then

$$\langle \gamma \wedge d\theta | v\rangle = \sum_\pi \sigma(\pi) \langle \gamma | v_{\pi(1)} \wedge \cdots \wedge v_{\pi(k-2)}\rangle \langle d\theta | v_{\pi(k-1)} \wedge v_{\pi(k)}\rangle$$

where the sum is extended to all the permutations π of $\{1, \ldots, k\}$ and $\sigma(\pi)$ is ± 1 accordingly with the parity of the permutation π; hence, $\forall \gamma \in \bigwedge^{k-2} \mathfrak{h}_1$, $\langle \gamma \wedge d\theta | v_1 \wedge \cdots \wedge v_k\rangle = 0$, is equivalent with $[v_i, v_j] = \langle d\theta | v_i \wedge v_j\rangle = 0$ for $1 \leqslant i < j \leqslant k$.

(3) $\Leftrightarrow$ (4). Let $v_j = \sum_{i=1}^{2n} v_{i,j} W_i \in \mathfrak{h}_1$. Put $\tilde{v}_j := (v_{1,j}, \dots, v_{2n,j}) \in \mathbb{R}^{2n}$. Then if A is the $2n \times k$ matrix $A = [v_{i,j}] = [\tilde{v}_1 \mid \cdots \mid \tilde{v}_k]$, then $A^T J A = [\langle J\tilde{v}_i, \tilde{v}_j\rangle_{\mathbb{R}^{2n}}]_{1\leqslant i<j\leqslant k}$; so that recalling Proposition 2.7 the required equivalence follows. $\square$

We show now that the spaces of integrable covectors are canonically isomorphic with the spaces defined by Rumin in [28]. Indeed Rumin's paper largely inspired the present one. We begin recalling Rumin's approach: First define $\mathcal{I}^*$ and $\mathcal{J}^* \subset \bigwedge^* \mathfrak{h}$, where $\mathcal{I}^*$ is the graded ideal generated by θ, that is $\mathcal{I}^* := \{\beta \wedge \theta + \gamma \wedge d\theta\colon\ \beta, \gamma \in \bigwedge^* \mathfrak{h}\}$ and $\mathcal{J}^*$ is the annihilator of $\mathcal{I}^*$, that is $\mathcal{J}^* := \{\alpha \in \bigwedge^* \mathfrak{h}\colon\ \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0\}$. Both $\mathcal{I}^*$ and $\mathcal{J}^*$ are graded, indeed $\mathcal{I}^* = \bigoplus_{k=1}^{2n+1} \mathcal{I}^k$ and $\mathcal{J}^* = \bigoplus_{k=1}^{2n+1} \mathcal{J}^k$, where $\mathcal{I}^k, \mathcal{J}^k \subset \bigwedge^k \mathfrak{h}$ and

$$\mathcal{I}^k = \left\{\beta \wedge \theta + \gamma \wedge d\theta\colon\ \beta \in \bigwedge^{k-1} \mathfrak{h},\ \gamma \in \bigwedge^{k-2} \mathfrak{h}\right\},$$

$$\mathcal{J}^k = \left\{\alpha \in \bigwedge^k \mathfrak{h}\colon\ \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0\right\}.$$

As Rumin observes, for $1 \leqslant k \leqslant n$ we have $\mathcal{I}^{2n+1-k} = \bigwedge^{2n+1-k} \mathfrak{h}$ and $\mathcal{J}^k = 0$.

The following identities, or natural isomorphisms, hold.

Theorem 2.9. *For* $1 \leqslant k \leqslant n$,

$$_H\bigwedge_k = \ker \mathcal{I}^k \quad \textit{and} \quad _H\bigwedge_{2n+1-k} \simeq \frac{\bigwedge_{2n+1-k} \mathfrak{h}}{\ker \mathcal{J}^{2n+1-k}}, \tag{7}$$

$$_H\bigwedge^k \simeq \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} \quad \textit{and} \quad _H\bigwedge^{2n+1-k} = \mathcal{J}^{2n+1-k}, \tag{8}$$

where $\ker \mathcal{I}^k = \{v \in \bigwedge_k \mathfrak{h}\colon\ \langle \varphi | v\rangle = 0\ \forall \varphi \in \mathcal{I}^k\}$ *and* $\ker \mathcal{J}^{2n+1-k}$ *is analogously defined.*

Proof. To prove the first equality in (7) notice that, if $v \in \bigwedge_k \mathfrak{h}$, the condition $\langle \beta \wedge \theta | v\rangle = 0$ for all $\beta \in \bigwedge^{k-1} \mathfrak{h}$ implies $v \in \bigwedge_k \mathfrak{h}_1$, hence we get $\ker \mathcal{I}^k = \{v \in \bigwedge_k \mathfrak{h}_1\colon\ \langle \gamma \wedge d\theta | v\rangle = 0\ \forall \gamma \in \bigwedge^{k-2} \mathfrak{h}\}$, and we conclude by the equivalence of (1) and (3) in Theorem 2.8.

To prove the second one in (7) recall that, by Definition 2.5, $_H\bigwedge_{2n+1-k} = *_H\bigwedge_k = * \ker \mathcal{I}^k$. Moreover, $\ker \mathcal{I}^k = \{v \in \bigwedge_k \mathfrak{h}\colon\ \langle \varphi^*, v\rangle = 0\ \forall \varphi \in \mathcal{I}^k\}$ where $\varphi^* \in \bigwedge_k \mathfrak{h}$ is such that $\langle \varphi | v\rangle = \langle \varphi^*, v\rangle$, $\forall v \in \bigwedge_k \mathfrak{h}$. Hence

$$*\bigl(\ker \mathcal{I}^k\bigr) = \left\{v \in \bigwedge_{2n+1-k} \mathfrak{h}\colon\ \bigl\langle *\varphi^*, v\bigr\rangle = 0\ \forall \varphi \in \mathcal{I}^k\right\}. \tag{9}$$

Now notice that

$$\varphi \in \mathcal{I}^k \quad \Longleftrightarrow \quad *\varphi^* \in \ker \mathcal{J}^{2n+1-k}; \tag{10}$$

indeed, $*\varphi^* \in \ker \mathcal{J}^{2n+1-k} \Leftrightarrow \langle \psi | *\varphi^*\rangle = 0$, $\forall \psi \in \mathcal{J}^{2n+1-k} \Leftrightarrow \langle *\psi | \varphi^*\rangle = 0$, $\forall \psi \in \mathcal{J}^{2n+1-k}$; hence $*\varphi^* \in \ker \mathcal{J}^{2n+1-k} \Leftrightarrow \langle \alpha | \varphi^*\rangle = 0$, $\forall \alpha \in *(\mathcal{J}^{2n+1-k}) = (\mathcal{I}^k)^\perp \Leftrightarrow \langle \alpha, \varphi\rangle = 0$, $\forall \alpha \in (\mathcal{I}^k)^\perp \Leftrightarrow \varphi \in \mathcal{I}^k$.

Finally, from (9) and (10) it follows

$$
\begin{aligned}
*\Big({}_H\textstyle\bigwedge_k\Big) &\equiv *\big(\ker\mathcal{I}^k\big)\\
&= \Big\{v\in \textstyle\bigwedge_{2n+1-k}\mathfrak{h}\colon\ \langle\psi,v\rangle = 0\ \forall\psi\in\ker\mathcal{J}^{2n+1-k}\Big\}\\
&= \big(\ker\mathcal{J}^{2n+1-k}\big)^{\perp}\simeq\frac{\bigwedge_{2n+1-k}\mathfrak{h}}{\ker\mathcal{J}^{2n+1-k}}.
\end{aligned}
$$

This concludes the proof of the second part of (7).

To prove (8), recall that by Definition 2.5, for $1\leqslant k\leqslant 2n+1$, ${}_H\bigwedge^k := \bigwedge^1({}_H\bigwedge_k)$. Now, given that for any two finite dimensional vector spaces V and W with V subspace of W, it holds that

$$
\bigwedge\nolimits^1\left(\frac{W}{V}\right)\simeq\ker(V)\quad\text{and}\quad\bigwedge\nolimits^1 V\simeq\frac{\bigwedge^1 W}{\ker(V)},
$$

we have, for $k=1,\dots,n$,

$$
\bigwedge\nolimits^1\big(\ker\mathcal{I}^k\big)\simeq\frac{\bigwedge^1\bigwedge_k\mathfrak{h}}{\ker(\ker\mathcal{I}^k)}\simeq\frac{\bigwedge^k\mathfrak{h}}{\mathcal{I}^k},
$$

and, for $k=n+1,\dots,2n+1$,

$$
\bigwedge\nolimits^1\left(\frac{\bigwedge_k\mathfrak{h}}{\ker\mathcal{J}^k}\right)\simeq\ker\big(\ker\mathcal{J}^k\big)=\mathcal{J}^k.\qquad\square
$$

Finally we observe that our previous algebraic construction yields canonically several bundles over $\mathbb{H}^n$. These are the bundles of k-vectors and k-covectors, still indicated as $\bigwedge_k\mathfrak{h}$ and $\bigwedge^k\mathfrak{h}$, the bundles $\bigwedge_k\mathfrak{h}_1$ and $\bigwedge^k\mathfrak{h}_1$ of the horizontal k-vectors and k-covectors and the bundles ${}_H\bigwedge_k$ and ${}_H\bigwedge^k$ of the integrable k-vectors and k-covectors. The fiber of $\bigwedge_k\mathfrak{h}$ over $p\in\mathbb{H}^n$ is denoted by $\bigwedge_{k,p}\mathfrak{h}$ and analogously for the other ones.

It is customary to call *horizontal bundle* $H\mathbb{H}^n$ the bundle generated by $X_1,\dots,X_n,Y_1,\dots,Y_n$, or, with our previous notations, $H\mathbb{H}^n := \bigwedge_1\mathfrak{h}_1$.

The inner product $\langle\cdot,\cdot\rangle$ on $\bigwedge_k\mathfrak{h}$ and on $\bigwedge^k\mathfrak{h}$ induces an inner product on each fiber of the previous bundles.

2.3. *Calculus on* $\mathbb{H}^n$

Definition 2.10. (Pansu [24]) Let $(\mathbb{G}^1,\cdot)$ and $(\mathbb{G}^2,\cdot)$ be Carnot groups with dilation automorphisms δ^1_λ and δ^2_λ. Let $\mathcal{U}$ be an open subset of $\mathbb{G}^1$, and $f:\mathcal{U}\to\mathbb{G}^2$. We say that f is P-differentiable at $p_0\in\mathcal{U}$ if there is a (unique) H-linear map $d_H f_{p_0}:\mathbb{G}^1\to\mathbb{G}^2$ such that

$$
d_H f_{p_0}(p) := \lim_{\lambda\to 0}\delta^2_{1/\lambda}\big(f(p_0)^{-1}\cdot f\big(p_0\cdot\delta^1_\lambda p\big)\big)
$$

uniformly for p in compact subsets of $\mathcal{U}$.

In the sequel, we shall deal only with the cases $\mathbb{G}^1 = \mathbb{R}^k$, $\mathbb{G}^2 = \mathbb{H}^n$, and $\mathbb{G}^1 = \mathbb{H}^n$, $\mathbb{G}^2 = \mathbb{R}^k$. The structure of the differential map in the first case has been already described in Proposition 2.7. In the second case, because of the commutativity of the target space, the differential can be thought as the k-uple of the P-differentials of the components of f. Again, the differential can be written in the form $d_H f_{p_0}(p) = A_{p_0} p'$, where A_{p_0} is a $k \times 2n$ matrix (see, e.g., [12, Proposition 2.5]). Thus, if $k = 1$, $d_H f_{p_0}$ can be identified with an element of $\bigwedge^1 \mathfrak{h}_1$.

Definition 2.11. If $f : \mathcal{U} \subset \mathbb{H}^n \to \mathbb{R}$ is differentiable at p, then the *horizontal gradient* of f at p is defined as

$$\nabla_H f(p) := d_H f(p)^* \in \bigwedge\nolimits_1 \mathfrak{h}_1$$

or equivalently as

$$\nabla_H f(p) = \sum_{j=1}^{n} \big(X_j f(p)\big) X_j + \big(Y_j f(p)\big) Y_j .$$

Definition 2.12. In the sequel, we shall use the following notations for function spaces. If $\mathcal{U} \subset \mathbb{H}^n$ and $\mathcal{V} \subset \mathbb{R}^k$ are open subsets, we denote

- $\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})$ is the vector space of continuous functions $f : \mathcal{U} \to \mathbb{R}$ such that the P-differential $d_H f$ is also continuous, $[\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$ is the set of k-uples $f = \{f_1, \dots, f_k\}$ such that each $f_i \in \mathcal{C}^1_{\mathbb{H}}(\mathcal{U})$, for $1 \leqslant i \leqslant k$.
- $\mathcal{C}^1(\mathcal{V}; \mathbb{H}^n)$ is the vector space of continuous functions $f : \mathcal{V} \to \mathbb{H}^n$ such that the P-differential $d_H f(p)$ depends continuously on $p \in \mathcal{V}$.
- $\mathrm{Lip}(\mathbb{R}^k; \mathbb{H}^n)$, $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k; \mathbb{H}^n)$, $\mathrm{Lip}(\mathbb{H}^n; \mathbb{R}^k)$, $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{H}^n; \mathbb{R}^k)$ are the vector spaces of Lipschitz continuous (locally Lipschitz continuous) functions, where the metric used in the definition are the cc-metric of the corresponding spaces.

3. Regular surfaces and regular graphs

3.1. Regular submanifolds in $\mathbb{H}^n$

Here we give the definition of $\mathbb{H}$-regular surfaces in the spirit illustrated in the introduction. We distinguish low dimensional from low codimensional surfaces, the first ones being images of open subset of Euclidean spaces while the second ones are level sets of intrinsically regular functions.

Definition 3.1. Let $1 \leqslant k \leqslant n$. A subset $S \subset \mathbb{H}^n$ is a *k-dimensional $\mathbb{H}$-regular surface* (or a $\mathcal{C}^1_{\mathbb{H}}$ surface of dimension k) if for any $p \in S$ there are open sets $\mathcal{U} \subset \mathbb{H}^n$, $\mathcal{V} \subset \mathbb{R}^k$ and a function $\varphi : \mathcal{V} \to \mathcal{U}$ such that $p \in \mathcal{U}$, φ is injective, φ is continuously P-differentiable with $d_H \phi$ injective, and

$$S \cap \mathcal{U} = \varphi(\mathcal{V}).$$

Definition 3.2. Let $1 \leqslant k \leqslant n$. A subset $S \subset \mathbb{H}^n$ is a *k-codimensional $\mathbb{H}$-regular surface* (or a $\mathcal{C}^1_{\mathbb{H}}$ surface of codimension k or a $\mathcal{C}^1_{\mathbb{H}}$ surface of topological dimension $(2n+1-k)$) if for any $p \in S$ there are an open set $\mathcal{U} \subset \mathbb{H}^n$ and a function $f : \mathcal{U} \to \mathbb{R}^k$ such that $p \in \mathcal{U}$, $f = (f_1, \dots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, $\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k \neq 0$ in $\mathcal{U}$ (equivalently, $d_H f$ is onto) and

$$S \cap \mathcal{U} = \{q \in \mathcal{U}:\ f(q) = 0\}.$$

Remark 3.3. For $k = 1$, Definition 3.1 defines horizontal, continuously differentiable, curves. On the other hand, Definition 3.1 cannot be extended to the case $k > n$. Indeed Ambrosio and Kirchheim prove (see [1] and also [19]), that, for $k > n$, the set of maps φ satisfying the assumptions of Definition 3.1 is empty. They show that, if $k > n$, $\mathbb{H}^n$ is *purely k-unrectifiable*, i.e. $\forall f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k, \mathbb{H}^n)$, $\forall A \subset \mathbb{R}^k$ it follows $\mathcal{H}^k_c(f(A)) = 0$.

In turn Definition 3.2, for $k = 1$, gives the notion of $\mathbb{H}$-regular hypersurface introduced in [10] and [11]. Definition 3.2—unlike the previous one—could be formally extended to $k > n$, but we restrict ourselves to $1 \leqslant k \leqslant n$ because only in this situation it is possible to prove (see below) that a $\mathcal{C}^1_{\mathbb{H}}$ surface of codimension k is locally a graph in a consistent suitable sense.

The surfaces of these two families are very different from each other. The first ones are particular Euclidean C^1 submanifolds, precisely for $k = n$ they are Legendrian submanifolds [5], on the contrary, the second ones may be very irregular from an Euclidean point of view (see [16]). We will prove that both k-dimensional and k-codimensional $\mathbb{H}$-regular surfaces are intrinsic regular surfaces as defined in the introduction. We begin recalling the definition of *Heisenberg tangent cone* to a set A in a point p.

Definition 3.4. Let $A \subset \mathbb{H}^n$. The *intrinsic* (*Heisenberg*) *tangent cone* to A in 0 is the set

$$\mathrm{Tan}_{\mathbb{H}}(A, 0) \stackrel{\mathrm{def}}{=} \left\{x = \lim_{h \to +\infty} \delta_{r_h} x_h \in \mathbb{H}^n,\ \text{with } r_h \to +\infty \text{ and } x_h \in A\right\}$$

and the cone in a point p is given as $\mathrm{Tan}_{\mathbb{H}}(A, p) \stackrel{\mathrm{def}}{=} \tau_p \mathrm{Tan}_{\mathbb{H}}(\tau_{-p} A, 0)$.

We prove, in Theorem 3.5, that a k-dimensional $\mathbb{H}$-regular surface S has an intrinsic tangent cone $\mathrm{Tan}_{\mathbb{H}}(S, p)$ at each point p and that $\mathrm{Tan}_{\mathbb{H}}(S, p)$ is the Euclidean tangent k-plane $\mathrm{Tan}(S, p)$ to S in p. Notice that this statement is far from being evident, because $\mathrm{Tan}_{\mathbb{H}}(S, p)$ is the limit of S under intrinsic dilations δ_λ, while $\mathrm{Tan}(S, p)$ is the limit under Euclidean dilations.

If $S = \{p:\ f(p) = 0\}$ is a k-codimensional $\mathbb{H}$-regular surfaces, in Proposition 3.29 we prove that the Heisenberg tangent cone $\mathrm{Tan}_{\mathbb{H}}(S, p)$ is always a $(2n + 1 - k)$-plane and that it is the translated in p of the kernel of the differential $d_H f_p$. On the contrary, the Euclidean tangent plane to S may never exist.

On the other side, not necessarily a k-dimensional, smooth, Euclidean submanifold of $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$ belongs to any of these families: clearly it does not for $1 \leqslant k \leqslant n$ because of the necessary condition of being tangent to $H\mathbb{H}^n$, but also for $n < k$ because of the possible presence of the so-called characteristic points.

