Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 55 (2008) 2662–2672
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Orienting Cayley graphs generated by transposition trees

Eddie Cheng*, László Lipták, Nart Shawash

Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA

Received 31 May 2007; received in revised form 7 September 2007; accepted 10 October 2007

Abstract

Day and Tripathi [K. Day, A. Tripathi, Unidirectional star graphs, Inform. Process. Lett. 45 (1993) 123–129] proposed an assignment of directions on the star graphs and derived attractive properties for the resulting directed graphs. Cheng and Lipman [E. Cheng, M.J. Lipman, On the Day–Tripathi orientation of the star graphs: Connectivity, Inform. Process. Lett. 73 (2000) 5–10; E. Cheng, M.J. Lipman, Connectivity properties of unidirectional star graphs, Congr. Numer. 150 (2001) 33–42] studied the connectivity properties of these unidirectional star graphs. The class of star graphs is a special case of Cayley graphs generated by transposition trees. In this paper, we give directions on these graphs and study the connectivity properties of the resulting unidirectional graphs.

Keywords: Interconnection networks; Unidirectional graphs; Cayley graphs; Connectivity

1. Introduction

Directed interconnection networks have gained much attention in the area of interconnection networks. Some recent research in this area includes [1,3–11]. In particular, [5] gave an application and an architectural model for the studies of unidirectional graph topologies as well as a comparison of the diameters among many known unidirectional interconnection networks. In addition, [6] proposed unidirectional hypercubes as the basis for high speed networking.

One of the most popular interconnection network is the star graph proposed by [12]. It has many advantages over the hypercube such as lower degree and a smaller diameter. Day and Tripathi [1] first proposed an orientation of the star graph so that the resulting graph is almost regular. One of the main criteria of a good unidirectional graph topology is that it has good connectivity properties. Indeed, [8] showed this for the unidirectional hypercube proposed in [6] and [2,3] showed this for the unidirectional star graph proposed in [1].

In this paper, we orient the edges of Cayley graphs generated by transposition trees and show that the resulting unidirectional graphs have good connectivity properties. This class of graphs is a common generalization of the star graphs and the bubble sort graphs [13] studied in [14]. Interests in them have recently been rejuvenated by [15].

* Corresponding author.
E-mail addresses: echeng@oakland.edu (E. Cheng), liptak@oakland.edu (L. Lipták), nmshawas@oakland.edu (N. Shawash).

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2007.10.016

2. Preliminaries

Basic terminology in graph theory can be found in [16]. Let H be a graph and X be a proper nonempty subset of the vertex set. Define $d_H(X)$ to be the number of edges with exactly one end in X. Let G be a directed graph and X be a proper nonempty subset of the vertex set. Define $\delta_G(X)$ (respectively, $\rho_G(X)$) to be the number of arcs leaving (respectively, entering) X, that is, the number of arcs with their head (respectively, tail) in $\overline{X}$. Observe that $\delta_G(X) = \rho_G(\overline{X})$. A graph H is *k-edge-connected* if the deletion of any $k-1$ edges will not disconnect the graph. This is equivalent to $d_H(X) \geq k$ for every $\emptyset \neq X \subsetneq V$. A directed graph G is *k-arc-connected* if the deletion of any $k-1$ arcs will not disconnect the graph, that is, the resulting graph is strongly connected. This is equivalent to $\delta_G(X) \geq k$ for every $\emptyset \neq X \subsetneq V$. The *edge-connectivity* (respectively, arc-connectivity) of a graph (respectively, directed graph) is r if it is r-edge-connected (respectively, r-arc-connected) but not $(r+1)$-edge-connected (respectively, $(r+1)$-arc-connected). A graph (respectively, directed graph) $G = (V, E)$ is k-connected with $k \leq |V| - 1$ if it remains connected (respectively, strongly connected) after deleting at most $k-1$ vertices. The *vertex-connectivity* or simply *connectivity* of a graph (or directed graph) is r if it is r-connected but not $(r+1)$-connected. A graph (respectively, directed graph) $G = (V, E)$ is *maximally connected* if it is r-connected where $r = \min\{d(v) : v \in V\}$ (respectively, $r = \min\{\rho(v), \delta(v) : v \in V\}$) Similarly, we can define the terms *maximally edge-connected* and *maximally arc-connected*. Given a digraph G, a *strong component* of G is a maximal strongly connected subgraph of G. Since vertex-connectivity is stronger than edge-connectivity and arc-connectivity, we will deal with vertex-connectivity in this paper. In fact, we study properties that are stronger than vertex-connectivity.

Let Γ be a finite group and let S be a set of elements of Γ such that the identity of the group does not belong to S. The *Cayley graph* for (Γ, S) is the directed graph with vertex set being the set of elements of Γ in which there is an arc from u to v if there is an $s \in S$ such that $u = vs$. The Cayley graph for (Γ, S) is connected if and only if S is a generating set for Γ. A Cayley graph is always vertex-transitive,[1] so it is maximally arc-connected if it is connected; however, its vertex-connectivity may be low. (Note that not all vertex-transitive graphs are Cayley graphs; for example, the Petersen graph.)

Throughout the paper, a permutation is shown in the form $[a_1, a_2, \ldots, a_n]$ as an arrangement. However, for simplicity, it is expressed as $a_1a_2\ldots a_n$ in figures. In addition, when we refer to a transposition (that is, a permutation where the symbols in every position is fixed, except positions i and j where the symbols are interchanged), we will use the cycle notation; in other words, we use the notation (ij) for this transposition. In this paper, we choose the finite group to be Γ_n, the symmetric group on $\{1, 2, \ldots, n\}$, and the generating set S to be a set of transpositions. The vertices of the corresponding Cayley graph are permutations, and since S only has transpositions, there is an arc from vertex u to vertex v if and only if there is an arc from v to u. Hence we can regard these Cayley graphs as undirected graphs by replacing each pair of arcs in opposite directions by an edge. With transpositions as the generating set, a simple way to depict S is via a graph with vertex set $\{1, 2, \ldots, n\}$ where there is an edge between i and j if and only if the transposition (ij) belongs to S. This graph is called the *transposition generating graph of* (Γ_n, S) or simply *(transposition) generating graph* if it is clear from the context. In fact, the star graphs were introduced via the generating graph $K_{1,n-1}$, where the centre is 1 and the leaves are $2, 3, \ldots, n$. Hence they are called star graphs, though perhaps the more descriptive term should be "star-generated graphs". In this paper, we consider connectivity properties of a class of good orientation of these graphs. We note that the Cayley graph generated by the transpositions in S is connected if and only if the generating graph corresponding to S is connected. Since an interconnection network needs to be connected, we require the transposition generating graph to be connected.

