Journal of Computational and Applied Mathematics 11 (1984) 209–224
North-Holland

Min-max vs. max-min flow control algorithms for optimal computer network capacity assignment

Carol A. NIZNIK *
Center for Brain Research, University of Rochester, School of Medicine, Rochester, NY 14627, U.S.A.

Received 15 January 1984

Abstract: The theory for optimally assigning capacities to the links of a store and forward computer communication network is developed by the minimization of the maximum of two groupings of all link capacities previously assigned to the network. The MIN-MAX and MAX-MIN Algorithms developed have both local and global properties. The basic mathematical structure of the local section of the MIN-MAX Algorithm is derived from the Lagrange Multiplier technique for minimizing convex functions, or the Kuhn–Tucker method for constrained minimization solutions, and the constraints imposed by the functional structure definitions. Since the MIN-MAX Algorithm attains the optimal minimized delay assignment by minimization of the Min-Max assignment, the aspects of its relation to the MAX-MIN Algorithms, the Min-Max inequality and the von Neumann Min-Max Theorem theory is explored. The global section of both algorithms offers an option for considering all possible, allowed link assignment combinations $(2^n - 2)$ of the n capacities available to further minimize delay.

Keywords: MIN-MAX algorithm, MAX-MIN algorithm, Lagrange multiplier, von Neumann minimax theorem, capacity, topology, congestion, convex function, concave function, Kuhn-Tucker method

1. Introduction

Flow control procedures, congestion removal in a fair manner in store and forward computer communication networks, are implemented within software algorithms for managing resources and preventing total utilization of the network by a single user, or a group of users. This paper deals with the development of the theory for generating a sectional capacity assignment algorithm during a specific time interval for application to the local and global computer network topologies. The algorithm formulations are specifically generated for networks where congestion has been isolated to a section of the network. The optimal reassignment of the capacities for the links of these network sections is determined from the theoretical areas of optimimization theory [9] and the game theory min-max inequality [4,20] (max-min theory [2,3,5]). The global option for both algorithms via the binomial theorem, allows the consideration of all possible groupings $(2^n - 2)$ of the n capacities to secure the most minimal delay. Previous research on link capacity

* Visiting 1983–1984.

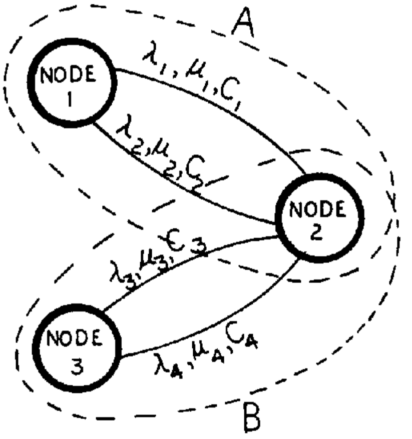

Fig. 1. Network example.

assignment [1,6,14,18,19] does not compute the minimal assignment options derived here. Therefore, the main contribution of this work is to offer the network designer the delay minimization options available from the MIN-MAX and MAX-MIN Algorithms developed [11].

2. Problem statement

The problem is to optimally assign capacities to the links of two sections of a computer communication network to offer effective flow control in these sections of the network by minimizing their maximum delay. This means that once the n links of concern in the network have been identified by their respective λ_i (interarrival rate in msg/sec), μ_i (service rate in bits/msg), γ (total external arrival rate in msg/sec), and C_i (link capacity in bits/sec), then the further grouping C_A and C_B ($\{C_i\}_A$, $\{C_i\}_B$) are elements of the space of all link capacities C_0 ($(C_1,\ldots,C_n) \in C_0$). Therefore, the difference between the MIN-MAX and MAX-MIN algorithmic methods [11] and those in previous store and forward computer network capacity assignment solutions [5,18] is basically in the tighter constraint on the delay minimization in each of the sections considered. Consult Fig. 1 for the sample network topology referenced in the derivations. To explore this problem from a theoretical point of view initially, the following assumptions are made for an M/M/1 queuing system at each computer node:

(1) each store and forward switching node has unlimited buffer storage,
(2) FCFS message priority discipline.
(3) independent, exponential interarrival distribution rate λ_i.
(4) independent exponential service rate, μ_i,
(5) constant average number of packets/msg,
(6) error-free link transmission,
(7) only queue mean waiting time delay (no node processing delay),
(8) full duplex link transmission.

3. Min-max algorithm derivation

The statement of the overall goal of the double constraint min-max allocation problem required to develop the MIN-MAX Algorithm is [6,18]

$$\min T(n) = T(C_1,\ldots,C_n) = \sum_{i=1}^{n} \left[\lambda_i/(\gamma(\mu_i C_i - \lambda_i))\right] \tag{1}$$

with,

$$\gamma = \sum_{l}\sum_{m} \gamma_{lm}, \quad l, m: C_i \in \Pi_{lm}$$

where,

γ_{lm} = external arrival rate at node l destined for node m, Π_{lm} = message path originating at node l and terminating at node m with, the global link delay for links $(i = 1,\ldots,k)$ in subnetwork A defined,

$$T_A = \sum_{i=1}^{k} \left[\lambda_i/(\gamma(\mu_i C_i - \lambda_i))\right] \tag{2}$$

and, the global link delay for links $(i = k+1,\ldots,n)$ in subnetwork B defined,

$$T_B = \sum_{i=k+1}^{n} \left[\lambda_i/(\gamma(\mu_i C_i - \lambda_i))\right] \tag{3}$$

subject to the constraint

$$\sum_{i=1}^{n} C_i = C_0 = C_A + C_B = \sum_{i=1}^{k} C_i + \sum_{i=k+1}^{n} C_i. \tag{4}$$

The heuristic explanation of the local MIN-MAX Algorithm Options (1, 2, 3 and 4) is based on the expression,

$$\min[\max[T_A, T_B]]. \tag{5}$$

Since T_A and T_B are convex, then $\max[T_A, T_B]$ are convex.