The following theorem provides a description of the class of the k-dimensional $\mathbb{H}$-regular surfaces.

Theorem 3.5. *If S is a k-dimensional $\mathbb{H}$-regular surface, $1 \leqslant k \leqslant n$, then*:

(1) *S is an Euclidean k-dimensional submanifold of $\mathbb{R}^{2n+1}$ of class $\mathcal{C}^1$.*
(2) *The Euclidean tangent bundle $\operatorname{Tan} S$ is a subbundle of $\bigwedge_k \mathfrak{h}_1$ and*

$$\operatorname{Tan}(S, p) = \operatorname{Tan}_{\mathbb{H}}(S, p)$$

for any point $p \in S$.
(3) *$\mathcal{S}^k_\infty \llcorner S$ is comparable with $\mathcal{H}^k_E \llcorner S$.*

Proof. Let $\mathcal{V} \subset \mathbb{R}^k$, $\mathcal{U} \subset \mathbb{H}^n$ be open sets such that $\phi : \mathcal{V} \to \mathcal{U}$, $\phi \in \mathcal{C}^1(\mathcal{V}; \mathbb{H}^n)$, ϕ injective, $d_H\phi$ injective and $S \cap \mathcal{U} = \phi(\mathcal{V})$. Assume $p = \phi(x) \in S \cap \mathcal{U}$ and $x \in \mathcal{V}$. To prove (1) it is enough to show that the Euclidean differential $d\phi_x$ exists for every $x \in \mathcal{V}$, depends continuously on x and that $d\phi_x$ is injective. Notice that $\phi \in \mathcal{C}^1(\mathbb{R}^k; \mathbb{H}^n)$ yields that $\phi \in \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}^k; \mathbb{H}^n)$ and this in turn implies that $\phi \in \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}^k; \mathbb{R}^{2n+1})$. Hence ϕ is Euclidean differentiable a.e. in $\mathcal{V}$. Let $x_0 \in \mathcal{V}$ be such that both $d\phi_{x_0}$ and $d_H\phi_{x_0}$ exist.

By Proposition 2.7, there exists a $2n \times k$ matrix A_{x_0} with $A^T_{x_0} J A_{x_0} = 0$, such that

$$d_H\phi_{x_0}(\xi) = (A_{x_0}\xi, 0),$$

for all $\xi \in \mathbb{R}^k$. By the very definition of P-differential, it is easy to see that the rows of A_{x_0} are just the first $2n$ rows of the (Euclidean) Jacobian matrix of $\phi = (\phi_1, \dots, \phi_{2n+1})$ in x_0. Because $d_H\phi_x$ is continuous and everywhere defined in $\mathcal{V}$, it follows that $\nabla\phi_j(x)$, $1 \leqslant j \leqslant 2n$, exist in $\mathcal{V}$ and are continuous in x.

Because the last component of $d_H\phi_x$ is zero, once more by the definition of P-differentiability, it follows

$$\nabla\phi_{2n+1}(x) = 2\sum_{j=1}^{n} \big(\phi_{j+n}(x)\nabla\phi_j(x) - \phi_j(x)\nabla\phi_{j+n}(x)\big), \tag{11}$$

for all $x \in \mathcal{V}$. This implies that $\nabla\phi_{2n+1}(x)$ is a continuous function and eventually that ϕ is continuously differentiable.

Because the rank of A_x equals k for any x, since $d_H\phi$ is 1–1, also the Jacobian matrix of ϕ is a $(2n+1) \times k$ matrix with rank k and the proof of (1) is completed.

Let us now prove $\operatorname{Tan}(S, p) = \operatorname{Tan}_{\mathbb{H}}(S, p)$ for any point $p \in S$.

First observe that, if $x \in \mathcal{V}$ and $p = \phi(x)$, an explicit computation, using (11), gives

$$p + d\phi_x(h) = p \cdot d_H\phi_x(h), \quad \text{for any } h \in \mathbb{R}^k.$$

Because $\operatorname{Tan}(S, p) = p + d\phi_x(\mathbb{R}^k)$, to achieve point (2) of the thesis, it is enough to show that

$$\operatorname{Tan}_{\mathbb{H}}(S, p) = p \cdot d_H\phi_x\big(\mathbb{R}^k\big). \tag{12}$$

Without loss of generality, we can assume $p = 0 = \phi(0)$, so that we have to prove that $d_H\phi_0(\mathbb{R}^k) = \operatorname{Tan}_{\mathbb{H}}(S, 0)$.

Let $\xi = d_H\phi_0(h)$ be given. Consider the points $p_n = \phi(\frac{1}{n}h)$ that belong to S for $n \in \mathbb{N}$ sufficiently large. By definition of P-differential

$$\delta_n(p_n) \to d_H\phi_0(h) = \xi \quad \text{as } n \to \infty,$$

so that $\xi \in \mathrm{Tan}_{\mathbb{H}}(S, 0)$ and $d_H\phi_0(\mathbb{R}^k) \subset \mathrm{Tan}_{\mathbb{H}}(S, 0)$.

To prove the reverse inclusion, let $\xi \in \mathrm{Tan}_{\mathbb{H}}(S, 0)$ be of the form $\xi = \lim_{h\to+\infty} \delta_{r_h} p_h$ with $r_h \to +\infty$ and $p_h \in S$. Since $r_h d_c(p_h, 0) = d_c(\delta_{r_h} p_h, 0) \to d_c(\xi, 0)$, necessarily $p_h \to 0$ as $h \to \infty$. Thus, by local inverse function theorem, we can assume without loss of generality that $p_h = \phi(z_h)$, with $z_h \in \mathbb{R}^k$, $z_h \to 0$ as $h \to \infty$. Notice now that there exist $c > 0$ and $\rho > 0$ such that

$$|z|_{\mathbb{R}^k} \leqslant c d_c\big(\phi(z), 0\big), \quad \text{provided} \quad |z|_{\mathbb{R}^k} \leqslant \rho. \tag{13}$$

Indeed, suppose by contradiction the statement is false: then there exists a sequence of points $w_h \in \mathbb{R}^k$ such that $w_h \to 0$ and

$$d_c\big(\phi(w_h), 0\big)/|w_h|_{\mathbb{R}^k} \to 0 \quad \text{as } h \to \infty.$$

Without loss of generality, we may assume $w_h/|w_h|_{\mathbb{R}^k} \to w$ as $h \to \infty$, with $|w| = 1$. Then, by definition of P-differential, because the convergence is required to be uniform with respect to the direction, we have

$$\begin{aligned} 0 &= \lim_{h\to\infty} \frac{d_c(\phi(w_h), 0)}{|w_h|_{\mathbb{R}^k}} = \lim_{h\to\infty} d_c\left(\delta_{1/|w_h|_{\mathbb{R}^k}}\left(\phi\left(|w_h|_{\mathbb{R}^k} \frac{w_h}{|w_h|_{\mathbb{R}^k}}\right)\right), 0\right) \\ &= d_c\big(d_H\phi(0)w, 0\big), \end{aligned}$$

that yields $w = 0$ because of the injectivity of $d_H\phi_0$ and hence a contradiction. Thus, we can apply (13) with $z = z_h$ for h sufficiently large, and we get $r_h|z_h| \leqslant c r_h d_c(p_h, 0) = c d_c(\delta_{r_h} p_h, 0) \leqslant C$, for $h \in \mathbb{N}$, and therefore we can assume $r_h z_h \to z_0$ as $h \to \infty$. Finally, once more by definition of P-differential, we get that $\xi \in d_H\phi_0(\mathbb{R}^k)$, because $\xi = \lim_{h\to+\infty} \delta_{r_h} p_h = \lim_{h\to+\infty} \delta_{r_h}\phi(\frac{1}{r_h} r_h z_h) = d_H\phi_0(z_0)$, achieving the proof of (2).

Finally we address the proof of (3). Assume that $\mathcal{U}$ is a bounded open set. Since $\mathcal{H}^k_\infty$ is comparable with $\mathcal{S}^k_\infty$, we will prove that there are positive constants c_1 and c_2, depending on $\mathcal{U}$, such that

$$c_1 \mathcal{H}^k_E(S \cap \mathcal{U}) \leqslant \mathcal{H}^k_\infty(S \cap \mathcal{U}) \leqslant c_2 \mathcal{H}^k_E(S \cap \mathcal{U}). \tag{14}$$

Because $d_H\phi_x(\mathbb{R}^k) \subset H\mathbb{H}^n_0$, it follows that, on $d_H\phi_x(\mathbb{R}^k) \equiv d\phi_x(\mathbb{R}^k)$, group translations and dilations coincide with Euclidean translations and dilations. Hence $\mathcal{H}^k_\infty \llcorner d_H\phi_x(\mathbb{R}^k)$ and $\mathcal{H}^k_E \llcorner d\phi_x(\mathbb{R}^k)$ are uniformly distributed measures on $d_H\phi_x(\mathbb{R}^k)$, so that (see, e.g., Theorem 3.4 in [21]) there is $c = c(x)$, positive and continuously dependent on x, such that

$$\mathcal{H}^k_E \llcorner d_H\phi_x\big(\mathbb{R}^k\big) = c(x)\, \mathcal{H}^k_\infty \llcorner d\phi_x\big(\mathbb{R}^k\big). \tag{15}$$

From the area formula in Carnot groups (see Theorem 4.3.4 in [19]) we have

$$H_\infty^k(S \cap \mathcal{U}) = \int_{\mathcal{V}} \frac{\mathcal{H}_\infty^k \llcorner d_H\phi_x(d_H\phi_x(B_\infty))}{\mathcal{L}^{2n+1}(B_\infty)}\, dx$$

where B_∞ is the unit ball with respect to the d_∞ metric. On the other hand, from the Euclidean area formula we have

$$H_E^k(S \cap \mathcal{U}) = \int_{\mathcal{V}} \frac{\mathcal{H}_E^k \llcorner d\phi_x(d\phi_x(B))}{\mathcal{L}^{2n+1}(B)}\, dx$$

where B is the unit ball with respect to the Euclidean metric. Using (15), the two area formulas entail the proof of (3). $\square$

3.2. *Foliations and graphs in a Lie group $\mathbb{G}$*

The Heisenberg group $\mathbb{H}^n$ and also any other Carnot group $\mathbb{G}$ is a product of subgroups in many different ways. Hence it makes sense in a natural way to speak of subsets that are graphs inside $\mathbb{G}$. The following definition seems to share with the usual Euclidean notion many good features.

Assume that the algebra $\mathfrak{g}$ of $\mathbb{G}$ is the direct sum of two subalgebras $\mathfrak{w}$ and $\mathfrak{v}$, that is

$$\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}.$$

Set now $\mathbb{G}_\mathfrak{w} := \exp \mathfrak{w}$, and $\mathbb{G}_\mathfrak{v} := \exp \mathfrak{v}$. We denote *system of coordinate planes* (i.e. left laterals) of $\mathbb{G}$ the double family $\mathcal{L}_\mathfrak{v}$ and $\mathcal{L}_\mathfrak{w}$ defined as

$$\mathcal{L}_\mathfrak{v}(p) := p \cdot \mathbb{G}_\mathfrak{v}, \quad \forall p \in \mathbb{G}_\mathfrak{w} \qquad \text{and} \qquad \mathcal{L}_\mathfrak{w}(q) := q \cdot \mathbb{G}_\mathfrak{w}, \quad \forall q \in \mathbb{G}_\mathfrak{v}.$$

Observe that each $x \in \mathbb{G}$ belongs exactly to one leaf in $\mathcal{L}_\mathfrak{v}$ and to one in $\mathcal{L}_\mathfrak{w}$ and that the leaves are invariant by translations, that is $x \in \mathcal{L}_\mathfrak{v}(p) \Rightarrow \tau_x \mathcal{L}_\mathfrak{v}(p) = \mathcal{L}_\mathfrak{v}(p)$. In particular, each $x \in \mathbb{G}$ can be written in a unique way as $x = x_\mathfrak{w} \cdot x_\mathfrak{v}$, with $x_\mathfrak{w} \in \mathbb{G}_\mathfrak{w}$ and $x_\mathfrak{v} \in \mathbb{G}_\mathfrak{v}$. It is easy to see that there is $c = c(\mathfrak{w}, \mathfrak{v}) > 1$ such that

$$c^{-1}\big(|x_\mathfrak{w}| + |x_\mathfrak{v}|\big) \leqslant |x| \leqslant |x_\mathfrak{w}| + |x_\mathfrak{v}|. \tag{16}$$

We propose the following

Definition 3.6 *(Graphs and Regular Graphs).* Assume $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}$, with $\mathfrak{v}$ and $\mathfrak{w}$ subalgebras. A set $S \subset \mathbb{G}$ is a *graph over* $\mathbb{G}_\mathfrak{w}$ *along* $\mathbb{G}_\mathfrak{v}$ if, for each $\xi \in \mathbb{G}_\mathfrak{w}$, $S \cap \mathcal{L}_\mathfrak{v}(\xi)$ contains at most one point. Equivalently if there is a function $\varphi : E \subset \mathbb{G}_\mathfrak{w} \to \mathbb{G}_\mathfrak{v}$ such that

$$S = \big\{\xi \cdot \varphi(\xi):\ \xi \in E\big\}$$

and we say that S *is the graph of* φ. If $\mathfrak{w}$ is also an ideal and S is as before we say that S is a *regular graph*.

When $\mathbb{G} \equiv \mathbb{H}^n$, the following special instance of Definition 3.6, similar to the notion of orthogonal graphs in Euclidean spaces, is available.

Definition 3.7 *(Orthogonal Graph).* With the notations of Definition 3.6, let $\mathbb{G} \equiv \mathbb{H}^n$, $(w_1, \dots, w_{2n+1-k})$, $(v_1, \dots, v_k)$ be bases, respectively, of $\mathfrak{w}$ and of $\mathfrak{v}$, $|v_1 \wedge \cdots \wedge v_k| = |w_1 \wedge \cdots \wedge w_{2n+1-k}| = 1$ and let

$$w_1 \wedge \cdots \wedge w_{2n+1-k} = *(v_1 \wedge \cdots \wedge v_k),$$

then we call S an *orthogonal graph over* $\mathbb{G}_\mathfrak{w}$.

We stress here that these intrinsic notions of graphs, adapted to the geometry of the group, are not a pointless generalization. From one side, the fact that a surface is locally a graph is, as usual, a powerful tool; here the fact that $\mathbb{H}$-regular surfaces are locally intrinsic graphs is a key tool in studying their local structure (see Sections 3.5 and 4). On the other side, one could not have used the usual Euclidean notion. Indeed, as the following example shows, $\mathbb{H}$-*regular surfaces* (*of low codimension*)*, in general, are not graphs in the usual Euclidean sense, while they are always, locally, graphs in the intrinsic Heisenberg sense*.

Example 3.8. See Figs. 1 and 2. In $\mathbb{H}^1$, with the notations of Definition 3.6, let $\mathfrak{v} = \mathrm{span}\{X\}$ and $\mathfrak{w} = \mathrm{span}\{Y, T\}$. Then $\mathbb{G}_\mathfrak{v} = \{(x, 0, 0)\colon\ x \in \mathbb{R}\}$ and $\mathbb{G}_\mathfrak{w} = \{(0, \eta, \tau)\colon\ \eta, \tau \in \mathbb{R}\}$. Then, fix $1/2 < \alpha < 1$, and take $\varphi : \mathbb{G}_\mathfrak{w} \to \mathbb{G}_\mathfrak{v}$ as $\varphi(0, \eta, \tau) = (|\tau|^\alpha, 0, 0)$. Define S as the graph of φ, precisely

$$S = \bigl\{\xi \cdot \varphi(\xi)\colon\ \xi \in \mathbb{G}_\mathfrak{w}\bigr\} = \bigl\{\bigl(|\tau|^\alpha, \eta, \tau + 2\eta|\tau|^\alpha\bigr)\colon\ \eta, \tau \in \mathbb{R}\bigr\}.$$

From Corollary 5.11 of [3], it follows that S is a $\mathbb{H}$-regular surface. But, as one can easily check, S is not an Euclidean graph in any neighborhood of the origin.

Notice that one could have defined intrinsic graphs in more general ways. For example, one can drop the assumption that $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras asking only that they are linear subspaces such that $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}$. Everything said up to now about graphs is true in this more general setting, but for the fact that the coordinate planes in $\mathcal{L}_\mathfrak{v}$ and $\mathcal{L}_\mathfrak{w}$ are not anymore cosets of $\mathbb{G}$. This more general setting has been taken by many authors, for example when sets (graphs) as $\{(x_1, x_2, f(x_1, x_2))\} \subset \mathbb{H}^1$ are studied. In our notation this amounts to the choice of $\mathfrak{v} = \mathrm{span}\{T\}$ and $\mathfrak{w} = \mathrm{span}\{X_1, Y_1\}$. Here clearly $\mathfrak{w}$ is not a subalgebra and $\exp \mathfrak{w}$ is not a group.

On the other hand, intrinsic graphs, as in Definition 3.6, enjoy some nice properties that are not anymore true admitting more general definitions. For example, if $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras the intrinsic Hausdorff dimensions of the coordinate planes add up correctly to the total homogeneous dimension of $\mathbb{H}^n$. This may be false in more general settings. Think again to $\mathbb{H}^1$ with, as before, $\mathfrak{v} = \mathrm{span}\{T\}$ and $\mathfrak{w} = \mathrm{span}\{X_1, Y_1\}$; then $\dim(\exp \mathfrak{v}) = 2$, $\dim(\exp \mathfrak{w}) = 3$ (at least in a generic non-characteristic point) while $\dim(\mathbb{H}^1) = 4$.

Moreover, if $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras and if S is an intrinsic graph over $\mathbb{G}_\mathfrak{w}$ then also any left translation of S along $\mathbb{G}_\mathfrak{w}$ is an intrinsic graph. Precisely, if $p \in \mathbb{G}_\mathfrak{w}$ then $\tau_p S = \{p \cdot \xi \cdot \varphi(\xi)\colon\ \xi \in E\} = \{\eta \cdot \varphi \circ \tau_{-p}(\eta)\colon\ \eta \in \tau_p E\}$. That is, as it happens with Euclidean graphs, if S is the graph of φ then $\tau_p S$ is the graph of $\varphi \circ \tau_{-p}$.