Some important properties of interconnection networks are low degree and high connectivity. The Cayley graph that we consider has $n!$ vertices. In this paper, we restrict ourselves to graphs obtained from transposition generating graphs that are trees. The Cayley graphs obtained by these generating trees are $(n-1)$-regular and bipartite. For convenience, we refer to a transposition generating graph that is a tree as a *(transposition) generating tree*. In this paper, we give an orientation of these graphs and we prove that they have good connectivity properties. This includes the star graph whose generating tree is $K_{1,n-1}$ and the bubble-sort graph whose generating tree is a path of length n usually labelled with the symbols $1, 2, \ldots, n$ in this order. Fig. 1 gives the star graph and Fig. 2 gives the bubble-sort

[1] Two vertices u and v are *equivalent* if there is an automorphism ϕ such that $\phi(u) = v$. A graph is *vertex-transitive* if every pair of vertices are equivalent.

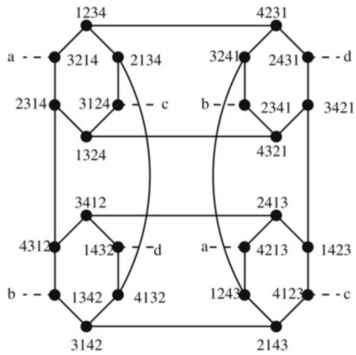

Fig. 1. Star graph.

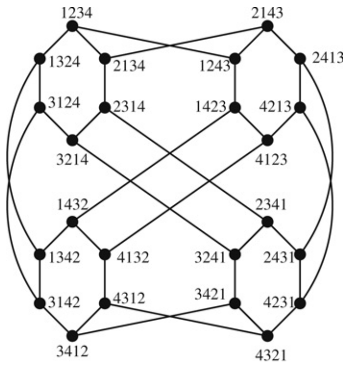

Fig. 2. Bubble-sort graph.

graph when $n = 4$. Throughout this paper, we use G_n to denote a graph generated by some transposition generating tree on n vertices. To avoid trivial cases, we assume that a generating tree has at least three vertices.

3. Orienting G_n

Let G_n be the Cayley graph generated by a transposition generating tree T_n. So T_n has $n-1$ edges. In the definition of a transposition generating tree, the vertices are labelled from 1 to n. Clearly, permuting these vertex labellings will not effect the Cayley graph generated by it via isomorphism. So we will label them in a convenient way. In fact, to describe the orientation of the edges, we will also label the edges using the symbols in the set $\{2, \ldots, n\}$. We now describe this labelling: Pick a leaf and its corresponding leaf-edge, label this leaf with n, label this leaf-edge with n and delete this leaf; pick a leaf and its corresponding leaf-edge in the resulting graph, label the leaf with $n-1$, label this leaf-edge $n-1$ and delete this leaf; repeat until all edges are labelled, and label this last vertex with 1. We will call an edge in G_n an i-*edge* if it is generated by the edge with label i in the transposition generating tree. Note that the vertex with label n is a leaf. We can now describe the orientation. Let π_0 and π_1 be adjacent corresponding to an i-edge. We may assume π_0 is even and π_1 is odd. Then the edge is oriented from π_0 to π_1 if i is even and the edge is oriented from π_1 to π_0 if i is odd. We note that for star graphs, we have only two ways to label $K_{1,n-1}$. Figs. 3 and 4 give the two possible labellings of $K_{1,5}$. Note that we did not label the edges. It is clear from the labelling rule that the label on an edge is the larger of the two labels of its end vertices. It is clear that the two graphs that they generate are isomorphic as undirected graphs. The next proposition addresses the orientation of these graphs:

Proposition 3.1. *Let T_n be the generating tree for the Cayley graph G_n where $n \geq 3$. Suppose L_n^1 and L_n^2 are two labellings of T_n such that they are identical except the vertices labelled* 1 *and* 2 *(that is, the end vertices of the edge labelled* 2*) are exchanged. Let $\overrightarrow{G}_n^1$ and $\overrightarrow{G}_n^2$ be the resulting orientation of G_n according to L_n^1 and L_n^2, respectively. Then $\overrightarrow{G}_n^1$ is isomorphic to the graph obtained from $\overrightarrow{G}_n^2$ by reversing the direction of every edge.*

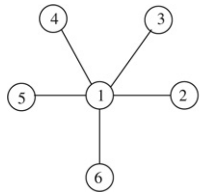

Fig. 3. $K_{1,5}$.

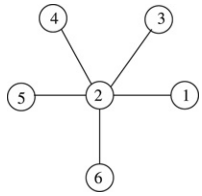

Fig. 4. $K_{1,5}$.

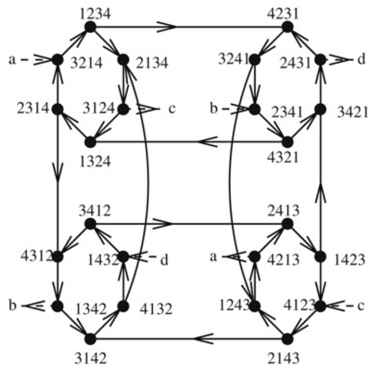

Fig. 5. An orientation of the graph in Fig. 1.

Proof. Clearly G_n^1 and G_n^2 are isomorphic via the bijection $\psi : V(G_n^1) \longrightarrow V(G_n^2)$ that maps $[a_1, a_2, a_3, \ldots, a_n]$ to $[a_2, a_1, a_3, \ldots, a_n]$. Let $\overleftarrow{G}_n^2$ be the graph obtained from $\overrightarrow{G}_n^2$ by reversing the direction of every edge. Then the same function will work, that is, the bijection $\phi : V(\overrightarrow{G}_n^1) \longrightarrow V(\overleftarrow{G}_n^2)$ that maps $[a_1, a_2, a_3, \ldots, a_n]$ to $[a_2, a_1, a_3, \ldots, a_n]$ is the desired isomorphism. There are three cases to check. Consider an edge e in L_n^1 and L_n^2. Note that the edge labellings of the two trees are identical.

