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Existence of nonoscillatory solutions to neutral dynamic equations on time scales ☆

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Abstract

In this paper, we give an analogue of the Arzela–Ascoli theorem on time scales. Then, we establish the existence of nonoscillatory solutions to the neutral dynamic equation $[x(t) + p(t)x(g(t))]^{\Delta} + f(t, x(h(t))) = 0$ on a time scale. To dwell upon the importance of our results, three interesting examples are also included.

Keywords: Neutral dynamic equations; Time scales; Nonoscillatory solutions; Existence

1. Introduction

Consider neutral functional dynamic equations of the form

$$\left[x(t) + p(t)x\left(g(t)\right)\right]^{\Delta} + f\left(t, x\left(h(t)\right)\right) = 0 \tag{1}$$

on a time scale $\mathbb{T}$. The motivation originates from Mathsen et al. [8], where some open problems were presented and one of them is under what conditions there will exist positive solutions to

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equation

$$\big[x(t)+p(t)x\big(g(t)\big)\big]^{\Delta}+q(t)x\big(h(t)\big)=0 \tag{2}$$

on a time scale. In this paper, we try to solve this problem and find some conditions for the existence of nonoscillatory solutions of (1). We remark that there have been a number of literatures to study the oscillatory behaviors for dynamic equations on time scales, see, e.g., Refs. [1–3,5–10]. However, there are few papers to discuss the existence of nonoscillatory solutions for neutral functional dynamic equations on time scales.

For convenience, we recall some concepts related to time scales. More details can be found in [1,2].

Definition 1. A time scale is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers with the topology and ordering inherited from $\mathbb{R}$. Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$ the forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T}\colon\ s > t\}$, the backward jump operator by $\rho(t) := \sup\{s \in \mathbb{T}\colon\ s < t\}$, and the graininess function by $\mu(t) := \sigma(t) - t$, where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered; otherwise, it is left-dense. The set $\mathbb{T}^{\kappa}$ is defined as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 2. For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, we define the delta-derivative $f^{\Delta}(t)$ of $f(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some δ) such that

$$\big|\big[f\big(\sigma(t)\big) - f(s)\big] - f^{\Delta}(t)\big[\sigma(t) - s\big]\big| \leqslant \varepsilon\big|\sigma(t) - s\big| \quad \text{for all } s \in U.$$

We say that f is delta-differentiable (or in short: differentiable) on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

It is easily seen that if f is continuous at $t \in \mathbb{T}$ and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Moreover, if t is right-dense then f is differential at t iff the limit

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

In addition, if $f^{\Delta} \geqslant 0$, then f is nondecreasing.

Definition 3. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, f is called right-dense continuous (rd-continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of f provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{k}$. By the antiderivative, the Cauchy integral of f is defined as $\int_a^b f(s)\,\Delta s = F(b) - F(a)$, and $\int_a^{\infty} f(s)\,\Delta s = \lim_{t\to\infty} \int_a^t f(s)\,\Delta s$.

Let $C_{rd}(\mathbb{T}, \mathbb{R})$ denote the set of all rd-continuous functions mapping $\mathbb{T}$ to $\mathbb{R}$. It is shown in [2] that every rd-continuous function has an antiderivative. Since we are interested in the nonoscillatory behavior of (1), we assume throughout that the time scale $\mathbb{T}$ under consideration satisfies $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$.

As usual, by a solution of (1) we mean a continuous function $x(t)$ which is defined on $\mathbb{T}$ and satisfies (1) for $t \geqslant t_1 \geqslant t_0$. A solution x of (1) is said to be eventually positive (or eventually negative) if there exists $c \in \mathbb{T}$ such that $x(t) > 0$ (or $x(t) < 0$) for all $t \geqslant c$ in $\mathbb{T}$. A solution x of (1) is said to be nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is oscillatory.

