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# Global solvability for a second order nonlinear neutral delay difference equation

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ABSTRACT

This paper studies the global existence of solutions of the second order nonlinear neutral delay difference equation

$$\Delta(a_n\Delta(x_n + bx_{n-\tau})) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0$$

with respect to all $b \in \mathbb{R}$. A few results on global existence of uncountably many bounded nonoscillatory solutions are established for the above difference equation. Several nontrivial examples which dwell upon the importance of the results obtained in this paper are also included.



## 1. Introduction and preliminaries

Recently, there has been increasing interest in the study of qualitative analysis of various second order difference equations, for example, see [1–12] and the references cited there.

Tang [9] discussed the existence of a bounded nonoscillatory solution for the second order linear delay difference equations

$$\Delta^2 x_n = p_n x_{n-k}, \quad n \geq 0, \tag{1.1}$$

$$\Delta^2 x_n = \sum_{i=1}^{\infty} p_i(n) x_{n-k_i}, \quad n \geq 0. \tag{1.2}$$

Zhang and Li [12] obtained some oscillation criteria for the second order advanced functional difference equation

$$\Delta(a_n\Delta x_n) + p_n x_{g(n)} = 0. \tag{1.3}$$

Thandapani et al. [10] considered necessary and sufficient conditions for the asymptotic behavior of nonoscillatory solutions of the difference equation

$$\Delta(a_n\Delta x_n) = q_n x_{n+1}, \quad n \geq 0, \tag{1.4}$$

and discussed a few sufficient conditions for the asymptotic behavior of certain types of nonoscillatory solutions of the second order difference equation

$$\Delta(a_n\Delta x_n) = q_n f(x_{n+1}), \quad n \geq 0. \tag{1.5}$$

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Li and Zhu [6] established the asymptotic behavior of the second order nonlinear difference equation

$$\Delta(r_{n-1}\Delta x_{n-1}) + q_n(\Delta x_n)^{\beta} - p_n x_n^{\alpha} = e_n, \quad n \ge 0. \tag{1.6}$$

Cheng et al. [2] and Zhang [11] discussed the asymptotic behaviors of solutions and nonoscillatory solutions for some special cases of Eq. (1.6), respectively. Recently, Jinfa [3] utilized the contraction principle to study the existence of a nonoscillatory solution for the second order neutral delay difference equation with positive and negative coefficients

$$\Delta^2 (x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \ge n_0 \tag{1.7}$$

under the condition $p \neq -1$. Migda and Migda [8] gave the asymptotic behavior of the second order neutral difference equation

$$\Delta^2 (x_n + px_{n-k}) + f(n, x_n) = 0, \quad n \ge 1. \tag{1.8}$$

Very recently, Meng and Yan [7] investigated the sufficient and necessary conditions of the existence of the bounded nonoscillatory solutions for the second order nonlinear neutral delay difference equation

$$\Delta^2 (x_n - px_{n-\tau}) = \sum_{i=1}^{m} q_i f_i(x_{n-\sigma_i}), \quad n \ge n_0. \tag{1.9}$$

However, to the best of our knowledge, neither did anyone investigate the global existence of nonoscillatory solutions for Eqs. (1.7)–(1.9) with respect to all $p \in \mathbb{R}$, nor did they discuss the existence of uncountably many bounded nonoscillatory solutions for Eqs. (1.1)–(1.9) and any other second order difference equations.

Motivated by the papers mentioned above, in this paper we investigate the following more general second order nonlinear neutral delay difference equation

$$\Delta\left(a_n \Delta(x_n + bx_{n-\tau})\right) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) = c_n, \quad n \ge n_0, \tag{1.10}$$

where $b \in \mathbb{R}$, $\tau, k \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{c_n\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $a_n > 0$ for $n \in \mathbb{N}_{n_0}$, $\bigcup_{l=1}^{k}\{d_{ln}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{Z}$, and $f : \mathbb{N}_{n_0} \times \mathbb{R}^k \to \mathbb{R}$ is a mapping. Using the contraction principle, we establish some global existence results of uncountably many bounded nonoscillatory solutions for Eq. (1.10) relative to all $b \in \mathbb{R}$. Our results sharp and improve Theorem 1 in [3]. To illustrate our results, seven examples are also included.