(1) Suppose that both end vertices of e have labels at least 3. Then since $[a_1, a_2, a_3, \ldots, a_n]$ and $[a_2, a_1, a_3, \ldots, a_n]$ have opposite parity and the label of e is the same in both L_n^1 and L_n^2, the result follows.

(2) Suppose that exactly one of the end vertices of e has label at least 3. Then we may assume the end vertices of e in L_n^1 are labelled 1 and i, and respectively labelled 2 and i in L_n^2 where $i \geq 3$. For notational convenience, assume $i = 3$. Now consider an edge in G_n generated by e; its two end vertices are $[a_1, a_2, a_3, a_4, \ldots, a_n]$ and $[a_3, a_2, a_1, a_4, \ldots, a_n]$. Now, under ϕ, this edge maps to one that with end vertices $[a_2, a_1, a_3, a_4, \ldots, a_n]$ and $[a_2, a_3, a_1, a_4, \ldots, a_n]$. Again, it is clear from our orientation rule that these two edges are oriented in the opposite direction.

(3) Suppose the end vertices of e are labelled 1 and 2. This is similar to the above cases. □

By Proposition 3.1, one can conclude that, by using this rule, there is essentially one way to orient the Cayley graph generated by $K_{1,n-1}$. Fig. 5 gives the orientation for the graph generated by $K_{1,3}$, that is, a star graph. This labelling generates an orientation that coincides with the orientation given in [1]. We observe that our orientation rule is determined by a *local orientation rule*, that is, the orientation of an edge is given by the labels of its two ends. We

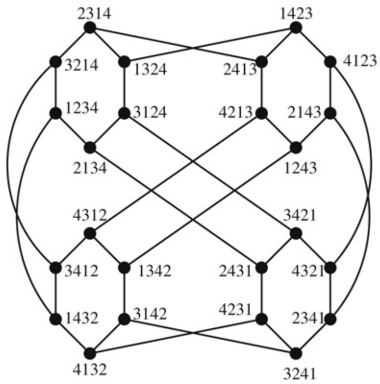

Fig. 6. Cayley graph generated by Fig. 7.

Fig. 7. P_4.

Fig. 8. P_4.

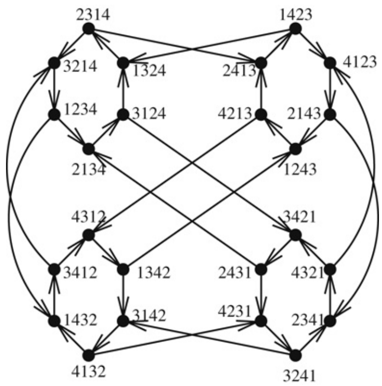

Fig. 9. Orientation of Fig. 6 via Fig. 7.

will denote the resulting directed graph by $\overrightarrow{G}_n$. We will show that $\overrightarrow{G}_n$ is maximally connected. In fact, we will show that $\overrightarrow{G}_n$ is "more than" maximally connected. The following result follows directly from the orientation rule.

Proposition 3.2. *Let $\overrightarrow{G}_n$ be an unidirectional Cayley graph generated by a transposition generating tree on n vertices. If π is an even permutation, then the outdegree and indegree of π are $\lceil \frac{n-1}{2} \rceil$ and $\lfloor \frac{n-1}{2} \rfloor$, respectively. If π is an odd permutation, then the indegree and outdegree of π are $\lceil \frac{n-1}{2} \rceil$ and $\lfloor \frac{n-1}{2} \rfloor$, respectively.*

A quick way to remember Proposition 3.2 is that the orientation is an *odd-more-in* orientation, that is, the indegree is greater than or equal to the outdegree for an odd vertex. Moreover, $\overrightarrow{G}_n$ is regular if n is odd.

As we noted earlier, there is essentially only one way to label $K_{1,n-1}$ for "distinct" orientation of the resulting Cayley graph. One may wonder whether this is true in general. We note that for $n = 4$, a generating tree is either $K_{1,3}$ or P_4, a path of length three. We claim that there are two ways to label P_4 so that the resulting orientations are "different." It is clear that any such labelling can be reduced to either the one in Fig. 7 or the one in Fig. 8. Of course, Figs. 7 and 8 generate isomorphic undirected graphs. In fact, Fig. 7 generates Fig. 6 and Fig. 8 generates Fig. 2. However, they generate nonisomorphic digraphs. To see this, observe that Fig. 9 is the orientation of Fig. 2 using Fig. 7, and Fig. 10 is the orientation of Fig. 6 using Fig. 8. Now, these two digraphs are fundamentally different as the

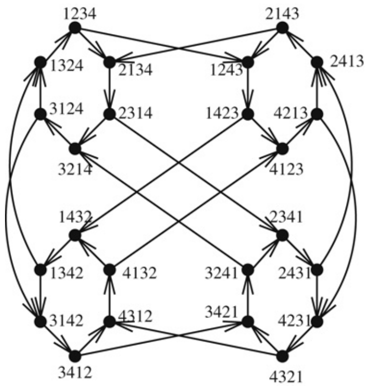

Fig. 10. Orientation of Fig. 2 via Fig. 8.

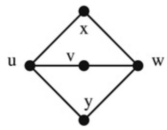

Fig. 11. Forbidden subgraph.

digraph in Fig. 9 contains directed 4-cycles but the digraph in Fig. 10 does not. In either digraph, every arc is either on a directed 4-cycle or on a directed 6-cycle. In fact, this is always true. We will give this side observation later. We now state some results that will be useful to us.