2. Preliminaries

For $T_0, T_1 \in \mathbb{T}$, let $[T_0, \infty)_{\mathbb{T}} := \{t \in \mathbb{T}\colon t \geqslant T_0\}$ and $[T_0, T_1]_{\mathbb{T}} := \{t \in \mathbb{T}\colon T_0 \leqslant t \leqslant T_1\}$. Further, let $C([T_0, \infty)_{\mathbb{T}}, \mathbb{R})$ denote all continuous functions mapping $[T_0, \infty)_{\mathbb{T}}$ into $\mathbb{R}$, and

$$BC[T_0, \infty)_{\mathbb{T}} := \Big\{x\colon x \in C\big([T_0, \infty)_{\mathbb{T}}, \mathbb{R}\big) \text{ and } \sup_{t \in [T_0, \infty)_{\mathbb{T}}} \big|x(t)\big| < \infty\Big\}. \tag{3}$$

Endowed on $BC[T_0, \infty)_{\mathbb{T}}$ with the norm $\|x\| = \sup_{t \in [T_0, \infty)_{\mathbb{T}}} |x(t)|$, $(BC[T_0, \infty)_{\mathbb{T}}, \|\cdot\|)$ is a Banach space. Let $X \subseteq BC[T_0, \infty)_{\mathbb{T}}$, we say X is uniformly Cauchy if for any given $\varepsilon > 0$, there exists $T_1 \in [T_0, \infty)_{\mathbb{T}}$ such that for any $x \in X$,

$$\big|x(t_1) - x(t_2)\big| < \varepsilon \quad \text{for all } t_1, t_2 \in [T_1, \infty)_{\mathbb{T}}.$$

X is said to be equi-continuous on $[a, b]_{\mathbb{T}}$ if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in X$ and $t_1, t_2 \in [a, b]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$,

$$\big|x(t_1) - x(t_2)\big| < \varepsilon.$$

The following is an analogue of the Arzela–Ascoli theorem on time scales.

Lemma 4. *Suppose that $X \subseteq BC[T_0, \infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[T_0, T_1]_{\mathbb{T}}$ for any $T_1 \in [T_0, \infty)_{\mathbb{T}}$. Then X is relatively compact.*

Proof. By the assumption of uniformly Cauchy, we see that for any $\varepsilon > 0$, there exists $T_1 \in [T_0, \infty)_{\mathbb{T}}$ such that for any $x \in X$,

$$\big|x(t_1) - x(t_2)\big| < \frac{\varepsilon}{3}, \quad t_1, t_2 \in [T_1, \infty)_{\mathbb{T}}. \tag{4}$$

Moreover, there exists $\alpha > 0$ such that $\|x\| \leqslant \alpha$ for all $x \in X$. Choose $N_1 + 1$ real numbers y_i $(i = 0, 1, 2, \ldots, N_1)$ so that $-\alpha = y_0 < y_1 < \cdots < y_{N_1} = \alpha$ and

$$|y_{i+1} - y_i| < \frac{\varepsilon}{3}, \quad 0 \leqslant i \leqslant N_1 - 1. \tag{5}$$

By the assumption of equi-continuity on $[T_0, T_1]_{\mathbb{T}}$, we see that for the above $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in X$,

$$\big|x(s) - x(t)\big| < \frac{\varepsilon}{3} \quad \text{for } |s - t| \leqslant \delta,\ s, t \in [T_0, T_1]_{\mathbb{T}}. \tag{6}$$

Note that we can insert N_2 numbers into the interval $[T_0, T_1]$ of $\mathbb{R}$ so that $T_0 = t_1 < t_2 < \cdots < t_{N_2 - 1} < t_{N_2} = T_1$ and

$$|t_{i+1} - t_i| \leqslant \delta, \quad 1 \leqslant i \leqslant N_2 - 1. \tag{7}$$

Now, we construct a continuous function class $\mathcal{U}$ on the interval $[T_0, T_1]$. For each $i \in \{1, 2, \dots, N_2 - 1\}$ and $j \in \{0, 1, \dots, N_1 - 1\}$, we define a function $u_{ij}(t)$ on $[t_i, t_{i+1}] \subset [T_0, T_1]$ to figure one of two diagonals of the rectangle domain: $t_i \leqslant t \leqslant t_{i+1}$ and $y_j \leqslant y \leqslant y_{j+1}$ as follows. That is,

$$u_{ij}(t) = y_j + \frac{y_{j+1} - y_j}{t_{i+1} - t_i}(t - t_i), \quad t \in [t_i, t_{i+1}],$$

or

$$u_{ij}(t) = y_{j+1} + \frac{y_j - y_{j+1}}{t_{i+1} - t_i}(t - t_i), \quad t \in [t_i, t_{i+1}].$$

