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Dynamic intuitionistic fuzzy multi-attribute decision making

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Abstract

The dynamic multi-attribute decision making problems with intuitionistic fuzzy information are investigated. The notions of intuitionistic fuzzy variable and uncertain intuitionistic fuzzy variable are defined, and two new aggregation operators: dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator are presented. Some methods, including the basic unit-interval monotonic (BUM) function based method, normal distribution based method, exponential distribution based method and average age method, are introduced to determine the weight vectors associated with these operators. A procedure based on the DIFWA operator is developed to solve the dynamic intuitionistic fuzzy multi-attribute decision making (DIF-MADM) problems where all the decision information about attribute values takes the form of intuitionistic fuzzy numbers collected at different periods, and a procedure based on the UDIFWA operator is developed for DIF-MADM under interval uncertainty in which all the decision information about attribute values takes the form of interval-valued intuitionistic fuzzy numbers collected at different periods. Finally, a practical case is used to illustrate the developed procedures.

Keywords: Dynamic intuitionistic fuzzy multi-attribute decision making (DIF-MADM); Dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator; Uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator

1. Introduction

Intuitionistic fuzzy set (IFS) [1] characterized by a membership function and a non-membership function, is an extension of Zadeh's fuzzy set [2] whose basic component is only a membership function. IFS has been proven to be highly useful to deal with uncertainty and vagueness, and a lot of work has been done to develop and enrich the IFS theory [3,4]. In many complex decision making problems, the decision information provided by a decision maker is often imprecise or uncertain [5] due to time pressure, lack of data, or the decision maker's limited attention and information processing capabilities. Accordingly, IFS is a very suitable tool to be used to describe the imprecise or uncertain decision information. Recently, some researchers have shown

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great interest in the IFS theory and applied it to the field of decision making. Gau and Buehrer [6] introduced the vague set, which is an equivalence of IFS [7]. Later, based on vague sets, Chen and Tan [8], and Hong and Choi [9] utilized the minimum and maximum operations to develop some approximate technique for handling multi-attribute decision making problems under fuzzy environment. Szmidt and Kacprzyk [10] proposed some solution concepts such as the intuitionistic fuzzy core and consensus winner in group decision making with intuitionistic (individual and social) fuzzy preference relations, and proposed a method to aggregate the individual intuitionistic fuzzy preference relations into a social fuzzy preference relation on the basis of fuzzy majority equated with a fuzzy linguistic quantifier. Atanassov et al. [11] proposed an intuitionistic fuzzy interpretation of multi-person multi-attribute decision making, in which each decision maker is asked to evaluate at least a part of the alternatives in terms of their performance with respect to each predefined attribute: the decision maker's evaluations are expressed in a pair of numeric values, interpreted in the intuitionistic fuzzy framework: these numbers express a "positive" and a "negative" evaluation, respectively. They also proposed a method for multi-person multi-attribute decision making, and presented some examples of the proposed method in the context of public relation and mass communication. Xu and Yager [12] developed some aggregation operators including the intuitionistic fuzzy weighted geometric operator, intuitionistic fuzzy ordered weighted geometric operator, and intuitionistic fuzzy hybrid geometric operator, which extend the traditional weighted geometric operator and ordered weighted geometric operator to accommodate the environment where the given arguments are IFSs. Moreover, we developed an approach, based on the intuitionistic fuzzy hybrid geometric operator, to multi-attribute decision making based on IFSs. Liu and Wang [13] gave an evaluation function for the decision making problem to measure the degrees to which alternatives satisfy and do not satisfy the decision maker's requirement. Then, they introduced the intuitionistic fuzzy point operators, and defined a series of new score functions for the multi-attribute decision making problems based on intuitionistic fuzzy point operators and evaluation function. Xu [14] defined some new intuitionistic preference relations, such as the consistent intuitionistic preference relation, incomplete intuitionistic preference relation and acceptable intuitionistic preference relation, and studied their properties. We also developed a method for group decision making based on intuitionistic preference relations and a method for group decision making based on incomplete intuitionistic preference relations, respectively. All these studies are focused on the decision making problems where all the original decision information are provided at the same period. However, in many decision areas, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation, etc., the original decision information are usually collected at different periods. Thus, it is necessary to develop some approaches to dealing with these issues. In this paper, we shall study the fuzzy multi-attribute decision making problems where all the attribute values are expressed in intuitionistic fuzzy numbers collected at different periods (for convenience, we call this kind of problems dynamic intuitionistic fuzzy multi-attribute decision making (DIF-MADM) problems). To do that, we first introduce the notion of intuitionistic fuzzy variable and develop an aggregation operator called dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator. Then, we introduce some methods such as the basic unit-interval monotonic (BUM) function based method, normal distribution based method, exponential distribution based method and average age method, to determine the weight vectors associated with the operator, and develop a procedure for DIF-MADM. Furthermore, we extend the developed operator and procedure to deal with the situations where all the attribute values are expressed in interval-valued intuitionistic fuzzy numbers collected at different periods. At last, an illustrative example is given.

2. Preliminaries

Let us first review some basic concepts related to IFSs [1].

Definition 1 [2]. Let a set Z be fixed, a fuzzy set F in Z is given by Zadeh [2] as follows:

$$F = \{< z, \mu_F(z) > | z \in Z\} \tag{1}$$

where

$$\mu_F : Z \to [0,1], \quad z \in Z \to \mu_Z(z) \in [0,1] \tag{2}$$

and $\mu_F(z)$ denotes the degree of membership of the element z to the set Z.

Definition 2 [1]. Let a set Z be fixed, an IFS A in Z is given by Atanassov [1] as an object having the following form:

$$A = \{< z, \mu_A(z), v_A(z) > | z \in Z\} \tag{3}$$

where the functions

$$\mu_A : Z \to [0,1], \quad z \in Z \to \mu_A(z) \in [0,1] \tag{4}$$

and

$$v_A : Z \to [0,1], \quad z \in Z \to v_A(z) \in [0,1] \tag{5}$$

with the condition

$$0 \leqslant \mu_A(z) + v_A(z) \leqslant 1, \quad \forall z \in Z \tag{6}$$

$\mu_A(z)$ and $v_A(z)$ denote the degree of membership and the degree of non-membership of the element $z \in Z$ to the set A, respectively. In addition, for each IFS A in Z, if

$$\pi_A(z) = 1 - \mu_A(z) - v_A(z) \tag{7}$$

then $\pi_A(z)$ is called the degree of indeterminacy of z to A [3], or called the degree of hesitancy of z to A [15]. Especially, if $\pi_A(z) = 0$, for all $z \in Z$, then the IFS A is reduced to a fuzzy set.

Clearly, a prominent characteristic of IFS is that it assigns to each element a membership degree, a non-membership degree and a hesitation degree, and thus, IFS constitutes an extension of Zadeh's fuzzy set which only assigns to each element a membership degree.

For convenience of computation, we call $\alpha = (\mu_\alpha, v_\alpha, \pi_\alpha)$ an intuitionistic fuzzy number (IFN), where

$$\mu_\alpha \in [0,1], \quad v_\alpha \in [0,1], \quad \mu_\alpha + v_\alpha \leqslant 1, \quad \pi_\alpha = 1 - \mu_\alpha - v_\alpha \tag{8}$$

For an IFN $\alpha = (\mu_\alpha, v_\alpha, \pi_\alpha)$, if the value μ_α gets bigger and the value v_α gets smaller, then the IFN α gets greater, and thus from (8), we know that $\alpha^+ = (1,0,0)$ and $\alpha^- = (0,1,0)$ are the largest and smallest IFNs, respectively.

Similar to the normalized Hamming distance between IFSs [15], below we define a distance measure between two IFNs.

Definition 3. Let $\alpha_1 = (\mu_{\alpha_1}, v_{\alpha_1}, \pi_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, v_{\alpha_2}, \pi_{\alpha_2})$ be two IFNs, then

$$d(\alpha_1, \alpha_2) = \frac{1}{2}(|\mu_{\alpha_1} - \mu_{\alpha_2}| + |v_{\alpha_1} - v_{\alpha_2}| + |\pi_{\alpha_1} - \pi_{\alpha_2}|) \tag{9}$$

is called the distance between α_1 and α_2.

3. Dynamic intuitionistic fuzzy weighted averaging operator

Information aggregation is an essential process and is also an important research topic in the field of information fusion. In [1], Atanassov defined some basic operations and relations over IFSs. De et al. [16] added some new operations such as concentration, dilation and normalization of IFSs. Xu and Yager [12] developed some geometric operators to aggregate intuitionistic fuzzy information. All these operations, relations and operators can only be used to deal with time independent arguments. However, if time is taken into account, for example, the argument information may be collected at different periods, then the aggregation operators and their associated weights should not be kept constant. As a result, in the following, based on (8), we first define the notion of intuitionistic fuzzy variable.

Definition 4. Let t be a time variable, then we call $\alpha(t) = (\mu_{\alpha(t)}, v_{\alpha(t)}, \pi_{\alpha(t)})$ an intuitionistic fuzzy variable, where

$$\mu_{\alpha(t)} \in [0,1], \quad v_{\alpha(t)} \in [0,1], \quad \mu_{\alpha(t)} + v_{\alpha(t)} \leqslant 1, \quad \pi_{\alpha(t)} = 1 - \mu_{\alpha(t)} - v_{\alpha(t)} \tag{10}$$

For an intuitionistic fuzzy variable $\alpha(t) = (\mu_{\alpha(t)}, v_{\alpha(t)}, \pi_{\alpha(t)})$, if $t = t_1, t_2, \ldots, t_p$, then $\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_n)$ indicate p IFNs collected at p different periods. Below we introduce some operations related to IFNs.