If S is a regular graph in the sense of Definition 3.6, it is possible to write explicitly how S behaves under a generic translation. Indeed:

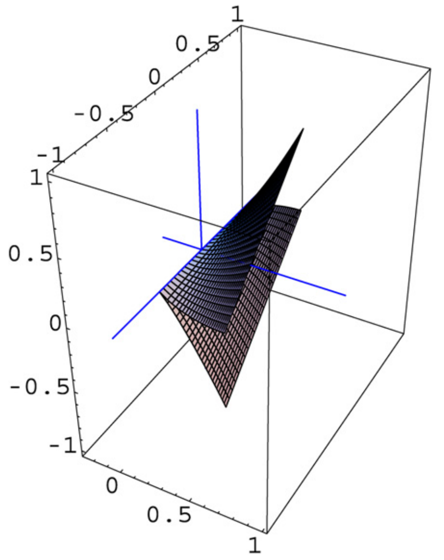

Fig. 1. The surface $S \subset \mathbb{H}^1$ of Example 3.8 when $\alpha = 2/3$.

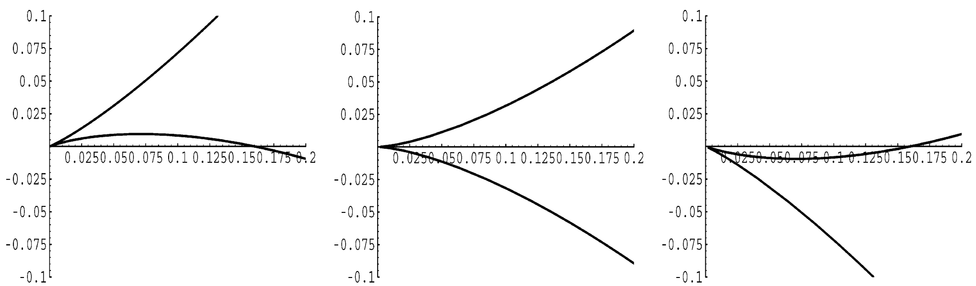

Fig. 2. Sections of S for $x = 0.2$, $x = 0$ and $x = -0.2$.

Proposition 3.9. *With notations of Definition* 3.6*, assume that* $S = \{\Phi(\xi) := \xi \cdot \varphi(\xi)\colon \xi \in E\}$ *is a regular graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$*. Let* $q = q_{\mathfrak{w}} \cdot q_{\mathfrak{v}} \in \mathbb{G}$*, with* $q_{\mathfrak{w}} \in \mathbb{G}_{\mathfrak{w}}$ *and* $q_{\mathfrak{v}} \in \mathbb{G}_{\mathfrak{v}}$*. Then the translated set* $\tau_q S$ *is again a regular graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$*, precisely*

$$\tau_q S = \left\{\Phi_q(\eta) := \eta \cdot \varphi_q(\eta)\colon\ \eta \in E' := q \cdot E \cdot (q_{\mathfrak{v}})^{-1} \subset \mathbb{G}_{\mathfrak{w}}\right\},$$

where $\varphi_q : E' \to \mathbb{G}_{\mathfrak{v}}$ *is defined as* $\varphi_q(\eta) := q_{\mathfrak{v}} \cdot \varphi(q^{-1} \cdot \eta \cdot q_{\mathfrak{v}})$*. In addition* $\Phi_q = \tau_{q^{-1}} \circ \Phi \circ \sigma_{q^{-1}}$*, where* $\sigma_p : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{w}}$ *is defined by* $\sigma_p(\eta) = p \cdot \eta \cdot p_{\mathfrak{v}}^{-1}$*.*

Proof. Because $\mathbb{G}_{\mathfrak{w}}$ is a normal subgroup of $\mathbb{G}$ then $E' = q_{\mathfrak{w}} \cdot q_{\mathfrak{v}} \cdot E \cdot q_{\mathfrak{v}}^{-1} \subset \mathbb{G}_{\mathfrak{w}}$. Given this, the proof is an elementary computation. □

If we assume that $\mathbb{G}$ is an homogeneous group, or in particular a Carnot group, so that a family of dilations $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is defined, we can study also the behavior of a graph under dilations. It is easy to see that:

Proposition 3.10. *With notations of Definition* 3.6*, assume that* $\mathbb{G}$ *is a homogeneous group with family of dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ *for* $\lambda > 0$*, and assume that* $\mathbb{G}_{\mathfrak{w}}$ *and* $\mathbb{G}_{\mathfrak{v}}$ *are subgroups invariant by dilations. If* $S = \{\xi \cdot \varphi(\xi) \colon \xi \in E\}$ *is a graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$*, then any dilated set* $\delta_\lambda S$ *is again a graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$*, precisely*

$$\delta_\lambda S = \{\eta \cdot \varphi_\lambda(\eta) \colon \eta \in E' := \delta_\lambda E \subset \mathbb{G}_{\mathfrak{w}}\},$$

where $\varphi_\lambda : E' \to \mathbb{G}_{\mathfrak{v}}$ *is defined as* $\varphi_\lambda = \delta_\lambda \circ \varphi \circ \delta_{1/\lambda}$.

Proof.

$$\begin{aligned}\delta_\lambda S &= \{\delta_\lambda(\eta \cdot \varphi(\eta)) \colon \eta \in E \subset \mathbb{G}_{\mathfrak{w}}\}\\ &= \{\delta_\lambda \eta \cdot (\delta_\lambda \circ \varphi \circ \delta_{1/\lambda})(\delta_\lambda \eta) \colon \eta \in E \subset \mathbb{G}_{\mathfrak{w}}\}. \qquad \square\end{aligned}$$

Remark 3.11. Definition 3.6 can be written 'in coordinates.' If $v_1, \dots, v_k$ and $w_1, \dots, w_{2n+1-k}$ are bases of $\mathfrak{v}$ and $\mathfrak{w}$, then φ can be associated with a k-uple $(\varphi_1, \dots, \varphi_k) : \tilde{E} \subset \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$, making the following diagram commutative:

$$\begin{tikzcd} \mathbb{G}_{\mathfrak{w}} \arrow[r, "\varphi"] & \mathbb{G}_{\mathfrak{v}} \\ \mathbb{R}^{2n+1-k} \arrow[u, "\exp"] \arrow[r, "(\varphi_1,\dots,\varphi_k)"'] & \mathbb{R}^k. \arrow[u, "\exp"'] \end{tikzcd}$$

That is,

$$\varphi(\xi) = \exp\left(\sum_{l=1}^{k} \varphi_l(\xi_1, \dots, \xi_{d-k}) v_l\right) \quad \text{for } \xi = \exp\left(\sum_{l=1}^{2n+1-k} \xi_l w_l\right) \in E$$

and S is a graph of $\varphi : E \to \mathbb{G}_{\mathfrak{v}}$ if

$$S = \left\{\xi \cdot \exp\left(\sum_{l=1}^{k} \varphi_l(\xi_1, \dots, \xi_{d-k}) v_l\right)\right\}. \tag{17}$$

Remark 3.12. When $\mathbb{G} \equiv \mathbb{H}^n$, if $\mathfrak{h} = \mathfrak{w} \oplus \mathfrak{v}$ and $\mathfrak{w}$, $\mathfrak{v}$ are subalgebras, then the larger one of the two algebras is necessarily an ideal, that is, in $\mathbb{H}^n$ graphs of codimension strictly smaller than $n+1$ are necessarily regular graphs. We are indebted with Adam Korányi for this remark and for the following elegant proof [17].

Assume that $\dim \mathfrak{w} \geqslant n+1$, then there are two cases:

(1) $\mathfrak{w}$ is not abelian. Then it contains some non-zero bracket, hence it contains T, hence it contains $\mathfrak{h}_2$ so that $\mathfrak{w}$ is an ideal.
(2) $\mathfrak{w}$ is abelian. Consider the bilinear form B on $\mathfrak{h}$ defined by

$$B(X, Y)T := [X, Y].$$

Observe that B restricted to $\mathfrak{h}_1$ is symplectic. Because B is invariant under the projection $P:\mathfrak{h}\to\mathfrak{h}_1$, then $P\mathfrak{w}$ is an isotropic subspace of $\mathfrak{h}_1$, hence $\dim\mathfrak{w}\leqslant n$. Clearly $\mathfrak{w}$ is a subspace of $P\mathfrak{w}+\mathfrak{h}_2$. Then

$$n+1\leqslant\dim\mathfrak{w}\leqslant\dim(P\mathfrak{w}+\mathfrak{h}_2)=\dim(P\mathfrak{w})+1\leqslant n+1.$$

Hence $\dim\mathfrak{w}=\dim(P\mathfrak{w}+\mathfrak{h}_2)$ so that $\mathfrak{w}=P\mathfrak{w}+\mathfrak{h}_2$ and, consequently, $\mathfrak{w}$ contain $\mathfrak{h}_2$ so that it is an ideal.

3.3. Implicit Function Theorem

In the first part of this section we prove a preliminary general version of the Implicit Function Theorem (IFT). The main assumption in IFT can be stated in geometrical terms as follows: There exist a foliation of $\mathbb{H}^n$ whose leaves are cosets of a subgroup $\mathbb{G}_{\mathfrak{v}}$ and are 'transverse' to the surface in object (this is the meaning of (18) and (19)). The thesis is that the surface is a graph associated with the assumed foliation (see the previous section for the relevant definitions). In Proposition 3.25 we will prove that the assumption of the existence of an adapted foliation holds true, at least locally, for any k-codimensional, $\mathbb{H}$-regular surface. This way we prove that, locally, these surfaces are always Heisenberg regular graphs.

The strategy of our proof, suggested by an argument used in [8] for codimension 1 surfaces in nilpotent groups, is simple: we change variables using exponential coordinates associated with the above mentioned foliation, this way, using in an essential way that the leaves of the foliation are cosets in $\mathbb{H}^n$, the assumptions of IFT in $\mathbb{H}^n$ become the assumptions of a classical IFT in Euclidean spaces.

Probably it is better to remark that here we are forced to use a version of classical IFT, that albeit being well known, it is not the one most frequently stated in Calculus books. Precisely, given $g:\mathbb{R}^d\to\mathbb{R}^k$ and the equation $g(x,y)=0$ we do not assume that g is globally C^1 but merely that it is globally continuous and C^1 only in the variables to be made explicit. Consequently the implicitly defined function $\phi:\mathbb{R}^{d-k}\to\mathbb{R}^k$, such that $g(x,\phi(x))=0$, turns out to be only continuous and not C^1.

In the second part of the section we provide a few results related with the regularity of the implicitly defined functions that will be used in the proof of the existence of the tangent plane to $\mathbb{H}$-regular surfaces.

We describe now the setting of the theorem.

Assume $1\leqslant k\leqslant n$, $v=v_1\wedge\cdots\wedge v_k\in{}_H\bigwedge_k$, $v\neq 0$. That is $v_1,\dots,v_k$ are linearly independent, left invariant vector fields in $\mathfrak{h}_1$ satisfying

$$[v_i,v_j]=0,\quad \text{for all } 1\leqslant i,j\leqslant k. \tag{18}$$

By definition, $*v\in{}_H\bigwedge_{2n+1-k}$ and there are $w_1,\dots,w_{2n-k}\in\mathfrak{h}_1$ with $w_{2n+1-k}=T$ such that $*v=w_1\wedge\cdots\wedge w_{2n+1-k}$.

Set $\mathfrak{v}:=\operatorname{span}\{v_1,\dots,v_k\}$, $\mathbb{G}_{\mathfrak{v}}=\exp\mathfrak{v}$, $\mathfrak{w}:=\operatorname{span}\{w_1,\dots,w_{2n+1-k}\}$, $\mathbb{G}_{\mathfrak{w}}=\exp\mathfrak{w}$.

Notice that $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras, $\mathfrak{w}$ is an ideal and $\mathfrak{w}\oplus\mathfrak{v}=\mathfrak{h}$. Hence $\mathbb{G}_{\mathfrak{v}}$ and $\mathbb{G}_{\mathfrak{w}}$ are subgroups, $\mathbb{G}_{\mathfrak{w}}$ is a normal subgroup, the family of cosets $\mathcal{L}_{\mathfrak{v}}(p):=p\cdot\mathbb{G}_{\mathfrak{v}}$, $p\in\mathbb{G}_{\mathfrak{w}}$, is a foliation of $\mathbb{H}^n$.

Proposition 3.13 *(Implicit Function Theorem). Let* $\mathcal{U} \subset \mathbb{H}^n$ *be an open set,* $p^0 \in \mathcal{U}$, $p^0 = p^0_{\mathfrak{w}} \cdot p^0_{\mathfrak{v}}$ *with* $p^0_{\mathfrak{w}} \in \mathbb{G}_{\mathfrak{w}}$ *and* $p^0_{\mathfrak{v}} \in \mathbb{G}_{\mathfrak{v}}$. *Assume*: $f = (f_1, \dots, f_k) : \mathcal{U} \to \mathbb{R}^k$ *is a continuous function with* $f(p^0) = 0$, $v_j f_i$ *are continuous functions in* $\mathcal{U}$ *for* $1 \leqslant i, j \leqslant k$ *and*

$$\det\bigl(\bigl[v_i f_j\bigl(p^0\bigr)\bigr]_{1 \leqslant i, j \leqslant k}\bigr) \neq 0. \tag{19}$$

Finally define $S := \{p \in \mathcal{U}\colon\ f(p) = 0\}$.

Then there are an open set $\mathcal{U}' \subset \mathcal{U}$, *with* $p^0 \in \mathcal{U}'$, *such that* $S \cap \mathcal{U}'$ *is a* $(2n+1-k)$*-dimensional continuous graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$, *that is, there are a relatively open set* $\mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}$, $p^0_{\mathfrak{w}} \in \mathcal{V}$ *and* $\varphi : \mathcal{V} \to \mathbb{G}_{\mathfrak{v}}$, *with* $\varphi(p^0_{\mathfrak{w}}) = p^0_{\mathfrak{v}}$, *such that*

$$S \cap \mathcal{U}' = \bigl\{\xi \cdot \varphi(\xi),\ \xi \in \mathcal{V}\bigr\}. \tag{20}$$

Proof. Let $d := 2n + 1$. Consider the one to one map $\psi : \mathbb{R}^{d-k} \times \mathbb{R}^k \to \mathbb{H}^n \simeq \mathbb{R}^d$, defined as

$$\psi(x_1, \dots, x_{d-k}, y_1, \dots, y_k) \overset{\text{def}}{=} \exp \sum_{l=1}^{d-k} x_l w_l \cdot \exp \sum_{l=1}^{k} y_l v_l. \tag{21}$$

Observe that ψ, as a map $\mathbb{R}^d \to \mathbb{R}^d$, is a global diffeomorphism. Moreover by definition, $\psi(\mathbb{R}^{d-k} \times \{0\}) = \mathbb{G}_{\mathfrak{w}}$ and $\psi(\{0\} \times \mathbb{R}^k) = \mathbb{G}_{\mathfrak{v}}$. We define $\psi_{\mathfrak{w}} : \mathbb{R}^{d-k} \to \mathbb{G}_{\mathfrak{w}}$ as $\psi_{\mathfrak{w}}(x_1, \dots, x_{d-k}) := \psi(x_1, \dots, x_{d-k}, 0)$ and $\psi_{\mathfrak{v}} : \mathbb{R}^k \to \mathbb{G}_{\mathfrak{v}}$ analogously. Let $(x^0_1, \dots, y^0_k) = \psi_{\mathfrak{v}}^{-1}(p^0)$. Define the map $g : \mathbb{R}^{d-k} \times \mathbb{R}^k \to \mathbb{R}^k$ as $g = f \circ \psi$, that is

$$g(x_1, \dots, y_k) = f\left(\exp \sum_{l=1}^{d-k} x_l w_l \cdot \exp \sum_{l=1}^{k} y_l v_l\right),$$

so that the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{H}^n & \supset & \mathcal{U} \arrow[r, "f"] & \mathbb{R}^k \\
\mathbb{R}^d \arrow[u, "\psi"] & \supset & \psi^{-1}(\mathcal{U}). \arrow[ur, "g"'] &
\end{tikzcd}$$

Clearly g is continuous in the open set $\psi^{-1}(\mathcal{U}) \subset \mathbb{R}^d$, $(x^0_1, \dots, y^0_k) \in \psi^{-1}(\mathcal{U})$ and $g(x^0_1, \dots, y^0_k) = 0$. The derivatives $\partial g_i / \partial y_j$ exist and are continuous in $\psi^{-1}(\mathcal{U})$. Indeed, using Campbell–Hausdorff formula, assumption (18) entails that

$$\frac{\partial g_i}{\partial y_j}(x_1, \dots, y_k) = (v_j f_i)\bigl(\psi(x_1, \dots, y_k)\bigr). \tag{22}$$

Hence, through (22), assumption (19) becomes

$$\det\left(\left[\frac{\partial g_i}{\partial y_j}\bigl(x^0_1, \dots, y^0_k\bigr)\right]_{1 \leqslant i, j \leqslant k}\right) \neq 0. \tag{23}$$

Then, as discussed at the beginning of this section, an application of classical Implicit Function Theorem to g yields that there are an open $\tilde{\mathcal{U}} \subset \psi^{-1}(\mathcal{U})$, such that $(x_1^0, \dots, y_k^0) \in \tilde{\mathcal{U}}$, an open $\tilde{\mathcal{V}} \subset \mathbb{R}^{d-k}$ with $(x_1^0, \dots, x_{d-k}^0) \in \tilde{\mathcal{V}}$ and a continuous $\mathbb{R}^k$ valued function $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_k) : \tilde{\mathcal{V}} \to \mathbb{R}^k$, such that

$$\begin{aligned}\tilde{S} &:= \bigl\{(x_1, \dots, y_k) \in \tilde{\mathcal{U}}:\ g(x_1, \dots, y_k) = 0\bigr\}\\ &= \bigl\{g\bigl(x_1, \dots, x_{d-k}, \tilde{\varphi}(x_1, \dots, x_{d-k})\bigr):\ (x_1, \dots, x_{d-k}) \in \tilde{\mathcal{V}}\bigr\}.\end{aligned}$$

Finally, assertion (20) follows with $\mathcal{U}' = \psi(\tilde{\mathcal{U}})$, $\mathcal{V} = \psi(\tilde{\mathcal{V}} \times \{0\})$ and

$$\varphi \stackrel{\text{def}}{=} \psi_{\mathfrak{v}} \circ \tilde{\varphi} \circ \psi_{\mathfrak{w}}^{-1} : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}. \qquad \square$$

The regularity of the implicitly defined function φ is a more delicate issue. One can address both the problems of Euclidean and of intrinsic regularity.