Lemma 3.3. *Let G_n be a Cayley graph obtained from a transposition generating tree T_n on $\{1, 2, \ldots, n\}$ with $n \geq 3$. Then*

(1) *G_n has girth*[2] 4 *unless G_n is the star graph (with generating tree $K_{1,n-1}$) which has girth* 6.
(2) *G_n does not have $K_{2,3}$, the graph in Fig. 11, as a subgraph.*

Proof. It is well known and indeed easy to check that star graphs have girth 6. If $n = 3$, then T_n is a path of length 3, so G_n is a 6-cycle, and hence the claim is true. Since G_n is generated by transpositions, a 4-cycle must be generated by transpositions that involve no more than five positions. We first consider a 4-cycle that involves exactly four positions in the permutations representing the vertices of the 4-cycle. The edges of this 4-cycle must correspond to the following transpositions in the given order: (ab), (cd), (ab), (cd), where a, b, c, d are distinct. (We note that (ab) is the cycle representation of the permutation in which the symbol in position a and position b. In other words, (ab) corresponds to the edge in the generating tree whose end vertices are labelled a and b respectively.) Let this be a *Type A 4-cycle*. Next, consider if a 4-cycle involves exactly three positions in the permutations representing the vertices of the 4-cycle. Then the edges of this 4-cycle must correspond to the transpositions in one of the following orders: (ab), (bc), (ab), (ac) or (ab), (bc), (ac), (bc), where a, b, c are distinct. Let such a cycle be a *Type B 4-cycle*. But if a Type B 4-cycle exists in G_n, then T_n contains a K_3 (the permutations (ab), (bc) and (ac)), a contradiction since T is a tree. So there are no Type B 4-cycles. Hence the cycles (u, v, w, x) and (u, v, w, y) in Fig. 11 are Type A 4-cycles. Since three vertices on a Type A 4-cycle uniquely determine the last vertex on this cycle, Fig. 11 cannot be a subgraph of G_n and the result is proved. □

Theorem 3.4. *Let $n \geq 3$ and G_n be the Cayley graph generated by a transposition tree on n vertices where a leaf has label n. Then the following holds:*

(1) *G_n is an $(n-1)$-regular bipartite graph on $n!$ vertices.*
(2) *G_n is vertex-transitive.*

[2] The girth of a graph is the length of a shortest cycle in the graph.

(3) *G_n is maximally connected and maximally edge-connected.*
(4) *Let H_i be the subgraph of G_n ($n \geq 4$) induced by the vertices with i in the last position for $1 \leq i \leq n$. Then H_i is isomorphic to some Cayley graph G_{n-1}. Moreover, there are $(n-2)!$ independent*[3] *edges between $V(H_i)$ and $V(H_j)$ for $1 \leq i < j \leq n$. In addition, every vertex in H_i has exactly one neighbour not in H_i via the n-edge for $1 \leq i \leq n$.*

The first statement of Theorem 3.4 is obvious. We note that G_n is not edge-transitive[4] in general. It is not difficult to see that G_n is indeed vertex-transitive. The edge-connectivity portion of Theorem 3.4 can be found in [17]: If $G = (V, E)$ is a connected vertex-transitive r-regular graph, then G has edge-connectivity r. The connectivity portion follows from a general result given in [18]: A minimal Cayley graph[5] is maximally connected. It is also proved directly in [19]. For additional structural properties for this class of Cayley graph, we refer the reader to [19–22]. The fourth statement of the theorem is easy to check.

Another proposition that will be helpful in our discussion will be the following result:

Proposition 3.5. *Let H_i be the (directed) subgraph of $\overrightarrow{G}_n$ ($n \geq 4$) induced by vertices with i in the last position for $1 \leq i \leq n$. Then there exists a Cayley graph G_{n-1} such that H_i has the same connectivity as $\overrightarrow{G}_{n-1}$. Moreover, for $1 \leq i < j \leq n$, among the $(n-2)!$ independent edges between $V(H_i)$ and $V(H_j)$ in G_n, exactly half of them are directed from $V(H_i)$ to $V(H_j)$ in $\overrightarrow{G}_n$.*

Proof. Let T_n be the generating tree for G_n. Consider a vertex $\pi = [a_1, a_2, \ldots, a_{n-1}, i]$ in H_i. Then π is even if and only if the number of transpositions required to take π to $[1, 2, \ldots, n]$ is even. Clearly H_i is isomorphic to some G_{n-1} as an undirected graph via the natural mapping $[a_1, a_2, \ldots, a_{n-1}, i] \mapsto [a_1, a_2, \ldots, a_{n-1}]$ and the mapping of the symbols $a_1, a_2, \ldots, a_{n-1}$ to $1, 2, \ldots, n-1$ in the usual way. To be precise, G_{n-1} is the graph generated by the graph obtained by deleting the leaf labelled n in T_n. Hence an i-edge in G_{n-1} is an i-edge in G_n where $1 \leq i \leq n-1$ and vice versa. Now, $[a_1, a_2, \ldots, a_{n-1}]$ is even if and only if the number of transpositions required to take it to $[1, 2, \ldots, i-1, i+1, \ldots, n]$ is even. But one can first go from $[a_1, a_2, \ldots, a_{n-1}, i]$ to $[1, 2, \ldots, n]$ and then to $[1, 2, \ldots, i-1, i+1, \ldots, n, i]$. Since the second sequence depends only on i, $[a_1, a_2, \ldots, a_{n-1}, i]$ and $[a_1, a_2, \ldots, a_{n-1}]$ will either have the same parity for every pair or opposite parity for every pair. Hence the natural mapping will either show H_i is isomorphic to such a $\overrightarrow{G}_{n-1}$ or isomorphic to the graph obtained by reversing the direction of every arc in such a $\overrightarrow{G}_{n-1}$. In either case, the first statement is proved.

To prove the second part, we first note that the edges whose end vertices are in different H_i's are precisely the edges generated by the leaf-edge of T_n with its leaf labelled n. Suppose this vertex's unique neighbour is labelled α. Since $n \geq 4$, there are two more symbols β and γ that are neither n nor α. For notational convenience, assume $\alpha = n-1$, $\beta = 1$ and $\gamma = 2$. Then an edge between H_i and H_j must have end vertices of the form $[a_1, a_2, \ldots, a_{n-2}, j, i]$ and $[a_1, a_2, \ldots, a_{n-2}, i, j]$. Hence, there are $(n-2)!$ of them. Now since $n \geq 4$, we can consider the two edges where the first arc is between the vertices $[a_1, a_2, a_3, \ldots, a_{n-2}, j, i]$ and $[a_1, a_2, a_3, \ldots, a_{n-2}, i, j]$, and the second arc is between the vertices $[a_2, a_1, a_3, \ldots, a_{n-2}, j, i]$ and $[a_2, a_1, a_3, \ldots, a_{n-2}, i, j]$. Clearly, the two arcs will be oriented in an opposite way. This construction pairs up all the arcs between H_i and H_j, and hence we are done. □

We will now digress and prove the next result that we stated earlier.