Definition 5. Let $\alpha(t_1) = (\mu_{\alpha_1(t_1)}, v_{\alpha_1(t_1)}, \pi_{\alpha_1(t_1)})$ and $\alpha(t_2) = (\mu_{\alpha_2(t_2)}, v_{\alpha_2(t_2)}, \pi_{\alpha_2(t_2)})$ be two IFNs, then

(1) $\alpha(t_1) \oplus \alpha(t_2) = (\mu_{\alpha(t_1)} + \mu_{\alpha(t_2)} - \mu_{\alpha(t_1)}\mu_{\alpha(t_2)}, v_{\alpha(t_1)}v_{\alpha(t_2)}, (1-\mu_{\alpha(t_1)})(1-\mu_{\alpha(t_2)}) - v_{\alpha(t_1)}v_{\alpha(t_2)})$.
(2) $\lambda\alpha(t_1) = (1-(1-\mu_{\alpha(t_1)})^{\lambda}, v^{\lambda}_{\alpha(t_1)}, (1-\mu_{\alpha(t_1)})^{\lambda} - v^{\lambda}_{\alpha(t_1)}), \lambda > 0$.

Definition 6. Let $\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)$ be a collection of IFNs collected at p different periods $t_k (k = 1, 2, \ldots, p)$, and $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^{\mathrm{T}}$ be the weight vector of the periods $t_k (k = 1, 2, \ldots, p)$, then we call

$$\mathrm{DIFWA}_{\lambda(t)}(\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)) = \lambda(t_1)\alpha(t_1) \oplus \lambda(t_2)\alpha(t_2) \oplus \cdots \oplus \lambda(t_p)\alpha(t_p) \tag{11}$$

a dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator.

By Definition 5, (11) can be rewritten as follows:

$$\mathrm{DIFWA}_{\lambda(t)}(\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)) = \left(1 - \prod_{k=1}^{p}(1-\mu_{\alpha(t_k)})^{\lambda(t_k)},\ \prod_{k=1}^{p} v^{\lambda(t_k)}_{\alpha(t_k)},\ \prod_{k=1}^{p}(1-\mu_{\alpha(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p} v^{\lambda(t_k)}_{\alpha(t_k)}\right) \tag{12}$$

where

$$\lambda(t_k) \geqslant 0, \quad k = 1, 2, \ldots, p, \quad \sum_{k=1}^{p} \lambda(t_k) = 1 \tag{13}$$

In what follows, we introduce some methods to determine the weight vector $\lambda(t)$ of the periods $t_k (k = 1, 2, \ldots, p)$:

(1) *BUM function based method* [17,18]: Let Q: $[0,1] \to [0,1]$ be a function having the following properties:

(i) $Q(0) = 0$.
(ii) $Q(1) = 1$.
(iii) $Q(x) \geqslant Q(y)$ if $x > y$.

Then Q is a basic unit-interval monotonic (BUM) function [17,18]. Using a BUM function, we can obtain the weight vector $\lambda(t)$ as follows:

$$\lambda(t_k) = Q\left(\frac{k}{p}\right) - Q\left(\frac{k-1}{p}\right), \quad k = 1, 2, \ldots, p \tag{14}$$

with the condition (13).

For example, if $Q(x) = x^r$, $r > 0$, then

$$\lambda(t_k) = \left(\frac{k}{p}\right)^r - \left(\frac{k-1}{p}\right)^r = \left(\frac{k}{p}\right)^r - \left(\frac{k}{p} - \frac{1}{p}\right)^r, \quad k = 1, 2, \ldots, p \tag{15}$$

Let

$$f(x) = x^r - \left(x - \frac{1}{p}\right)^r, \quad x \geqslant \frac{1}{p} \tag{16}$$

then

$$f'(x) = rx^{r-1} - r\left(x - \frac{1}{p}\right)^{r-1} = r\left(x^{r-1} - \left(x - \frac{1}{p}\right)^{r-1}\right) \tag{17}$$

thus,

(i) If $r > 1$, then $f'(x) > 0$, i.e., $f(x)$ is a strictly monotonic increasing function.
(ii) If $r = 1$, then $f'(x) = 0$, i.e., $f(x)$ is a constant function.
(iii) If $r < 1$, then $f'(x) < 0$, i.e., $f(x)$ is a strictly monotonic decreasing function.

Therefore, by (14), we have

(i) If $r > 1$, then $\lambda(t_{k+1}) > \lambda(t_k)$, $k = 1, 2, \ldots, p-1$, i.e., the sequence $\{\lambda(t_k)\}$ is a monotonic increasing sequence. Especially, if $r = 2$, then

$$\lambda(t_{k+1}) - \lambda(t_k) = \left(\frac{k+1}{p}\right)^2 - \left(\frac{k}{p}\right)^2 - \left(\frac{k}{p}\right)^2 + \left(\frac{k-1}{p}\right)^2 = \frac{2}{p^2}, \quad k = 1, 2, \ldots, p-1 \tag{18}$$

i.e., the sequence $\{\lambda(t_k)\}$ is an increasing arithmetic sequence.

(ii) If $r = 1$, then

$$\lambda(t_k) = \frac{k}{p} - \frac{k-1}{p} = \frac{1}{p}, \quad k = 1, 2, \ldots, p \tag{19}$$

thus $\lambda(t) = (1/p, 1/p, \ldots, 1/p)^{\mathrm{T}}$.

(iii) If $r < 1$, then $\lambda(t_{k+1}) < \lambda(t_k)$, $k = 1, 2, \ldots, p-1$, i.e., the sequence $\{\lambda(t_k)\}$ is a monotonic decreasing sequence.

(2) *Normal distribution based method* [19]: The normal distribution is one of the most commonly observed and is the starting point for modeling many natural processes. It is usually found in events that are the aggregation of many smaller, but independent random events. Below we first review the concept of normal distribution (or so-called Gaussian distribution).

The normal probability density function of normal distribution for a variable x is defined as follow:

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty \tag{20}$$

where μ is a mean and $\sigma(\sigma > 0)$ is a standard deviation.

We can utilize the normal distribution based method to determine the weight vector $\lambda(t)$ [19]:

$$\lambda(t_k) = \frac{1}{\sqrt{2\pi}\sigma_p} \mathrm{e}^{-\frac{(k-\mu_p)^2}{2\sigma_p^2}}, \quad k = 1, 2, \ldots, p \tag{21}$$

where μ_p is the mean of the collection of $1, 2, \ldots, p$, and $\sigma_p(\sigma_p > 0)$ is the standard deviation of the collection of $1, 2, \ldots, p$. μ_p and σ_p are obtained by using the following formulas, respectively:

$$\mu_p = \frac{1}{p}\frac{p(1+p)}{2} = \frac{1+p}{2} \tag{22}$$

$$\sigma_p = \sqrt{\frac{1}{p}\sum_{k=1}^{p}(k-\mu_p)^2} \tag{23}$$

By (13) and (21), we have

$$\lambda(t_k) = \frac{\mathrm{e}^{-\frac{(k-\mu_p)^2}{2\sigma_p^2}}}{\sum_{j=1}^{p}\mathrm{e}^{-\frac{(j-\mu_p)^2}{2\sigma_p^2}}}, \quad k = 1, 2, \ldots, p \tag{24}$$

The normal distribution based method has the following properties [19]:

(i) The weights $\lambda(t_k)(k = 1, 2, \ldots, p)$ are symmetrical, i.e.,

$$\lambda(t_k) = \lambda(t_{p+1-k}), \quad k = 1, 2, \ldots, p \tag{25}$$

(ii) It assigns the largest weights to the mean period, and the further the period t_k departs from the mean period, the smaller the weight assigned to the period t_k.

(3) *Exponential distribution based method* [20]: The exponential distribution is a memory-less continuous distribution. The exponential distribution is often used to model the time between random arrivals of events

that occur at a constant average rate. The normal probability density function of exponential distribution for a variable x is defined as follows:

$$h(x) = \frac{1}{\mu}\mathrm{e}^{-\frac{x}{\mu}}, \quad x > 0 \tag{26}$$

where μ is the mean time between failures.

To generate the weight vector $\lambda(t)$ using the normal probability density function of exponential distribution, (26) can be rewritten as follows:

$$\lambda(t_k) = \frac{1}{\mu_p}\mathrm{e}^{-\frac{k}{\mu_p}}, \quad k = 1, 2, \ldots, p \tag{27}$$

where μ_p is shown as in (22).

By (13) and (27), we have

$$\lambda(t_k) = \frac{\mathrm{e}^{-\frac{k}{\mu_p}}}{\sum_{j=1}^{p}\mathrm{e}^{-\frac{j}{\mu_p}}}, \quad k = 1, 2, \ldots, p \tag{28}$$

From (28), we know that the sequence $\{\lambda(t_k)\}$ is a monotonic decreasing sequence, that is, the larger k, the smaller the weight assigned to the period t_k.

If we use the inverse form of exponential distribution to determine the weight vector $\lambda(t)$, then

$$\lambda(t_k) = \frac{1}{\mu_p}\mathrm{e}^{\frac{k}{\mu_p}}, \quad k = 1, 2, \ldots, p \tag{29}$$

By (13) and (29), we have

$$\lambda(t_k) = \frac{\mathrm{e}^{\frac{k}{\mu_p}}}{\sum_{j=1}^{p}\mathrm{e}^{\frac{j}{\mu_p}}}, \quad k = 1, 2, \ldots, p \tag{30}$$

where the sequence $\{\lambda(t_k)\}$ is a monotonic increasing sequence, that is, the larger k, the greater the weight assigned to the period t_k.