On the other hand, using similar arguments and techniques, the results presented in this paper could be extended to second order nonlinear neutral delay differential equations. Of course, we shall continue to study these possible extensions in the future.

Throughout this paper, we assume that $\Delta$ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}$ and $\mathbb{N}$ stand for the sets of all integers and positive integers, respectively,

$$N_a = \{n : n \in \mathbb{N} \text{ with } n \ge a\}, \quad Z_a = \{n : n \in \mathbb{Z} \text{ with } n \ge a\}, \quad a \in \mathbb{Z},$$
$$\alpha = \inf\{n - d_{ln} : 1 \le l \le k, n \in \mathbb{N}_{n_0}\}, \quad \beta = \min\{n_0 - \tau, \alpha\},$$
$$\lim_{n\to\infty} (n - d_{ln}) = +\infty, \quad 1 \le l \le k,$$

$l_\beta^\infty$ denotes the Banach space of all bounded sequences on $\mathbb{Z}_\beta$ with norm

$$\|x\| = \sup_{n\in\mathbb{Z}_\beta} |x_n| \quad \text{for } x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in l_\beta^\infty$$

and

$$A(N, M) = \left\{x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in l_\beta^\infty : N \le x_n \le M,\ n \in Z_\beta\right\} \quad \text{for } M > N > 0.$$

It is easy to see that $A(N, M)$ is a bounded closed and convex subset of $l_\beta^\infty$.

By a solution of Eq. (1.10), we mean a sequence $\{x_n\}_{n\in Z_\beta}$ with a positive integer $T \ge n_0 + \tau + |\alpha|$ such that Eq. (1.10) is satisfied for all $n \ge T$. As is customary, a solution of Eq. (1.10) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

## 2. Existence of uncountable bounded nonoscillatory solutions

Now we investigate the existence of uncountable bounded nonoscillatory solutions for Eq. (1.10).

**Theorem 2.1.** *Let $M$ and $N$ be two positive constants with $M > N$ and $b = -1$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$|f(n, u_1, u_2, \ldots, u_k) - f(n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k)| \le P_n \max\{|u_l - \bar{u}_l| : 1 \le l \le k\}, \quad n \in \mathbb{N}_{n_0}, u_l, \bar{u}_l \in [N, M], 1 \le l \le k; \tag{2.1}$$

$$|f(n, u_1, u_2, \ldots, u_k)| \le Q_n, \quad n \in \mathbb{N}_{n_0}, u_l \in [N, M], 1 \le l \le k; \tag{2.2}$$

$$\sum_{i=1}^{\infty} \sum_{s=n_0+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \max\{P_t, Q_t, |c_t|\} < +\infty. \tag{2.3}$$

*Then Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

**Proof.** Set $L \in (N, M)$. Now we construct a contraction mapping $S_L : A(N, M) \to A(N, M)$ and prove that its fixed point is a bounded nonoscillatory solution of Eq. (1.10). It follows from (2.3) that there exist $\theta \in (0, 1)$ and $T \ge n_0 + \tau + |\alpha|$ satisfying

$$\theta = \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{P_t}{a_s}; \tag{2.4}$$

$$\sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{Q_t + |c_t|}{a_s} \le \min\{M - L, L - N\}, \quad n \ge T + 1. \tag{2.5}$$

Define a mapping $S_L : A(N, M) \to l_\beta^\infty$ by

$$(S_L x)_n = \begin{cases} L + \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right), & n \ge T + 1, \\ (S_L x)_{T+1}, & \beta \le n < T + 1 \end{cases} \tag{2.6}$$

for $x \in A(N, M)$. In terms of (2.1), (2.4) and (2.6), we gain that for $x, y \in A(N, M)$ and $n \ge T + 1$

$$\begin{aligned} |(S_L x)_n - (S_L y)_n| &\le \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{P_t}{a_s} \max\{|x_{t-d_{lt}} - y_{t-d_{lt}}| : 1 \le l \le k\} \\ &\le \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{P_t}{a_s} \|x - y\|. \end{aligned}$$