Let $\mathcal{U}$ be the set of all possible continuous functions on $[T_0, T_1] = \bigcup_{i=1}^{N_2-1}[t_i, t_{i+1}]$ connecting such $u_{ij}(t)$ as above from $[t_1, t_2]$ to $[t_{N_2-1}, t_{N_2}]$. It is clear that $\mathcal{U}$ is finite. For each $u(t) \in \mathcal{U}$, we define a function $\bar{u}(t)$ on $[T_0, \infty)_{\mathbb{T}}$ by

$$\bar{u}(t) = \begin{cases} u(t), & t \in [T_0, T_1]_{\mathbb{T}}, \\ u(T_1), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Let L be the set of all possible functions $\bar{u}(t)$ defined as above, then L is finite. We claim that L is a finite ε-net for X. In fact, in light of (5), (6) and the definition of $\bar{u}(t)$, for any $x \in X$, we can choose $\bar{u}(t) \in L$ such that

$$\left|\bar{u}(t) - x(t)\right| < \frac{\varepsilon}{3}, \quad t \in [T_0, T_1]_{\mathbb{T}}. \tag{8}$$

When $t \in [T_1, \infty)_{\mathbb{T}}$, from (4) and (8) we have

$$\left|\bar{u}(t) - x(t)\right| = \left|u(T_1) - x(t)\right| \leqslant \left|x(T_1) - x(t)\right| + \left|u(T_1) - x(T_1)\right| < \frac{2\varepsilon}{3}. \tag{9}$$

From (8) and (9), we see that

$$\|\bar{u} - x\| = \sup_{t \in [T_0, \infty)_{\mathbb{T}}} \left|\bar{u}(t) - x(t)\right| \leqslant \frac{2\varepsilon}{3}.$$

It follows that L is a finite ε-net for X. Thus, X is relatively compact. The proof is complete. □

In next section, we will employ Kranoselskii's fixed point theorem (see [4]) to establish the existence of nonoscillatory solutions for (1). For the sake of convenience, we state here this theorem as follows.

Lemma 5. *Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Suppose further that there exist two operators $U, S : X \to \Omega$ such that*

(i) *$Ux + Sy \in X$ for all $x, y \in X$;*
(ii) *U is a contraction mapping;*
(iii) *S is completely continuous.*

Then $U + S$ has a fixed point in X.

It is obvious that the conclusion of Lemma 5 holds when the operator $U = 0$. Hence we have

Corollary 6. *Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Suppose further that there exists an operator $S : X \to \Omega$ such that*

(i) *$Sx \in X$ for all $x \in X$;*
(ii) *S is completely continuous.*

Then S has a fixed point in X.

3. Main results

Throughout this section, we will assume in (1) that

(H1) $g, h \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $g(t) \leqslant t$, $\lim_{t\to\infty} g(t) = \infty$, $\lim_{t\to\infty} h(t) = \infty$, and there exists $\{c_k\}_{k\geqslant 0}$ such that $\lim_{k\to\infty} c_k = \infty$ and $g(c_{k+1}) = c_k$.
(H2) $p \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and there exists a constant p_0 with $|p_0| < 1$ such that $\lim_{t\to\infty} p(t) = p_0$.
(H3) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $f(t, x)$ is nondecreasing in x and $xf(t, x) > 0$ for $t \in \mathbb{T}$ and $x \neq 0$.

We note by the assumptions above that if $x(t)$ is an eventually negative solution of (1), then $y(t) = -x(t)$ satisfies

$$\bigl[y(t) + p(t)y\bigl(g(t)\bigr)\bigr]^{\Delta} - f\bigl(t, -y\bigl(h(t)\bigr)\bigr) = 0.$$

We further note that $\bar{f}(t, u) := -f(t, -u)$ is nondecreasing in the second variable and $u\bar{f}(t, u) > 0$ for $t \in \mathbb{T}$ and $u \neq 0$. Hence, in the following we will restrict our attentions to eventually positive solutions of (1).

In the sequel, we use the notation

$$z(t) = x(t) + p(t)x\bigl(g(t)\bigr). \tag{10}$$

Theorem 7. *If $x(t)$ is an eventually positive solution of* (1)*, then either* $\lim_{t\to\infty} x(t) = a > 0$ *or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is an eventually positive solution of (1). In view of the conditions (H1) and (H2), there exist $T_1 \in \mathbb{T}$ and $|p_0| \leqslant p_1 < 1$ such that $x(h(t)) > 0, x(g(t)) > 0$ and $|p(t)| \leqslant p_1$ for all $t \in [T_1, \infty)_{\mathbb{T}}$. Then, from (1) we have $z^{\Delta}(t) < 0$ on $[T_1, \infty)_{\mathbb{T}}$, which means that $z(t)$ is decreasing on $[T_1, \infty)_{\mathbb{T}}$.