The possible conditions involved in solving Eq. (5) are: *Option 1,* $T_A > T_B$, *Option 2,* $T_B > T_A$, *Option 3,* $T_A = T_B$. If the delays obtained for Options 1 and 2 are not sufficiently minimal for the avoidance of congestion, the additional *Option 4* obtains a further minimization of the capacity assignment delay based on the goal statement

$$\min_{\{C_A, C_B\}} \max \left[\min_{\{C_1,\ldots,C_k\}} T_A \;, \min_{\{C_{k+1},\ldots,C_n\}} T_B \right]. \tag{6}$$

Option 5, the Global Option, performs Options 1, 2, 3 or 4 for all possible groupings of the n capacity links examined to compute the lowest minimum delay for the network. Further description of the local and global options follows:

Local MIN-MAX Algorithm Options

Objective 1: Assign capacities to links of subnetwork A to minimize T_A.
Option 1: $T_A > T_B$. Using the Lagrange Multiplier minimization technique, and assuming that

$T_A > T_B$, formulate [6],

$$G = T_A + \alpha\left[\sum_{i=1}^{k} C_i - C_A\right] \tag{7}$$

where,

α = Lagrange Multiplier subject to the constraint,

$$\sum_{i=1}^{k} C_i = C_A + C_0 - C_B, \tag{8}$$

to solve for $\{C_i\}_A = (C_1, \ldots, C_k)$ and hold the $\{C_i\}_B = (C_{k+1}, \ldots, C_n)$ constant.

Substituting for T_A from Eq. (2) in G, and setting $\partial G/\partial C_i = 0$ for $i = 1, \ldots, k$, α is computed in terms of C_i:

$$C_i = (\lambda_i/\mu_i) + \left(\sqrt{\lambda_i\mu_i/\gamma}\,/\sqrt{\alpha\mu_i}\right). \tag{9}$$

Then, obtaining the constraint in Eq. (8) by setting $\partial G/\partial\alpha = 0$,

$$\sum_{i=1}^{k} C_i = \sum_{i=1}^{k}\left[(\lambda_i/\mu_i) + \left(\sqrt{\lambda_i\mu_i/\gamma}\,/\mu_i\sqrt{\alpha}\right)\right] = C_A \tag{10}$$

and recognizing,

$$(\sqrt{\alpha})^{-1} = \left(C_A - \sum_{i=1}^{k}\lambda_i/\mu_i\right)\Big/\sum_{i=1}^{k}\left(\sqrt{\lambda_i\mu_i/\gamma}\,/\mu_i\right) \tag{11}$$

the expression for $\{C_i\}_A$ is computed by substituting Eq. (1) into Eq. (3),

$$\{C_i\}_A = \left[\sqrt{\lambda_i\mu_i/\gamma}\,)/\mu_i\right]\left[C_A - \sum_{j=1}^{k}(\lambda_j/\mu_j)\right]\Big/\sum_{j=1}^{k}\lambda_j/\mu_j\gamma. \tag{12}$$

Objective 2: Assign capacities to all links of subnetwork B to minimize T_B.

Option 2: $T_B > T_A$. Similarly, use the Lagrange Multiplier as in Option 1 to compute $\{C_i\}_B$ and hold the $\{C_i\}_A$ constant. The Lagrangian equation is,

$$G = T_B + \alpha\left[\sum_{i=k+1}^{n} C_i - C_B\right] \tag{13}$$

subject to the constraint

$$\sum_{i=k+1}^{n} C_i = C_B = C_0 - C_A. \tag{14}$$

Then,

$$\{C_i\}_B = (\lambda_i/\mu_i) + \sqrt{\lambda_i/\gamma\mu_i}\left[\left[C_B - \sum_{j=k+1}^{n}(\lambda_j/\mu_j)\right]\Big/\sum_{j=k+1}^{n}\sqrt{\lambda_j/\mu_j\gamma}\right]. \tag{15}$$

Objective 3: Assign capacities to links of both subnetworks A and B to minimize T_A and T_B.

Option 3: $T_A = T_B$. Here, the goal is to minimize $T(n)$, subject to the constraints described by Eq. (4) and $T_A = T_B$ written,

$$\sum_{i=1}^{k} (\lambda_i/(\gamma(\mu_i C_i - \lambda_i))) = \sum_{i=k+1}^{n} (\lambda_i/(\gamma(\mu_i C_i - \lambda_i))), \tag{16}$$

Forming the Lagrangian for the equality constraints in Eqs. (4) and (16),

$$G = T_A + T_B + \beta\left[\sum_{i=1}^{n} C_i - C_0\right] + \alpha[T_A - T_B]. \tag{17}$$

Taking the $\partial G/\partial C_i = 0$, $i = 1,\ldots,k$,

$$\{C_i\}_A = \left(\sqrt{(\alpha+1)/\beta}\sqrt{\lambda_i/\mu_i\gamma}\right) + (\lambda_i/\mu_i) \tag{18}$$

and

$$\lambda_i\mu_i/\gamma(\mu_i C_i - \lambda_i)^2 = \beta/(\alpha+1). \tag{19}$$

Similarly,

$$\frac{\partial G}{\partial C_i} = 0 \quad \text{for } i = k+1,\ldots,n$$

yields

$$\{C_i\}_B = \left(\sqrt{(1-\alpha)/\beta}\sqrt{\lambda_i/\mu_i\gamma}\right) + (\lambda_i/\mu_i) \tag{20}$$

and

$$\lambda_i\mu_i/\gamma(\mu_i C_i - \lambda_i)^2 = \beta/(1-\alpha). \tag{21}$$

The other two solution equations result from $\partial G/\partial\beta$ (Eq. (4)) and $\partial G/\partial\alpha$ (Eq. (16)). The expression for $1/\sqrt{\beta}$ is obtained by substituting Eqs. (19) and (20) into Eq. (4),

$$\sum_{i=1}^{n} C_i = C_0 = \sqrt{(\alpha+1/\beta}\left[\sum_{i=1}^{k}\sqrt{\lambda_i/\mu_i\gamma}\right] + \sum_{i=1}^{k}(\lambda_i/\mu_i) + \sqrt{(1-\alpha)/\beta}\left[\sum_{i=k=1}^{n}\sqrt{\lambda_i/\mu_i\gamma}\right] + \sum_{i=k+1}^{n}(\lambda_i/\mu_i)$$

resulting in

$$1/\sqrt{\beta} = \left[C_0 - \sum_{i=1}^{k}(\lambda_i/\mu_i) - \sum_{i=k+1}^{n}(\lambda_i/\mu_i)\right]\Big/\left[\sqrt{\alpha+1}\left[\sum_{i=1}^{k}\sqrt{\lambda_i/\mu_i\gamma}\right] + \sqrt{1-\alpha}\left[\sum_{i=k+1}^{n}\sqrt{\lambda_i/\mu_i\gamma}\right]\right]. \tag{22}$$

Also, the expression for $\sqrt{(1-\alpha)/(\alpha+1)}$ is derived with the following steps by creating the equality in Eq. (16) from Eqs. (19) and (21):

Step 1: Multiply the numerator of both sides of Eqs. (19) and (21) by λ_i and the denominator by γ.