This leads to

$$\|S_L x - S_L y\| \le \theta \|x - y\|, \quad x, y \in A(N, M). \tag{2.7}$$

In view of (2.2), (2.5) and (2.6), we infer that for any $x \in A(N, M)$ and $n \ge T$

$$|(S_L x)_n - L| \le \sum_{i=1}^{\infty} \sum_{s=T+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{Q_t + |c_t|}{a_s} \le \min\{M - L, L - N\},$$

which yields that $S_L(A(N, M)) \subseteq A(N, M)$. Hence (2.7) means that $S_L$ is a contraction mapping and it has a unique fixed point $x \in A(N, M)$. It follows that for $n \ge T + \tau + 1$

$$x_n = L + \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right),$$

$$x_{n-\tau} = L + \sum_{i=1}^{\infty} \sum_{s=n+(i-1)\tau}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right),$$

which give that

$$\Delta(x_n - x_{n-\tau}) = \sum_{t=n}^{\infty} \frac{1}{a_n} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right),$$

$$\Delta\left(a_n \Delta(x_n - x_{n-\tau})\right) = -f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) + c_n$$

for $n \ge T + \tau + 1$. That is, $x$ a bounded nonoscillatory solution of Eq. (1.10).

Let $L_1, L_2 \in (N, M)$ and $L_1 \ne L_2$. For each $j \in \{1, 2\}$, we choose a constant $\theta_j \in (0, 1)$, a positive integer $T_j \ge n_0 + \tau + |\alpha|$ and a mapping $S_{L_j}$ satisfying (2.4)–(2.6), where $\theta, L$ and $T$ are replaced by $\theta_j, L_j$ and $T_j$, respectively, and

$\sum_{i=1}^{\infty}\sum_{s=T_3+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s} < \frac{|L_1-L_2|}{2M}$ for some $T_3 > \max\{T_1, T_2\}$. Obviously, the contraction mappings $S_{L_1}$ and $S_{L_2}$ have the unique fixed points $x, y \in A(N, M)$, respectively. That is, $x$ and $y$ are bounded nonoscillatory solutions of Eq. (1.10) in $A(N, M)$. In order to show that Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$, we prove only that $x \neq y$. In fact, by (2.6) we gain that for $n \geq T_3 + 1$

$$x_n = L_1 + \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right),$$

$$y_n = L_2 + \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left(f(t, y_{t-d_{1t}}, y_{t-d_{2t}}, \ldots, y_{t-d_{kt}}) - c_t\right).$$

It follows that

$$\begin{aligned}|x_n - y_n| &\geq |L_1 - L_2| - \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\\ &\quad\times\left|f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - f(t, y_{t-d_{1t}}, y_{t-d_{2t}}, \ldots, y_{t-d_{kt}})\right|\\ &\geq |L_1 - L_2| - \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}\|x-y\|\\ &\geq |L_1 - L_2| - 2M\sum_{i=1}^{\infty}\sum_{s=T_3+i\tau}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}\\ &> 0, \quad n \geq T_3 + 1,\end{aligned}$$

that is, $x \neq y$. This completes the proof. □

**Theorem 2.2.** *Let $M$ and $N$ be two positive constants with $M > N$ and $b = 1$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (2.1) and (2.2) and*

$$\sum_{s=n_0}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\max\{P_t, Q_t, |c_t|\} < +\infty. \tag{2.8}$$

*Then Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

**Proof.** Let $L \in (N, M)$. Now we construct a contraction mapping $S_L : A(N, M) \to A(N, M)$ and prove that its fixed point is a bounded nonoscillatory solution of Eq. (1.10). Clearly (2.8) implies that there exist $\theta \in (0, 1)$ and $T \geq n_0 + \tau + |\alpha|$ satisfying

$$\theta = \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}; \tag{2.9}$$

$$\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{Q_t + |c_t|}{a_s} \leq \min\{M - L, L - N\}. \tag{2.10}$$