Example 3.14. Let $f : \mathbb{H}^1 \to \mathbb{R}$ be defined as $f(x) = x_1 - 1$. Then $S = \{x \in \mathbb{H}^1:\ x_1 = 1\}$ is 1-codimensional $\mathbb{H}$-regular surface. The function φ is constant: $\varphi(\xi_1, \xi_2) = (1, 0, 0)$ while Φ : $\mathbb{G}_{\mathfrak{w}} \to \mathbb{H}^n$, defined as $\Phi(\xi) := \xi \cdot \varphi(\xi)$—even if it is $\mathcal{C}^\infty$ in Euclidean sense from $\mathbb{R}^2 \to \mathbb{R}^3$—is not Lipschitz as a map from $\mathbb{G}_{\mathfrak{v}} \to \mathbb{G}_{\mathfrak{w}}$.

More generally, if the defining function f is Euclidean regular—say $\mathcal{C}^\infty$—then both φ and Φ are Euclidean $\mathcal{C}^\infty$ and, consequently, $\varphi \in \mathcal{C}^\infty(\mathbb{H}^n, \mathbb{R}^k)$. Here the fact that φ is $\mathbb{R}^k$ valued plays a key role, indeed, as the previous example shows, in general $\Phi \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{H}^n, \mathbb{H}^n)$.

If we do not assume Euclidean regularity on f, in general the implicitly defined functions φ and Φ do not have any Euclidean regularity.

Example 3.15. Let $f : \mathbb{H}^1 \to \mathbb{R}$ be defined as

$$f(x) = x_1 - \sqrt{x_1^4 + x_2^4 - x_3^\alpha}$$

with $1 < \alpha < 2$. Then S is a 1-codimensional $\mathbb{H}$-regular surface. In this case φ, as a map from $\mathbb{R}^2$ to $\mathbb{R}$, is not Euclidean Lipschitz continuous in 0.

Notice that a much more dramatic example, in this line, is exhibited in [16] where the corresponding function φ is non-differentiable almost everywhere.

In the Euclidean setting, a $\mathcal{C}^1$-surface is locally the graph of a $\mathcal{C}^1$-function and vice versa. In $\mathbb{H}^n$ the characterization of those functions φ whose graphs are $\mathbb{H}$-regular surfaces is a hard problem, surprisingly somehow connected with the regularity of solutions of non-linear diffusion equations. This problem is addressed in a forthcoming paper by Ambrosio, Serra Cassano and Vittone [3]. In particular, as it is shown in that paper, in general it is false that φ is a Lipschitz function from $\mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$. Nevertheless, it is true that, if $\varphi(p) = 0$, then $|\varphi(q)| \leqslant c d_c(p, q)$ (see Corollary 3.18). This fact is a key point in our proof of the existence of the tangent plane to any k-codimensional regular surface.

Proposition 3.16. *Given the same hypotheses and notations of Proposition* 3.13 *we assume also that there are* $\alpha \in (0, 1]$ *and* $c_\alpha > 0$*, such that*

$$\big|f\big(\xi_2 \cdot \varphi(\xi_1)\big)\big|_{\mathbb{R}^k} \leqslant c_\alpha \big(d_c(\xi_1, \xi_2)\big)^\alpha \tag{24}$$

for $\xi_1, \xi_2 \in \mathcal{V}$ *with* $\xi_2 \cdot \varphi(\xi_1) \in \mathcal{U}'$*. Then there is* $c > 0$*,* c *depending on* c_α*,* $\mathfrak{v}$ *and* $\mathfrak{w}$*, such that*

$$d_c\big(\varphi(\xi_1), \varphi(\xi_2)\big) \leqslant c\big(d_c(\xi_1, \xi_2)\big)^\alpha. \tag{25}$$

Proof. First observe that (23) yields that there is $r > 0$ such that the map

$$y_1, \ldots, y_k \mapsto f\big(\xi \cdot \phi_{\mathfrak{v}}(y_1, \ldots, y_k)\big),$$

from $\mathbb{R}^k$ to $\mathbb{R}^k$, is invertible in $\psi^{-1}(B(p^0, r) \cap \mathcal{U}')$, for each fixed $\xi \in \mathbb{G}_{\mathfrak{w}}$, when ξ is close to $p^0_{\mathfrak{w}}$. Moreover the inverse map is bounded, that is there is $c_1 > 0$ such that

$$\big|\psi^{-1}(\eta_2) - \psi^{-1}(\eta_1)\big|_{\mathbb{R}^k} \leqslant c_1 \big|f(\xi \cdot \eta_2) - f(\xi \cdot \eta_1)\big|_{\mathbb{R}^k}$$

when η_1 and η_2 are sufficiently close to $p^0_{\mathfrak{v}}$. Observe also that assumption (18) yields that the map $\psi_{\mathfrak{v}} : \mathbb{R}^k \to \mathbb{G}_{\mathfrak{v}}$ is globally biLipschitz.

Hence there is a relatively open set $\mathcal{V}' \subset \mathbb{G}_{\mathfrak{w}}$, with $p^0_{\mathfrak{w}} \equiv \psi_{\mathfrak{w}}(x^0_1, \ldots, x^0_{d-k}) \in \mathcal{V}'$, and there is a constant $c_2 > 0$ such that, keeping in mind that $f(\xi_1 \cdot \varphi(\xi_1)) = 0$, for $\xi_1, \xi_2 \in \mathcal{V}'$,

$$\begin{aligned}\big|f\big(\xi_1 \cdot \varphi(\xi_2)\big)\big|_{\mathbb{R}^k} &= \big|f\big(\xi_1 \cdot \varphi(\xi_2)\big) - f\big(\xi_1 \cdot \varphi(\xi_1)\big)\big|_{\mathbb{R}^k} \\ &\geqslant c_2 d_c\big(\varphi(\xi_1), \varphi(\xi_2)\big).\end{aligned}$$

On the other side, once more because $f(\xi_2 \cdot \varphi(\xi_2)) = 0$, from assumption (24) we get

$$\begin{aligned}\big|f\big(\xi_1 \cdot \varphi(\xi_2)\big)\big|_{\mathbb{R}^k} &= \big|f\big(\xi_1 \cdot \varphi(\xi_2)\big) - f\big(\xi_2 \cdot \varphi(\xi_2)\big)\big|_{\mathbb{R}^k} \\ &\leqslant c_\alpha \big(d_c(\xi_1, \xi_2)\big)^\alpha.\end{aligned}$$

Hence we get (25). □

Remark 3.17. Hypothesis (24) is not an easy one to verify. A special instance of it, that we will use later, is the following: If $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, then $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{U}; \mathbb{R}^k)$ hence there is $L = L(\mathcal{V}) > 0$ such that for $\xi_1, \xi_2 \in \mathcal{V}$,

$$\begin{aligned}\big|f\big(\xi_2 \cdot \varphi(\xi_1)\big)\big|_{\mathbb{R}^k} &= \big|f\big(\xi_2 \cdot \varphi(\xi_1)\big) - f\big(\xi_1 \cdot \varphi(\xi_1)\big)\big|_{\mathbb{R}^k} \\ &\leqslant L d_c\big(\xi_2 \cdot \varphi(\xi_1), \xi_1 \cdot \varphi(\xi_1)\big).\end{aligned}$$

Now if

$$d_c\big(\xi_2 \cdot \varphi(\xi_1), \xi_1 \cdot \varphi(\xi_1)\big) = d_c(\xi_2, \xi_1) \tag{26}$$

then (24) holds with $\alpha = 1$. Notice that (26) trivially holds when $\varphi(\xi_1) = 0$.

Corollary 3.18. *Given the same assumptions and notations of Theorem* 3.13, *assume also that* $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{U}, \mathbb{R}^k)$. *Then, for any relatively compact* $\mathcal{V}' \subset \mathcal{V}$, *there is a positive constant* c *such that the implicitly defined function* φ *satisfies*

$$d_c\big(\varphi(\xi_1), \varphi(\xi_0)\big) \leqslant c d_c\big(\xi_1 \cdot \varphi(\xi_0), \xi_0 \cdot \varphi(\xi_0)\big) \tag{27}$$

for all $\xi_0, \xi_1 \in \mathcal{V}'$. *Moreover if* $\varphi(\xi_0) = 0$, *that is if* $\xi_0 \in S \cap \mathcal{U} \cap \mathbb{G}_{\mathfrak{w}}$, *then* (27) *becomes*

$$\big|\varphi(\xi)\big| \leqslant c d_c(\xi, \xi_0), \quad \forall \xi \in \mathcal{V}'. \tag{28}$$

Proof. If $p = \xi_0 \cdot \varphi(\xi_0) \in S$ then, working as in Proposition 3.9—here we use that $\mathbb{G}_{\mathfrak{w}}$ is a normal subgroup of $\mathbb{H}^n$ because $\mathfrak{w}$ is an ideal in $\mathfrak{h}$—we get $\tau_{p^{-1}} S = \{\eta \cdot \varphi_{p^{-1}}(\eta)\colon \eta \in E'\}$, where

$$\varphi_{p^{-1}}(\eta) := \varphi(\xi_0)^{-1} \cdot \varphi\big(p \cdot \eta \cdot \varphi(\xi_0)^{-1}\big).$$

Now $\varphi_{p^{-1}}(0) = 0$ hence, keeping in mind the preceding remark, from Theorem 3.13 we get $|\varphi_{p^{-1}}(\xi)| \leqslant c|\xi|$, for all $\xi \in \mathbb{G}_{\mathfrak{w}} \cap \mathcal{V}'$, that is

$$\big|\varphi(\xi_0)^{-1} \cdot \varphi\big(p \cdot \xi \cdot \varphi(\xi_0)^{-1}\big)\big| = d_c\big(\varphi\big(p \cdot \xi \cdot \varphi(\xi_0)^{-1}\big), \varphi(\xi_0)\big) \leqslant c|\xi|.$$

Putting now $\xi_1 := p \cdot \xi \cdot \varphi(\xi_0)^{-1}$ we get (27) and (28). $\square$

Coherently with our purpose, previous results were stated in an intrinsic form, that is in coordinate free formulation. Later on we need also identities written 'in coordinates.' To this end we define a function $\tilde{\Phi}$ that is nothing but the function Φ seen in exponential coordinates.

Definition 3.19. Keeping the notations in Proposition 3.13 we define, as before,

$$\Phi : \mathcal{V} \to \mathbb{H}^n \quad \text{as} \quad \Phi(\xi) = \xi \cdot \varphi(\xi)$$

and $\tilde{\Phi} : \tilde{\mathcal{V}} \to \mathbb{R}^{2n+1}$ by the commutative diagram

$$\begin{tikzcd}
\mathbb{G}_{\mathfrak{w}} \arrow[r, "\Phi"] & \mathbb{H}^n \\
\mathbb{R}^{2n+1-k} \arrow[u, "\psi_{\mathfrak{w}}"] \arrow[r, "\tilde{\Phi}"'] & \mathbb{R}^d. \arrow[u, "\psi"']
\end{tikzcd}$$

Hence, if $x = (x_1, \ldots, x_{d-k})$,

$$\tilde{\Phi}(x) = \big(x_1, \ldots, x_{d-k}, \varphi_1(x_1, \ldots, x_{d-k}), \ldots, \varphi_k(x_1, \ldots, x_{d-k})\big),$$

where $\varphi_1, \ldots, \varphi_k$ have been defined in (17).

We evaluate here the Jacobian of the map ψ defined in (21).

Proposition 3.20. *Let $1 \leqslant k \leqslant n$, with the same notations of Proposition* 3.13*, we assume $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$ and $w = w_1 \wedge \cdots \wedge w_{2n+1-k} \in {}_H\bigwedge_{2n+1-k}$, with $w_1 \wedge \cdots \wedge w_{2n-k} \in \bigwedge_{2n-k} \mathfrak{h}_1$ and $w_{2n+1-k} = T$. Then*

$$|\det J_\psi| = |w_1 \wedge \cdots \wedge w_{2n+1-k} \wedge v_1 \wedge \cdots \wedge v_k| \tag{29}$$

and

$$|\det J_{\psi_{\mathfrak{w}}}| = |w_1 \wedge \cdots \wedge w_{2n+1-k}|. \tag{30}$$

Hence in particular if we choose $w = *v$ and $|v| = 1$ we have

$$|\det J_\psi| = 1. \tag{31}$$

Proof. Let $d = 2n + 1$, then, for $\ell = 1, \ldots, d - k$,

$$\begin{aligned}\psi(\xi,\eta) &:= \exp\left(\sum_{j=1}^{d-k} \xi_j w_j\right) \cdot \exp\left(\sum_{j=1}^{k} \eta_j v_j\right)\\ &= \exp\left(\sum_j \eta_j v_j\right) \cdot \exp\left(\sum_{j\neq\ell} \xi_j w_j\right) \cdot \exp(\xi_\ell w_\ell) + \alpha_\ell T,\end{aligned}$$

α_ℓ depend on all the variables ξ and η but not on ξ_{d-k}. Hence, because v_j and w_j are invariant by translations, we have

$$\begin{aligned}&\frac{\partial\psi}{\partial\xi_\ell} = w_\ell + \frac{\partial\alpha_\ell}{\partial\xi_\ell}T, \quad \text{for } \ell \neq d-k, \qquad \text{and} \qquad \frac{\partial\psi}{\partial\xi_{d-k}} = T;\\ &\frac{\partial\psi}{\partial\eta_j} = v_j, \quad \text{for } j = 1, \ldots, k.\end{aligned}$$

Hence

$$\begin{aligned}|\det J_\psi| = \left|\frac{\partial\psi}{\partial\xi_1} \wedge \cdots \wedge \frac{\partial\psi}{\partial\eta_k}\right| &= \left|\left(w_1 + \frac{\partial\alpha_1}{\partial\xi_1}T\right) \wedge \cdots \wedge T \wedge v_1 \wedge \cdots \wedge v_k\right|\\ &= |w_1 \wedge \cdots \wedge w_{d-k} \wedge v_1 \wedge \cdots \wedge v_k|\\ &= \left|\left\langle *(w_1 \wedge \cdots \wedge w_{d-k}), v_1 \wedge \cdots \wedge v_k\right\rangle\right|.\end{aligned}$$

The proof of (30) follows analogously. □

The following result is well known.

Lemma 3.21. *Let $\xi = \xi_1 \wedge \cdots \wedge \xi_k$, $\eta = \eta_1 \wedge \cdots \wedge \eta_k \in \bigwedge_k \mathfrak{h}$ be simple k-vectors in $\mathbb{R}^{2n+1}$. Then*

$$\langle \xi, \eta \rangle_{\bigwedge_k \mathbb{R}^{2n+1}} = \det\left[\langle \xi_i, \eta_j \rangle_{\mathbb{R}^{2n+1}}\right]_{i,j=1,\ldots,k}. \tag{32}$$

Lemma 3.22. *Let $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in \bigwedge_k \mathfrak{h}$ and $\eta = \eta_1 \wedge \cdots \wedge \eta_{d-k} \in \bigwedge_{d-k} \mathfrak{h}$ be simple. If*

$$\langle \xi_i, \eta_j \rangle_{\mathbb{R}^{2n+1}} = 0 \quad \text{for } i = 1, \dots, k,\ j = 1, \dots, d-k, \tag{33}$$

then ξ and $*\eta$ are linearly dependent, where here the $*$ operator is the Hodge operator associated with the Euclidean scalar product in $\mathbb{R}^{2n+1}$.

Proof. Put $d := 2n + 1$. Since $\langle \cdot, \cdot \rangle_{\bigwedge_k \mathbb{R}^d}$ is a positive definite scalar product in $\bigwedge_k \mathfrak{h}$, we need only to show that

$$\left| \langle \xi, *\eta \rangle_{\bigwedge_k \mathbb{R}^d} \right| = \left(\langle \xi, \xi \rangle_{\bigwedge_k \mathbb{R}^d} \right)^{1/2} \left(\langle *\eta, *\eta \rangle_{\bigwedge_k \mathbb{R}^d} \right)^{1/2}.$$

First notice that, by definition,

$$\begin{aligned} \langle *\eta, *\eta \rangle_{\bigwedge_k \mathbb{R}^d} dp_1 \wedge \cdots \wedge dp_{2n+1} &= (-1)^{k(d-k)} (*\eta) \wedge \eta = \eta \wedge *\eta \\ &= \langle \eta, \eta \rangle_{\bigwedge_{d-k} \mathbb{R}^d} dp_1 \wedge \cdots \wedge dp_{2n+1}, \end{aligned}$$

so that we have to show that

$$\left| \langle \xi, *\eta \rangle_{\bigwedge_k \mathbb{R}^d} \right| = \left(\langle \xi, \xi \rangle_{\bigwedge_k \mathbb{R}^d} \right)^{1/2} \left(\langle \eta, \eta \rangle_{\bigwedge_{d-k} \mathbb{R}^d} \right)^{1/2}.$$

Denote now by $C = [c_{i,j}]_{i,j=1,\dots,d}$ the $d \times d$ matrix with rows ordinately given by $\xi_1, \dots, \xi_k$, $\eta_1, \dots, \eta_{d-k}$, i.e. if $\xi_i = (\xi_{i1}, \dots, \xi_{id})$ and $\eta_i = (\eta_{i1}, \dots, \eta_{id})$, then

$$c_{i,j} = \begin{cases} \xi_{ij} & \text{if } i = 1, \dots, k,\ j = 1, \dots, d, \\ \eta_{ij} & \text{if } i = k+1, \dots, d,\ j = 1, \dots, d. \end{cases}$$

Keeping in mind (33) and (32), we have

$$\begin{aligned} (\det C)^2 &= \det C\, {}^t C \\ &= \det \begin{bmatrix} \langle \xi_1, \xi_1 \rangle_{\mathbb{R}^d} & \cdots & \langle \xi_1, \xi_k \rangle_{\mathbb{R}^d} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \langle \xi_k, \xi_1 \rangle_{\mathbb{R}^d} & \cdots & \langle \xi_k, \xi_k \rangle_{\mathbb{R}^d} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \langle \eta_1, \eta_1 \rangle_{\mathbb{R}^d} & \cdots & \langle \eta_1, \eta_{d-k} \rangle_{\mathbb{R}^d} \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \langle \eta_{d-k}, \eta_1 \rangle_{\mathbb{R}^d} & \cdots & \langle \eta_{d-k}, \eta_{d-k} \rangle_{\mathbb{R}^d} \end{bmatrix} \\ &= \langle \xi, \xi \rangle_{\bigwedge_k \mathbb{R}^d} \langle \eta, \eta \rangle_{\bigwedge_{d-k} \mathbb{R}^d}, \end{aligned}$$

and the lemma is proved. □

Proposition 3.23. *We keep the notations of Proposition* 3.13*, of Definition* 3.19 *and we set*

$$\Delta(p) = \left| \det\left(\left[v_i f_j(p) \right]_{1 \leqslant i,j \leqslant k} \right) \right|.$$

Assume now that f is continuously differentiable in the Euclidean sense. Then the implicitly defined function $\Phi \circ \psi_{\mathfrak{w}}$—that is continuously differentiable by usual Euclidean Implicit Function Theorem—satisfies the identity

$$\begin{aligned}
&\left|\nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p)\right|_{\bigwedge_k \mathbb{R}^d} \\
&\quad = \frac{\Delta}{|\det J_\psi|}\left|\frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi)\right|_{\bigwedge_k \mathbb{R}^d}
\end{aligned}$$

where $\xi = (\Phi \circ \psi_{\mathfrak{w}})^{-1}(p)$ *and* $\Delta = \Delta(\Phi \circ \psi_{\mathfrak{w}}(\xi))$.