Proposition 3.6. *Let $\overrightarrow{G}_n$ be the unidirectional Cayley graph generated by a labelling of a transposition generating tree T_n on n vertices. Then every arc in $\overrightarrow{G}_n$ is either on a directed 4-cycle or on a directed 6-cycle.*

Proof. Consider an edge e in G_n that is generated by the edge corresponding to the edge/transposition (ab) in T_n. As seen in the proof of Lemma 3.3, we can pick another edge (cd) in T_n, such that a, b, c, d are distinct and that the edge e is in a 4-cycle. Now it is clear from our orientation rule that this 4-cycle is a directed 4-cycle if and only if the edges (ab) and (cd) in T_n have labels of different parity. Without loss of generality, assume the edge (ab) in T_n has an even label. If there is an edge (cd) in T_n with an odd label where a, b, c, d are distinct, then we are done. So all the edges

[3] A set of edges are *independent* if no two of them are incident to the same vertex.

[4] Two edges (u, v) and (x, y) are *equivalent* if there is an automorphism ϕ such that $\phi(u) = x$ and $\phi(v) = y$, or $\phi(u) = y$ and $\phi(v) = x$. A graph is *edge-transitive* if every pair of edges are equivalent.

[5] A Cayley graph is *minimal* if the set S generates Γ, but no strict subset of it does.

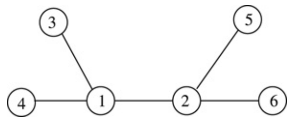

Fig. 12.

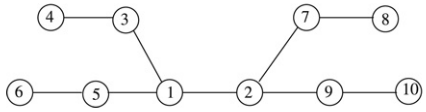

Fig. 13.

of T_n with an odd label must be incident to either a or b. Without loss of generality, assume an edge with an odd label is (ac). So using (ab) and (ac) gives us a 6-cycle. Again, it is clear from our orientation rule that this 6-cycle is a directed 6-cycle since the edges (ab) and (ac) have labels of opposite parity. □

We note that the proof of Proposition 3.6 gives a necessary and sufficient condition of whether an arc is on a directed 4-cycle. Consider Fig. 12 as a labelled generating tree. (Again, the label on an edge is the maximum of the labels of its end vertices.) Then an arc generated by edge 2 is not on a 4-cycle as there is no odd labelled edge in Fig. 12 that is independent with edge 2. Fig. 13 provides another example.

4. Connectivity properties

Theorem 4.1. *Let $\overrightarrow{G}_n$ be the unidirectional Cayley graph generated by a labelling of a transposition generating tree T_n on n vertices where $n \geq 3$. Then $\overrightarrow{G}_n$ is maximally connected, that is, it has connectivity $\lfloor \frac{n-1}{2} \rfloor$.*

Proof. We apply induction on n. Since $\overrightarrow{G}_3$ is a directed 6-cycle, the result is true for $n = 3$. For $n = 4$, one can check that the result is true for the graphs in Figs. 5, 9 and 10. We assume that the result is true for every $\overrightarrow{G}_{n-1}$ where $n \geq 5$ is fixed. We now consider a particular $\overrightarrow{G}_n$. Let H_i be the subgraph of $\overrightarrow{G}_n$ with i in the last position for $1 \leq i \leq n$. Then H_i and $\overrightarrow{G}_{n-1}$ have the same connectivity. Recall that every vertex in H_i has exactly one neighbour not in H_i. Let T be a set of vertices in $\overrightarrow{G}_n$ such that $|T| \leq \lfloor \frac{n-1}{2} \rfloor - 1$. Let $T_i = V(H_i) \cap T$ and $t_i = |T_i|$ for $1 \leq i \leq n$. We want to show that $\overrightarrow{G}_n \setminus T$ is strongly connected. Suppose n is even. Then $\lfloor \frac{n-1}{2} \rfloor - 1 = \lfloor \frac{n-2}{2} \rfloor - 1 = \frac{n-2}{2} - 1$. Since H_i is $\lfloor \frac{n-2}{2} \rfloor$-connected, $H_i \setminus T_i$ is strongly connected for $1 \leq i \leq n$. Let $1 \leq i, j \leq n$ be arbitrary. Since $(n-2)!/2 - (\lfloor \frac{n-1}{2} \rfloor - 1) \geq 1$, there is at least one arc remaining from $V(H_i)$ to $V(H_j)$ and at least one arc remaining from $V(H_j)$ to $V(H_i)$. Hence $\overrightarrow{G}_n \setminus T$ is strongly connected.

Suppose n is odd with $n = 2k + 1$. Since $n \geq 5$, $k \geq 2$. Moreover, $\lfloor \frac{n-1}{2} \rfloor = k$ and $\lfloor \frac{n-2}{2} \rfloor = k - 1$. We consider two cases. The first case is $t_i \leq k - 2$ for all $1 \leq i \leq n$. In this case, $H_i \setminus T_i$ is strongly connected for $1 \leq i \leq n$. Let $1 \leq i, j \leq n$ be arbitrary. Since $(n-2)!/2 - (\lfloor \frac{n-1}{2} \rfloor - 1) = (2k-1)!/2 - (k-1) \geq 1$, there is at least one arc remaining from $V(H_i)$ to $V(H_j)$ and at least one arc remaining from $V(H_j)$ to $V(H_i)$. Hence $\overrightarrow{G}_n \setminus T$ is strongly connected.

In the second case, without loss of generality, we may assume $t_1 = k - 1$ and $t_j = 0$ for $2 \leq j \leq n$. It is clear that $W = \overrightarrow{G}_n \setminus V(H_1)$ is strongly connected. Let Y be the strong component of $\overrightarrow{G}_n \setminus T$ containing W. Let C be a strong component in $H_1 \setminus T_1$. We want to show that C is part of Y. It is enough to show that there is a directed path from a vertex in $V(C)$ to a vertex in $V(W)$ and a directed path from a vertex in $V(W)$ to a vertex in $V(C)$. Suppose C has at least two vertices. Pick an arc $\pi_1 \to \pi_2$. Let π_3 and π_4 be the unique neighbours of π_1 and π_2, respectively, that are in W. Note that π_3 and π_4 are distinct. Since $t_j = 0$ for $2 \leq j \leq n$, $\pi_3, \pi_4 \notin T$. If π_1 is even, then π_2 is odd, π_3 is odd and π_4 is even. Since (π_1, π_3) and (π_2, π_4) are n-edges in G_n and n is odd, we have $\pi_3 \to \pi_1$ and $\pi_2 \to \pi_4$. Hence C is part of Y. The case π_1 being odd is similar. Suppose C has only one vertex π_1. We further assume that π_1 is even. Let π_2 be the unique neighbour of π_1 in W. Note that $\pi_2 \notin T$ and it is odd. Moreover, we have the arc $\pi_2 \to \pi_1$ since n is odd. There are k distinct odd vertices $y_1, y_2, \ldots, y_k \in H_1$ such that $\pi_1 \to y_i$ is an arc for $1 \leq i \leq k$. Since $T_1 = k-1$, we may assume that $y_1 \notin T$. Let z be the unique neighbour of y_1 in W. Since n is odd, we have the arc $y_1 \to z$. Hence we have the directed path $\pi_1 \to y_1 \to z$ in $\overrightarrow{G}_n \setminus T$. Therefore C is part of Y. The case π_1 being odd is similar. □