Define a mapping $S_L : A(N, M) \to l_\beta^\infty$ by

$$(S_Lx)_n = \begin{cases} L - \displaystyle\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{1}{a_s}\left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right), & n \geq T,\\ (S_Lx)_T, \quad \beta \leq n < T \end{cases} \tag{2.11}$$

for $x \in A(N, M)$. Using (2.1), (2.9) and (2.11), we conclude that for $x, y \in A(N, M)$ and $n \geq T$

$$\begin{aligned}|(S_Lx)_n - (S_Ly)_n| &\leq \sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\frac{P_t}{a_s}\max\left\{|x_{t-d_{lt}} - y_{t-d_{lt}}| : 1 \leq l \leq k\right\}\\ &\leq \theta\|x - y\|,\end{aligned}$$

which yields (2.7). Note that (2.2), (2.10) and (2.11) ensure that for any $x \in A(N, M)$ and $n \geq T$

$$|(S_Lx)_n - L| \leq \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{Q_t + |c_t|}{a_s} \leq \min\{M - L, L - N\},$$

which means that $S_L(A(N,M)) \subseteq A(N,M)$. That is, (2.7) ensures that $S_L$ is a contraction mapping and it has a unique fixed point $x \in A(N,M)$. It follows that for $n \geq T+\tau$

$$x_n = L - \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \dots, x_{t-d_{kt}}) - c_t\right),$$

$$x_{n-\tau} = L - \sum_{i=1}^{\infty} \sum_{s=n+2(i-1)\tau}^{n+(2i-1)\tau-1} \sum_{t=s}^{\infty} \frac{1}{a_s} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \dots, x_{t-d_{kt}}) - c_t\right),$$

which imply that

$$\Delta(x_n + x_{n-\tau}) = \sum_{t=n}^{\infty} \frac{1}{a_n} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \dots, x_{t-d_{kt}}) - c_t\right), \quad n \geq T+\tau$$

and

$$\Delta(a_n \Delta(x_n + x_{n-\tau})) = -f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \dots, x_{n-d_{kn}}) + c_n, \quad n \geq T+\tau.$$

Therefore $x$ a bounded nonoscillatory solution of Eq. (1.10). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

**Theorem 2.3.** *Let $|b| \in [0, \frac{1}{2})$, $M$ and $N$ be two positive constants with $M(1-2|b|) > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (2.1), (2.2) and (2.8). Then Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.*

**Proof.** Let $L \in (N + |b|M, M(1-|b|))$. Now we construct a contraction mapping $S_L : A(N,M) \to A(N,M)$ and verify that its fixed point is a bounded nonoscillatory solution of Eq. (1.10). It follows from (2.8) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\alpha|$ satisfying

$$\theta = |b| + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{P_t}{a_s}; \tag{2.12}$$

$$\sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{Q_t + |c_t|}{a_s} \leq \min\{M(1-|b|) - L, L - |b|M - N\}, \quad n \geq T+1. \tag{2.13}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$(S_L x)_n = \begin{cases} L - bx_{n-\tau} - \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{1}{a_s} \left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \dots, x_{t-d_{kt}}) - c_t\right), & n \geq T+1, \\ (S_L x)_{T+1}, \quad \beta \leq n \leq T \end{cases} \tag{2.14}$$

for $x \in A(N,M)$. On account of (2.1), (2.12) and (2.14), we derive that for $x, y \in A(N,M)$ and $n \geq T+1$

$$\begin{aligned} |(S_L x)_n - (S_L y)_n| &\leq |b|\|x-y\| + \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \frac{P_t}{a_s} \max\left\{|x_{t-d_{lt}} - y_{t-d_{lt}}| : 1 \leq i \leq k\right\} \\ &\leq \theta\|x-y\|, \end{aligned}$$

which implies (2.7). It follows from (2.2), (2.13) and (2.14) that for any $x \in A(N,M)$ and $n \geq T+1$

$$\begin{aligned} |(S_L x)_n - L| &\leq |b|M + \sum_{s=T}^{\infty} \sum_{t=s}^{\infty} \frac{Q_t + |c_t|}{a_s} \\ &\leq |b|M + \min\{M(1-|b|) - L, L - |b|M - N\}, \end{aligned}$$

which yields that $S_L(A(N,M)) \subseteq A(N,M)$. Thus (2.7) guarantees that $S_L$ is a contraction mapping and it has a unique fixed point $x \in A(N,M)$. It is clear that $x$ is a bounded nonoscillatory solution of Eq. (1.10). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

**Theorem 2.4.** *Let $b < -1$, $M$ and $N$ be two positive constants with $M > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (2.1), (2.2) and (2.8). Then Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.*