Step 2: Multiply both sides of the Step 1 alterations of Eqs. (19) and (21) by $1/\mu_i$.

Step 3: Form the square root of both sides of the Step 2 alterations of Eqs. (19) and (21).
Step 4: Sum both sides of Step 3 equations over their respective indices.
Step 5: Equate equations in Step 4 per the Eq. (16) constraint, with the result

$$\sqrt{(1-\alpha)/(\alpha+1)} = \frac{\sum_{i=k+1}^{n} \sqrt{\lambda_i/\mu_i}}{\sum_{i=1}^{k} \sqrt{\lambda_i/\mu_i}}. \tag{23}$$

The following final C_i solutions for $i = 1,\ldots,k$ ($\{C_i\}_A$) and $i = k+1,\ldots,n$ ($\{C_i\}_B$) are generated by substituting the expressions for $1/\sqrt{\beta}$ Eq. (22) and $\sqrt{(1-\alpha)/(\alpha+1)}$ Eq. (23) into Eqs. (18) and (20) for $i = 1,\ldots,k$,

$$\{C_i\}_A = \frac{\left[C_0 - \sum_{j=1}^{n} (\lambda_j/\mu_j)\right]\sqrt{\lambda_i/\mu_i}}{\left[\sum_{j=1}^{k} \sqrt{\lambda_j/\mu_j}\right] + \left[\sum_{j=k+1}^{n} \sqrt{\lambda_j/\mu_j}\right] / \sum_{j=1}^{k} \sqrt{\lambda_j/\mu_j}} + (\lambda_i/\mu_i) \tag{24}$$

and for $i = k+1,\ldots,n$,

$$\{C_i\}_B = \frac{\left[C_0 - \sum_{j=1}^{n} (\lambda_j/\mu_j)\right]\sqrt{\lambda_i/\mu_i}}{\left\{\left[\sum_{j=1}^{k} \sqrt{\lambda_j/\mu_j}\right]^2 / \sum_{j=k+1}^{n} \sqrt{\lambda_j/\mu_j}\right\} + \left[\sum_{j=k+1}^{n} \sqrt{\lambda_j/\mu_j}\right]} + (\lambda_i/\mu_i). \tag{25}$$

Objective 4: Maximize the minimum delay between the minimization of T_A and T_B.

Option 4: This further minimization technique for initial conditions $T_A < T_B$ and $T_B < T_A$ requires that the point where $T_A = T_b$ be reached (refer to Eq. (16)). Therefore, Eq. (6) in an operations research interpretation is equivalent to

$$\max_{\{C_A, C_B\}} \min \left[\min_{\{C_1,\ldots,C_k\}} T_A \;,\; \min_{\{C_{k+1},\ldots,C_N\}} T_B \right] \tag{26}$$

because, both expressions yield the same capacity assignment for $\{C_i\}_A$ and $\{C_i\}_B$. The derivation of the optimal C_A and C_B selection for $T_A = T_B$ is determined by the following procedure:

(a) substitute the values of $\{C_i\}_A$ (Eq. (12)) into Eq. (2) and $\{C_i\}B$ (Eq. (15)) into Eq. (3),

$$\sum_{i=1}^{k} \left[\lambda_i/\left[\left(\gamma\sqrt{\lambda_i\mu_i/\gamma}\right)/\mu_i\right]\left[\left(C_A - \sum_{j=1}^{k}\left[(\lambda_j/\mu_j))/\sum_{j=1}^{k}\sqrt{\lambda_j/\mu_j\gamma}\right]\right]\right.$$

$$= \sum_{i=k+1}^{n} \left[\lambda_i/\left[\gamma\sqrt{\lambda_i\mu_i/\gamma}\left[\left[(C_0 - C_A) - \sum_{j=k+1}^{n} (\lambda_j/\mu_j)\right] \Big/ \sum_{j=k+1}^{n} \sqrt{\lambda_j/\mu_j\gamma}\right]\right]\right]. \tag{27}$$

Now, moving terms constant with respect to the i indexed summations outside of these summations and cross multiplying,

$$\left[\sum_{j=1}^{n}\sqrt{\lambda_j/\mu_j\gamma}\right]\left[(C_0-C_A)-\sum_{j=k+1}^{n}(\lambda_j/\mu_j)\right]\left[\sum_{i=1}^{k}\left(\lambda_i/\sqrt{\lambda_i\mu_i\gamma}\right)\right]$$
$$=\left[\sum_{j=k+1}^{n}\sqrt{\lambda_j/\mu_j\gamma}\right]\left[C_A-\sum_{j=1}^{k}(\lambda_j/\mu_j)\right]\left[\sum_{i=k+1}^{n}\left(\lambda_i/\sqrt{\lambda_i\mu_i\gamma}\right)\right]; \tag{28}$$

(b) multiplying out terms in Eqs. (28) and rearranging to solve for C_A,

$$C_A=(C_A)_N/(C_A)_D \tag{29}$$

where

$$(C_A)_N=\left[C_0-\sum_{j=k+1}^{n}\lambda_j/\mu_j\right]\left[\sum_{j=1}^{k}\sqrt{\lambda_j/\mu_j\gamma}\right]^2+\left[\sum_{j=k+1}^{n}\sqrt{\lambda_j/\mu_j\gamma}\right]^2\left[\sum_{j=1}^{k}\lambda_j/\mu_j\right],$$

$$(C_A)_D=\left[\sum_{j=k+1}^{n}\sqrt{\lambda_j/\mu_j\gamma}\right]^2+\left[\sum_{j=1}^{k}\sqrt{\lambda_j/\mu_j\gamma}\right]^2.$$

c_A	c_B		
c_1c_2	c_3c_4	2^n-2	2^n
c_3c_4	c_1c_2		
c_1c_3	c_2c_4		
c_1c_4	c_2c_3		
c_1	$c_2c_3c_4$		
c_2	$c_1c_3c_4$		
c_3	$c_2c_4c_1$		
c_4	$c_2c_3c_1$		
$c_2c_3c_4$	c_1		
$c_1c_3c_4$	c_2		
$c_2c_4c_1$	c_3		
$c_2c_3c_1$	c_4		
c_2c_4	c_1c_3		
c_2c_3	c_1c_4		
$c_1c_2c_3c_4$		2	
	$c_1c_2c_3c_4$		

Fig. 2. Global algorithm example ($n = 4$).