We claim that $z(t) \geqslant 0$ eventually. Otherwise, $\lim_{t\to\infty} z(t) < 0$ or $\lim_{t\to\infty} z(t) = -\infty$, which implies that there exists $T_2 \geqslant T_1$ such that

$$x(t) < -p(t)x\bigl(g(t)\bigr) < p_1 x\bigl(g(t)\bigr) \quad \text{for } t \in [T_2, \infty)_{\mathbb{T}}.$$

By (H1), we can choose some positive integer k_0 such that $c_k \geqslant T_2$ for all $k \geqslant k_0$. Then for any $k \geqslant k_0 + 1$, we have

$$\begin{aligned} x(c_k) &< p_1 x\bigl(g(c_k)\bigr) = p_1 x(c_{k-1}) < p_1^2 x\bigl(g(c_{k-1})\bigr) = p_1^2 x(c_{k-2}) \\ &< \cdots < p_1^{k-k_0} x\bigl(g(c_{k_0+1})\bigr) = p_1^{k-k_0} x(c_{k_0}). \end{aligned}$$

The inequality above implies that $\lim_{k\to\infty} x(c_k) = 0$. It follows from (10) that $\lim_{k\to\infty} z(c_k) = 0$ and then contradicts $\lim_{t\to\infty} z(t) < 0$ or $\lim_{t\to\infty} z(t) = -\infty$.

Now we have that $\lim_{t\to\infty} z(t) = b \geqslant 0$, where b is finite. We assert that $x(t)$ is bounded. If it is not true, there exists $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ such that

$$x(t_k) = \max_{t_0 \leqslant s \leqslant t_k} x(s) \quad \text{and} \quad \lim_{k\to\infty} x(t_k) = \infty.$$

Since $g(t) \leqslant t$ and

$$z(t_k) = x(t_k) + p(t_k)x\big(g(t_k)\big) \geqslant \big(1 - \big|p(t_k)\big|\big)x(t_k),$$

it follows from (H2) that $\lim_{k\to\infty} z(t_k) = \infty$, which contradicts the conclusion that $\lim_{t\to\infty} z(t) = b \geqslant 0$ and b is finite. Hence, $x(t)$ is bounded.

Next, we assume that

$$\limsup_{t\to\infty} x(t) = \bar{x}, \qquad \liminf_{t\to\infty} x(t) = \underline{x}.$$

If $0 \leqslant p_0 < 1$, we have

$$b \geqslant \bar{x} + p_0\underline{x} \quad \text{and} \quad b \leqslant \underline{x} + p_0\bar{x},$$

which implies that $\bar{x} \leqslant \underline{x}$. Thus $\bar{x} = \underline{x}$ when $0 \leqslant p_0 < 1$. If $-1 < p_0 < 0$, we have

$$b \geqslant \bar{x} + p_0\bar{x} \quad \text{and} \quad b \leqslant \underline{x} + p_0\underline{x},$$

which implies that $\bar{x} \leqslant \underline{x}$. Thus $\bar{x} = \underline{x}$ when $-1 < p_0 < 0$.

To sum up, we see that $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = b/(1 + p_0)$. The proof is complete. □

We remark that Theorem 7 gives a classification scheme for the eventually positive solutions of (1). Next we will give the existence criteria for each type of solutions.

Theorem 8. *Equation* (1) *has an eventually positive solution* $x(t)$ *with* $\lim_{t\to\infty} x(t) = a > 0$ *if and only if there exits some constant* $K > 0$ *such that*

$$\int_{t_0}^{\infty} f(s, K)\,\Delta s < \infty. \tag{11}$$

Proof. Suppose that $x(t)$ is an eventually positive solution of (1) satisfying $\lim_{t\to\infty} x(t) = a > 0$, then $\lim_{t\to\infty} z(t) = (1 + p_0)a$ and there exists $T_1 \in \mathbb{T}$ such that $x(h(t)) \geqslant a/2$ for $t \in [T_1, \infty)_{\mathbb{T}}$. From (1), we obtain that

$$z(t) - z(T_1) = -\int_{T_1}^{t} f\big(s, x\big(h(s)\big)\big)\,\Delta s, \tag{12}$$

which implies that

$$\int_{T_1}^{\infty} f\big(s, x\big(h(s)\big)\big)\,\Delta s < \infty. \tag{13}$$

In view of (H3) and (13), we see that $\int_{T_1}^{\infty} f(s, a/2)\,\Delta s < \infty$ and then (11) holds.