We have shown that $\overrightarrow{G}_n$ is maximally connected. At first glance, it seems that it is the best possible result. However, it is important to study the resulting disconnected directed graph when a sufficient number of vertices have been deleted. A graph (respectively, directed graph) with connectivity r is *loosely super connected* if for any r vertices deleted, the resulting graph (respectively, directed graph) is either still connected (respectively, strongly connected) or has at most one component (respectively, strong component) of size greater than one. The notion of "superness" was first introduced in [23]. One can immediately see that this is an important concept in the study of interconnection networks as this provides an insight into the severity of a disconnection. "Superness" suggests that the "core" of the network remains intact if the network is minimally disconnected. It is shown in [19] that G_n has this property; we will show that $\overrightarrow{G}_n$ also has this property.

Lemma 4.2. *Suppose $\overrightarrow{G}_n$ is the unidirectional Cayley graph generated by a labelling of a transposition generating tree T_n on n vertices where $n \ge 3$. Let T be a set of vertices in $\overrightarrow{G}_n$. If C is a strong component of $\overrightarrow{G}_n \setminus T$, then T is either a singleton or it is of size at least 4 and has a directed path of length 3.*

Proof. Suppose C is not a singleton. Let $\pi_1 \to \pi_2$ be an arc in C. Since C is strongly connected, there is a directed path from π_2 to π_1. The result now follows from the fact that G_n has girth at least four. □

Theorem 4.3. *Suppose $\overrightarrow{G}_n$ is the unidirectional Cayley graph generated by a labelling of a transposition generating tree T_n on n vertices where $n \ge 3$. Then $\overrightarrow{G}_n$ is loosely super connected.*

Proof. Suppose we delete $\lfloor \frac{n-1}{2} \rfloor$ vertices. Let this set of vertices be T. We first consider the case that n is even. For $n = 4$, we can check that the statement is true for every possible $\overrightarrow{G}_4$, that is, the graphs in Figs. 5, 9 and 10. From now on, we assume $n = 2k$ with $k \ge 3$. So $|T| = k - 1$. Let H_i be the subgraph of $\overrightarrow{G}_n$ with i in the last position for $1 \le i \le n$. Then H_i and $\overrightarrow{G}_{n-1}$ have the same connectivity (by Proposition 3.2). So the connectivity is $\lfloor \frac{2k-2}{2} \rfloor = k-1$. As before, let $T_i = V(H_i) \cap T$ and $t_i = |T_i|$ for $1 \le i \le n$.

Suppose $t_i \le k - 2$ for all $1 \le i \le n$. Then $H_i \setminus T_i$ is strongly connected for $1 \le i \le n$. Let $1 \le i, j \le n$ be arbitrary. If we can show that there is at least one arc from $V(H_j)$ to $V(H_i)$ and at least one arc from $V(H_i)$ to $V(H_j)$, then we can conclude that $\overrightarrow{G}_n \setminus T$ is strongly connected. This can be accomplished by showing $(2k-2)!/2 > t_i + t_j$. Since $t_i + t_j \le k - 1$ and $k \ge 3$, we are done as $(2k-2)!/2 > k - 1$.

Suppose there is a $t_i \ge k - 1$. Without loss of generality, let this be t_1. Then $t_1 = k - 1$ and $T = T_1$. It is clear that $W = \overrightarrow{G}_n \setminus V(H_1)$ is strongly connected. Let Y be the strong component in $\overrightarrow{G}_n \setminus T$ containing W. Let C be a strong component in $H_1 \setminus T_1$. We want to show that C is part of Y or C is a singleton. Suppose C is not a singleton. Then by Lemma 4.2, it has a directed path $\pi_1 \to \pi_2 \to \pi_3 \to \pi_4$ where $\pi_1, \pi_2, \pi_3, \pi_4$ are distinct vertices in C. Let $\pi_5, \pi_6, \pi_7, \pi_8$ be the unique neighbours of $\pi_1, \pi_2, \pi_3, \pi_4$ in W through an n-edge in G_n, respectively. Note that $\pi_5, \pi_6, \pi_7, \pi_8$ are distinct. Suppose π_1 is even. Then π_3, π_6, π_8 are even and $\pi_2, \pi_4, \pi_5, \pi_7$ are odd. Since n is even, we have the following arcs: $\pi_1 \to \pi_5$,$\pi_6 \to \pi_2$,$\pi_3 \to \pi_7$, and $\pi_8 \to \pi_4$. The arcs $\pi_1 \to \pi_5$ and $\pi_6 \to \pi_2$ show that C is part of Y. The case π_1 being odd is similar.

We now consider the case that n is odd. We can check that the statement is true for $\overrightarrow{G}_3$ (one possibility). From now on, we assume $n = 2k + 1$ with $k \ge 2$. So $|T| = k$. Let H_i be the subgraph of $\overrightarrow{G}_n$ with i in the last position for $1 \le i \le n$. Then H_i and $\overrightarrow{G}_{n-1}$ have the same connectivity, which is $\lfloor \frac{2k-1}{2} \rfloor = k - 1$. Let $T_i = V(H_i) \cap T$ and $t_i = |T_i|$ for $1 \le i \le n$.

Suppose $t_i \le k - 2$ for all $1 \le i \le n$. Then $H_i \setminus T_i$ is strongly connected for $1 \le i \le n$. Let $1 \le i, j \le n$ be arbitrary. Since $k \ge 2$, we have $(2k-1)!/2 > k$. This together with $t_i + t_j \le k$ implies $(2k-1)!/2 > t_i + t_j$.