**Proof.** Put $L \in (M(1+b), N(1+b))$. Now we construct a contraction mapping $S_L : A(N, M) \to A(N, M)$ and show that its fixed point is a bounded nonoscillatory solution of Eq. (1.10). In terms of (2.8), we choose $\theta \in (0, 1)$ and $T \geq n_0 + \tau + |\alpha|$ satisfying

$$\theta = -\frac{1}{b}\left(1 + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}\right); \tag{2.15}$$

$$\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{Q_t + |c_t|}{a_s} \leq \min\{L - (1+b)M, N(1+b) - L\}, \quad n \geq T+1. \tag{2.16}$$

Define a mapping $S_L : A(N, M) \to l_\beta^\infty$ by

$$(S_L x)_n = \begin{cases} \frac{L}{b} - \frac{x_{n+\tau}}{b} - \frac{1}{b}\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right), & n \geq T+1, \\ (S_L x)_{T+1}, & \beta \leq n \leq T \end{cases} \tag{2.17}$$

for $x \in A(N, M)$. Thus (2.1) together with (2.15) and (2.17) implies that for $x, y \in A(N, M)$ and $n \geq T+1$

$$\begin{aligned} |(S_L x)_n - (S_L y)_n| &\leq -\frac{\|x-y\|}{b} - \frac{1}{b}\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}\max\left\{|x_{t-d_{lt}} - y_{t-d_{lt}}| : 1 \leq l \leq k\right\} \\ &\leq \theta\|x-y\|, \end{aligned}$$

which gives (2.7). By virtue of (2.2), (2.16) and (2.17), we conclude that for any $x \in A(N, M)$ and $n \geq T+1$

$$\begin{aligned} (S_L x)_n &\leq \frac{L}{b} - \frac{M}{b} - \frac{1}{b}\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{Q_t + |c_t|}{a_s} \\ &\leq \frac{L}{b} - \frac{M}{b} - \frac{1}{b}\min\{L - (1+b)M, N(1+b) - L\} \\ &\leq M \end{aligned}$$

and

$$\begin{aligned} (S_L x)_n &\geq \frac{L}{b} - \frac{N}{b} + \frac{1}{b}\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\frac{Q_t + |c_t|}{a_s} \\ &\geq \frac{L}{b} - \frac{N}{b} + \frac{1}{b}\min\{L - (1+b)M, N(1+b) - L\} \\ &\geq N, \end{aligned}$$

which imply that $S_L(A(N, M)) \subseteq A(N, M)$. Thus (2.7) gives that $S_L$ is a contraction mapping and hence it has a unique fixed point $x \in A(N, M)$. It is easy to verify that $x$ is a bounded nonoscillatory solution of Eq. (1.10).

Put $L_1, L_2 \in (M(1+b), N(1+b))$ and $L_1 \neq L_2$. For each $j \in \{1, 2\}$, we select a constant $\theta_j \in (0, 1)$, a positive integer $T_j \geq n_0 + \tau + |\alpha|$ and a mapping $S_{L_j}$ satisfying (2.15)–(2.17), where $\theta$, $L$ and $T$ are replaced by $\theta_j$, $L_j$ and $T_j$, respectively, and $\sum_{s=T_3}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s} < \frac{|L-K|}{2M}$ for some $T_3 > \max\{T_1, T_2\}$. Note that the contraction mappings $S_{L_1}$ and $S_{L_2}$ have the unique fixed points $x, y \in A(N, M)$, respectively, and $x$ and $y$ are bounded nonoscillatory solutions of Eq. (1.10) in $A(N, M)$. In order to show that Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$, we need to prove that $x \neq y$. In fact, by (2.17) we gain that for $n \geq T_3 + 1$

$$\begin{aligned} x_n &= \frac{L_1}{b} - \frac{x_{n+\tau}}{b} - \frac{1}{b}\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left(f(t, x_{t-d_{1t}}, x_{t-d_{2t}}, \ldots, x_{t-d_{kt}}) - c_t\right), \\ y_n &= \frac{L_2}{b} - \frac{y_{n+\tau}}{b} - \frac{1}{b}\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{1}{a_s}\left(f(t, y_{t-d_{1t}}, y_{t-d_{2t}}, \ldots, y_{t-d_{kt}}) - c_t\right). \end{aligned}$$