Clearly, the weights generated by exponential distribution based method are similar to those generated by the BUM function based method.

(4) *Average age method* [21]: We can associate with a set of weights $\lambda(t_k)(k = 1, 2, \ldots, p)$ a concept of the average age of the data. Assume $\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p)$ are the weights with t_p being the most recent and t_1 being the earliest. Using this we can calculate

$$\bar{t} = \sum_{k=1}^{p}(p - k)\lambda(t_k) \tag{31}$$

where $\bar{t}$ indicates the average age of the data. We note that for the BUM approach the area under Q can be used to approximate $\bar{t}$:

$$\bar{t} \approx (p - 1)\int_0^1 Q(x)\mathrm{d}x \tag{32}$$

More generally, we can obtain the weights by specifying a value for $\bar{t}$ and then find a set of weights that satisfies the following mathematical programming model for the $\lambda(t_k)$:

$$\begin{aligned}
\text{Minimize:} \quad & \sum_{k=1}^{p}(\lambda(t_k))^2 \\
\text{Subject to:} \quad & \sum_{k=1}^{p}(p - k)\lambda(t_k) = \bar{t} \\
& \sum_{k=1}^{p}\lambda(t_k) = 1, \quad \lambda(t_k) \geqslant 0, \quad k = 1, 2, \ldots, p
\end{aligned}$$

To solve this model, we construct the Lagrange function:

$$L(\lambda(t), \eta_1, \eta_2) = \sum_{k=1}^{p} (\lambda(t_k))^2 - 2\eta_1 \left(\sum_{k=1}^{p} (p-k)\lambda(t_k) - \bar{t} \right) - 2\eta_2 \left(\sum_{k=1}^{p} \lambda(t_k) - 1 \right) \tag{33}$$

where $\lambda(t) = (\lambda(t_1), \lambda(t_2), \dots, \lambda(t_p))^{\mathrm{T}}$, η_1 and η_2 are the Lagrange multipliers.

Differentiating (33) with respect to $\lambda(t_k)(k = 1, 2, \dots, p)$, η_1 and η_2, and setting these partial derivatives equal to zero, the following set of equations is obtained:

$$\frac{\partial L(\lambda(t), \eta_1, \eta_2)}{\partial \lambda(t_k)} = 2\lambda(t_k) - 2\eta_1(p-k) - 2\eta_2 = 0 \tag{34}$$

$$\frac{\partial L(\lambda(t), \eta_1, \eta_2)}{\partial \eta_1} = -2\left(\sum_{k=1}^{p} (p-k)\lambda(t_k) - \bar{t} \right) = 0 \tag{35}$$

$$\frac{\partial L(\lambda(t), \eta_1, \eta_2)}{\partial \eta_2} = -2\left(\sum_{k=1}^{p} \lambda(t_k) - 1 \right) = 0 \tag{36}$$

Simplifying (34)–(36), we have

$$\lambda(t_k) = (p-k)\eta_1 + \eta_2 \tag{37}$$

$$\sum_{k=1}^{p} (p-k)\lambda(t_k) = \bar{t} \tag{38}$$

$$\sum_{k=1}^{p} \lambda(t_k) = 1 \tag{39}$$

Combining (37)–(39), it follows that

$$\eta_1 \sum_{k=1}^{p} (p-k)^2 + \eta_2 \sum_{k=1}^{p} (p-k) = \bar{t} \tag{40}$$

$$\eta_1 \sum_{k=1}^{p} (p-k) + \eta_2 p = 1 \tag{41}$$

By solving (40) and (41), we get

$$\eta_1 = \frac{12\bar{t} - 6(p-1)}{p(p-1)^2} \tag{42}$$

$$\eta_2 = \frac{4(p-1) - 6\bar{t}}{p(p-1)} \tag{43}$$

and thus, by (34), we have

$$\lambda(t_k) = \frac{(12\bar{t} - 6p + 6)(p-k) + 4(p-1)^2 - 6\bar{t}(p-1)}{p(p-1)^2}, \quad k = 1, 2, \dots, p \tag{44}$$

Since $\lambda(t_k) \geqslant 0$, for all k, then

$$\frac{(12\bar{t} - 6p + 6)(p-k) + 4(p-1)^2 - 6\bar{t}(p-1)}{p(p-1)^2} \geqslant 0, \quad k = 1, 2, \dots, p \tag{45}$$

i.e,

$$(3p - 6k + 3)\bar{t} \geqslant (p-1)(p - 3k + 2), \quad k = 1, 2, \dots, p \tag{46}$$

thus,

(i) If $(3p-6k+3)=0$, i.e., $k=\frac{p+1}{2}$, then (46) holds, for all $\bar{t}$.
(ii) If $(3p-6k+3)>0$, i.e., $k<\frac{p+1}{2}$, then (46) holds, for $\bar{t}\geqslant\frac{p-1}{3}$.
(iii) If $(3p-6k+3)<0$, i.e., $k>\frac{p+1}{2}$, then (45) holds, for $\bar{t}\leqslant\frac{2(p-1)}{3}$.

Therefore, we can obtain the weights $\lambda(t_k)(k=1,2,\ldots,p)$ by using (44) with the following condition:

$$\frac{p-1}{3}\leqslant\bar{t}\leqslant\frac{2(p-1)}{3} \tag{47}$$

If let

$$g(x)=\frac{(12\bar{t}-6p+6)(p-x)+4(p-1)^2-6\bar{t}(p-1)}{p(p-1)^2} \tag{48}$$

then

$$g'(x)=-\frac{(12\bar{t}-6p+6)}{p(p-1)^2} \tag{49}$$

thus,

(i) If $\frac{p-1}{3}\leqslant\bar{t}<\frac{p-1}{2}$, then $g'(x)>0$, i.e., $g(x)$ is a strictly monotonic increasing function.
(ii) If $\bar{t}=\frac{p-1}{2}$, then $g'(x)=0$, i.e., $g(x)$ is a constant function.
(iii) If $\frac{p-1}{2}<\bar{t}\leqslant\frac{2(p-1)}{3}$, then $g'(x)<0$, i.e., $g(x)$ is a strictly monotonic decreasing function.

Therefore, by (44), we have

(i) If $\frac{p-1}{3}\leqslant\bar{t}<\frac{p-1}{2}$, then $\lambda\ (t_{k+1})>\lambda(t_k)$, $k=1,2,\ldots,p-1$, i.e., the sequence $\{\lambda(t_k)\}$ is a monotonic increasing sequence. Also since

$$\begin{aligned}\lambda(t_{k+1})-\lambda(t_k)&=\frac{(12\bar{t}-6p+6)(p-(k+1))+4(p-1)^2-6\bar{t}(p-1)}{p(p-1)^2}\\&\quad-\frac{(12\bar{t}-6p+6)(p-k)+4(p-1)^2-6\bar{t}(p-1)}{p(p-1)^2}\\&=-\frac{(12\bar{t}-6p+6)}{p(p-1)^2}>0,\quad k=1,2,\ldots,p-1\end{aligned} \tag{50}$$

then the sequence $\{\lambda(t_k)\}$ is an increasing arithmetic sequence.

(ii) If $\bar{t}=\frac{p-1}{2}$, then

$$\lambda(t_k)=\frac{(12\bar{t}-6p+6)(p-k)+4(p-1)^2-6\bar{t}(p-1)}{p(p-1)^2}=\frac{1}{p},\quad k=1,2,\ldots,p \tag{51}$$

thus $\lambda(t)=(1/p,1/p,\ldots,1/p)^{\mathrm{T}}$.

(iii) If $\frac{p-1}{2}<\bar{t}\leqslant\frac{2(p-1)}{3}$, then $\lambda\ (t_{k+1})<\lambda(t_k)$, $k=1,2,\ldots,p-1$, i.e., the sequence $\{\lambda(t_k)\}$ is a monotonic decreasing sequence. Similar to (50), we have $\lambda(t_{k+1})-\lambda(t_k)<0$, $k=1,2,\ldots,p-1$, thus the sequence $\{\lambda(t_k)\}$ is a decreasing arithmetic sequence.