Then, C_B is computed from Eq. (4), $\{C_i\}_A$ from Eq. (9), and $\{C_i\}_B$ from Eq. (15).

Option 5: Global MIN-MAX or MAX-MIN Algorithm. This formulation computes an overall minimal delay for the network by the computation of the previous four options for all distinct double groupings of the n capacity links examined for either the MIN-MAX Algorithm or the MAX-MIN Algorithm (refer to Section 4.) Refer to Fig. 2 for the combinations of capacities in the groupings C_A, C_B possible when there are a total of four capacities. The binomial coefficient theorem for two groupings composed together of n distinct entries yield 2^n distinct combinations. Of the 2^n distinct combinations, only $2^n - 2$ groupings entries yield 2^n distinct combinations. Of the 2^n distinct combinations, only $2^n - 2$ groupings are of interest in the algorithm since there will be two combinations of the two groupings where no capacities are in either grouping C_A or C_B. Refer to Fig. 3 for the overall MIN-MAX Algorithm flow chart.

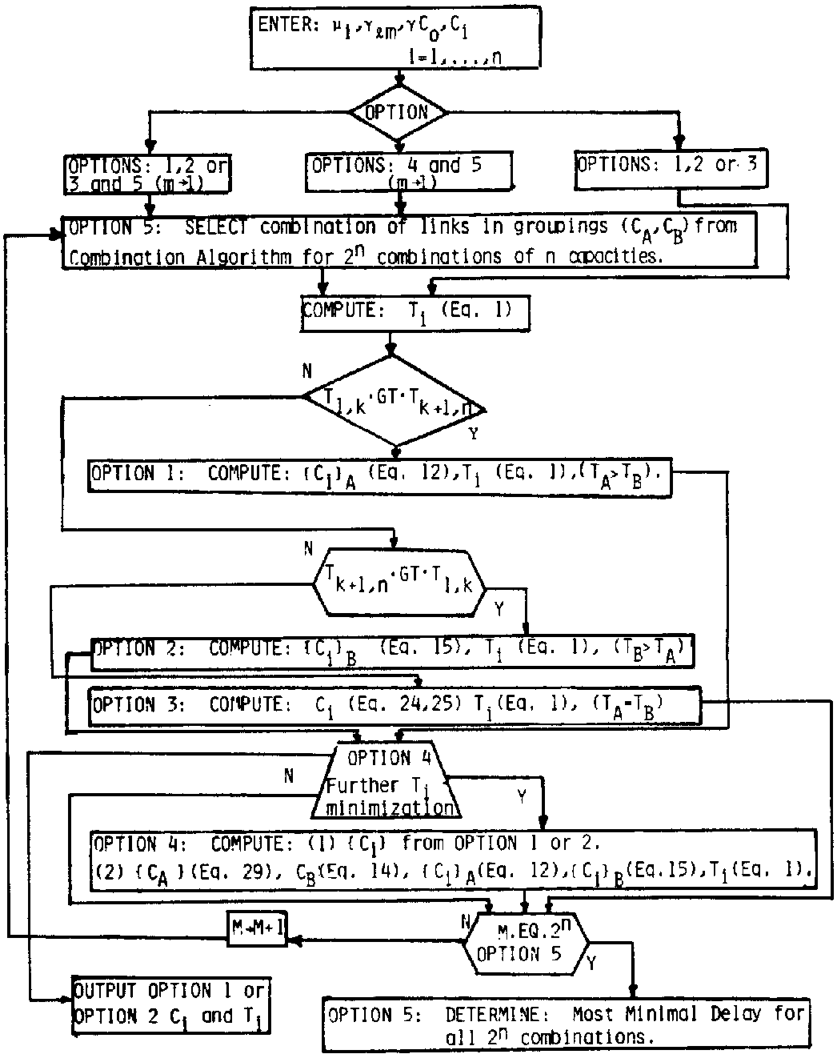

Fig. 3. MIN-MAX capacity assignment algorithm.

4. Max-min algorithm derivation

An approach from game theory is examined to obtain a purely mathematical structure for min-max theory and its comparison to the MIN-MAX Algorithm in Section 3. The von Neumann Minimax Theorem [16], the min-max inequality [4,7,8,20] and the theory of max-min by Danskin [2,3], provide the mathematical basis for the MAX-MIN Algorithm [11] offering other minimization options. Refer to [20] for the formalism illustrating the development of these three mathematical tools from game theory. The min-max inequality is stated as [4,19],

$$\max_{A} \min_{B} f(A, B) \leqslant \min_{B} \max_{A} f(A, B). \tag{30}$$

The von Neumann Minimax Theorem states that the equality,

$$\max_{A} \min_{B} f(A, B) = \min_{B} \max_{A} f(A, B) = f(\bar{A}, \bar{B}) \tag{31}$$

exists when the saddle point $(\bar{A}, \bar{B})$ exists on $U \times V$. $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ are nonempty compact, convex sets where $f: U \times V \rightarrow \mathbb{R}$ are continuous. For the saddle point $(\bar{A}, \bar{B})$ to exist, for each fixed $B, f(A, B)$ is concave on U and for each fixed $A, f(A, B)$ is convex on V. For the capacity assignment problem posed here, this inequality means that maximizing over A (B constant) the most minimal assignment over B (A constant) results in a smaller or equal delay function $f(A, B)$, as minimizing the maximum over the A subset. In terms of game theory, this means that the B player acts in full cognizance of the A player's initial conditions ($\{C_i\}A$). After the B player has made a move (here reassigned $\{C_i\}B$), the A player can react to the selection of the worst case (most maximal) of the B player's functions.