It follows that for $n \geq T_3 + 1$

$$\begin{aligned} \left|x_n - y_n + \frac{x_{n+\tau} - y_{n+\tau}}{b}\right| &\geq -\frac{|L_1 - L_2|}{b} + \frac{1}{b}\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}\max\left\{|x_{t-d_{lt}} - y_{t-d_{lt}}| : 1 \leq l \leq k\right\} \\ &\geq -\frac{|L_1 - L_2|}{b} + \frac{1}{b}\sum_{s=n+\tau}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}\|x-y\| \end{aligned}$$

$$\geq -\frac{|L_1 - L_2|}{b} + \frac{2M}{b}\sum_{s=T_3}^{\infty}\sum_{t=s}^{\infty}\frac{P_t}{a_s}$$
$$> 0,$$

that is, $x \neq y$. This completes the proof. □

The proofs of Theorems 2.5–2.7 below are analogous to that of Theorems 2.3 and 2.4, and hence are omitted.

**Theorem 2.5.** *Let $b \in [0, 1)$, $M$ and $N$ are two positive constants with $M(1-b) > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (2.1), (2.2) and (2.8). Then Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

**Theorem 2.6.** *Let $b \in (-1, 0]$, $M$ and $N$ be two positive constants with $M > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (2.1), (2.2) and (2.8). Then Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

**Theorem 2.7.** *Let $b > 1$, $M$ and $N$ be two positive constants with $M > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (2.1), (2.2) and (2.8). Then Eq. (1.10) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

**Remark 2.1.** Theorems 2.1 and 2.7 are generalizations of Theorem 1 in [3].

## 3. Examples

Now we construct seven examples to explain the results presented in Section 2.

**Example 3.1.** Consider the second order nonlinear neutral delay difference equation

$$\Delta(n^4\Delta(x_n - x_{n-\tau})) + \frac{(x_{n-5})^3}{n^3 + |x_{n+2}|} = \frac{(-1)^{n-1}}{n(2n-1)}, \quad n \geq n_0 = 1, \tag{3.1}$$

where $\tau \in \mathbb{N}_{n_0}$ is fixed. Let $M$ and $N$ be two positive constants with $M > N$ and

$$k = 2, \quad a_n = n^4, \quad b = -1, \quad d_{1n} = 5, \quad d_{2n} = -2, \quad c_n = \frac{(-1)^{n-1}}{n(2n-1)},$$
$$f(n, u, v) = \frac{u^3}{n^3 + |v|}, \quad P_n = \frac{M^2(4M + 3n^2)}{(n^3 + N)^2}, \quad Q_n = \frac{M^3}{n^3 + N}, \quad n \geq 1,\ u, v \in \mathbb{R}.$$

It is easy to see that the conditions (2.1)–(2.3) are satisfied. Thus Theorem 2.1 implies that Eq. (3.1) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. But Theorem 1 in [3] is not valid for Eq. (3.1).

**Example 3.2.** Consider the second order nonlinear neutral delay difference equation

$$\Delta((n+1)^{\frac{3}{2}}\Delta(x_n + x_{n-\tau})) + \frac{(x_{n(n-1)})^5}{n^2 + (x_{n(n-1)})^2} = \frac{(-1)^{n-1}\ln n}{n^{\frac{3}{2}}}, \quad n \geq n_0 = 1, \tag{3.2}$$

where $\tau \in \mathbb{N}_{n_0}$ is fixed. Let $M$ and $N$ be two positive constants with $M > N$ and

$$k = 1, \quad a_n = (n+1)^{\frac{3}{2}}, \quad b = 1, \quad d_{1n} = n(2-n), \quad c_n = \frac{(-1)^{n-1}\ln n}{n^{\frac{3}{2}}},$$
$$f(n, u) = \frac{u^5}{n^2 + u^2}, \quad P_n = \frac{M^4(5n^2 + 3M^2)}{(n^2 + N^2)^2}, \quad Q_n = \frac{M^5}{n^2 + N^2}, \quad n \geq 1, u \in \mathbb{R}.$$

Obviously, the conditions (2.1), (2.2) and (2.8) hold. Therefore Theorem 2.2 guarantees that Eq. (3.2) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. But Theorem 1 in [3] is inapplicable for Eq. (3.2).