4. A procedure for DIF-MADM

In this section, we consider the DIF-MADM problems where all the attribute values are expressed in intuitionistic fuzzy numbers, which are collected at different periods. The following notations are used to depict the considered problems:

- $X = \{x_1, x_2, \ldots, x_n\}$: A discrete set of n feasible alternatives.
- $G = \{G_1, G_2, \ldots, G_m\}$: A finite set of attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$, where $w_j \geqslant 0$, $j = 1, 2, \ldots, m$, $\sum_{j=1}^{m} w_j = 1$.
- There are p periods $t_k (k = 1, 2, \ldots, p)$, whose weight vector is $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^{\mathrm{T}}$, where $\lambda(t_k) \geqslant 0$, $k = 1, 2, \ldots, p$, $\sum_{k=1}^{p} \lambda(t_k) = 1$.
- $R(t_k) = (r_{ij}(t_k))_{n\times m}$: An intuitionistic fuzzy decision matrix of the period t_k, where $r_{ij}(t_k) = (\mu_{r_{ij}(t_k)}, v_{r_{ij}(t_k)}, \pi_{r_{ij}(t_k)})$ is an attribute value, denoted by an IFN, $\mu_{r_{ij}(t_k)}$ indicates the degree that the alternative x_i should satisfy the attribute G_j at the period t_k, $v_{r_{ij}(t_k)}$ indicates the degree that the alternative x_i should not satisfy the attribute G_j at the period t_k, and $\pi_{r_{ij}(t_k)}$ indicates the degree of indeterminacy of the alternative x_i to the attribute G_j, such that

$$\mu_{r_{ij}(t_k)} \in [0,1], \quad v_{r_{ij}(t_k)} \in [0,1], \quad \mu_{r_{ij}(t_k)} + v_{r_{ij}(t_k)} \leqslant 1, \quad \pi_{r_{ij}(t_k)} = 1 - \mu_{r_{ij}(t_k)} - v_{r_{ij}(t_k)},$$
$$i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m \tag{52}$$

Based on the above decision information, in what follows, we propose a practical procedure to rank and select the most desirable alternative(s):

Procedure I

Step 1. Utilize the DIFWA operator:

$$\begin{aligned} r_{ij} &= \mathrm{DIFWA}_{\lambda(t)}(r_{ij}(t_1), r_{ij}(t_2), \ldots, r_{ij}(t_p)) \\ &= \left(1 - \prod_{k=1}^{p}(1-\mu_{r_{ij}(t_k)})^{\lambda(t_k)}, \; \prod_{k=1}^{p} v_{r_{ij}(t_k)}^{\lambda(t_k)}, \; \prod_{k=1}^{p}(1-\mu_{r_{ij}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p} v_{r_{ij}(t_k)}^{\lambda(t_k)}\right) \end{aligned} \tag{53}$$

to aggregate all the intuitionistic fuzzy decision matrices $R(t_k) = (r_{ij}(t_k))_{m\times n}$ $(k = 1, 2, \ldots, p)$ into a complex intuitionistic fuzzy decision matrix $R = (r_{ij})_{n\times m}$, where $r_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij})$, $\mu_{ij} = 1 - \prod_{k=1}^{p}(1-\mu_{r_{ij}(t_k)})^{\lambda(t_k)}$, $v_{ij} = \prod_{k=1}^{p} v_{r_{ij}(t_k)}^{\lambda(t_k)}$, $\pi_{ij} = \prod_{k=1}^{p}(1-\mu_{r_{ij}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p} v_{r_{ij}(t_k)}^{\lambda(t_k)}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Step 2. Define $\alpha^+ = (\alpha_1^+, \alpha_2^+, \ldots, \alpha_m^+)^{\mathrm{T}}$ and $\alpha^- = (\alpha_1^-, \alpha_2^-, \ldots, \alpha_m^-)^{\mathrm{T}}$ as the intuitionistic fuzzy ideal solution (IFIS) and the intuitionistic fuzzy negative ideal solution (IFNIS), respectively, where $\alpha_i^+ = (1,0,0)(i = 1, 2, \ldots, m)$ are the m largest IFNs, and $\alpha_i^- = (0,1,0)(i = 1, 2, \ldots, m)$ are the m smallest IFNs. Furthermore, for convenience of depiction, we denote the alternatives $x_i (i = 1, 2, \ldots, n)$ by $x_i = (r_{i1}, r_{i2}, \ldots, r_{im})^{\mathrm{T}}$, $i = 1, 2, \ldots, n$.

Step 3. Calculate the distance between the alternative x_i and the IFIS α^+ and the distance between the alternative x_i and the IFNIS α^-, respectively:

$$\begin{aligned} d(x_i, \alpha^+) &= \sum_{j=1}^{m} w_j d(r_{ij}, \alpha_j^+) = \frac{1}{2}\sum_{j=1}^{m} w_j(|\mu_{ij} - 1| + |v_{ij} - 0| + |\pi_{ij} - 0|) \\ &= \frac{1}{2}\sum_{j=1}^{m} w_j(1 - \mu_{ij} + v_{ij} + \pi_{ij}) = \frac{1}{2}\sum_{j=1}^{m} w_j(1 - \mu_{ij} + v_{ij} + 1 - \mu_{ij} - v_{ij}) \\ &= \sum_{j=1}^{m} w_j(1 - \mu_{ij}) \end{aligned} \tag{54}$$

$$\begin{aligned} d(x_i, \alpha^-) &= \sum_{j=1}^{m} w_j d(r_{ij}, \alpha_j^-) = \frac{1}{2}\sum_{j=1}^{m} w_j(|\mu_{ij} - 0| + |v_{ij} - 1| + |\pi_{ij} - 0|) \\ &= \frac{1}{2}\sum_{j=1}^{m} w_j(1 + \mu_{ij} - v_{ij} + \pi_{ij}) = \frac{1}{2}\sum_{j=1}^{m} w_j(1 + \mu_{ij} - v_{ij} + 1 - \mu_{ij} - v_{ij}) \\ &= \frac{1}{2}\sum_{j=1}^{m} w_j(1 - v_{ij}) \end{aligned} \tag{55}$$

where $r_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Step 4. Calculate the closeness coefficient of each alternative:

$$c(x_i) = \frac{d(x_i, \alpha^-)}{d(x_i, \alpha^+) + d(x_i, \alpha^-)}, \quad i = 1, 2, \ldots, n \tag{56}$$

Since

$$d(x_i, \alpha^+) + d(x_i, \alpha^-) = \sum_{j=1}^{m} w_j(1 - \mu_{ij}) + \sum_{j=1}^{m} w_j(1 - v_{ij}) = \sum_{j=1}^{m} w_j(2 - \mu_{ij} - v_{ij}) = \sum_{j=1}^{m} w_j(1 + \pi_{ij}) \tag{57}$$

then, (56) can be rewritten as

$$c(x_i) = \frac{\sum_{j=1}^{m} w_j(1 - v_{ij})}{\sum_{j=1}^{m} w_j(1 + \pi_{ij})}, \quad i = 1, 2, \ldots, n \tag{58}$$

Step 5. Rank all the alternatives $x_i (i = 1, 2, \ldots, n)$ according to the closeness coefficients $c(x_i)(i = 1, 2, \ldots, n)$, the greater the value $c(x_i)$, the better the alternative x_i.

Step 6. End.

5. A procedure for DIF-MADM under interval uncertainty

In [22], Atanassov and Gargov generalized IFS and defined the notion of the interval-valued IFS (IVIFS), which is characterized by a membership function and a non-membership function whose values are intervals rather than exact numbers.

Definition 7 [22]. Let a set Z be fixed, an IVIFS $\widetilde{A}$ over Z is an object having the form:

$$\widetilde{A} = \{< z, \tilde{\mu}_{\widetilde{A}}(z), \tilde{v}_{\widetilde{A}}(z) > | z \in Z\} \tag{59}$$

where $\tilde{\mu}_{\widetilde{A}}(z) = [\tilde{\mu}^L_{\widetilde{A}}(z), \tilde{\mu}^U_{\widetilde{A}}(z)] \subset [0, 1]$ and $\tilde{v}_{\widetilde{A}}(z) = [\tilde{v}^L_{\widetilde{A}}(z), \tilde{v}^U_{\widetilde{A}}(z)] \subset [0, 1]$ are intervals, $\tilde{\mu}^L_{\widetilde{A}}(z) = \inf \tilde{\mu}_{\widetilde{A}}(z)$, $\tilde{\mu}^U_{\widetilde{A}}(z) = \sup \tilde{\mu}_{\widetilde{A}}(z)$, $\tilde{v}^L_{\widetilde{A}}(z) = \inf \tilde{v}_{\widetilde{A}}(z)$, $\tilde{v}^U_{\widetilde{A}}(z) = \sup \tilde{v}_{\widetilde{A}}(z)$, and for every $z \in Z$:

$$\tilde{\mu}^U_{\widetilde{A}}(z) + \tilde{v}^U_{\widetilde{A}}(z) \leqslant 1 \tag{60}$$

Let $\tilde{\pi}_{\widetilde{A}}(z) = [\tilde{\pi}^L_{\widetilde{A}}(z), \tilde{\pi}^U_{\widetilde{A}}(z)]$, where

$$\tilde{\pi}^L_{\widetilde{A}}(z) = 1 - \tilde{\mu}^U_{\widetilde{A}}(z) - \tilde{v}^U_{\widetilde{A}}(z), \quad \tilde{\pi}^U_{\widetilde{A}}(z) = 1 - \tilde{\mu}^L_{\widetilde{A}}(z) - \tilde{v}^L_{\widetilde{A}}(z), \quad \text{for all } z \in Z \tag{61}$$

Here, we call the triple $(\tilde{\mu}_{\widetilde{A}}(z), \tilde{v}_{\widetilde{A}}(z), \tilde{\pi}_{\widetilde{A}}(z))$ an interval-valued intuitionistic fuzzy number (IVIFN). For convenience, we denote an IVIFN by $\tilde{\alpha} = (\tilde{\mu}_{\tilde{\alpha}}, \tilde{v}_{\tilde{\alpha}}, \tilde{\pi}_{\tilde{\alpha}})$, where

$$\begin{aligned} &\tilde{\mu}_{\tilde{\alpha}} = [\tilde{\mu}^L_{\tilde{\alpha}}, \tilde{\mu}^U_{\tilde{\alpha}}] \subset [0, 1], \quad \tilde{v}_{\tilde{\alpha}} = [\tilde{v}^L_{\tilde{\alpha}}, \tilde{v}^U_{\tilde{\alpha}}] \subset [0, 1], \quad \tilde{\mu}^U_{\tilde{\alpha}} + \tilde{v}^U_{\tilde{\alpha}} \leqslant 1, \\ &\tilde{\pi}_{\tilde{\alpha}} = [\tilde{\pi}^L_{\tilde{\alpha}}, \tilde{\pi}^U_{\tilde{\alpha}}] = [1 - \tilde{\mu}^U_{\tilde{\alpha}} - \tilde{v}^U_{\tilde{\alpha}}, 1 - \tilde{\mu}^L_{\tilde{\alpha}} - \tilde{v}^L_{\tilde{\alpha}}] \end{aligned} \tag{62}$$

Obviously, by (62), we know that $\tilde{\alpha}^+ = ([1, 1], [0, 0], [0, 0])$ and $\tilde{\alpha}^- = ([0, 0], [1, 1], [0, 0])$ are, respectively, the largest and smallest IVIFNs.