Now, define,

$$A = \{C_i\}_A, B = \{C_i\}_B \quad \text{and}$$
$$f(A, B) = f(\{C_i\}_A, \{C_i\}_B) = T(\{C_i\}_A, \{C_i\}_B) \quad \text{(Eq. (1))}$$

with the constraint $\sum_{i=1}^{n} C_i = C_0$.

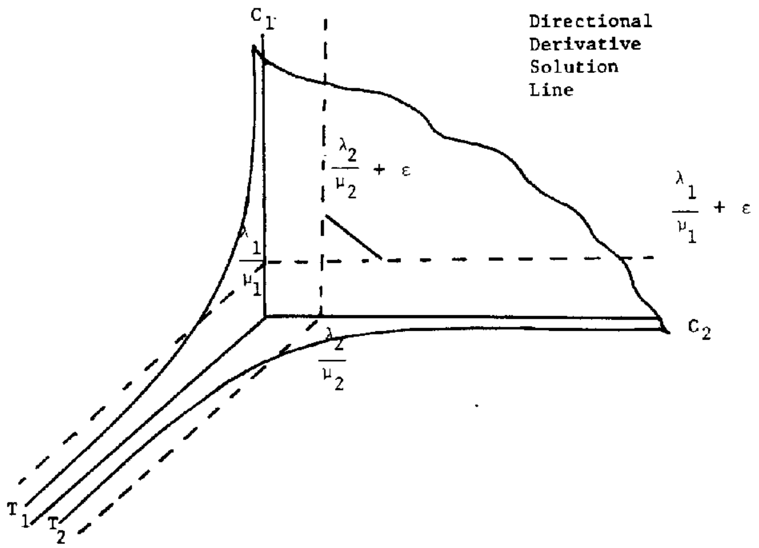

Fig. 4. MAX-MIN algorithm solution space.

To attain the most minimal delay, the minimizing delay, the minimizing capacity assignment $(\{C_i\}_B)$ occurs after the maximizing assignment of $\{C_i\}_A$ has been initially fixed. Then, to achieve a maximum, the $\{C_i\}_A$ are assigned the maximum values,

$$\{C_i\}_{A_{\mathrm{MAX}}} = ((\lambda_i/\mu_i) + \epsilon) \tag{32}$$

because of the nature of the T_i curve for each C_i. Referring to Fig. 4, it is noted that the function T_i attains a minimum but not a maximum. Therefore, a specific saddle point doesn't exist for the von Neumann Minimax Theorem unless Eq. (32) has an ϵ defined. Then, a plane of defined saddle points can exist in a specific region. The conditions for the existence of the directional derivative require that the function is continuous in the region of the maximum. Therefore, the maximation is performed on the $\{C_i\}_A$ for a selected maximum $((\lambda_i/\mu_i) + \epsilon)$. For the options listed in the MIN-MAX Algorithm, the following format for the MAX-MIN Algorithm is observed to provide the most minimal delay assignment of capacities:

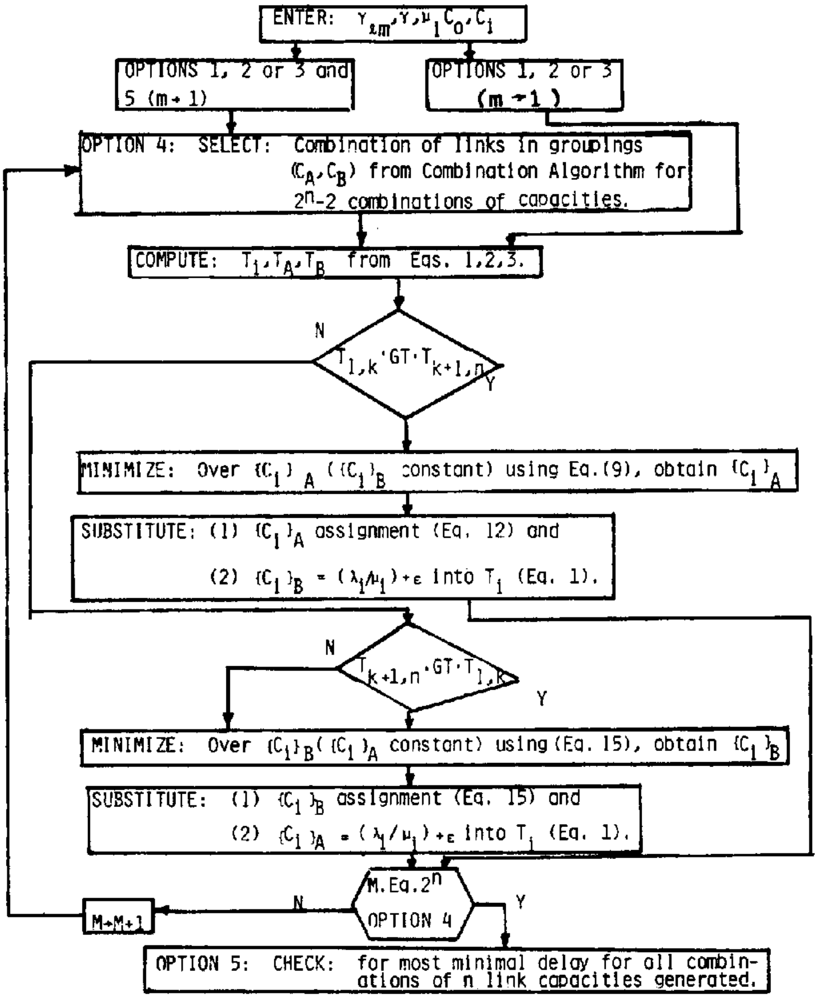

Fig. 5. MAX-MIN capacity assignment algorithm.