Proof. Let $d = 2n+1$. Since $f_i(\Phi \circ \psi_{\mathfrak{w}}(\xi)) \equiv 0$ for $\xi \in \tilde{\mathcal{V}}$, we have

$$\left\langle \nabla f_i(\Phi \circ \psi_{\mathfrak{w}}), \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_j}\right\rangle_{\mathbb{R}^d} \equiv 0$$

for $i = 1, \dots, k$ and $j = 1, \dots, d-k$. By Lemma 3.22, this implies that, for $\xi \in \tilde{\mathcal{V}}$,

$$\nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p) = \lambda(p) * \left(\frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi)\right),$$

where, as in Lemma 3.22 and through all this proof, $*$ denotes the Hodge operator with respect to the Euclidean scalar product.

To evaluate $\lambda(p)$, from the above identity, setting $dV := dp_1 \wedge \cdots \wedge dp_d$ and $v = v_1 \wedge \cdots \wedge v_k$, we get

$$\begin{aligned}
&\left\langle v, \nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p)\right\rangle_{\bigwedge_k \mathbb{R}^d} dV \\
&\quad = \lambda(p)\left\langle v, *\left(\frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_1} \wedge \cdots \wedge \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_{d-k}}\right)(\xi)\right\rangle_{\bigwedge_k \mathbb{R}^d} dV \\
&\quad = \lambda(p)\left(v_1 \wedge \cdots \wedge v_k \wedge \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi)\right).
\end{aligned} \tag{34}$$

By Lemma 3.21, we can also write

$$\begin{aligned}
\left|\left\langle v, \nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p)\right\rangle_{\bigwedge_k \mathbb{R}^d}\right| &= \left|\det\left[\langle v_i, \nabla f_j\rangle_{\mathbb{R}^d}\right]_{i,j=1,\dots,k}\right| \\
&= \left|\det[v_i f_j]_{i,j=1,\dots,k}\right| = \Delta.
\end{aligned} \tag{35}$$

By Definition 3.19, $v_\ell(\Phi \circ \psi_{\mathfrak{w}}(\xi)) = v_\ell(\psi \circ \tilde{\Phi}(\xi)) = J_\psi(\tilde{\Phi}(\xi)) e_{d-k+\ell}$, for $\ell = 1, \dots, k$. Indeed, for any point $(x, y) = (x_1, \dots, x_{d-k}, y_1, \dots, y_k)$, we can always write

$$\psi(x,y) = \exp \sum_{j=1}^{d-k} x_j w_j \cdot \exp\left(\sum_{i \neq \ell} y_i v_i + y_\ell v_\ell\right) = \exp(y_\ell v_\ell) \cdot \left(\exp \sum_{j=1}^{d-k} x_j w_j \cdot \exp \sum_{i \neq \ell} y_i v_i\right),$$

so that

$$\frac{\partial \psi}{\partial y_\ell} = v_\ell\big(\psi(x,y)\big).$$

Analogously

$$\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_j}(\xi)=J_\psi\big(\tilde\Phi(\xi)\big)\frac{\partial\tilde\Phi}{\partial\xi_j}(\xi),\quad \text{for } j=1,\dots,d-k.$$

Hence

$$\begin{aligned}&v_1\wedge\cdots\wedge v_k\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\\&\quad=\det J_\psi\big(\tilde\Phi(\xi)\big)\cdot\left(e_{d-k+1}\wedge\cdots\wedge e_d\wedge\frac{\partial\tilde\Phi}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial\tilde\Phi}{\partial\xi_{d-k}}(\xi)\right).\end{aligned}\tag{36}$$

On the other hand, by construction, $\frac{\partial\tilde\Phi}{\partial\xi_j}(\xi)=\sum_{\ell=1}^k\frac{\partial\phi_\ell}{\partial\xi_j}(\xi)e_{d-k+\ell}+e_j$. Using this and keeping into account Proposition 3.20, (36) becomes

$$\begin{aligned}&v_1\wedge\cdots\wedge v_k\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\\&\quad=\det J_\psi(e_{d-k+1}\wedge\cdots\wedge e_d\wedge e_1\wedge\cdots\wedge e_{d-k})\\&\quad=\varepsilon_I\det J_\psi dV,\end{aligned}\tag{37}$$

where ε_I is 1 or -1 according to the parity of the permutation $(d-k+1,\dots,d,1,\dots,d-k)$. Thus, combining (34), (35), and (37), we get $\Delta=|\lambda||\det J_\psi|$, and, consequently,

$$\begin{aligned}&\big|\nabla f_1(p)\wedge\cdots\wedge\nabla f_k(p)\big|\\&\quad=\frac{|\Delta|}{|\det J_\psi|}\left|*\left(\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\right)\right|\\&\quad=\frac{|\Delta|}{|\det J_\psi|}\left|\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\right|.\qquad\Box\end{aligned}\tag{38}$$

3.4. Regular surfaces locally are graphs

In this section we prove that k-codimensional $\mathbb{H}$-regular surfaces, locally, are graphs in the Heisenberg sense. That is we have to show that assumptions of Proposition 3.13 hold true. In particular, if we assume, accordingly with the notations of Proposition 3.13, that the surface S is locally defined by the equation $S=\{p\in\mathcal{U}:\ f(p)=0\}$, we have to check that if $\nabla_H f_1\wedge\cdots\wedge\nabla_H f_k\neq 0$, then there exist k, linearly independent, horizontal vectors $v_1,\dots,v_k$ such that

$$[v_i,v_j]=0,\qquad\text{for } 1\leqslant i,j\leqslant k,\tag{39}$$

$$\det\big([v_i f_j]_{1\leqslant i,j\leqslant k}\big)\neq 0.\tag{40}$$

Notice that this problem does not appear when $k=1$; indeed if $\nabla_H f\neq 0$ then there is at least one $i\in\{1,\dots,2n\}$ with $W_i f\neq 0$ and we can take $v_1=W_i$.

When $k>1$, condition $\nabla_H f_1\wedge\cdots\wedge\nabla_H f_k\neq 0$ yields the existence of k vectors in $X_1,\dots,Y_n$ such that (40) holds but not necessarily (39). For instance consider the following example:

Example 3.24. Let $f = (f_1, f_2): \mathbb{H}^2 \to \mathbb{R}^2$ be defined as

$$f(p_1, \dots, p_5) = (p_1, p_3).$$

Then S is the 2-codimensional plane $S = \{p_1 = p_3 = 0\}$. Writing explicitly the 2×4 matrix associated with $d_H f$, we see that all 2×2 minors vanish but for

$$\begin{bmatrix} X_1 f_1, & Y_1 f_1 \\ X_1 f_2, & Y_1 f_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{41}$$

Clearly, the choice $v_1 = X_1$ and $v_2 = Y_1$ satisfies (40) but not (39). Hence we cannot foliate $\mathbb{H}^2$ using integral surfaces of v_1 and v_2, by Frobenius Theorem. Nevertheless an adapted foliation, satisfying both (39) and (40), exists: indeed it is enough to take

$$v_1 \overset{\text{def}}{=} X_1 + X_2, \qquad v_2 \overset{\text{def}}{=} Y_1 - Y_2. \tag{42}$$

Clearly this is a typical non-Euclidean phenomenon. In the following part of this section we prove that the procedure in (42) can be generalized.

Proposition 3.25. *For $2 < k \leqslant n$, let $f = (f_1, \dots, f_k): \mathbb{H}^n \to \mathbb{R}^k$, $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$. If there is $p^0 \in \mathbb{H}^n$, such that*

$$\operatorname{rank}\big[W_i f_j\big(p^0\big)\big]_{1 \leqslant i \leqslant 2n, 1 \leqslant j \leqslant k} = k, \tag{43}$$

then there are an open $\mathcal{U} \ni p^0$ and a simple, integrable k-vector $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$ such that, for all $p \in \mathcal{U}$,

$$\det\big[v_i f_j(p)\big]_{1 \leqslant i, j \leqslant k} \neq 0.$$

Proposition 3.25 will be proved using the following algebraic lemma. Its elegant proof, much simpler than the combinatorial one we had in the first version of this paper, was provided us by Fulvio Ricci [26].

Lemma 3.26. *Let V be a real vector space of dimension $2n$ endowed with a symplectic form ω, and let W be a linear subspace of V of dimension $d \geqslant n = \frac{1}{2} \dim V$.*

Then W admits a totally isotropic complementary subspace W', i.e. there exists a linear subspace W' such that

$$V = W \oplus W' \quad \textit{and} \quad \omega \equiv 0 \quad \textit{on } W'. \tag{44}$$

Proof. Set

$$W_0 := \operatorname{rad} W = \big\{w \in W \colon \omega(w, w') = 0 \text{ for all } w' \in W\big\},$$

and let W_1 be a complementary subspace of W_0 in W. Then ω is non-degenerate in W_1. Indeed, let $w \in W_1$ be such that $\omega(w, \cdot) \equiv 0$ on W_1; keeping in mind that a generic point $w' \in W$ can be written in the form $w' = w_0 + w_1$, with $w_0 \in W_0$ and $w_1 \in W_1$, it follows that $\omega(w, w') = 0$ for all $w' \in W$. This implies that w belongs to W_0 and hence $w = 0$.

Thus W_1 admits a symplectic basis $\{e_1, \dots, e_k, f_1, \dots, f_k\}$ such that

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \quad \omega(e_i, f_j) = \delta_{ij} \quad \text{for } i, j = i, \dots, k. \tag{45}$$

Let now $\{e_{k+1}, \dots, e_{k+\ell}\}$ be a basis of W_0. Then, the system

$$\{e_1, \dots, e_{k+\ell}, f_1, \dots, f_k\}$$

still satisfies (45), and let us show that it can be completed to be a symplectic basis of all V. To this end, choose first f_{k+1} such that

$$\omega(e_{k+1}, f_{k+1}) = 1$$

and

$$\omega(e_j, f_{k+1}) = 0, \quad \text{for } j \neq k+1, \qquad \omega(f_{k+1}, f_j) = 0, \quad \text{for } j \leqslant k.$$

Successively, repeat this construction to define $f_{k+2}, \dots, f_{k+\ell}$. Finally, we add $n - (k + \ell)$ couples $\{e_{k+\ell+1}, f_{k+\ell+1}\}, \{e_{k+\ell+2}, f_{k+\ell+2}\}, \dots$ still satisfying (45). We can now distinguish three cases:

(1) If $k + \ell = n$, then it is enough to choose $W' = \operatorname{span}\{f_{k+1}, \dots, f_n\}$.
(2) If $k = 0$, then necessarily $\ell = n$, since $d = \ell$. Hence we have $W = \operatorname{span}\{e_1, \dots, e_n\}$, and we can take $W' = \operatorname{span}\{f_1, \dots, f_n\}$.
(3) Finally, suppose $k \geqslant 1$, and $k + \ell < n$. Since $d \geqslant n$, necessarily $k \geqslant p := n - (k + \ell)$. Then for $1 \leqslant j \leqslant p$, we replace the couple $\{e_{k+\ell+j}, f_{k+\ell+j}\}$ by the couple $\{e'_{k+\ell+j}, f'_{k+\ell+j}\}$, where

$$e'_{k+\ell+j} := e_{k+\ell+j} + e_j, \qquad f'_{k+\ell+j} := f_{k+\ell+j} + f_j,$$

and we set eventually

$$W' = \operatorname{span}\bigl\{f_{k+1}, \dots, f_{k+\ell}, e'_{k+\ell+1}, \dots, e'_n, f'_{k+\ell+j}, \dots, f'_n\bigr\}.$$

It is easy to check directly that again W' satisfies (44) in the statement of the lemma. □

Proof of Proposition 3.25. As pointed out in Section 2.3, the P-differential $d_H f_{p^0}$ can be identified with a linear map from $\mathfrak{h}_1$ to $\mathbb{R}^k$. Because of (43), the rank of such a map is k. Thus, in Lemma 3.26 choose $V := \mathfrak{h}_1$, $\omega := d\theta$, $W := \ker d_H f_{p^0}$. Clearly, the assumptions of Lemma 3.26 are satisfied, since the dimension of W is $2n - k$, and thus we can find k linearly independent vectors $v_1, \dots, v_k$, spanning a complementary subspace W' of $\ker d_H f_{p^0}$, such that $d\theta(v_i \wedge v_j) = 0$ for $i, j = 1, \dots, k$. By Theorem 2.8, the k-vector $v := v_1 \wedge \cdots \wedge v_k$ belongs to ${}_H\bigwedge_k$, and clearly $d_H f_{p^0}(v_1), \dots, d_H f_{p^0}(v_k)$ are linearly independent. Thus the assertion follows noticing that $v_i f_j(p^0) = d_H f_{j,p^0}(v_i)$, for $i, j = 1, \dots, k$. □

Theorem 3.27. *Let $S \subset \mathbb{H}^n$ be a k-codimensional $\mathbb{H}$-regular surface, $1 \leqslant k \leqslant n$. Then S is locally a regular graph, that is, for each $p \in S$ there are an open subset $\mathcal{U} \subset \mathbb{H}^n$, with $p \in \mathcal{U}$, a simple k-vector $v \in {}_H\bigwedge_k$, a simple $(2n+1-k)$-vector w and a continuous $\varphi : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ such that*

$$S \cap \mathcal{U} = \{\xi \cdot \varphi(\xi) \colon\ \xi \in \mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}\}.$$

Moreover, it is possible to choose v and w such that $|v| = |w| = 1$.

Proof. The statement follows combining Propositions 3.13 and 3.25. □

3.5. Tangent group to a $\mathbb{H}$-regular surface

We have already proved, in Theorem 3.5, that a low dimensional $\mathbb{H}$-regular surface S has a tangent plane at every point $p \in S$. The tangent plane in p is a coset of a subgroup of $\mathbb{H}^n$ contained in the horizontal fibre through 0. We prove here the analogous statement for low codimensional surfaces.

Definition 3.28. Let $S = \{x \colon\ f(x) = 0\}$ be a k-codimensional $\mathbb{H}$-regular surface in $\mathbb{H}^n$ (with $1 \leqslant k \leqslant n$). The *tangent group* to S in p, indicated as $T^g_{\mathbb{H}}S(p)$, is the subgroup of $\mathbb{H}^n$ defined as

$$T^g_{\mathbb{H}}S(p) \stackrel{\text{def}}{=} \{x \in \mathbb{H}^n \colon\ d_H f_p(x) = 0\}.$$

The *group normal* (or *horizontal normal*) $n_{\mathbb{H}}(p) \in \bigwedge_{k,p} \mathfrak{h}_1$ is defined by

$$n_{\mathbb{H}}(p) \stackrel{\text{def}}{=} \frac{\nabla_H f_1(p) \wedge \cdots \wedge \nabla_H f_k(p)}{|\nabla_H f_1(p) \wedge \cdots \wedge \nabla_H f_k(p)|}.$$

The $(2n+1-k)$-vector $t_{\mathbb{H}}(p) \in \bigwedge_{2n+1-k,p} \mathfrak{h}$ defined as

$$t_{\mathbb{H}}(p) \stackrel{\text{def}}{=} *n_{\mathbb{H}}(p)$$

will be said to be the *group tangent* to S in p.

The group tangent vector is never horizontal. It can always be written in the form $t_{\mathbb{H}}(p) = \xi \wedge T$, where $\xi \in \bigwedge_{2n-k,p} \mathfrak{h}_1$ (6). Moreover, if $t_{\mathbb{H}}(p) = v_1 \wedge \cdots \wedge v_{2n+1-k}$, then $T^g_{\mathbb{H}}S(p) = \exp(\operatorname{span}\{v_1, \ldots, v_{2n+1-k}\})$.

As in the Euclidean setting, a $\mathbb{H}$-*orientation of S* will be identified with a continuous horizontal group vector field, or, equivalently, with a continuous group tangent vector field. If they exist, then S is said to be $\mathbb{H}$-orientable.

Finally notice that the definitions of $t_{\mathbb{H}}$ and of $n_{\mathbb{H}}$ are good ones. Indeed, as proved in the following proposition, they do not depend on the defining function f.

Proposition 3.29. *If S is a k-codimensional $\mathbb{H}$-regular surface, $1 \leqslant k \leqslant n$, and $p \in S$, then*

$$\operatorname{Tan}_{\mathbb{H}}(S, p) = \tau_p T^g_{\mathbb{H}}S(p). \tag{46}$$

Proof. Since $T^g_{\mathbb{H}}(\tau_{-p}S)(0) = T^g_{\mathbb{H}}S(p)$ it is enough to prove that if $0 \in S$ then

$$\mathrm{Tan}_{\mathbb{H}}(S,0) = T^g_{\mathbb{H}}S(0).$$

With the same notations as used in Proposition 3.13, fix $r_0 > 0$ such that $B(0,r_0) \subset \mathcal{U}'$ and $S \cap B(0,r_0) = \{x \in B(0,r_0)\colon f(x) = 0\} = \{\Phi(\xi) := \xi \cdot \varphi(\xi)\colon \xi \in \mathcal{V}\}$. For $r \geqslant 1$, define $S_r := \delta_r S$ and $f_r := rf \circ \delta_{1/r}$. Clearly,

$$S_r \cap B(0, rr_0) = \bigl\{x\colon \delta_{1/r}x \in S \cap B(0,r_0)\bigr\} = \bigl\{x \in B(0,rr_0)\colon f_r(x) = 0\bigr\}$$
$$= \bigl\{\delta_r \Phi(\xi)\colon \xi \in \mathcal{V}\bigr\} = \bigl\{\Phi_r(\xi)\colon \xi \in \delta_r\mathcal{V}\bigr\},$$

where we have defined $\Phi_r := \delta_r \circ \Phi \circ \delta_{1/r}$. Notice also that $f_r \in \mathcal{C}^1_{\mathbb{H}}(B(0,rr_0))$ and for any left invariant, horizontal vector field W and for all $x \in B(0,rr_0)$, $Wf_r(x) = Wf(\delta_{1/r}x)$.