Suppose there is a $t_i \ge k - 1$. Without loss of generality, let this be t_1. Then t_1 is either $k - 1$ or k. It is clear that $W = \overrightarrow{G}_n \setminus V(H_1)$ is strongly connected. Let Y be the strong component in $\overrightarrow{G}_n \setminus T$ containing W. Let C be a strong component in $H_1 \setminus T_1$. We want to show that C is part of Y or C is a singleton. Suppose C is not a singleton. Then by Lemma 4.2, it has a directed path $\pi_1 \to \pi_2 \to \pi_3 \to \pi_4$ where $\pi_1, \pi_2, \pi_3, \pi_4$ are distinct vertices in C. Let $\pi_5, \pi_6, \pi_7, \pi_8$ be the unique neighbours of $\pi_1, \pi_2, \pi_3, \pi_4$ in W through an n-edge in G_n, respectively. Note that $\pi_5, \pi_6, \pi_7, \pi_8$ are distinct. Suppose π_1 is odd. Then π_3, π_6, π_8 are odd and $\pi_2, \pi_4, \pi_5, \pi_7$ are even. Since n is odd, we have the following arcs: $\pi_1 \to \pi_5$,$\pi_6 \to \pi_2$, $\pi_3 \to \pi_7$, and $\pi_8 \to \pi_4$. Since at most one of $\pi_5, \pi_6, \pi_7, \pi_8$ is in T, there are arcs from C to W and W to C, and we are done. The case π_1 being even is similar. □

Consider deleting a vertex in $\overrightarrow{G}_3$. Then the resulting directed graph is a directed path of length five, and hence it has five singleton strong components. So we need to consider a property stronger than loosely super connectedness.

A graph (respectively, directed graph) with connectivity r is *tightly super connected* if, for any r vertices deleted, the resulting graph (respectively, directed graph) is either still connected (respectively, strongly connected) or has exactly two components (respectively, strong components), one of which is a singleton. In [19], it is shown that for $n \geq 3$, G_n is tightly super connected. The next result shows that $\overrightarrow{G}_n$ has the same property.

Theorem 4.4. *Let $\overrightarrow{G}_n$ be the unidirectional Cayley graph generated by a labelling of a transposition generating tree T_n on n vertices where $n \geq 8$. Then $\overrightarrow{G}_n$ is tightly super connected.*

Proof. Suppose we delete $\lfloor \frac{n-1}{2} \rfloor$ vertices. By Theorem 4.3, the resulting graph has at most one non-singleton strong component. We only have to show that there is at most one singleton strong component. We follow the notation we have defined in the proof of Theorem 4.3. We first consider the case when $n = 2k$ with $k \geq 4$. Suppose there are two singleton strong components in $\overrightarrow{G}_n \setminus T$, say one with vertex a and one with vertex b. Then from the proof of Theorem 4.3, without loss of generality, we may assume $t_1 = k-1$, and both a and b are singleton strong components in $H_1 \setminus T_1$. Suppose a is even. Then there is a vertex y in W such that $a \to y$ is an arc. (This is a directed n-edge with n even.) Now a has $k-1$ predecessors in H_1. If one of them, say π, is not in T, then π is odd and there is an even vertex z in W such that $z \to \pi$ is an arc. Hence we have the directed path $z \to \pi \to a$ and the arc $a \to y$. Therefore a belongs to Y. Suppose a is odd. Then there is a vertex y in W such that $y \to a$ is an arc. Now a has $k-1$ successors in H_1. If one of them, say π, is not in T, then π is even and there is an odd vertex z in W such that $\pi \to z$ is an arc. Hence we have the directed path $a \to \pi \to z$ and the arc $y \to a$. Therefore a belong to Y. Hence if a is a strong component in $\overrightarrow{G}_n \setminus T$, T contains $k-1$ neighbours of a; let A be such a set. Similarly, T contains $k-1$ neighbours of b; let B be such a set. Since $|T| = k-1$, $T = A = B$. By Lemma 3.3, $A \cap B$ has at most two vertices. This is impossible since $k \geq 4$.

We now consider the case when $n = 2k+1 \geq 9$. Suppose there are two singleton strong components in $\overrightarrow{G}_n \setminus T$, say one with vertex a and one with vertex b. Then we may assume t_1 is $k-1$ or k, and both a and b are singleton strong components in $H_1 \setminus T_1$. We first consider the case when $t_1 = k$, so $T = T_1$. Suppose a is odd. Then there is a vertex y in W such that $a \to y$ is an arc. (This is a directed n-edge with n odd.) Now a has k predecessors in H_1. If one of them, say π, is not in T, then π is even and there is an odd vertex z in W such that $z \to \pi$ is an arc. Hence we have the directed path $z \to \pi \to a$ and the arc $a \to y$. Hence a belongs to Y. Suppose a is even. Then there is a vertex y in W such that $y \to a$ is an arc. Now a has k successors in H_1. If one of them, say π, is not in T, then π is odd and there is an even vertex z in W such that $\pi \to z$ is an arc. Hence we have the directed path $a \to \pi \to z$ and the arc $y \to a$. Hence a belongs to Y. Since a is a strong component in $\overrightarrow{G}_n \setminus T$, T contains k neighbours of a; let A be such a set. Similarly, T contains k neighbours of b; let B be such a set. By Lemma 3.3, $A \cap B$ has at most two vertices. This is impossible since $k \geq 3$ and $|T| = k$.

We now consider the case when $t_1 = k-1$. Let g be the unique vertex in $T \setminus T_1$. Suppose a is odd. Then there is a vertex y in W such that $a \to y$ is an arc. We consider whether $y = g$. If $y \neq g$, then we have a directed path from a to a vertex in W. Now a has k predecessors in H_1. Every one of them is even and hence has an odd predecessor in $\overrightarrow{G}_n \setminus H_1$ and these odd vertices are all distinct and at most one of them is $g \in T$. This will give a directed path from a vertex in W to a (which implies a is part of Y) unless T contains a vertex from each of these directed paths. Hence $k-1$ predecessors of a are in T. Note that this implies a is of distance 2 from g via a vertex in H_1 and every other vertex in T_1 is a predecessor of a. If $y = g$, then a has k predecessors in H_1. Every one of them is even and hence has an odd predecessor in W and these odd vertices are all distinct. Moreover, these odd vertices are distinct from g. This gives a directed path from a vertex in W to a since $|T_1| = k-1$. Now a has $k-1$ even successors in H_1. If one of them is not in T, then we can find k odd successors from this even vertex (distance 2 from a) in H_1. At least one of them is not in T and it has a successor in W (distance 3 from a) which is not in T, as $y = g$. This gives a directed path from a to a vertex in W (and hence a is part of Y) unless the $k-1$ successors of a belong to T. Note that in this case, every vertex in T is a successor of a. The case a being even is similar. To sum up, T_1 contains $k-1$ neighbours of a and g is of at most distance 2 from a in G_n. In a similar way, T_1 contains $k-1$ neighbours of b and g is of at most distance 2 from b in G_n. Since $|T_1| = k-1$, every vertex in T_1 is a neighbour of a and b. Since $k \geq 4$, this is impossible by Lemma 3.3. □