In what follows, we define a distance measure between IVIFNs.

Definition 8. Let $\tilde{\alpha}_1 = ([\tilde{\mu}^L_{\tilde{\alpha}_1}, \tilde{\mu}^U_{\tilde{\alpha}_1}], [\tilde{v}^L_{\tilde{\alpha}_1}, \tilde{v}^U_{\tilde{\alpha}_1}], [\tilde{\pi}^L_{\tilde{\alpha}_1}, \tilde{\pi}^U_{\tilde{\alpha}_1}])$ and $\tilde{\alpha}_2 = ([\tilde{\mu}^L_{\tilde{\alpha}_2}, \tilde{\mu}^U_{\tilde{\alpha}_2}], [\tilde{v}^L_{\tilde{\alpha}_2}, \tilde{v}^U_{\tilde{\alpha}_2}], [\tilde{\pi}^L_{\tilde{\alpha}_2}, \tilde{\pi}^U_{\tilde{\alpha}_2}])$ be two IVIFNs, then

$$d(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{4}(|\tilde{\mu}^L_{\tilde{\alpha}_1} - \tilde{\mu}^L_{\tilde{\alpha}_2}| + |\tilde{\mu}^U_{\tilde{\alpha}_1} - \tilde{\mu}^U_{\tilde{\alpha}_2}| + |\tilde{v}^L_{\tilde{\alpha}_1} - \tilde{v}^L_{\tilde{\alpha}_2}| + |\tilde{v}^U_{\tilde{\alpha}_1} - \tilde{v}^U_{\tilde{\alpha}_2}|) + |\tilde{\pi}^L_{\tilde{\alpha}_1} - \tilde{\pi}^L_{\tilde{\alpha}_2}| + |\tilde{\pi}^U_{\tilde{\alpha}_1} - \tilde{\pi}^U_{\tilde{\alpha}_2}|) \tag{63}$$

is called the distance between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

Similar to Definitions 4–6, we have

Definition 9. Let t be a time variable, then we call $\tilde{\alpha}(t) = ([\tilde{\mu}^L_{\tilde{\alpha}(t)}, \tilde{\mu}^U_{\tilde{\alpha}(t)}], [\tilde{v}^L_{\tilde{\alpha}(t)}, \tilde{v}^U_{\tilde{\alpha}(t)}], [\tilde{\pi}^L_{\tilde{\alpha}(t)}, \tilde{\pi}^U_{\tilde{\alpha}(t)}])$ an uncertain intuitionistic fuzzy variable, where

$$[\tilde{\mu}^L_{\tilde{\alpha}(t)}, \tilde{\mu}^U_{\tilde{\alpha}(t)}] \subset [0,1], [\tilde{v}^L_{\tilde{\alpha}(t)}, \tilde{v}^U_{\tilde{\alpha}(t)}] \subset [0,1], \quad \tilde{\mu}^U_{\tilde{\alpha}(t)} + \tilde{v}^U_{\tilde{\alpha}(t)} \leqslant 1$$
$$[\tilde{\pi}^L_{\tilde{\alpha}(t)}, \tilde{\pi}^U_{\tilde{\alpha}(t)}] = [1 - \tilde{\mu}^U_{\tilde{\alpha}(t)} - \tilde{v}^U_{\tilde{\alpha}(t)}, 1 - \tilde{\mu}^L_{\tilde{\alpha}(t)} - \tilde{v}^L_{\tilde{\alpha}(t)}] \tag{64}$$

Let $\tilde{\alpha}(t) = ([\tilde{\mu}^L_{\tilde{\alpha}(t)}, \tilde{\mu}^U_{\tilde{\alpha}(t)}], [\tilde{v}^L_{\tilde{\alpha}(t)}, \tilde{v}^U_{\tilde{\alpha}(t)}], [\tilde{\pi}^L_{\tilde{\alpha}(t)}, \tilde{\pi}^U_{\tilde{\alpha}(t)}])$ be an uncertain intuitionistic fuzzy variable, then $\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \ldots, \tilde{\alpha}(t_n)$ denote p IVIFNs collected at p different periods.

Now we introduce the following operations related to IVIFNs:

Definition 10. Let $\tilde{\alpha}(t_k) = ([\tilde{\mu}^L_{\tilde{\alpha}(t_k)}, \tilde{\mu}^U_{\tilde{\alpha}(t_k)}], [\tilde{v}^L_{\tilde{\alpha}(t_k)}, \tilde{v}^U_{\tilde{\alpha}(t_k)}], [\tilde{\pi}^L_{\tilde{\alpha}(t_k)}, \tilde{\pi}^U_{\tilde{\alpha}(t_k)}])$ $(k = 1, 2)$ be two IVIFNs, then

$$\begin{aligned}(1)\ \ \tilde{\alpha}(t_1) \oplus \tilde{\alpha}(t_2) = (&[\tilde{\mu}^L_{\alpha(t_1)} + \tilde{\mu}^L_{\alpha(t_2)} - \tilde{\mu}^L_{\alpha(t_1)}\tilde{\mu}^L_{\alpha(t_2)}, \tilde{\mu}^U_{\alpha(t_1)} + \tilde{\mu}^U_{\alpha(t_2)} - \tilde{\mu}^U_{\alpha(t_1)}\tilde{\mu}^U_{\alpha(t_2)}], [\tilde{v}^L_{\alpha(t_1)}\tilde{v}^L_{\alpha(t_2)}, \tilde{v}^U_{\alpha(t_1)}v^U_{\alpha(t_2)}],\\ &[(1 - \tilde{\mu}^U_{\alpha(t_1)})(1 - \tilde{\mu}^U_{\alpha(t_2)}) - \tilde{v}^U_{\alpha(t_1)}\tilde{v}^U_{\alpha(t_2)}, (1 - \tilde{\mu}^L_{\alpha(t_1)})(1 - \tilde{\mu}^L_{\alpha(t_2)}) - \tilde{v}^L_{\alpha(t_1)}\tilde{v}^L_{\alpha(t_2)}])\end{aligned}$$
$$\begin{aligned}(2)\ \ \lambda\tilde{\alpha}(t_1) = (&[1 - (1 - \tilde{\mu}^L_{\tilde{\alpha}(t_1)})^\lambda, 1 - (1 - \tilde{\mu}^U_{\tilde{\alpha}(t_1)})^\lambda], [(\tilde{v}^L_{\tilde{\alpha}(t_1)})^\lambda, (\tilde{v}^U_{\tilde{\alpha}(t_1)})^\lambda],\\ &[(1 - \tilde{\mu}^U_{\tilde{\alpha}(t_1)})^\lambda - (\tilde{v}^U_{\tilde{\alpha}(t_1)})^\lambda, (1 - \tilde{\mu}^L_{\tilde{\alpha}(t_1)})^\lambda - (\tilde{v}^L_{\tilde{\alpha}(t_1)})^\lambda]), \quad \lambda > 0\end{aligned}$$

Definition 11. Let $\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \ldots, \tilde{\alpha}(t_p)$ be a collection of IVIFNs collected at p different periods $t_k (k = 1, 2, \ldots, p)$, and $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^{\mathrm{T}}$ be the weight vector of the periods $t_k (k = 1, 2, \ldots, p)$, which can be obtained by the methods proposed in Section 3, then we call

$$\mathrm{UDIFWA}_{\lambda(t)}(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \ldots, \tilde{\alpha}(t_p)) = \lambda(t_1)\tilde{\alpha}(t_1) \oplus \lambda(t_2)\tilde{\alpha}(t_2) \oplus \cdots \oplus \lambda(t_p)\tilde{\alpha}(t_p) \tag{65}$$

an uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator, which can be rewritten as follows:

$$\begin{aligned}&\mathrm{UDIFWA}_{\lambda(t)}(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \ldots, \tilde{\alpha}(t_p)) = \left(\left[1 - \prod_{k=1}^p (1 - \tilde{\mu}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}, 1 - \prod_{k=1}^p (1 - \tilde{\mu}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}\right],\right.\\ &\left.\left[\prod_{k=1}^p (\tilde{v}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^p (\tilde{v}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}\right], \left[\prod_{k=1}^p (1 - \tilde{\mu}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^p (\tilde{v}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^p (1 - \tilde{\mu}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^p (\tilde{v}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}\right]\right)\end{aligned} \tag{66}$$

with the condition (13).