Option 1: $T_A > T_B$. Evaluate T for $\{C_i\}_A$ at $i = k+1,\dots,n$ and $\{C_i\}_B = ((\lambda_i/\mu_i)+\epsilon)$ for $i = 1,\dots,k$,

$$\max_{\{C_i\}_B} \min_{\{C_i\}_A} f(\{C_i\}_A, \{C_i\}_B) \leqslant \min_{\{C_i\}_A} \max_{\{C_i\}_B} f(\{C_i\}_A, \{C_i\}_B). \tag{33}$$

Option 2: $T_B > T_A$. Evaluate T for $\{C_i\}_B$ at $j = 1,\dots,k$ and $\{C_i\}_A = ((\lambda_i/\mu_i)+\epsilon)$ at $i = k+1,\dots,n$,

$$\max_{\{C_i\}_A} \min_{\{C_i\}_B} f(\{C_i\}_A, \{C_i\}_B) \leqslant \min_{\{C_i\}_B} \max_{\{C_i\}_A} f(\{C_i\}_A, \{C_i\}_B). \tag{34}$$

Option 3: $T_A = T_B$. Either Eq. (33) or Eq. (34).

Option 4: Global Algorithm. Referring to Fig. 5 for the MAX-MIN Algorithm format, it is observed that the similarity to the MIN-MAX Algorithm is in the use of Eqs. (12) and (15) for obtaining the constrained minimization of $f(\{C_i\}_A, \{C_i\}_B)$. The MAX-MIN Algorithm will, in most cases, compute a less optimal solution than the MIN-MAX Algorithm due to the maximization.

5. Min-max and max-min solution for any number of capacity groupings

In this section, the further derivation of the MIN-MAX and MAX-MIN Algorithm Theory for the generalization of any number of groupings of capacities is formulated. A reformulation of the basic theory for two groupings of capacities from Sections 2 and 3 is provided before proceeding with the additional formalism for the general case of G_K groupings.

5.1. Generalization of two groupings of capacities

Let $z = (C_1,\dots,C_k)$, $w = (C_{k+1},\dots,C_n)$ be the capacities for two groups of links. The constraint is that,

$$C_1 + \cdots + C_k + C_{k+1} + \cdots + C_n = C_0. \tag{35}$$

The objective is to minimize over the choice of capacities $C_1,\dots,C_n$ the maximum of (T_1, T_2), where

$$T_1(z) \triangleq \sum_{i=1}^{k} \lambda_i \frac{1}{\gamma(\mu_i C_i - \lambda_i)}, \qquad T_2(w) \triangleq \sum_{i=k+1}^{n} \lambda_i \frac{1}{\gamma(\mu_i C_i - \lambda_i)}. \tag{36}$$

Therefore, it can now be stated that, $\min_{C_i} \max[T_1, T_2]$. Let $0 \leqslant x \leqslant 1$, be such that

$$\sum_{i=1}^{k} C_i = xC_0, \qquad \sum_{i=k+1}^{n} C_i = (1-x)C_0. \tag{37}$$

For fixed x, the minimization of T_1, T_2 is the standard capacity assignment problem, and we have to minimize

$$\min_{z,w} \max[T_1(z), T_2(w)], \tag{38}$$

The overall problem is

$$\min_x \left\{ \min_z \min_w \max[T_1(z), T_2(w)] \right\}. \tag{39}$$

Note that T_1 depends only on z, T_2 only on w. So,

$$\min_x \left\{ \min_z \min_w \max[T_1(z), T_2(w)] \right\} = \min_x \left\{ \max\left[\min_z T_1(z), \min_w T_2(w) \right] \right\}. \tag{40}$$

Each of the $\min_z T_1(z)$, $\min_w T_2(w)$ is a 'standard' capacity assignment problem, solved with one Lagrange multiplier, denoted as α^2. Then

$$\frac{\partial}{\partial C_i}\left[T_1(z) + \frac{\alpha^2}{\gamma} \sum_{i=1}^{k} C_i \right] = 0, \tag{41}$$

$$\frac{1}{\gamma} \frac{\lambda_i}{(\mu_i C_i - \lambda_i)^2} = \frac{\alpha^2}{\gamma} \to C_i^* = \frac{\lambda_i}{\mu} + \frac{\lambda_i^{1/2}}{\alpha}. \tag{42}$$

From the constraint: $\sum_{i=1}^{k} C_i = xC_0$, we find

$$\frac{1}{\alpha} = \left(xC_0 - \sum_{i=1}^{k} \frac{\lambda_i}{\mu_i} \right) \left(\sum_{i=1}^{k} \lambda_i^{1/2} \right)^{-1}, \tag{43}$$

$$\min_z T_1(z) = \gamma^{-1} \left(\sum_{i=1}^{k} \lambda_i^{1/2} \right)^2 \left[xC_0 - \sum_{i=1}^{k} \lambda_i \mu_i^{-1} \right]^{-1}. \tag{44}$$

Similarly,

$$\min_w T_2(w) = \gamma^{-1} \left(\sum_{i=k+1}^{n} \lambda_i^{1/2} \right) \left[(1-x)C_0 - \sum_{i=k+1}^{n} \lambda_i \mu_i^{-1} \right]^{-1}. \tag{45}$$

Note that the system is stable, and the solution is valid only if $C_i^* > \lambda_i/\mu_i$, $i = 1, \ldots, n$, and hence

$$xC_0 > \sum_{i=1}^{k} \lambda_i/\mu_i, \qquad (1-x)C_0 > \sum_{i=k+1}^{n} \frac{\lambda_i}{\mu_i} \tag{46}$$

The region of admissible x's is: $0 < x_0 < x < x_1 < 1$ where

$$x_0 = C_0^{-1} \sum_{i=1}^{k} \lambda_i \mu_i^{-1}, \qquad x_1 = C_0^{-1} \sum_{i=k+1}^{n} \lambda_i \mu_i^{-1}. \tag{47}$$

Within the above region, $\min_z T_1(z)$ is monotone decreasing with x and $\min_w T_2(w)$ is monotone increasing. Hence, if we define: x_s by the equation

$$\min_z T_A(z) = \min_w T_A(w) \tag{48}$$

we have

$$\left(\sum_{i=1}^{k} \lambda_i^{1/2} \right) \left(x_s C_0 - \sum_{i=1}^{k} \lambda_i \mu_i^{-1} \right)^{-1} = \left(\sum_{i=k+1}^{n} \lambda_i^{1/2} \right) \left((1-x_s)C_0 - \sum_{i=k+1}^{n} \lambda_i \mu_i^{-1} \right)^{-1}, \tag{49}$$