5. Conclusion

In this paper, we proved that $\overrightarrow{G}_n$ is not only maximally connected but also tightly super connected. So even when the graph is disconnected when $\lfloor \frac{n-1}{2} \rfloor$ vertices are deleted, it has two strong components, with one of them being a singleton. Hence the "core" of the graph is still intact. This means $\overrightarrow{G}_n$ has good structural properties. Moreover, this generalized the result for the Day–Tripathi unidirectional star graph given in [3]. We note that Theorem 4.1 is not surprising as it follows from Nash–Williams [24] that such an orientation exists to give maximal connectivity. However, the orientation that we gave here is a local orientation rule. Moreover, this orientation has stronger connectivity properties as given in Theorems 4.3 and 4.4. In addition, Proposition 3.6 implies that the diameter of $\overrightarrow{G}_n$ is at most 5 times the diameter of G_n. So our orientation rule produces maximum connectivity and low diameter in the resulting directed graph, which is a difficult task for general graphs.

One may wonder why we labelled the generating trees as stated. If n is even, then $\overrightarrow{G}_n$ is almost regular. In our proof, we require each H_i to have the same connectivity as $\overrightarrow{G}_{n-1}$ for some G_{n-1}. In particular, H_i needs to be regular. So the edges between the H_j's should be k-edges where k is even. A simple way to guarantee this is for the chosen leaf-edge to have label n.

Finally, one may wonder whether $n \geq 8$ is necessary in Theorem 4.4. We have already seen that the result is not true if $n = 3$. For example, the result is true for $n \geq 4$ for the Day–Tripathi unidirectional star graph as shown in [3]. One can indeed do a tighter analysis on the possible orientation of 4-cycles to present the cases for $n = 4, 5, 6, 7$. However, we feel that it is not worthwhile to lengthen the paper for several small cases.

Acknowledgements

We would like to thank the two anonymous referees for a number of helpful comments and suggestions.

References

[1] K. Day, A. Tripathi, Unidirectional star graphs, Inform. Process. Lett. 45 (1993) 123–129.
[2] E. Cheng, M.J. Lipman, On the Day–Tripathi orientation of the star graphs: Connectivity, Inform. Process. Lett. 73 (2000) 5–10.
[3] E. Cheng, M.J. Lipman, Connectivity properties of unidirectional star graphs, Congr. Numer. 150 (2001) 33–42.
[4] E. Cheng, M.J. Lipman, Orienting split-stars and alternating group graphs, Networks 35 (2000) 139–144.
[5] S.C. Chern, J.S. Jwo, T.C. Tuan, Uni-directional alternating group graphs, in: Lecture Notes in Comput. Sci., 959, Springer, Berlin, 1995, pp. 490–495.
[6] C.H. Chou, D.H.C. Du, Unidirectional hypercubes, in: Proc. Supercomputing'90, 1990, pp. 254–263.
[7] F. Comellas, M.A. Fiol, Vertex-symmetric digraphs with small diameter, Discrete Appl. Math. 58 (1995) 1–11.
[8] J.S. Jwo, T.C. Tuan, On container length and connectivity in unidirectional hypercubes, Networks 32 (1998) 307–317.
[9] E. Cheng, M.J. Lipman, Orienting the arrangement graphs, Congr. Numer. 142 (2000) 97–112.
[10] E. Cheng, M.J. Lipman, Unidirectional (n, k)-star graphs, J. Interconnection Networks 3 (2002) 19–34.
[11] E. Cheng, W.A. Lindsey, D.E. Steffy, Maximal vertex-connectivity of $\overrightarrow{S_{n,k}}$, Networks 46 (2005) 154–162.
[12] S.B. Akers, D. Harel, B. Krishnamurthy, The star graph: An attractive alternative to the n-cube, in: Proc. Int'l Conf. Parallel Processing, 1987, pp. 393–400.
[13] S.B. Akers, B. Krishnamurthy, A group theoretic model for symmetric interconnection networks, in: Pro. Int'l. Conf. Parallel Proc., 1986, pp. 216–223.
[14] M. Tchuente, Generation of permutations by graphical exchanges, Ars Combin. 14 (1982) 115–122.
[15] T. Araki, Hyper hamiltonian laceability of Cayley graphs generated by transpositions, Networks 48 (2006) 121–124.
[16] D.B. West, Introduction to Graph Theory, Prentice Hall, 1996.
[17] L. Lovász, Combinatorial Problems and Exercises, 2nd ed., Akadémiai Kiadó, Budapest, 1993.
[18] C.D. Godsil, GRR's for non-solvable groups, Coll. Math. Soc. János Bolyai 25 (1981) 221–239.
[19] E. Cheng, L. Lipták, Structural properties of Cayley graphs generated by transposition trees, Congr. Numer. 180 (2006) 81–96.
[20] E. Cheng, L. Lipták, A supplement to the paper "Structural properties of Cayley graphs generated a by transposition trees", Technical Report 2007–6, Oakland University, 2007.
[21] E. Cheng, L. Lipták, Fault resiliency of Cayley graphs generated by transpositions, Internat. J. Found. Comput. Sci. 18 (2007) 1005–1022.
[22] E. Cheng, L. Lipták, Linearly many faults in Cayley graphs generated by transposition trees, Inform. Sci. 177 (2007) 4877–4882.
[23] D. Bauer, F. Boesch, C. Suffel, R. Tindell, Connectivity extremal problems and the design of reliable probabilistic networks, in: The Theory and Application of Graphs, Wiley, New York, 1981, pp. 89–98.
[24] C.St.J.A. Nash-Williams, On orientations, connectivity and odd-vertex pairings in finite graph, Canad. J. Math. 12 (1960) 555–567.