Define now $f_\infty : \mathbb{H}^n \to \mathbb{R}^k$ as $f_{\infty,i}(x) = \langle \nabla_H f_i(0), \pi x\rangle_0$, for $i = 1,\dots,k$. Observe that, because $f \in \mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)$, $f_r \to f_\infty$ as $r \to +\infty$ uniformly on each compact subset of $\mathbb{H}^n$ and that, by definition of tangent group,

$$T^g_{\mathbb{H}}S(0) = \bigl\{x\colon f_\infty(x) = 0\bigr\} = \bigl\{\Phi_\infty(\xi)\colon \xi \in \mathbb{G}_{\mathfrak{w}}\bigr\},$$

where $\Phi_\infty : \mathbb{G}_{\mathfrak{w}} \to T^g_{\mathbb{H}}S(0)$ is implicitly defined by $f_\infty(\Phi_\infty(\xi)) = 0$, but can be also explicitly written solving the equation $f_\infty(\xi \cdot \exp(\sum_{l=1}^k \lambda_l V_l)) = 0$ with respect to ξ.

We want to prove that, for each $\xi \in \mathbb{G}_{\mathfrak{w}}$,

$$\Phi_r(\xi) \to \Phi_\infty(\xi) \quad \text{as } r \to +\infty. \tag{47}$$

First observe that, for each fixed ξ, $r \mapsto \Phi_r(\xi)$ is bounded for $r \to +\infty$. Indeed, from the Lipschitz continuity of φ (see (25)) it follows $|\Phi(\xi)|_c = |\xi \cdot \varphi(\xi)|_c \leqslant |\xi|_c + |\varphi(\xi)|_c \leqslant (1+c)|\xi|_c$, where c is the constant in Corollary 3.18. Hence

$$\bigl|\Phi_r(\xi)\bigr|_c = \bigl|(\delta_r \circ \Phi \circ \delta_{1/r})(\xi)\bigr|_c = r\bigl|\Phi(\delta_{1/r}\xi)\bigr|_c \leqslant r(1+c)|\delta_{1/r}\xi|_c = (1+c)|\xi|_c.$$

Hence, for each fixed ξ, the limit class of $\Phi_r(\xi)$ as $r \to +\infty$, is not empty. Moreover, if $\Phi_{r_h}(\xi) \to l(\xi)$ as $r_h \to +\infty$, because $f_r \to f_\infty$ as $r \to +\infty$ uniformly on compact subsets, it follows that $l(\xi) = \Phi_\infty(\xi)$, and we have proved (47).

Since for r large $\Phi_r(\xi) \in S_r$, from (47) it follows

$$T^g_{\mathbb{H}}S(0) \subset \mathrm{Tan}_{\mathbb{H}}(S,0).$$

To prove the opposite inequality, assume $p_h \in S_{r_h}$ and $p_h \to p$ as $r_h \to +\infty$. For $h \geqslant h_0$, $p_h \in S_{r_h} \cap B(0, r_h r_0)$, hence $p_h = \Phi_{r_h}(\xi_h)$ with $\xi_h \in \mathbb{G}_{\mathfrak{w}}$. But, $0 = f_{r_h}(\Phi_{r_h}(\xi_h)) \to f_\infty(p)$, hence $f_\infty(p) = 0$ and $p \in T^g_{\mathbb{H}}S(0)$. $\square$

4. Surface measures and their representation

4.1. Low codimensional $\mathbb{H}$-regular surfaces

Theorem 4.1. *Let S be a k-codimensional $\mathbb{H}$-regular surface, $1 \leqslant k \leqslant n$. By Theorem 3.27 and with the notations therein, we know that S is locally an orthogonal graph, that is, for each $p_0 \in S$ there are an open set $\mathcal{U} \subset \mathbb{H}^n$, with $p_0 \in \mathcal{U}$, a function $f = (f_1, \dots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, a simple k-vector $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$, with $|v| = 1$, a simple $(2n+1-k)$-vector*

$$w \stackrel{\text{def}}{=} *v \in {}_H\bigwedge\nolimits_{2n+1-k},$$

a relatively open $\mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}$ and a continuous $\varphi : \mathcal{V} \to \mathbb{G}_{\mathfrak{v}}$ such that

$$S \cap \mathcal{U} = \bigl\{x \in \mathcal{U}\colon\ f(x) = 0\bigr\} = \bigl\{\Phi(\xi) \stackrel{\text{def}}{=} \xi \cdot \varphi(\xi),\ \xi \in \mathcal{V}\bigr\}.$$

Now, if we put

$$\Delta(p) \stackrel{\text{def}}{=} \bigl|\det\bigl[v_i f_j(p)\bigr]_{1 \leqslant i,j \leqslant k}\bigr| \neq 0, \quad \textit{for } p \in \mathcal{U},$$

then

$$\mathcal{S}^{Q-k}_{\infty} \llcorner (S \cap \mathcal{U}) = \Phi_\sharp\biggl(\biggl(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi\biggr)\mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}}\biggr). \tag{48}$$

Here $\Phi_\sharp \mu$ is the image measure of the measure μ [21, Definition 1.17]. *Notice also that, since $\mathbb{G}_{\mathfrak{w}}$ is a linear space, $\mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}} = \mathcal{L}^{2n+1-k} \llcorner \mathbb{G}_{\mathfrak{w}}$, the $(2n+1-k)$-dimensional Lebesgue measure.*

Remark 4.2. If we assume simply that $S \cap \mathcal{U}$ is a regular graph (and not an orthogonal graph) then formula (48) takes the following more general form:

$$\mathcal{S}^{Q-k}_{\infty} \llcorner (S \cap \mathcal{U}) = \frac{|\det J_\psi|}{|\det J_{\psi_{\mathfrak{w}}}|}\Phi_\sharp\biggl(\biggl(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi\biggr)\mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}}\biggr)$$

and recalling the computations in Proposition 3.20,

$$= \frac{|v \wedge w|}{|w|}\Phi_\sharp\biggl(\biggl(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi\biggr)\mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}}\biggr).$$

Proof. Let $d = 2n + 1$. We need the following Differentiation Theorem whose proof can be found in Federer's book (see [9, Theorems 2.10.17 and 2.10.18]).

Theorem 4.3 *(Differentiation Theorem). Let μ be a regular measure and ζ the valuation function defined in* (4) *and used in the definition of the measure $\mathcal{S}^{Q-k}_{\infty}$. If*

$$\lim_{r \to 0} \frac{\mu(B_\infty(x,r))}{(\zeta(B_\infty(x,r)))^{Q-k}} = \mathfrak{s}(x), \quad \textit{for } \mu\textit{-a.e. } x \in \mathbb{H}^n, \tag{49}$$

then

$$\mu = \mathfrak{s}(x)\mathcal{S}_\infty^{Q-k}. \tag{50}$$

We are going to apply Theorem 4.3 to the measure $\mu = \mu_S$ defined as

$$\mu_S(\mathcal{O}) \stackrel{\text{def}}{=} \int\limits_{\Phi^{-1}(\mathcal{O})} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}}$$

for any Borel set $\mathcal{O} \subset \mathbb{H}^n$. By Theorem 4.3, identity (48) follows from

$$\begin{aligned}
&\lim_{r\to 0} \frac{1}{r^{Q-k}} \int\limits_{\Phi^{-1}(B_\infty(p,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&\quad = 2\omega_{Q-k-2}
\end{aligned} \tag{51}$$

and from the definition of the valuation function ζ in (4).

Hence we shall prove (51).

Step 1. Without loss of generality, in (51) we can assume $p = 0$. Indeed defining, as in Proposition 3.9,

$$\begin{aligned}
\sigma_p : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{w}} \quad &\text{as} \quad \sigma_p(\eta) := p \cdot \eta \cdot p_{\mathfrak{v}}^{-1}, \\
\Phi_p : \mathbb{G}_{\mathfrak{w}} \to \mathbb{H}^n \quad &\text{as} \quad \Phi_p := \Phi_p \circ \sigma_p,
\end{aligned}$$

we have that the Jacobian of σ_p from $\mathbb{G}_{\mathfrak{w}} \simeq \mathbb{R}^{d-k}$ to itself is identically 1, hence

$$\begin{aligned}
&\int\limits_{\Phi^{-1}(B_\infty(p,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&\quad = \int\limits_{\sigma_p^{-1} \circ \Phi^{-1}(B_\infty(p,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \circ \sigma_p \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&\quad = \int\limits_{(\Phi_{p^{-1}})^{-1}(B_\infty(0,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \tau_{p^{-1}} \circ \Phi_{p^{-1}} \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&\quad = \int\limits_{(\Phi_{p^{-1}})^{-1}(B_\infty(0,r))} \frac{|\nabla_H (f_1 \circ \tau_p) \wedge \cdots \wedge \nabla_H (f_k \circ \tau_p)|}{\Delta_p} \circ \Phi_{p^{-1}} \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}},
\end{aligned}$$

where $\Delta_p := |\det([v_i(f_j \circ \tau_p)]_{1 \leqslant i,j \leqslant k})|$.

Remember that $\tau_{p^{-1}}(S) = \{x\colon (f \circ \tau_p)(x) = 0\}$. Hence, the limit in (51) equals the same limit when we replace S by $\tau_{p^{-1}}(S)$ and accordingly p with 0. This concludes the proof of Step 1.

Set now, for $\rho > 0$, $f_{1/\rho} \overset{\text{def}}{=} \frac{1}{\rho} f \circ \delta_\rho$ and $\Phi_{1/\rho} \overset{\text{def}}{=} \delta_{1/\rho} \circ \Phi \circ \delta_\rho$. Then

$$\delta_{1/\rho} S = \{x \in \delta_{1/\rho}\mathcal{U}\colon\ f_{1/\rho}(x) = 0\} = \{\Phi_{1/\rho}(\xi)\colon\ \xi \in \delta_{1/\rho}\mathcal{V}\}.$$

Define, analogously to μ_S,

$$\mu_{(\delta_{1/\rho}S)}\big(B_\infty(0,r)\big) \overset{\text{def}}{=} \int\limits_{\Phi_{1/\rho}^{-1}(B_\infty(0,r))} \frac{|\nabla_H f_{1/\rho,1} \wedge \cdots \wedge \nabla_H f_{1/\rho,k}|}{\Delta_{1/\rho}} \circ \Phi_{1/\rho}\, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}},$$

and observe that

Step 2.

$$\frac{\mu_S(B_\infty(0,\rho))}{\rho^{Q-k}} = \mu_{(\delta_{1/\rho}S)}\big(B_\infty(0,1)\big), \quad \text{for } \rho > 0. \tag{52}$$

Indeed, given the homogeneity of the horizontal vector fields with respect to group dilations, (52) follows by the change of variables $x' = \delta_\rho(x)$. Indeed, the Jacobian of this transformation from $\mathbb{G}_{\mathfrak{w}}$ to itself is equal to ρ^{k-Q}, since $T \in \mathfrak{w}$, and $\Phi^{-1}(B_\infty(0,\rho)) = \delta_\rho(\Phi_{1/\rho}^{-1}(B_\infty(0,1)))$.

Step 3. We can prove that

$$\begin{aligned}
&\lim_{\rho\to 0} \mu_{(\delta_{1/\rho}S)}\big(B_\infty(0,1)\big)\\
&\quad= \int\limits_{\Phi_\infty^{-1}(B_\infty(0,1))} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \circ \Phi_\infty\, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}}\\
&\quad= \int\limits_{\Phi_\infty^{-1}(B_\infty(0,1))} \frac{|\nabla_H f_1(0) \wedge \cdots \wedge \nabla_H f_k(0)|}{\Delta_\infty} \circ \Phi_\infty\, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}},
\end{aligned} \tag{53}$$

where, as in Proposition 3.29, $\Phi_\infty : \mathbb{G}_{\mathfrak{w}} \to T_{\mathbb{H}}^{g} S(0) = \mathrm{Tan}_{\mathbb{H}}(S,0)$ is implicitly defined by the equation $f_\infty(\Phi_\infty(\xi)) = 0$, $f_{\infty,i}(x) = d_H f_{i\,0}(x)$, for $i = 1, \ldots, k$, and Δ_∞ is defined accordingly.

Indeed, let $\psi_{1,\varepsilon}$ and $\psi_{2,\varepsilon}$ be non-negative Lipschitz continuous functions supported, respectively, in an ε-neighborhood of $B_\infty(0,1)$ and in $B_\infty(0,1)$ and such that $\psi_{1,\varepsilon} \equiv 1$ on $B_\infty(0,1)$ and $\psi_{2,\varepsilon} \equiv 1$ on $B_\infty(0,1-\varepsilon)$. Then

$$\begin{aligned}
&\int\limits_{\mathbb{G}_{\mathfrak{w}}} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \psi_{2,\varepsilon} \circ \Phi_\infty\, d\mathcal{H}_E^{d-k}\\
&\quad\leqslant \liminf_{\rho\to 0} \mu_{(\delta_{1/\rho}S)}\big(B_\infty(0,1)\big) \leqslant \limsup_{\rho\to 0} \mu_{(\delta_{1/\rho}S)}\big(B_\infty(0,1)\big)\\
&\quad\leqslant \int\limits_{\mathbb{G}_{\mathfrak{w}}} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \psi_{1,\varepsilon} \circ \Phi_\infty\, d\mathcal{H}_E^{d-k},
\end{aligned}$$

thanks to the uniform convergence of $\nabla_H f_{1/\rho} \to \nabla_H f_\infty$ and of $\Phi_{1/\rho} \to \Phi_\infty$. Letting now $\varepsilon \to 0$, we get eventually (53). This concludes the proof of Step 3.

The function $f_\infty = d_H f_0$ is an H-linear map, hence, as a map from $\mathbb{R}^d \to \mathbb{R}^k$, it does not depend on the variable p_{2n+1}. It follows that

$$|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}| = |\nabla f_{\infty,1} \wedge \cdots \wedge \nabla f_{\infty,k}|_{\bigwedge_k \mathbb{R}^d}.$$

Remember that the first norm in the preceding inequality is the norm induced in $\bigwedge_k \mathfrak{h}_1$ by the norm in $\bigwedge_1 \mathfrak{h}_1$. Moreover notice that f_∞ is Euclidean smooth, so that we can apply Proposition 3.23. Starting from (53), with $U = B_\infty(0,1)$, we get

$$\begin{aligned}
&\int\limits_{\Phi_\infty^{-1}(U)} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \circ \Phi_\infty \, d\mathcal{H}_E^{d-k} \llcorner G_{\mathfrak{w}} \\
&\quad = \int\limits_{(\Phi_\infty \circ \psi_{\mathfrak{w}})^{-1}(U)} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \circ (\Phi_\infty \circ \psi_{\mathfrak{w}}) |\det J_{\psi_{\mathfrak{w}}}| \, d\mathcal{H}_E^{d-k} \\
&\quad = \int\limits_{(\Phi_\infty \circ \psi_{\mathfrak{w}})^{-1}(U)} \frac{|\nabla f_{\infty,1} \wedge \cdots \wedge \nabla f_{\infty,k}|_{\bigwedge_k \mathbb{R}^d}}{\Delta_\infty} \circ (\Phi_\infty \circ \psi_{\mathfrak{w}}) |\det J_{\psi_{\mathfrak{w}}}| \, d\mathcal{H}_E^{d-k}
\end{aligned}$$

and using Proposition 3.23

$$\begin{aligned}
&\quad = \int\limits_{(\Phi_\infty \circ \psi_{\mathfrak{w}})^{-1}(U)} \left| \frac{\partial (\Phi_\infty \circ \psi_{\mathfrak{w}})}{\partial \xi_1} \wedge \cdots \wedge \frac{\partial (\Phi_\infty \circ \psi_{\mathfrak{w}})}{\partial \xi_{d-k}} \right|_{\bigwedge_k \mathbb{R}^d} \frac{|\det J_{\psi_{\mathfrak{w}}}|}{|\det J_\psi|} \, d\mathcal{H}_E^{d-k}, \\
&\quad = \frac{|\det J_{\psi_{\mathfrak{w}}}|}{|\det J_\psi|} \mathcal{H}_E^{d-k}\big(\mathrm{Tan}_{\mathbb{H}}(S,0) \cap B_\infty(0,1)\big) = \frac{|w|}{|v \wedge w|} 2\omega_{Q-2},
\end{aligned}$$

from Proposition 3.20. $\square$

As in [11, Corollary 3.7], the following corollary follows:

Corollary 4.4. *If S is k-codimensional $\mathbb{H}$-regular surface with $1 \leqslant k \leqslant n$, then the Hausdorff dimension of S with respect to the cc-distance d_c, or any other metric comparable with it, is $Q - k$.*

Another consequence of area formula (48) is that surface measure $\mathcal{S}_\infty^{2n+2-k}$ restricted to a k-codimensional $\mathbb{H}$-regular surface is a doubling measure with respect to the d_∞ distance.

Proposition 4.5. *Let S be a k-codimensional $\mathbb{H}$-regular surface, $1 \leqslant k \leqslant n$. Then the surface measure $\mathcal{S}_\infty^{2n+2-k} \llcorner S$ is a doubling measure locally on S, i.e. $\forall p \in S$ there is a bounded open set $\mathcal{O} \subset \mathbb{H}^n$, $p \in \mathcal{O}$ and there are $r_0 = r_0(S, \mathcal{O}) > 0$, $c_0 = c_0(S, \mathcal{O}) > 1$, such that $\forall q \in S \cap \mathcal{O}$, $\forall r$, $0 < r < r_0$,*

$$\mathcal{S}_\infty^{2n+2-k}\big(S \cap B_\infty(q, 2r)\big) \leqslant c_0 \mathcal{S}_\infty^{2n+2-k}\big(S \cap B_\infty(q, r)\big). \tag{54}$$

Proof. We keep the notations of Theorem 4.1. Assume first that $0 \in S$ and let us prove (54) with $q = 0$.