**Example 3.3.** Consider the second order nonlinear neutral delay difference equation

$$\Delta\left(n^2\Delta\left(x_n + \frac{1}{4}x_{n-\tau}\right)\right) + \frac{(x_{n-(-1)^n})^2}{n^2} = \frac{1 - \frac{1}{n}}{n(n+1)}, \quad n \geq n_0 = 1, \tag{3.3}$$

where $\tau \in \mathbb{N}_{n_0}$ is fixed. Let $M$ and $N$ be two positive constants with $M > 2N$ and

$$k = 1, \quad a_n = n^2, \quad b = \frac{1}{4}, \quad d_{1n} = (-1)^n, \quad c_n = \frac{1-\frac{1}{n}}{n(n+1)},$$
$$f(n,u) = \frac{u^2}{n^2}, \quad P_n = \frac{2M}{n^2}, \quad Q_n = \frac{M^2}{n^2}, \quad n \ge 1, u \in \mathbb{R}.$$

Clearly, the conditions (2.1), (2.2) and (2.8) are satisfied. Thus Theorem 2.3 guarantees that Eq. (3.3) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. However, Theorem 1 in [3] is not applicable for Eq. (3.3).

**Example 3.4.** Consider the second order nonlinear neutral delay difference equation

$$\Delta\left(n^2 \ln n \Delta(x_n - 50x_{n-\tau})\right) + \frac{(-1)^n (x_{n^2})^2}{n^{\frac{3}{2}}} = \frac{-\sin(1-n^3)}{n\sqrt{n+3}}, \quad n \ge n_0 = 1, \tag{3.4}$$

where $\tau \in \mathbb{N}_{n_0}$ is fixed. Let $M$ and $N$ be two positive constants with $M > N$ and

$$k = 1, \quad a_n = n^2 \ln n, \quad b = -50, \quad d_{1n} = n(1-n), \quad c_n = \frac{-\sin(1-n^3)}{n\sqrt{n+3}},$$
$$f(n,u) = \frac{(-1)^n u^2}{n^{\frac{3}{2}}}, \quad P_n = \frac{2M}{n^{\frac{3}{2}}}, \quad Q_n = \frac{M^2}{n^{\frac{3}{2}}} \quad n \ge 1, \ u \in \mathbb{R}.$$

It is clear that the conditions (2.1), (2.2) and (2.8) hold. Hence Theorem 2.4 ensures that Eq. (3.4) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. However, Theorem 1 in [3] is not valid for Eq. (3.4).

**Example 3.5.** Consider the second order nonlinear neutral delay difference equation

$$\Delta\left(n \ln^2 n \Delta\left(x_n + \frac{4}{5}x_{n-\tau}\right)\right) + \frac{\left(x_{\frac{1}{2}n(n+3)}\right)^3}{n(n+4)} = \frac{(-1)^{n^2}}{n^3+1}, \quad n \ge n_0 = 1, \tag{3.5}$$

where $\tau \in \mathbb{N}_{n_0}$ is fixed. Let $\varepsilon_0 = \frac{1}{6}$, $M$ and $N$ be two positive constants with $M > N$ and

$$k = 1, \quad a_n = n \ln^2 n, \quad b = \frac{4}{5}, \quad d_{1n} = \frac{1}{2}n(n+3), \quad c_n = \frac{(-1)^{n^2}}{n^3+1},$$
$$f(n,u) = \frac{u^3}{n(n+4)}, \quad P_n = \frac{3M^2}{n(n+4)}, \quad Q_n = \frac{M^3}{n(n+4)}, \quad n \ge 1, u \in \mathbb{R}.$$

It is a simple matter to verify that the conditions (2.1), (2.2) and (2.8) hold. Thus Theorem 2.5 gives that Eq. (3.5) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. However, Theorem 1 in [3] is not applicable for Eq. (3.5).