Below we consider the DIF-MADM problems under interval uncertainty where all the attribute values are expressed in IVIFNs, which are collected at different periods. The following notations are used to depict the considered problems:

Let X, G, w, and $\lambda(t)$ be presented as in Section 4, and let $\widetilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{n \times m}$ be an uncertain intuitionistic fuzzy decision matrix of the period t_k, where $\tilde{r}_{ij}(t_k) = ([\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)}], [\tilde{v}^L_{\tilde{r}_{ij}(t_k)}, \tilde{v}^U_{\tilde{r}_{ij}(t_k)}], [\tilde{\pi}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\pi}^U_{\tilde{r}_{ij}(t_k)}])$ is an attribute value, denoted by an IVIFN, where $[\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)}]$ indicates the uncertain degree that the alternative x_i should satisfy the attribute G_j at the period t_k, $[\tilde{v}^L_{\tilde{r}_{ij}(t_k)}, \tilde{v}^U_{\tilde{r}_{ij}(t_k)}]$ indicates the uncertain degree that the alternative x_i should not satisfy the attribute G_j at the period t_k, and $[\tilde{\pi}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\pi}^U_{\tilde{r}_{ij}(t_k)}]$ indicates the range of indeterminacy of the alternative x_i to the attribute G_j, such that

$$\begin{aligned}&[\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)}] \in [0,1], \quad [\tilde{v}^L_{\tilde{r}_{ij}(t_k)}, \tilde{v}^U_{\tilde{r}_{ij}(t_k)}] \in [0,1], \quad \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)} + \tilde{v}^U_{\tilde{r}_{ij}(t_k)} \leqslant 1, \quad [\tilde{\pi}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\pi}^U_{\tilde{r}_{ij}(t_k)}]\\ &= [1 - \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)} - \tilde{v}^U_{\tilde{r}_{ij}(t_k)}, 1 - \tilde{\mu}^L_{\tilde{r}_{ij}(t_k)} - \tilde{v}^L_{\tilde{r}_{ij}(t_k)}], \quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m\end{aligned} \tag{67}$$

Similar to Section 4, a procedure for solving the above problems can be described as follows:

Procedure II

Step 1. Utilize the UDIFWA operator:

$$\tilde{r}_{ij} = \mathrm{UDIFWA}_{\lambda(t)}(\tilde{r}_{ij}(t_1), \tilde{r}_{ij}(t_2), \ldots, \tilde{r}_{ij}(t_p)) = \left(\left[1 - \prod_{k=1}^{p} (1 - \tilde{\mu}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)}, 1 - \prod_{k=1}^{p} (1 - \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)} \right], \right.$$
$$\left. \left[\prod_{k=1}^{p} (\tilde{v}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^{p} (\tilde{v}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)} \right], \left[\prod_{k=1}^{p} (1 - \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p} (\tilde{v}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^{p} (1 - \tilde{\mu}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p} (\tilde{v}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)} \right] \right) \tag{68}$$

to aggregate all the uncertain intuitionistic fuzzy decision matrices $\widetilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{n \times m}$ $(k = 1, 2, \ldots, p)$ into a complex uncertain intuitionistic fuzzy decision matrix $\widetilde{R} = (\tilde{r}_{ij})_{n \times m}$, where $\tilde{r}_{ij} = ([\tilde{\mu}^L_{ij}, \tilde{\mu}^U_{ij}], [\tilde{v}^L_{ij}, \tilde{v}^U_{ij}], [\tilde{\pi}^L_{ij}, \tilde{\pi}^U_{ij}]), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m.$

Step 2. Define $\tilde{\alpha}^+ = (\tilde{\alpha}^+_1, \tilde{\alpha}^+_2, \ldots, \tilde{\alpha}^+_m)^{\mathrm{T}}$ and $\tilde{\alpha}^- = (\tilde{\alpha}^-_1, \tilde{\alpha}^-_2, \ldots, \tilde{\alpha}^-_m)^{\mathrm{T}}$ as the uncertain intuitionistic fuzzy ideal solution (UIFIS) and the uncertain intuitionistic fuzzy negative ideal solution (UIFNIS), respectively, where $\tilde{\alpha}^+_i = ([1, 1], [0, 0], [0, 0])(i = 1, 2, \ldots, m)$ are the m largest IVIFNs, and $\tilde{\alpha}^-_i = ([0, 0], [1, 1], [0, 0])(i = 1, 2, \ldots, m)$ are the m smallest IVIFNs. Moreover, we denote the alternatives $x_i (i = 1, 2, \ldots, n)$ by $x_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{im})^{\mathrm{T}}, i = 1, 2, \ldots, n.$

Step 3. Calculate the distance between the alternative x_i and the UIFIS $\tilde{\alpha}^+$ and the distance between the alternative x_i and the UIFNIS $\tilde{\alpha}^-$, respectively:

$$\begin{aligned} d(x_i, \tilde{\alpha}^+) &= \sum_{j=1}^{m} w_j d(\tilde{r}_{ij}, \tilde{\alpha}^+_j) \\ &= \frac{1}{4} \sum_{j=1}^{m} w_j (|\tilde{\mu}^L_{ij} - 1| + |\tilde{\mu}^U_{ij} - 1| + |\tilde{v}^L_{ij} - 0| + |\tilde{v}^U_{ij} - 0| + |\tilde{\pi}^L_{ij} - 0| + |\tilde{\pi}^U_{ij} - 0|) \\ &= \frac{1}{4} \sum_{j=1}^{m} w_j [2 - (\tilde{\mu}^L_{ij} + \tilde{\mu}^U_{ij}) + \tilde{v}^L_{ij} + \tilde{v}^U_{ij} + \tilde{\pi}^L_{ij} + \tilde{\pi}^U_{ij}] \\ &= \frac{1}{4} \sum_{j=1}^{m} w_j [2 - (\tilde{\mu}^L_{ij} + \tilde{\mu}^U_{ij}) + \tilde{v}^L_{ij} + \tilde{v}^U_{ij} + 1 - \tilde{\mu}^U_{ij} - \tilde{v}^U_{ij} + 1 - \tilde{\mu}^L_{ij} - \tilde{v}^L_{ij}] \\ &= \frac{1}{4} \sum_{j=1}^{m} w_j [4 - 2(\tilde{\mu}^L_{ij} + \tilde{\mu}^U_{ij})] \\ &= \frac{1}{2} \sum_{j=1}^{m} w_j [2 - (\tilde{\mu}^L_{ij} + \tilde{\mu}^U_{ij})] \end{aligned} \tag{69}$$

$$\begin{aligned} d(x_i, \tilde{\alpha}^-) &= \sum_{j=1}^{m} w_j d(\tilde{r}_{ij}, \tilde{\alpha}^-_j) \\ &= \frac{1}{4} \sum_{j=1}^{m} w_j (|\tilde{\mu}^L_{ij} - 0| + |\tilde{\mu}^U_{ij} - 0| + |\tilde{v}^L_{ij} - 1| + |\tilde{v}^U_{ij} - 1| + |\tilde{\pi}^L_{ij} - 0| + |\tilde{\pi}^U_{ij} - 0|) \\ &= \frac{1}{4} \sum_{j=1}^{m} w_j [2 + \tilde{\mu}^L_{ij} + \tilde{\mu}^U_{ij} - (\tilde{v}^L_{ij} + \tilde{v}^U_{ij}) + 1 - \tilde{\mu}^U_{ij} - \tilde{v}^U_{ij} + 1 - \tilde{\mu}^L_{ij} - \tilde{v}^L_{ij}] \\ &= \frac{1}{4} \sum_{j=1}^{m} w_j [4 - 2(\tilde{v}^L_{ij} + \tilde{v}^U_{ij})] = \frac{1}{2} \sum_{j=1}^{m} w_j [2 - (\tilde{v}^L_{ij} + \tilde{v}^U_{ij})] \\ &= \frac{1}{2} \sum_{j=1}^{m} w_j [2 - (\tilde{v}^L_{ij} + \tilde{v}^U_{ij})] \end{aligned} \tag{70}$$

where $\tilde{r}_{ij} = ([\tilde{\mu}^L_{ij}, \tilde{\mu}^U_{ij}], [\tilde{v}^L_{ij}, \tilde{v}^U_{ij}], [\tilde{\pi}^L_{ij}, \tilde{\pi}^U_{ij}]), \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m.$

Step 4. Calculate the closeness coefficient of each alternative:

$$c(x_i) = \frac{d(x_i, \tilde{\alpha}^-)}{d(x_i, \tilde{\alpha}^+) + d(x_i, \tilde{\alpha}^-)}, \quad i = 1, 2, \ldots, n \tag{71}$$

Since

$$\begin{aligned} d(x_i, \tilde{\alpha}^+) + d(x_i, \tilde{\alpha}^-) &= \frac{1}{2}\sum_{j=1}^m w_j[2 - (\tilde{\mu}_{ij}^L + \tilde{\mu}_{ij}^U)] + \frac{1}{2}\sum_{j=1}^m w_j[2 - (\tilde{v}_{ij}^L + \tilde{v}_{ij}^U)] \\ &= \frac{1}{2}\sum_{j=1}^m w_j[2 - (\tilde{\mu}_{ij}^L + \tilde{\mu}_{ij}^U) - (\tilde{v}_{ij}^L + \tilde{v}_{ij}^U)] = \frac{1}{2}\sum_{j=1}^m w_j[4 - (\tilde{\mu}_{ij}^L + \tilde{\mu}_{ij}^U) - (\tilde{v}_{ij}^L + \tilde{v}_{ij}^U)] \\ &= \frac{1}{2}\sum_{j=1}^m w_j[2 + (\tilde{\pi}_{ij}^L + \tilde{\pi}_{ij}^U)] \end{aligned} \tag{72}$$

then, (71) can be rewritten as

$$c(x_i) = \frac{\sum_{j=1}^m w_j[2 - (\tilde{v}_{ij}^L + \tilde{v}_{ij}^U)]}{\sum_{j=1}^m w_j[2 + (\tilde{\pi}_{ij}^L + \tilde{\pi}_{ij}^U)]}, \quad i = 1, 2, \ldots, n \tag{73}$$

Step 5. Rank all the alternatives $x_i (i = 1, 2, \ldots, n)$ according to the closeness coefficients $c(x_i) (i = 1, 2, \ldots, n)$, the greater the value $c(x_i)$, the better the alternative x_i.