Observe that, by (16) and by Corollary 3.18, there is a constant $c_1 > 1$ such that

$$c_1^{-1}|\xi| \leqslant \big|\xi \cdot \varphi(\xi)\big| \leqslant c_1|\xi|, \quad \forall \xi \in \mathbb{G}_{\mathfrak{w}}, \tag{55}$$

hence, for all $r > 0$,

$$S \cap B_\infty(0, r) \subset \Phi\big(\mathbb{G}_{\mathfrak{w}} \cap B_\infty(0, c_1 r)\big) \subset S \cap B_\infty\big(0, c_1^2 r\big). \tag{56}$$

Observe also that there are an open set $\mathcal{O} \ni 0$ and two positive constants c_2 and c_3 such that in $S \cap \mathcal{O}$ we have

$$c_2 < \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi < c_3. \tag{57}$$

Hence, choosing r_0 small so that $S \cap B_\infty(0, 2c_1 r_0) \subset S \cap \mathcal{O}$, for $0 < r < r_0$, recalling (56) and (57), we have

$$\begin{aligned}
\mathcal{S}_\infty^{Q-k}\big(S \cap B_\infty(0, 2r)\big) &\leqslant \mathcal{S}_\infty^{Q-k}\big(\Phi\big(\mathbb{G}_{\mathfrak{w}} \cap B_\infty(0, 2c_1 r)\big)\big) \\
&= \int\limits_{\mathbb{G}_{\mathfrak{w}} \cap B_\infty(0, 2c_1 r)} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \, d\mathcal{H}_E^{2n+1-k} \\
&\leqslant c_0 \int\limits_{\mathbb{G}_{\mathfrak{w}} \cap B_\infty(0, r/c_1)} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \, d\mathcal{H}_E^{2n+1-k} \\
&= c_0 \mathcal{S}_\infty^{Q-k}\big(\Phi\big(\mathbb{G}_{\mathfrak{w}} \cap B_\infty(0, r/c_1)\big)\big) \\
&\leqslant c_0 \mathcal{S}_\infty^{Q-k}\big(S \cap B_\infty(0, r)\big).
\end{aligned} \tag{58}$$

Let us drop now the assumption $0 \in S$.

For $q \in S$, consider the translated $\mathbb{H}$-regular surface $\tau_{q^{-1}} S$; then $0 \in \tau_{q^{-1}} S$ and because

$$\mathcal{S}_\infty^{2n+2-k}\big(S \cap B_\infty(q, r)\big) = \mathcal{S}_\infty^{2n+2-k}\big(\tau_{q^{-1}} S \cap B_\infty(0, r)\big),$$

the thesis (54) follows from (58) considering that $\tau_{q^{-1}} S = \{x \in \mathbb{H}^n \colon (f \circ \tau_q)(x) = 0\}$. $\square$

4.2. Low codimensional Euclidean regular surfaces

Recall that regular surfaces in general are not Euclidean regular. In fact, as we already stressed, recently Kirchheim and Serra Cassano provided an example of a 1-codimensional $\mathbb{H}$-regular surface S in $\mathbb{H}^1$ that has Euclidean Hausdorff dimension 2.5 and hence it is not a 2-dimensional Euclidean rectifiable set. Thus, the topological dimension of S equals 2, its Euclidean Hausdorff dimension equals 2.5 and its intrinsic Hausdorff dimension equals 3.

On the contrary, if S is a k-codimensional Euclidean $\mathcal{C}^1$ submanifold of $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$, $1 \leqslant k \leqslant n$, then the surface measure $\mathcal{H}_E^{2n+1-k} \llcorner S$ is locally finite and its density with respect to the spherical Hausdorff measure $\mathcal{S}_\infty^{Q-k} \llcorner S$ can be explicitly computed. This is the content of Theorem 4.7. Formula (61) was proved for codimension 1 surfaces by the authors in [10], and, once more for codimension 1 surfaces but with the $\mathbb{H}$-perimeter in place of the Hausdorff measure, by Capogna, Danielli and Garofalo in [7].

Lemma 4.6. *Let S be an $\mathbb{H}$-regular surface of codimension k and suppose, in addition, that S is also an Euclidean $\mathcal{C}^1$-manifold. With the notations of Theorem* 4.1, *we have*

$$\mathcal{S}_\infty^{2n+2-k} \llcorner S = \Bigg(\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n\rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \Bigg)^{1/2} \mathcal{H}_E^{2n+1-k} \llcorner S, \tag{59}$$

where

$$n = n_1 \wedge \cdots \wedge n_k = \frac{\nabla f_1 \wedge \cdots \wedge \nabla f_k}{|\nabla f_1 \wedge \cdots \wedge \nabla f_k|_{\bigwedge_k \mathbb{R}^{2n+1}}} = \frac{\nabla f}{|\nabla f|_{\bigwedge_k \mathbb{R}^{2n+1}}}$$

is a continuous Euclidean unit normal k-vector field and $W_1 = X_1, \dots, W_{2n} = Y_n$.

Proof. Denote by $\Theta : \bigwedge_1 \mathfrak{h}_1 \to \mathbb{R}^{2n}$ the map that associates with an horizontal vector its canonical coordinates with respect to the orthonormal basis $X_1, \dots, X_n, Y_1, \dots, Y_n$. Clearly, Θ is a vector space isomorphism and an isometry. We still denote by Θ the induced operator acting from $\bigwedge_k \mathfrak{h}_1$ to $\bigwedge_k \mathbb{R}^{2n}$. We have, for $1 \leqslant j \leqslant k$, $\Theta(\nabla_H f_j) = (W \bullet \nabla f_j)$ where we have set

$$(W \bullet \nabla f) \overset{\text{def}}{=} \big(\langle X_1, \nabla f\rangle_{\mathbb{R}^{2n+1}}, \dots, \langle Y_n, \nabla f\rangle_{\mathbb{R}^{2n+1}}\big) \in \bigwedge\nolimits_1 \mathbb{R}^{2n}.$$

Notice that, thanks to the assumed Euclidean regularity of f, the local parametrization Φ of S is continuously differentiable in the Euclidean sense. Hence

$$\begin{aligned}
&|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|_{\bigwedge_k \mathfrak{h}_1} \\
&\quad = \big|(W \bullet \nabla f_1) \wedge \cdots \wedge (W \bullet \nabla f_k)\big|_{\bigwedge_k \mathbb{R}^{2n}} \\
&\quad = \Bigg(\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, \nabla f_1 \wedge \cdots \wedge \nabla f_k\rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \Bigg)^{1/2} \\
&\quad = \Bigg(\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n\rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \Bigg)^{1/2} |\nabla f|_{\bigwedge_k \mathbb{R}^{2n+1}}
\end{aligned}$$

and by (38), it follows

$$\begin{aligned}
&= \Bigg(\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n\rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \Bigg)^{1/2} \\
&\quad \times \frac{\Delta}{|\det J_\psi|} \left| \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial(\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi) \right|.
\end{aligned}$$

Replacing in (48) we obtain eventually (59). □

Lemma 4.6 cannot be applied immediately to Euclidean regular surfaces because an Euclidean regular S may be not $\mathbb{H}$-regular. Indeed, even if S is locally the zero set of a function $f \in [\mathcal{C}^1(\mathbb{R}^{2n+1})]^k \subset [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$ with non-vanishing Euclidean gradient, nevertheless the non-degeneracy condition $\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k \neq 0$ may fail to hold at some points. Usually these points are named *characteristic points* of S. Equivalently, a point $p \in S$ is a characteristic point if $\mathrm{Tan}(S, p) \subset H\mathbb{H}^n_p$. We denote by $C(S)$ the set of these points.

There are many results estimating the *size* of the closed set $C(S)$ inside S. These results vary both because of the different regularity assumptions on the surfaces and because different surface measures (Euclidean versus intrinsic) are used to estimate the size of $C(S)$. Balogh (see [6]) was the first one to prove that, in the Heisenberg groups, the intrinsic $(Q-1)$-Hausdorff measure of the characteristic set of an Euclidean $\mathcal{C}^1$ hypersurface vanishes. Recently, Magnani [18, 2.16] extended Balogh's estimate to Euclidean $\mathcal{C}^1$-submanifolds, of arbitrary codimension, in general Carnot groups. The result of Magnani, in the particular setting of Heisenberg groups, states that if S is an Euclidean $\mathcal{C}^1$-submanifold, of codimension k, $1 \leqslant k \leqslant n$, in $\mathbb{H}^n$ then

$$\mathcal{S}_\infty^{Q-k}\big(C(S)\big) = 0. \tag{60}$$

Since a $\mathcal{C}^1$-submanifold S in $\mathbb{H}^n$ can be written as $S = C(S) \cup (S \setminus C(S))$ and $S \setminus C(S)$ is a $\mathbb{H}$-regular surface, then, by Lemma 4.6, we have

Theorem 4.7. *If S is an Euclidean $\mathcal{C}^1$-submanifold of codimension k, $1 \leqslant k \leqslant n$, in $\mathbb{H}^n$, then*

$$\begin{aligned}\mathcal{S}_\infty^{2n+2-k} \llcorner S &= \Bigg(\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n\rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}}\Bigg)^{1/2} \mathcal{H}_E^{2n+1-k} \llcorner S \\ &= \Bigg(\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n} \big(\det\big[\langle W_{i_\ell}, n_j\rangle_{\mathbb{R}^{2n+1}}\big]_{\ell,j=1,\dots,k}\big)^2\Bigg)^{1/2} \mathcal{H}_E^{2n+1-k} \llcorner S, \end{aligned} \tag{61}$$

where $n = n_1 \wedge \cdots \wedge n_k$ is a continuous Euclidean unit normal k-vector field and $W = (X_1, \dots, Y_n)$.

5. Appendix: Rectifiable sets and Federer–Fleming currents

5.1. Rectifiable sets

A general theory of rectifiable sets and of Federer–Fleming currents in Heisenberg groups is well beyond the scope of the present paper. We limit ourselves here to suggest the features of a possible theory and to indicate some of the main problems of a future theory.

With the intrinsic notion of regular submanifolds proposed before, it is possible, following the usual approach as in [9] or [29], to give an (intrinsic) notion of rectifiable sets in $\mathbb{H}^n$. In the spirit of this section, we do not propose the most general definition and we say

Definition 5.1. $M \subset \mathbb{H}^n$ is a k-dimensional $\mathbb{H}$-rectifiable set if

$$M \subset M_0 \cup \Bigg(\bigcup_{j=1}^{+\infty} S_j\Bigg),$$

and, respectively,

- *for* $1 \leqslant k \leqslant n$*:* M is $\mathcal{S}_\infty^k$-measurable, $\mathcal{S}_\infty^k(M) < \infty$, S_j are k-dimensional $\mathbb{H}$-regular surfaces and $\mathcal{S}_\infty^k(M_0) = 0$;
- *for* $n+1 \leqslant k \leqslant 2n$*:* M is $\mathcal{S}_\infty^{k+1}$-measurable, $\mathcal{S}_\infty^{k+1}(M) < \infty$, S_j are $(2n+1-k)$-codimensional $\mathbb{H}$-regular surfaces and $\mathcal{S}_\infty^{k+1}(M_0) = 0$.

As one can easily guess, the differences between low dimensional and low codimensional $\mathbb{H}$-regular surfaces induce analogous differences between low dimensional and low codimensional $\mathbb{H}$-rectifiable sets. Precisely, when compared with Euclidean rectifiable sets, low dimensional $\mathbb{H}$-rectifiable ones are in a strict subclass of Euclidean rectifiable sets, while the low codimensional $\mathbb{H}$-rectifiable ones form a strictly larger class.

Proposition 5.2. *If* $1 \leqslant k \leqslant n$*,* k*-dimensional* $\mathbb{H}$*-rectifiable sets are* k*-dimensional rectifiable sets* (*in the usual Euclidean sense*)*.*

Proof. It is enough to observe that $\mathcal{H}_E^k \ll \mathcal{S}_\infty^k$, for $1 \leqslant k$ and to recall that low dimensional $\mathbb{H}$-regular surfaces are Euclidean regular submanifolds of the same dimension. □

Remark 5.3. The vertical T axis in $\mathbb{H}^1 \equiv \mathbb{R}^3$ gives the simplest example of an Euclidean 1-dimensional rectifiable set that is not 1-dimensional $\mathbb{H}$-rectifiable.

Proposition 5.4. *If* $1 \leqslant k \leqslant n$*,* $(2n+1-k)$*-dimensional Euclidean rectifiable sets are* k*-codimensional* $\mathbb{H}$*-rectifiable sets.*

Proof. It is enough to observe that

$$\mathcal{S}_\infty^{k+1} \ll \mathcal{H}_E^k, \quad \text{for } 1 \leqslant k$$

and to recall that if S is a k-codimensional Euclidean regular surfaces in $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ then

$$\mathcal{S}_\infty^{2n+2-k}\big(C(S)\big) = 0$$

where $C(S)$ is the characteristic set of S. □

Remark 5.5. If $1 \leqslant k \leqslant n$, k-dimensional $\mathbb{H}$-rectifiable sets coincide with the $\mathbb{R}^k$-rectifiable sets defined in [25] or with the $(\mathbb{R}^k, \mathbb{H}^n)$-rectifiable sets defined in [19]. The proof is not difficult using Theorem 3.5.

On the contrary, if $n+1 \leqslant k \leqslant 2n$, we do not know whether our k-dimensional $\mathbb{H}$-rectifiable sets coincide or not with corresponding classes defined by the previous authors.

Theorem 5.6. *Let* M *be a* k*-dimensional* $\mathbb{H}$*-rectifiable set,* $1 \leqslant k \leqslant 2n+1$*.*

If $1 \leqslant k \leqslant n$*, then* M *has an* (*intrinsic*) *tangent cone* $\mathrm{Tan}_{\mathbb{H}}(M, p)$ (*see Definition* 3.4) *for* $\mathcal{S}_\infty^k$*-almost every* $p \in M$*. Moreover* $\mathrm{Tan}_{\mathbb{H}}(M, p)$ *coincides with the Euclidean tangent plane in* p *to* S_j*, when* $p \in S_j$*.*

If $n+1 \leqslant k \leqslant 2n+1$*, then* M *has an* (*intrinsic*) *tangent cone* $\mathrm{Tan}_{\mathbb{H}}(M, p)$ *for* $\mathcal{S}_\infty^{k+1}$*-almost every* $p \in M$*. Moreover* $\mathrm{Tan}_{\mathbb{H}}(M, p)$ *coincides with the tangent subgroup* $T_{\mathbb{H}}^g S_j(p)$ *in* p *to* S_j*, when* $p \in S_j$*.*

Proof. The first statement follows easily from Proposition 5.2 and Theorem 3.5.

The second statement can be proved as in Theorem 11.6 of [29] using Proposition 4.5. □

Remark 5.7. By a standard argument, see, e.g., Remark 11.7 in [29], if M is a $\mathbb{H}$-rectifiable set, we can assume that M can be written as

$$M = M_0 \cup \left(\bigcup_{j=1}^{\infty} E_j \right),$$

where the E_j are pairwise disjointed Borel subsets of the $\mathbb{H}$-regular surfaces S_j, as in Definition 5.1. Hence any $\mathbb{H}$-rectifiable set M can be oriented by choosing orientations of the regular surfaces S_j.

5.2. Currents

We give, as anticipated, a definition of (Federer–Fleming) currents as duals of the intrinsic complex of differential forms on $\mathbb{H}^n$ introduced by Rumin in [28]. As a consequence we show that oriented $\mathbb{H}$-regular surfaces and, more generally, oriented $\mathbb{H}$-rectifiable sets can be naturally identified with currents defined in this way.

Let $\mathcal{U}$ be an open subset of $\mathbb{H}^n$ and let $\mathcal{D}^*(\mathcal{U}) = \mathcal{D}^0(\mathcal{U}) \oplus \cdots \oplus \mathcal{D}^{2n+1}(\mathcal{U})$ be the graded algebra of $\mathcal{C}^\infty$ differential forms on $\mathbb{R}^{2n+1}$ with compact support in $\mathcal{U}$.

Definition 5.8. Following Rumin [28] we denote by $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$ (*Heisenberg k-differential forms*) the space of compactly supported smooth sections respectively of ${}_H\bigwedge_k \equiv \frac{\bigwedge_k \mathfrak{h}}{\mathcal{I}^k}$, when $1 \leqslant k \leqslant n$ and of ${}_H\bigwedge_k \equiv \mathcal{J}^k$ when $n+1 \leqslant k \leqslant 2n+1$. These spaces are endowed with the natural topology induced by that of $\mathcal{D}^k(\mathcal{U})$. We denote by $\mathcal{D}^*_{\mathbb{H}}(\mathcal{U}) = \mathcal{D}^0_{\mathbb{H}}(\mathcal{U}) \oplus \cdots \oplus \mathcal{D}^{2n+1}_{\mathbb{H}}(\mathcal{U})$ the graded algebra of all Heisenberg differential forms with compact support, where $\mathcal{D}^0_{\mathbb{H}}(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U})$.

The following theorem is proved in [28].

Theorem 5.9 *(Rumin). There is a linear second order differential operator $D : \mathcal{D}^n_{\mathbb{H}}(\mathcal{U}) \to \mathcal{D}^{n+1}_{\mathbb{H}}(\mathcal{U})$ such that the following sequence has the same cohomology as the De Rham complex on $\mathcal{U}$:*

$$0 \to \mathcal{D}^0_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}^n_{\mathbb{H}}(\mathcal{U}) \xrightarrow{D} \mathcal{D}^{n+1}_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}^{2n+1}_{\mathbb{H}}(\mathcal{U}) \to 0$$

where d is the operator induced by the external differentiation from $\mathcal{D}^k(\mathcal{U}) \to \mathcal{D}^{k+1}(\mathcal{U})$, with $k \neq n$.

Definition 5.10. We call Heisenberg k-current, $1 \leqslant k \leqslant 2n+1$, any continuous linear functional on $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$ and we denote by $\mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ the set of all Heisenberg k-currents.

Proposition 5.11. *If $1 \leqslant k \leqslant n$, any $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ can be identified with an Euclidean k-current $\tilde{T} \in \mathcal{D}_k(\mathcal{U})$, setting*

$$\tilde{T}(\omega) \stackrel{\text{def}}{=} T\big([\omega]\big), \quad \forall \omega \in \mathcal{D}^k(\mathcal{U}).$$

Inversely, if $S \in \mathcal{D}_k(\mathcal{U})$ is such that $S(\alpha \wedge \theta) = 0$ for any $\alpha \in \mathcal{D}^{k-1}(\mathcal{U})$ and $S(\beta \wedge d\theta) = 0$ for any $\beta \in \mathcal{D}^{k-2}(\mathcal{U})$ if $k \geqslant 2$, then S induces a Heisenberg k-current $\hat{S} \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ setting

$$\hat{S}\big([\omega]\big) \stackrel{\text{def}}{=} S(\omega), \quad \forall[\omega] \in \mathcal{D}_{\mathbb{H}}^k(\mathcal{U}).$$

Obviously, with our previous notations, $\tilde{\hat{S}} = S$.