Then

$$\max\left[\min_z T_1(z), \min_w T_2(w) \right] \tag{50}$$

is equal to $\min_z T_1(z)$ for $x_0 < x \leqslant x_s$, and equal to $\min_w T_2(w)$ for $x_s < x < x$. Also, $\min_z T_1(z)$ is non-decreasing in x, and $\min_w T_2(w)$ is monotone increasing in x. Thus, the minimum of the above expression is achieved for $x = x_s$.

$$x_s = C_0^{-1}\left[\left(\sum_{i=1}^{k}\lambda_1^{1/2}\right)^2 + \left(\sum_{i=k+1}^{k}\lambda_i^{1/2}\right)^2\right]^{-1} \cdot\left[\sum_{i=1}^{k}\lambda_i\mu_i^{-1}\left(\sum_{i=k+1}^{n}\lambda_i^{1/2}\right)^2 - \sum_{i=k+1}^{n}\lambda_i\mu_i^{-1}\left(\sum_{i=1}^{k}\lambda_i^{1/2}\right)\right]^2 \tag{51}$$

and the capacity assignment problem has been resolved for two groups.

5.2. Generalization for m groupings of capacities

For m groups $G_1,\ldots,G_m$ of delays, where $C_{01} + \cdots + C_{0n} = C_0$; $z_k = (C_{ij}, i \in G_k)$,

$$T_k(z_k) \triangleq \sum_{i\in G_k}\lambda_i\frac{1}{\gamma[\mu_i C_{0i} - \lambda_i]}, \quad k = 1,\ldots,m. \tag{52}$$

Let $x_1 + \cdots + x_m = 1$, $x_i \geqslant 0$. Then, the objective is

$$\min_{C_{01},\ldots,C_{0n}} \max[T_1(z_1), T_2(z_2),\ldots,T_m(z_m)]. \tag{53}$$

Considering the subproblems: min $T_k(z_k)$, under $\sum_{i\in G_k}C_i = x_k C_0$, (standard capacity assignment subproblem), the solution to the subproblem, as before for group G_k is

$$C_i^* = \frac{\lambda_i}{\mu_i} + \frac{\lambda_i^{1/2}}{\alpha_k} \tag{54}$$

where

$$\frac{1}{\alpha_k} = \left[x_k C_{0i} - \sum_{i\in G_k}\lambda_i\mu_i^{-1}\right]\left[\sum_{i\in G_k}\lambda_i^{1/2}\right]^{-1} \tag{55}$$

and therefore,

$$\min_{z_k} T_k(z_k) = \gamma^{-1}\left(\sum_{i\in G_k}\lambda_i^{1/2}\right)^2\left[x_k C_0 - \sum_{i\in G_k}\lambda_i\mu_i^{-1}\right]^{-1}, \quad k = 1. \tag{56}$$

Going back to the overall problem expressed in Eq. (38),

$$\begin{aligned}&\min_{C_{01},\ldots,C_{0n}} \max_{G_j}[T_1(z_1),\ldots,T_m(z_m)]\\ &= \min_{x_1,\ldots,x_m}\left\{\min_{z_1}\min_{z_2},\ldots,\min_{z_m}\left\{\max_{G_j}[T_1(z_1),\ldots,T_m(z_m)]\right\}\right\}\\ &= \min_{x_1,\ldots,x_m}\left\{\max\left[\min_{z_1}T_1(z_1), \min_{z_2}T_2(z_2),\ldots,\min_{z_m}T_m(z_m)\right]\right\}\\ &= \min_{x_1,\ldots,x_m}\left\{\max\left[\frac{A_1}{x_1C_{01} - B_1},\ldots,\frac{A_m}{x_mC_{0m} - B_m}\right]\right\}\end{aligned} \tag{57}$$

where

$$A_k = \gamma^{-1}\left(\sum_{i\in G_k}\lambda_i^{1/2}\right)^2, \qquad B_k = \sum_{i\in G_k}\lambda_i\mu_i^{-1}. \tag{58}$$

Refer to Fig. 6 for the graphical description of B_k.

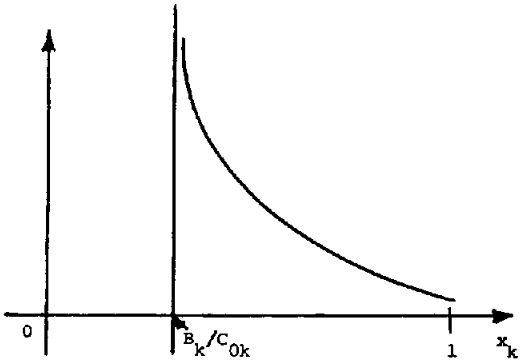

Fig. 6. Graphical description of minimizing x_i.

General properties for G_k

Each $A_k/(x_k C_{0k} - B_k) = g_k(x)$ is *concave*. Therefore,

$$\max_{1 \leqslant k \leqslant m} \left\{ \frac{A_k}{x_k C_{0k} - B_k} \right\} \triangleq s(x)$$

is also a concave function. This means that for $\{x; \Sigma_{k=1}^{m} x_k = 1, x_k \geqslant 0\}$, which is a *convex region*, it has a unique minimum. The following is the proof that $s(x)$ is concave. Let x^1, x^2 be two values of $x = (x_1, \ldots x_m)$. Then, $s(x) = \max_k\{g_k(x)\}$ where

$$g_k(\alpha x^1 + (1-\alpha)x^2) \leqslant \alpha g_k(x^1) + (1-\alpha) g_k(x^2). \tag{59}$$

This implies that

$$\max_k \{g_k(x)\} \leqslant \alpha \max_k g_k(x^1) + (1-\alpha) \max_k g_k(x^2). \tag{60}$$

To find the optimum x, the minimizing x must be found. We separate into regions in the same manner as the saddle points are determined for the MAX-MIN Algorithm in Section 4. For, $m = 3$, $\alpha_k = A_k/C_{0k}$ $b_k = B_k/C_{0k}$. Then (see also Fig. 7)

$$\frac{\alpha_1}{x_1 - b_1} > \frac{\alpha_2}{x_2 - b_2}, \qquad \frac{\alpha_1}{x_1 - b_1} > \frac{\alpha_3}{x_3 - b_3}.$$