Step 6. End.

6. Case illustration

The following practical case was adapted from [23]. Located in Central China and the middle reaches of the Changjiang (Yangtze) River, Hubei Province is distributed in a transitional belt where physical conditions and landscapes are on the transition from north to south and from east to west. Thus, Hubei Province is well known as "a land of rice and fish" since the region enjoys some of the favorable physical conditions, with a diversity of natural resources and the suitability for growing various crops. At the same time, however, there are also some restrictive factors for developing agriculture such as a tight man–land relation between, a constant degradation of natural resources and a growing population pressure on land resource reserve. Despite cherishing a burning desire to promote their standard of living, people living in the area are frustrated because they have no ability to enhance their power to accelerate economic development because of a dramatic decline in quantity and quality of natural resources and a deteriorating environment. Based on the distinctness and differences in environment and natural resources, Hubei Province can be roughly divided into seven agroecological regions: x_1 – Wuhan–Ezhou–Huanggang; x_2 – Northeast of Hubei; x_3 – Southeast of Hubei; x_4 – Jianghan region; x_5 – North of Hubei; x_6 – Northwest of Hubei; x_7 – Southwest of Hubei. In order to prioritize these agroecological regions $x_i (j = 1, 2, \ldots, 7)$ with respect to their comprehensive functions, a committee has been set up to provide assessment information on x_i $(i = 1, 2, \ldots, 7)$. The attributes which are considered here in assessment of $x_i (i = 1, 2, \ldots, 7)$ are: (1) G_1 is ecological benefit; (2) G_2 is economic benefit; and (3) G_3 is social benefit. The committee evaluates the performance of agroecological regions x_i $(i = 1, 2, \ldots, 7)$ in the years 2004–2006 according to the attributes $G_j (j = 1, 2, 3)$, and constructs, respectively, the intuitionistic fuzzy decision matrices $R(t_k) (k = 1, 2, 3$, here, t_1 denotes the year "2004", t_2 denotes the year "2005", and t_3 denotes the year "2006") as listed in Tables 1–3. Let $\lambda(t) = (1/6, 2/6, 3/6)^{\mathrm{T}}$ be the weight vector of the years $t_k (k = 1, 2, 3)$, and $w = (0.3, 0.4, 0.3)^{\mathrm{T}}$ be the weight vector of the attributes $G_j (j = 1, 2, 3)$.

Now we utilize the proposed procedure I to prioritize these agroecological regions:

Step 1. Utilize the DIFWA operator (53) to aggregate all the intuitionistic fuzzy decision matrices $R(t_k)$ into a complex intuitionistic fuzzy decision matrix R (see Table 4).

Step 2. Denote the IFIS α^+, IFNIS α^-, and the alternatives $x_i (i = 1, 2, \ldots, 7)$ by

$$\alpha^+ = ((1,0,0),(1,0,0),(1,0,0))^{\mathrm{T}}, \quad \alpha^- = ((0,1,0),(0,1,0),(0,1,0))^{\mathrm{T}}$$
$$x_1 = ((0.806, 0.100, 0.094),(0.874, 0.126, 0.000),(0.849, 0.112, 0.039))^{\mathrm{T}}$$

$$x_2 = ((0.849, 0.151, 0.000), (0.569, 0.159, 0.272), (0.594, 0.214, 0.192))^{\mathrm{T}}$$
$$x_3 = ((0.452, 0.482, 0.066), (0.755, 0.151, 0.094), (0.725, 0.126, 0.149))^{\mathrm{T}}$$
$$x_4 = ((0.859, 0.100, 0.041), (0.792, 0.141, 0.067), (0.838, 0.162, 0.000))^{\mathrm{T}}$$
$$x_5 = ((0.569, 0.218, 0.213), (0.748, 0.229, 0.023), (0.640, 0.178, 0.182))^{\mathrm{T}}$$
$$x_6 = ((0.289, 0.648, 0.063), (0.441, 0.200, 0.359), (0.390, 0.224, 0.386))^{\mathrm{T}}$$
$$x_7 = ((0.387, 0.470, 0.143), (0.601, 0.337, 0.062), (0.536, 0.464, 0.000))^{\mathrm{T}}$$

Table 1
Intuitionistic fuzzy decision matrix $R(t_1)$

	G_1	G_2	G_3
x_1	(0.8, 0.1, 0.1)	(0.9, 0.1, 0.0)	(0.7, 0.2, 0.1)
x_2	(0.7, 0.3, 0.0)	(0.6, 0.2, 0.2)	(0.6, 0.3, 0.1)
x_3	(0.5, 0.4, 0.1)	(0.7, 0.3, 0.0)	(0.6, 0.1, 0.3)
x_4	(0.9, 0.1, 0.0)	(0.7, 0.1, 0.2)	(0.8, 0.2, 0.0)
x_5	(0.6, 0.1, 0.3)	(0.8, 0.2, 0.0)	(0.5, 0.1, 0.4)
x_6	(0.3, 0.6, 0.1)	(0.5, 0.4, 0.1)	(0.4, 0.5, 0.1)
x_7	(0.5, 0.2, 0.3)	(0.4, 0.6, 0.0)	(0.5, 0.5, 0.0)

Table 2
Intuitionistic fuzzy decision matrix $R(t_2)$

	G_1	G_2	G_3
x_1	(0.9, 0.1, 0.0)	(0.8, 0.2, 0.0)	(0.8, 0.1, 0.1)
x_2	(0.8, 0.2, 0.0)	(0.5, 0.1, 0.4)	(0.7, 0.2, 0.1)
x_3	(0.5, 0.5, 0.0)	(0.7, 0.2, 0.1)	(0.8, 0.2, 0.0)
x_4	(0.9, 0.1, 0.0)	(0.9, 0.1, 0.0)	(0.7, 0.3, 0.0)
x_5	(0.5, 0.2, 0.3)	(0.6, 0.3, 0.1)	(0.6, 0.2, 0.2)
x_6	(0.4, 0.6, 0.0)	(0.3, 0.4, 0.3)	(0.5, 0.5, 0.0)
x_7	(0.3, 0.5, 0.2)	(0.5, 0.3, 0.2)	(0.6, 0.4, 0.0)

Table 3
Intuitionistic fuzzy decision matrix $R(t_3)$

	G_1	G_2	G_3
x_1	(0.7, 0.1, 0.2)	(0.9, 0.1, 0.0)	(0.9, 0.1, 0.0)
x_2	(0.9, 0.1, 0.0)	(0.6, 0.2, 0.2)	(0.5, 0.2, 0.3)
x_3	(0.4, 0.5, 0.1)	(0.8, 0.1, 0.1)	(0.7, 0.1, 0.2)
x_4	(0.8, 0.1, 0.1)	(0.7, 0.2, 0.1)	(0.9, 0.1, 0.0)
x_5	(0.6, 0.3, 0.1)	(0.8, 0.2, 0.0)	(0.7, 0.2, 0.1)
x_6	(0.2, 0.7, 0.1)	(0.5, 0.1, 0.4)	(0.3, 0.1, 0.6)
x_7	(0.4, 0.6, 0.0)	(0.7, 0.3, 0.0)	(0.5, 0.5, 0.0)

Table 4
Complex intuitionistic fuzzy decision matrix R

	G_1	G_2	G_3
x_1	(0.806, 0.100, 0.094)	(0.874, 0.126, 0.000)	(0.849, 0.112, 0.039)
x_2	(0.849, 0.151, 0.000)	(0.569, 0.159, 0.272)	(0.594, 0.214, 0.192)
x_3	(0.452, 0.482, 0.066)	(0.755, 0.151, 0.094)	(0.725, 0.126, 0.149)
x_4	(0.859, 0.100, 0.041)	(0.792, 0.141, 0.067)	(0.838, 0.162, 0.000)
x_5	(0.569, 0.218, 0.213)	(0.748, 0.229, 0.023)	(0.640, 0.178, 0.182)
x_6	(0.289, 0.648, 0.063)	(0.441, 0.200, 0.359)	(0.390, 0.224, 0.383)
x_7	(0.387, 0.470, 0.143)	(0.601, 0.337, 0.062)	(0.536, 0.464, 0.000)

and utilize (58) to calculate the closeness coefficient of each alternative:

$$c(x_1) = 0.852, \quad c(x_2) = 0.709, \quad c(x_3) = 0.687, \quad c(x_4) = 0.833, \quad c(x_5) = 0.700,$$
$$c(x_6) = 0.515, \quad c(x_7) = 0.548$$

Step 3. Rank all the alternatives $x_i (i = 1, 2, \ldots, 7)$ according to the closeness coefficients $c(x_i)(i = 1, 2, \ldots, 7)$:

$$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3 \succ x_7 \succ x_6$$

and thus the agroecological region with the most comprehensive functions is Wuhan–Ezhou–Huanggang.

If the committee evaluates the performance of agroecological regions $x_i (i = 1, 2, \ldots, 7)$ in the years 2004–2006 according to the attributes $G_j (j = 1, 2, 3)$, and constructs, respectively, the uncertain intuitionistic fuzzy decision matrices $\widetilde{R}(t_k)(k = 1, 2, 3)$ as listed in Tables 5–7.

In such case, we can utilize the proposed procedure II to prioritize these agroecological regions.