Definition 5.12. Let T be k-dimensional $\mathbb{H}$-current in an open set $\mathcal{U} \subset \mathbb{H}^n$, then the *mass* $\mathbf{M}_{\mathcal{V}}(T)$ of T in $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ open, is

$$\mathbf{M}_{\mathcal{V}}(T) \stackrel{\text{def}}{=} \sup\big\{T(\alpha)\colon\ \alpha \in \mathcal{D}_{\mathbb{H}}^k(\mathcal{V}),\ |\alpha| \leqslant 1\big\}.$$

Remark 5.13. In the last few years a very general theory of currents in metric spaces was developed by Ambrosio and Kirchheim in [2]. As pointed by the same authors in [1], this approach, when particularized to Heisenberg groups with Carnot–Carathéodory distance, is not satisfactory. Indeed, they prove the non-existence of rectifiable $(2n+1-k)$-currents (in their sense) in $\mathbb{H}^n$ when $k < n$. This depends, once more, on the non-existence of Lipschitz injective maps from $\mathbb{R}^{2n+1-k}$ to $\mathbb{H}^n$ when $k < n$.

On the contrary, there are plenty of Heisenberg $(2n+1-k)$-currents given as integration on $\mathbb{H}$-regular surfaces of codimension $k < n$, as we shall see below (see Proposition 5.15). These Heisenberg currents carried by $\mathbb{H}$-regular surfaces play a major role in applications since most naturally they will be the building blocks of Heisenberg rectifiable currents (whose theory has to be developed).

On the other hand, Ambrosio and Kirchheim (see Theorem 4.5 in [2]) proved that rectifiable metric k-currents in $\mathbb{H}^n$, when $k \leqslant n$, are carried by k-dimensional rectifiable sets of $\mathbb{H}^n$. These sets are, up to negligible subsets, countable unions of Lipschitz images of Borel sets in $\mathbb{R}^k$. Since our k-dimensional $\mathbb{H}$-regular surfaces, with $k \leqslant n$, are intrinsically $\mathcal{C}^1$ images of open sets in $\mathbb{R}^k$, it turns out again that our Heisenberg currents given by integration on $\mathbb{H}$-regular surfaces of dimension $k \leqslant n$ play the role of building blocks for a theory of Heisenberg rectifiable currents of low dimension.

Remark 5.14. The perimeter measure of 1-codimensional surfaces (see, e.g., [10]) can be seen as the mass of the boundary of suitable $(2n+1)$-dimensional $\mathbb{H}$-currents—this fact was observed by Magnani in [20]. Indeed, if $F = (F_1, \dots, F_{2n})$ is an horizontal vector field in an open subset of $\mathbb{H}^n$, and if we identify F with the form $\sum_{j=1}^{2n} F_j\, dx_j \in \bigwedge^1 \mathfrak{h}_1$, then we have

$$\operatorname{div}_{\mathbb{H}} F = *d(*F),$$

where $*$ denotes the Hodge operator defined in Definition 2.3. Thus, we can argue e.g. as in [29, Remark 27.7].

As in the Euclidean setting, we notice that Heisenberg currents are generalizations of Heisenberg regular submanifolds, in the sense that any oriented $\mathbb{H}$-regular surface induces, by integration, in a natural way a k-dimensional Heisenberg current.

Proposition 5.15. *Let $S \subset \mathcal{U}$ be a $\mathbb{H}$-regular surface, oriented by a group tangent k-vector field $t_{\mathbb{H}}$. Then, if S is k-dimensional, $1 \leqslant k \leqslant n$, the map*

$$\omega \to [\![S]\!](\omega) \stackrel{\text{def}}{=} \int_S \langle \omega | t_{\mathbb{H}} \rangle \, d\mathcal{S}_\infty^k$$

from $\mathcal{D}_{\mathbb{H}}^k$ to $\mathbb{R}$ is a Heisenberg k-current with locally finite mass and

$$\mathbf{M}_{\mathcal{V}}\big([\![S]\!]\big) = \mathcal{S}_\infty^k(S \cap \mathcal{V}), \quad \text{if } \mathcal{V} \Subset \mathcal{U}.$$

If S is k-codimensional, $1 \leqslant k \leqslant n$, the map

$$\omega \to [\![S]\!](\omega) \stackrel{\text{def}}{=} \int_S \langle \omega | t_{\mathbb{H}} \rangle \, d\mathcal{S}_\infty^{Q-k}$$

from $\mathcal{D}_{\mathbb{H}}^{2n+1-k}$ to $\mathbb{R}$ is a Heisenberg $(2n+1-k)$-current with locally finite mass. Precisely, if $\mathcal{V} \Subset \mathcal{U}$,

$$\mathbf{M}_{\mathcal{V}}\big([\![S]\!]\big) = \int_{S \cap \mathcal{V}} \big|\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}}(t_{\mathbb{H}})\big| \, d\mathcal{S}_\infty^{Q-k}, \tag{62}$$

where $\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}} : \bigwedge_{2n+1-k} \mathfrak{h}_1 \to {}_H\bigwedge_{2n+1-k}$ is the orthogonal projection with respect to the Riemannian scalar product defined in Section 2.2.

Corollary 5.16. *There exists a geometric constant $c_{n,k} \in (0,1)$ such that for any k-codimensional $\mathbb{H}$-regular surface S, $1 \leqslant k \leqslant n$, we have*

$$c_{n,k}\mathcal{S}_\infty^{Q-k}(S \cap \mathcal{V}) \leqslant \mathbf{M}_{\mathcal{V}}\big([\![S]\!]\big) \leqslant \mathcal{S}_\infty^{Q-k}(S \cap \mathcal{V}),$$

for every Borel set $\mathcal{V}$.

Proof. By (62), it is enough to show that

$$c_{n,k} \stackrel{\text{def}}{=} \inf\Big\{\big|\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}}(v)\big| \colon v \in \bigwedge\nolimits_{2n+1-k} \mathfrak{h}, \ v \text{ simple}, \ |v| = 1\Big\} > 0.$$

Indeed, by Proposition 3.25, $|\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}}(v)| > 0$ for all $v \in \bigwedge_{2n+1-k} \mathfrak{h}$, v simple, $|v| = 1$. Then the assertion follows by the compactness of the set of simple vectors of unit norm. □

Example 5.17. We stress that the mass of the current carried by a k-codimensional $\mathbb{H}$-regular surface S can be different (though equivalent) from its $\mathcal{S}_\infty^{Q-k}$-measure. Clearly, by (62) this does not happen when $t_{\mathbb{H}} \in {}_H\bigwedge_{2n+1-k}$ on S. On the other hand, if for instance we consider the surface S of Example 3.24, then a direct computation shows that, taking $t_{\mathbb{H}} = W_2 \wedge W_4 \wedge T$, we obtain

$$\mathbf{M}_{\mathcal{V}}\big([\![S]\!]\big) = \frac{1}{\sqrt{2}}\mathcal{S}_\infty^{Q-k}(S \cap \mathcal{V}).$$

More generally any oriented $\mathbb{H}$-rectifiable sets is naturally associated with a Heisenberg current.

Definition 5.18. With the notations of Remark 5.7, let $M = M_0 \cup (\bigcup_{j=1}^{\infty} E_j)$ be a k-dimensional oriented $\mathbb{H}$-rectifiable set, and denote $t_{\mathbb{H}}$ the unit tangent k-vector field giving the orientation. Then, setting respectively $\mu_k := \mathcal{S}_\infty^k$ if $1 \leqslant k \leqslant n$, and $\mu_k := \mathcal{S}_\infty^{k+1}$ if $n+1 \leqslant k \leqslant 2n+1$, the map

$$\omega \to [\![M]\!](\omega) \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \int_{S_j} \langle \omega | t_{\mathbb{H}} \rangle \, d\mu_k$$

from $\mathcal{D}_{\mathbb{H}}^k$ to $\mathbb{R}$ is a Heisenberg k-current with locally finite mass.

Thanks to Rumin's result, the operators d and D act in the complex as external differentiation does in De Rham complex, and we can give the following (obvious) definition.

Definition 5.19. Let T be a Heisenberg k-current in an open set $\mathcal{U} \subset \mathbb{H}^n$ with $1 \leqslant k \leqslant d$. Then we define the Heisenberg $(k-1)$-current $\partial_{\mathbb{H}} T$, the *Heisenberg boundary of* T, by the identity

$$\partial_{\mathbb{H}} T(\alpha) = T(d\alpha) \quad \text{if } k \neq n+1$$

and

$$\partial_{\mathbb{H}} T(\alpha) = T(D\alpha) \quad \text{if } k = n+1.$$

The following trivial statement says that—also when boundaries are concerned—low dimension $\mathbb{H}$-currents are but particular Euclidean currents.

Proposition 5.20. *If $1 \leqslant k \leqslant n$, the Heisenberg boundary $\partial_{\mathbb{H}} T$ of $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ can be identified as in Proposition* 5.11 *with the Euclidean $(k-1)$-current $\partial \tilde{T}$.*

Proof. Let us notice first that $\partial \tilde{T}(\alpha \wedge \theta) = 0$ for any $\alpha \in \mathcal{D}^{k-1}(\mathcal{U})$ and $\partial \tilde{T}(\beta \wedge \theta) = 0$ for any $\beta \in \mathcal{D}^{k-2}(\mathcal{U})$ if $k \geqslant 2$. Indeed (e.g.)

$$\partial \tilde{T}(\alpha \wedge \theta) = \tilde{T}(d\alpha \wedge \theta) + \tilde{T}(\alpha \wedge d\theta) = T\big([d\alpha \wedge \theta]\big) + T\big([\alpha \wedge d\theta]\big) = T\big([0]\big) + T\big([0]\big) = 0.$$

Thus, $\partial \tilde{T}$ induces a $(k-1)$-dimensional $\mathbb{H}$-current T'. On the other hand, for any $[\omega] \in \mathcal{D}_{\mathbb{H}}^{k-1}$, we have $T'([\omega]) = \partial \tilde{T}(\omega) = \tilde{T}(d\omega) = T([d\omega]) = T(d[\omega]) = \partial_{\mathbb{H}} T([\omega])$, so that $T' = \partial_{\mathbb{H}} T$. $\square$

When $k \geqslant n+1$, the structure of the boundary of a current is much more difficult to describe, even in the simplest situation of a current carried by a low codimensional $\mathbb{H}$-regular surface. As an example, consider the case $n = 1$, and let S be a 1-codimensional $\mathbb{H}$-regular (hyper)surface. We want to state here something similar to Stokes formula that yields that the boundary of a 2-dimensional current in $\mathbb{R}^3$ carried by a sufficiently regular portion of a 2-dimensional Euclidean differentiable manifold (a 2-dimensional oriented Euclidean differentiable manifold with boundary) is carried by the boundary itself, endowed with a suitable induced orientation.

First of all, we cannot think in general of a portion of a $\mathbb{H}$-regular hypersurface—whatever regularity we assume for the boundary—as a differentiable manifold with boundary, since, as we pointed out repeatedly, $\mathbb{H}$-regular surfaces may be very "bad" from the Euclidean point of view [16]. On the other hand, even when dealing with (Euclidean) smooth hypersurfaces with boundary, the mass of the boundary of the associated current may be not locally finite, unless the topological boundary is a horizontal curve.

Let us start by illustrating the last phenomenon: if $[\omega] \in \mathcal{D}^1_{\mathbb{H}}(\mathbb{H}^1)$ we can alway choose ω to be its horizontal representative $\omega = \omega_1 dp_1 + \omega_2 dp_2$. In this case, accordingly with Rumin's theorem [28], the operator D has the form

$$D[\omega] = d(\omega + \tilde{\omega}\theta),$$

where $\tilde{\omega} \in \mathcal{C}^\infty(\mathbb{H}^1)$, is chosen in order to have $d(\omega + \tilde{\omega}\theta) \in \mathcal{D}^2_{\mathbb{H}}(\mathbb{H}^1)$, i.e. such that $d(\omega + \tilde{\omega}\theta) \wedge \theta = 0$. An explicit computation (see also [14, Section 6]) shows that

$$\tilde{\omega} = \frac{1}{4}(W_2\omega_1 - W_1\omega_2).$$

Consider now the 2-dimensional $\mathbb{H}$-current $[\![S]\!]$ carried by the hypersurface $S = \{p_1 = 0, p_2 > 0\}$ oriented by $W_2 \wedge T$. Let t_0 be the boundary of S, i.e. $t_0 = \{p_1 = p_2 = 0\}$. If $[\omega] \in \mathcal{D}^1_{\mathbb{H}}(\mathbb{H}^1)$, with $\omega = \omega_1 dp_1 + \omega_2 dp_2$, by definition and by Stokes theorem (keeping also in mind that $\mathcal{S}^3_\infty \llcorner S = \mathcal{H}^2_E \llcorner S$, by (48)), we have

$$\begin{aligned}
\partial_{\mathbb{H}}[\![S]\!]\big([\omega]\big) &\stackrel{\text{def}}{=} \int_S \big\langle D\big([\omega]\big)\big| W_2 \wedge T\big\rangle\, d\mathcal{H}^2_E = \int_S \big\langle d(\omega + \tilde{\omega}\theta)\big| W_2 \wedge T\big\rangle\, d\mathcal{H}^2_E \\
&= \int_{t_0} \langle \omega + \tilde{\omega}\theta | T\rangle\, d\mathcal{H}^1_E = \frac{1}{4}\int_{t_0} (\partial_2\omega_1 - \partial_1\omega_2)\, d\mathcal{H}^1_E.
\end{aligned}$$

Clearly, the above quantity can be made arbitrary large still keeping $|[\omega]| \leqslant 1$. This shows that $\partial_{\mathbb{H}}[\![S]\!]$, though being a well defined current in our sense, has no locally finite mass.

An analysis of the example above shows quickly that the reason making the boundary of the current carried by S not being of finite mass relies precisely on the fact that the operator D is a second order differential operator because of the derivatives of ω hidden in $\tilde{\omega}$. These derivatives remain in the integration after applying Stokes theorem. Thus, we can expect the boundary of the current carried by a smooth 2-dimensional Euclidean manifold S with boundary ∂S to have finite mass if (and only if) ∂S is horizontal, since in this case $\langle \tilde{\omega}\theta | t_{\mathbb{H}}\rangle \equiv 0$, and no derivatives are left after applying Stokes theorem.

This is coherent with our definition of $\mathbb{H}$-regular surface, providing a further evidence for it: the boundary of an hypersurface in $\mathbb{H}^1$ according with our definition has finite mass only if the boundary is a 1-dimensional surface again in our sense (i.e. an horizontal curve). We stress that, if $n > 1$, this phenomenon is typical of n-codimensional $\mathbb{H}$-regular surfaces, since the surface itself and its boundary belong to the two different classes of $\mathbb{H}$-surfaces: the surface is a low codimensional, whereas the boundary is low dimensional. This is clearly strictly connected with the fact that the derivation in Rumin's complex is a second order operator only when we pass from dimension n to dimension $n + 1$. For instance, if we consider in $\mathbb{H}^2$ the 1-codimensional $\mathbb{H}$-regular surface $S = \{p_1 = 0, p_2 > 0\}$ oriented by $W_2 \wedge W_3 \wedge W_4 \wedge T$, again classical Stokes theorem yields that now $\partial[\![S]\!]$ is carried by the 2-codimensional $\mathbb{H}$-regular surface $\{p_1 = p_2 = 0\}$ oriented by $W_3 \wedge W_4 \wedge T$, despite of the analogy with the preceding example. This because in $\mathbb{H}^2$ both $\{p_1 = 0, p_2 > 0\}$ and $\{p_1 = p_2 = 0\}$ are low codimensional $\mathbb{H}$-regular surfaces.

If $k < n$, the above example can be easily generalized to that of a continuously differentiable $(2n+1-k)$-manifold with boundary $S \subset \mathbb{H}^n$ that locally has the form $\{f_1(p) = \cdots = f_k(p) = 0, f_{k+1}(p) \geqslant 0\}$, with (for sake of simplicity)

$$\det[W_i f_j]_{1\leqslant i,j\leqslant k} \neq 0 \quad \text{and} \quad \det[W_i f_j]_{1\leqslant i,j\leqslant k+1} \neq 0.$$

So far, we have dealt with Euclidean regular hypersurfaces in $\mathbb{H}^1$. If we want a more intrinsic result—still for 1-codimensional hypersurfaces in $\mathbb{H}^1$—we have to deal with pieces of 1-codimensional $\mathbb{H}$-regular hypersurfaces that are sufficiently regular. The following result can be derived from Theorem 5.4 in [14].

Theorem 5.21. *Let $S \subset \mathbb{H}^1$ be a $\mathbb{H}$-regular $\mathbb{H}$-oriented hypersurface, and let $V \subset S$ be the closure of a relatively open subset V_0 of S. We assume that V is a topological 2-manifold with boundary ∂V that is a finite union of disjoint simple closed $\mathbf{C}^1$-piecewise horizontal curves. Then $\partial[\![V_0]\!]$ is carried by $\partial_{\mathbb{H}} V$ and has finite mass.*

We stress that in the above theorem no non-intrinsic regularity is assumed. Indeed we only require V is a *topological 2-manifold with boundary*, and $\mathbb{H}$-regular hypersurface are topological 2-manifolds.

The proof of Theorem 5.4 in [14], relies on an approximation procedure and is much easier when both the surface and its boundary belong to the class of low codimensional $\mathbb{H}$-regular surfaces. Thus our previous remark concerning $(2n + 1 - k)$-currents carried by continuously differentiable $(2n + 1 - k)$-manifolds with boundary, of the form $\{f_1(p) = \cdots = f_k(p) = 0, f_{k+1}(p) \geqslant 0\}$, can be extended to the case when $f_1, \ldots, f_{k+1}$ are $\mathcal{C}^1_{\mathbb{H}}$-functions. Indeed, if J_ε is a Friedrichs' mollifier and we put $f_{i,\varepsilon} = f_i * J_\varepsilon$ for $i = 1, \ldots, k+1$, then $f_{i,\varepsilon} \to f$ and $W_k f_{i,\varepsilon} \to f$ as $\varepsilon \to 0$, uniformly on compact sets, for $k = 1, \ldots, 2n$, as proved in [10, Theorem 6.5, Step 1].

Acknowledgments

It is a duty as well a pleasure to thank here several friends that contributed to this paper. A special thank to Martin Reimann who raised to our attention Rumin's paper and to Maria Carla Tesi and Nicoletta Tchou because the div–curl problem in Heisenberg groups [14] was one of our original motivations. Giovanna Citti, Mariano Giaquinta, Adam Korányi, Pierre Pansu and Fulvio Ricci contributed to improve several parts. Luigi Ambrosio for his painstakingly accurate reading of the first version of the paper. Finally, we had fruitful discussions with Sorin Dragomir, Luca Migliorini, Michel Rumin and with Valentino Magnani, who also studied, differently, the subject of surface area of submanifolds in Heisenberg groups [20].

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