**Example 3.6.** Consider the second order nonlinear neutral delay difference equation

$$\Delta\left((2n + \cos n)^2 \Delta\left(x_n - \frac{6}{7}x_{n-\tau}\right)\right) + \frac{\left(x_{n(n+1)}\right)^2}{(n+2)(n+5)} = \frac{n - \frac{1}{n}}{n^3}, \quad n \ge n_0 = 1, \tag{3.6}$$

where $\tau \in \mathbb{N}_{n_0}$ is fixed. Let $M$ and $N$ be two positive constants with $M > N$ and

$$k = 1, \quad a_n = (2n + \cos n)^2, \quad b = -\frac{6}{7}, \quad d_{1n} = -n^2, \quad c_n = \frac{n-\frac{1}{n}}{n^3},$$
$$f(n,u) = \frac{u^2}{(n+2)(n+5)}, \quad P_n = \frac{2M}{(n+2)(n+5)}, \quad Q_n = \frac{M^2}{(n+2)(n+5)}, \quad n \ge 1, u \in \mathbb{R}.$$

It is easy to verify that the conditions (2.1), (2.2) and (2.8) hold. Thus Theorem 2.6 yields that Eq. (3.6) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. However, Theorem 1 in [3] is inapplicable for Eq. (3.6).

**Example 3.7.** Consider the second order nonlinear neutral delay difference equation

$$\Delta\left((n+1)^2 \Delta(x_n + 30x_{n-\tau})\right) + \frac{(-1)^{n-1}\left(x_{n(n-2)}\right)^2}{n^2+3n+10} = \frac{1}{(n+1)\ln^2 n}, \quad n \ge n_0 = 1, \tag{3.7}$$

where $\tau \in \mathbb{N}_{n_0}$ is fixed. Let $M$ and $N$ be two positive constants with $M > N$ and

$$k = 1, \quad a_n = (n+1)^2, \quad b = 30, \quad d_{1n} = n(3-n), \quad c_n = \frac{1}{(n+1)\ln^2 n},$$
$$f(n,u) = \frac{(-1)^n u^2}{n^2+3n+10}, \quad P_n = \frac{2M}{n^2+3n+10}, \quad Q_n = \frac{M^2}{n^2+3n+10}, \quad n \ge 1, u \in \mathbb{R}.$$

It follows that the conditions (2.1), (2.2) and (2.8) hold. Thus Theorem 2.7 guarantees that Eq. (3.7) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. However, Theorem 1 in [3] is not valid for Eq. (3.7).

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## References

[1] R.P. Agarwal, Difference Equations and Inequalities, second edition, Marcel Dekker, New York, 2000.
[2] S.S. Cheng, H.J. Li, W.T. Patula, Bounded and zero convergent solutions of second order difference equations, J. Math. Anal. Appl. 141 (1989) 463–483.
[3] C. Jinfa, Existence of a nonoscillatory solution of a second-order linear neutral difference equation, Appl. Math. Lett. 20 (2007) 892–899.
[4] V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic, Dordrecht, 1993.
[5] W.T. Li, S.S. Cheng, Oscillation criteria for a nonlinear difference equation, Comput. Math. Appl. 36 (1998) 87–94.
[6] X. Li, D. Zhu, New results for the asymptotic behavior of a nonlinear second-order difference equation, Appl. Math. Lett. 16 (2003) 627–633.
[7] Q. Meng, J. Yan, Bounded oscillation for second-order nonlinear difference equations in critical and non-critical states, J. Comput. Appl. Math. 211 (2008) 156–172.
[8] M. Migda, J. Migda, Asymptotic properties of solutions of second-order neutral difference equations, Nonlinear Anal. 63 (2005) e789–e799.
[9] X.H. Tang, Bounded oscillation of second-order neutral difference equations of unstable type, Comput. Math. Appl. 44 (2002) 1147–1156.
[10] E. Thandapani, M.M.S. Manuel, J.R. Graef, P.W. Spikes, Monotone properties of certain classes of solutions of second-order difference equations, Comput. Math. Appl. 36 (2001) 291–297.
[11] B.G. Zhang, Oscillation and asymptotic behavior of second order difference equations, J. Math. Anal. Appl. 173 (1993) 58–68.
[12] Z.G. Zhang, Q.L. Li, Oscillation theorems for second-order advanced functional difference equations, Comput. Math. Appl. 36 (1998) 11–18.