So

$$\alpha_1 x_2 - \alpha_1 b_2 > \alpha_2 x_1 - \alpha_2 b_1, \qquad \alpha_1 x_3 - \alpha_1 b_3 > \alpha_3 x_1 - \alpha_3.$$

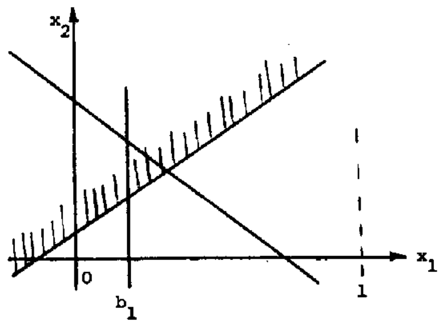

Fig. 7. Graphical description of x_1 and x_2.

Then,

$$x_3 = 1 - x_1 - x_2 \rightarrow \alpha_1(1 - x_1 - x_2) - \alpha_1 b_3 > \alpha_3 x_1 - \alpha_3 b_1,$$
$$x_1(\alpha_3 + \alpha_1) + x_2\alpha_1 < \alpha_1 + \alpha_3 b_1 - \alpha_1 b_3,$$

therefore,

$$\alpha_2 x_1 - \alpha_1 x_2 < \alpha_2 b_1 - \alpha_1 b_2. \tag{61}$$

The optimum solution is, therefore, given by

$$\frac{A_1}{x_1 C_{01} - B_1} = \frac{A_2}{x_2 C_{02} - B_2} = \cdots = \frac{A_m}{x_m C_{0m} - B_m} = \alpha, \qquad \sum_{i=1}^{m} x_i = 1, \tag{62}$$

$$x_i C_{0i} - B_i = \frac{A_i}{\alpha}, \qquad x_i = \left(\frac{A_i}{\alpha} + B_i\right)\frac{1}{C_{0i}}, \tag{63}$$

$$\frac{1}{\alpha}\sum \frac{A_i}{C_{0i}} + \sum B_i = 1, \qquad \frac{1}{\alpha} = \left(1 - \sum_{i=1}^{m} B_i\right)\left(\sum \frac{A_i}{C_{0i}}\right)^{-1}. \tag{64}$$

6. Conclusion

In addition to the use of these algorithms when all parameters have initial condition values, the C_i can be generated by using the C_i expressions for the options considered. The methods proposed here for min-max and max-min criterion solutions offer further mathematical structure and therefore ensure more uniformity in the treatment of average transmission delay assignments for computer networks. These criteria offer an alternative to those presently used in conventional computer networking capacity assignment solutions by Kleinrock [6] and Schwartz [18]. An important practical application of this theory is the reallocation of capacity to ensure survivability during network component error, failure, or total destruction. Also, when designing computer network congestion control consistent with security and privacy, game theory capacity reallocation algorithms for subnetwork areas can enable or prevent certain types of message flow. Further numerical experiments are required on specific network problems to determine the optimality of the various options under these conditions for the initial research results.

Acknowledgment

I acknowledge the helpful comments of L. Ellis and D. Kazakos concerning the MIN-MAX and MAX-MIN Algorithms.

References

[1] K.M. Chandy, J. Hogarth and C.H. Sauer, Selecting capacities in computer communication systems, *IEEE Trans. Software Engrg.* 3 (4) (1978).
[2] J.M. Danskin, *The Theory of Max-Min* (Springer, New York, 1967).

[3] J.M. Danskin, The theory of max-min with applications, *J. SIAM Appl. Math.* **14** (4) July (1966) 641–667.
[4] V.F. Dem'yannov and V.N. Malozemov, *Introduction to Mini-Max* (Wiley, New York, 1974).
[5] D. Kazakos, Maxi-min linear discrimination, Part I, *IEEE Trans. Systems, Man Cybernet.* 7 (9) (1977) 661–669.
[6] L. Kleinrock, *Queueing Systems, Volume 1: Theory, Queueing Systems, Volume 2: Computer Applications* (Wiley, New York, 1975, 1976).
[7] P.R. Kumar, Optimal mixed strategies in dynamic games, UMBC Math. Research Report, 1979.
[8] P.R. Kumar, The tank versus gun problem: An optimality theory in mixed strategies, *Proc. MTNS '79,* Delft University of Technology, Delft, Netherlands, 1979.
[9] D.G. Luenberger, *Introduction to Linear and Non-linear Programming* (Addison-Wesley, Reading, MA, 1973) 233–236.
[10] C.A. Niznik, A bounded optimization algorithm for capacity assignment in computer communication networks, *Internat. J. Systems Sci.* **11** (1) (1980) 57–64.
[11] C.A. Niznik, A min-max flow control algorithm for optimal computer network capacity assignment, *Proc. ICC'80,* (1980) 23.5.1–23.5.5.
[12] C.A. Niznik, *Measures of congestion for computer communication networks,* Ph.D. Dissertation, Dept. of Electrical Engineering, SUNY at Buffalo, Amherst, NY, 1978.
[13] A. Osyczka, The min-max approach to a multicriterion network optimization problem for engineering design, *Comput. Methods Appl. Mech. Engrg.* **15** (1978) 309–333.
[14] A. Osyczka, The min-max approach to a multicriterion network optimization problem, *Proc. MTNS '79,* Delft University of Technology, Delft, Netherlands, 1979.
[15] E. Protonotarios and E. Sykas, Line capacity assignment in computer networks, *INFO II*, Patras, Greece, 1979.
[16] A.W. Roberts and D.E. Varberg, *Convex Functions* (Academic Press, New York, 1973) 128–136.
[17] J. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1970).
[18] M. Schwartz, *Computer Communication Network Design and Analysis* (Prentice-Hall, Englewood Cliffs, NJ, 1977).
[19] D.F. Stanat and G.A. Mago, Minimixing maximum flows in linear graphs, *Networks* **9** (4) (1979) 333–362.
[20] N.N. Vorob'ev, *Game Theory Lectures for Economists and Systems Scientists* (Springer, New York, 1977) 8–26.