To do so, we first utilize the UDIFWA operator (64) to aggregate all the uncertain intuitionistic fuzzy decision matrices $\widetilde{R}(t_k)$ into a complex uncertain intuitionistic fuzzy decision matrix $\widetilde{R}$ (see Table 8): and then denote the UIFIS α^+, UIFNIS α^-, and the alternatives $x_i (i = 1, 2, \ldots, 7)$ by

Table 5
Uncertain intuitionistic fuzzy decision matrix $\widetilde{R}(t_1)$

	G_1	G_2	G_3
x_1	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.6, 0.8], [0.0, 0.2], [0.0, 0.4])
x_2	([0.6, 0.7], [0.2, 0.3], [0.0, 0.2])	([0.5, 0.7], [0.2, 0.3], [0.0, 0.3])	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])
x_3	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])	([0.4, 0.6], [0.1, 0.2], [0.2, 0.5])
x_4	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.6, 0.8], [0.0, 0.1], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])
x_5	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])
x_6	([0.2, 0.3], [0.5, 0.6], [0.1, 0.3])	([0.3, 0.5], [0.4, 0.5], [0.0, 0.3])	([0.4, 0.6], [0.3, 0.4], [0.0, 0.3])
x_7	([0.4, 0.5], [0.3, 0.4], [0.1, 0.3])	([0.2, 0.5], [0.3, 0.5], [0.0, 0.5])	([0.4, 0.7], [0.2, 0.3], [0.0, 0.4])

Table 6
Uncertain intuitionistic fuzzy decision matrix $\widetilde{R}(t_2)$

	G_1	G_2	G_3
x_1	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.7, 0.9], [0.0, 0.1], [0.0, 0.3])
x_2	([0.5, 0.7], [0.1, 0.2], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])
x_3	([0.3, 0.5], [0.1, 0.3], [0.2, 0.6])	([0.4, 0.5], [0.1, 0.3], [0.2, 0.5])	([0.3, 0.6], [0.3, 0.4], [0.0, 0.4])
x_4	([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])
x_5	([0.5, 0.7], [0.2, 0.3], [0.0, 0.3])	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])	([0.4, 0.6], [0.2, 0.3], [0.1, 0.4])
x_6	([0.3, 0.4], [0.4, 0.6], [0.0, 0.3])	([0.2, 0.4], [0.5, 0.6], [0.0, 0.3])	([0.4, 0.5], [0.4, 0.5], [0.0, 0.2])
x_7	([0.3, 0.5], [0.3, 0.5], [0.0, 0.4])	([0.4, 0.6], [0.3, 0.4], [0.0, 0.3])	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])

Table 7
Uncertain intuitionistic fuzzy decision matrix $\widetilde{R}(t_3)$

	G_1	G_2	G_3
x_1	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])	([0.7, 0.9], [0.0, 0.1], [0.0, 0.3])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])
x_2	([0.4, 0.6], [0.1, 0.2], [0.2, 0.5])	([0.5, 0.7], [0.1, 0.2], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])
x_3	([0.2, 0.4], [0.2, 0.3], [0.3, 0.6])	([0.3, 0.6], [0.2, 0.3], [0.1, 0.5])	([0.4, 0.6], [0.2, 0.4], [0.0, 0.4])
x_4	([0.7, 0.8], [0.0, 0.1], [0.1, 0.3])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.4, 0.7], [0.2, 0.3], [0.0, 0.4])
x_5	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])	([0.4, 0.5], [0.1, 0.2], [0.3, 0.5])	([0.6, 0.7], [0.2, 0.3], [0.0, 0.2])
x_6	([0.2, 0.3], [0.5, 0.6], [0.1, 0.3])	([0.3, 0.5], [0.3, 0.4], [0.1, 0.4])	([0.3, 0.6], [0.2, 0.4], [0.0, 0.5])
x_7	([0.5, 0.6], [0.3, 0.4], [0.0, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.2, 0.4])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])

Table 8
Complex uncertain intuitionistic fuzzy decision matrix $\widetilde{R}$

	G_1	G_2	G_3
x_1	([0.676, 0.782], [0, 0.218], [0.000, 0.324])	([0.738, 0.888], [0, 0.112], [0.000, 0.262])	([0.743, 0.888], [0, 0.112], [0.000, 0.257])
x_2	([0.472, 0.654], [0.112, 0.214], [0.132, 0.416])	([0.536, 0.700], [0.112, 0.245], [0.055, 0352])	([0.525, 0.627], [0.141, 0.330], [0.043, 0.334])
x_3	([0.271, 0.452], [0.159, 0.315], [0.233, 0.570])	([0.371, 0.569], [0.159, 0.300], [0.131, 0.470])	([0.368, 0.600], [0.204, 0.356], [0.044, 0.428])
x_4	([0.670, 0.771], [0, 0.141], [0.088, 0.330])	([0.743, 0.859], [0, 0.126], [0.015, 0.257])	([0.472, 0.700], [0.141, 0.280], [0.020, 0.387])
x_5	([0.500, 0.654], [0.178, 0.300], [0.046, 0.322])	([0.497, 0.638], [0.100, 0.229], [0.333, 0.403])	([0.510, 0.640], [0.200, 0.315], [0.045, 0.290])
x_6	([0.235, 0.335], [0.464, 0.600], [0.065, 0.301])	([0.268, 0.469], [0.373, 0.475], [0.056, 0.359])	([0.352, 0.569], [0.270, 0.431], [0.000, 0.378])
x_7	([0.423, 0.553], [0.300, 0.431], [0.016, 0.277])	([0.273, 0.450], [0.346, 0.464], [0.086, 0.381])	([0.576, 0.710], [0.141, 0.270], [0.020, 0.283])

$$\tilde{\alpha}^+ = (([1,1],[0,0],[0,0]),([1,1],[0,0],[0,0]),([1,1],[0,0],[0,0]))^{\mathrm{T}}$$

$$\tilde{\alpha}^- = (([0,0],[1,1],[0,0]),([0,0],[1,1],[0,0]),([0,0],[1,1],[0,0]))^{\mathrm{T}}$$

$$x_1 = (([0.676,0.782],[0.000,0.218],[0.000,0.324]),([0.738,0.888],[0.000,0.112],[0.000,0.262]),\\ ([0.743,0.888],[0.000,0.112],[0.000,0.257]))^{\mathrm{T}}$$

$$x_2 = (([0.472,0.654],[0.112,0.214],[0.132,0.416]),([0.536,0.700],[0.112,0.245],[0.055,0.352]),\\ ([0.525,0.627],[0.141,0.330],[0.043,0.334]))^{\mathrm{T}}$$

$$x_3 = (([0.271,0.452],[0.159,0.315],[0.233,0.570]),([0.371,0.569],[0.159,0.300],[0.131,0.470]),\\ ([0.368,0.600],[0.204,0.356],[0.044,0.428]))^{\mathrm{T}}$$

$$x_4 = (([0.670,0.771],[0.000,0.141],[0.088,0.330]),([0.743,0.859],[0.000,0.126],[0.015,0.257]),\\ ([0.472,0.700],[0.141,0.280],[0.020,0.387]))^{\mathrm{T}}$$

$$x_5 = (([0.500,0.654],[0.178,0.300],[0.046,0.322]),([0.497,0.638],[0.100,0.229],[0.333,0.403]),\\ ([0.510,0.640],[0.200,0.315],[0.045,0.290]))^{\mathrm{T}}$$

$$x_6 = (([0.235,0.335],[0.464,0.600],[0.065,0.301]),([0.268,0.469],[0.373,0.475],[0.056,0.359]),\\ ([0.352,0.569],[0.270,0.431],[0.000,0.378]))^{\mathrm{T}}$$

$$x_7 = (([0.423,0.553],[0.300,0.431],[0.016,0.277]),([0.273,0.450],[0.346,0.464],[0.086,0.381]),\\ ([0.576,0.710],[0.141,0.270],[0.020,0.283]))^{\mathrm{T}}$$

By (73), we calculate the closeness coefficient of each alternative as follows:

$$c(x_1) = 0.814,\quad c(x_2) = 0.663,\quad c(x_3) = 0.574,\quad c(x_4) = 0.794,\quad c(x_5) = 0.627,\\ c(x_6) = 0.474,\quad c(x_7) = 0.564$$

and rank all the alternatives $x_i (i = 1, 2, \ldots, 7)$ according to the values $c(x_i)(i = 1, 2, \ldots, 7)$:

$$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3 \succ x_7 \succ x_6$$

thus the best alternative is also x_1 (Wuhan–Ezhou–Huanggang).

7. Concluding remarks

In this paper, we have focused on the dynamic intuitionistic fuzzy multi-attribute decision making (DIF-MADM) problems, which occur in many decision areas, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation. Some aggregation operators such as the dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator have been proposed to aggregate dynamic or uncertain dynamic intuitionistic fuzzy information. We have utilized some well known functions including the basic unit-interval monotonic (BUM) function, normal distribution function exponential distribution function, and a mathematical programming model to determine the weights associated with these two operators. In the process of aggregating information, these operators can avoid losing the original intuitionistic fuzzy information and thus ensure the veracity and rationality of the aggregated results. Moreover, based on the DIFWA and UDIFWA operators respectively, we have developed two procedures for solving the DIF-MADM problems where all the attribute values are expressed in intuitionistic fuzzy numbers or interval-valued intuitionistic fuzzy numbers. In the procedures, we have extended the technique for order performance by similarity to ideal solution (TOPSIS) to intuitionistic fuzzy environment, and used the extended TOPSIS to rank and select the optimal alternative. To verify the effectiveness and practicality of the developed procedures, we have applied them to prioritize a set of agroecological regions in Hubei Province, China.

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