Conversely, suppose that there exits some constant $K > 0$ such that (11) holds. There will be two cases to be considered: $0 \leqslant p_0 < 1$ and $-1 < p_0 < 0$.

In case $0 \leqslant p_0 < 1$, take p_1 so that $p_0 < p_1 < (1+4p_0)/5 < 1$, then

$$p_0 > \frac{5p_1 - 1}{4}.$$

Since $\lim_{t\to\infty} p(t) = p_0$ and (11) holds, we can choose $T_0 \in \mathbb{T}$ large enough such that

$$\frac{5p_1 - 1}{4} \leqslant p(t) \leqslant p_1 < 1, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{14}$$

and

$$\int_{T_0}^{\infty} f(s, K)\,\Delta s \leqslant \frac{(1-p_1)K}{8}. \tag{15}$$

Furthermore, from (H1) we see that there exists $T_1 \in \mathbb{T}$ with $T_1 > T_0$ such that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define the Banach space $BC[T_0, \infty)_{\mathbb{T}}$ as in (3) and let

$$X = \left\{ x \in BC[T_0, \infty)_{\mathbb{T}} \colon \frac{K}{2} \leqslant x(t) \leqslant K \right\}. \tag{16}$$

It is easy to verify that X is a bounded, convex and closed subset of $BC[T_0, \infty)_{\mathbb{T}}$. By (H3), we have that for any $x \in X$,

$$f\big(t, x\big(h(t)\big)\big) \leqslant f(t, K), \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{17}$$

Now we define two operators U and $S : X \to BC[T_0, \infty)_{\mathbb{T}}$ as follows:

$$(Ux)(t) = \begin{cases} \frac{3Kp_1}{4} - p(t)x(g(T_1)), & t \in [T_0, T_1]_{\mathbb{T}}, \\ \frac{3Kp_1}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases} \tag{18}$$

and

$$(Sx)(t) = \begin{cases} \frac{3K}{4} + \int_{T_1}^{\infty} f(s, x(h(s)))\,\Delta s, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \frac{3K}{4} + \int_{t}^{\infty} f(s, x(h(s)))\,\Delta s, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \tag{19}$$

Next, we will show that U and S satisfy the conditions in Lemma 5.

(i) We first prove that $Ux + Sy \in X$ for any $x, y \in X$. Note that for any $x, y \in X$, $K/2 \leqslant x \leqslant K$ and $K/2 \leqslant y \leqslant K$. For any $x, y \in X$ and $t \in [T_1, \infty)_{\mathbb{T}}$, in view of (14), we have

$$\begin{aligned} (Ux)(t) + (Sy)(t) &= \frac{3(1+p_1)K}{4} - p(t)x\big(g(t)\big) + \int_t^{\infty} f\big(s, y\big(h(s)\big)\big)\,\Delta s \\ &\geqslant \frac{3(1+p_1)K}{4} - p_1K \\ &= \frac{(3-p_1)K}{4} \geqslant \frac{K}{2}. \end{aligned}$$

Also, by (14) and (15), we have

$$\begin{aligned} (Ux)(t) + (Sy)(t) &\leqslant \frac{3(1+p_1)K}{4} - \frac{p(t)K}{2} + \frac{(1-p_1)K}{8} \\ &\leqslant \frac{3(1+p_1)K}{4} - \frac{5p_1 - 1}{4} \times \frac{K}{2} + \frac{(1-p_1)K}{8} \\ &= K. \end{aligned}$$

Similarly, we can prove that $K/2 \leqslant (Ux)(t) + (Sy)(t) \leqslant K$ for any $x, y \in X$ and $t \in [T_0, T_1]_{\mathbb{T}}$. Hence, $Ux + Sy \in X$ for any $x, y \in X$.

(ii) We prove that U is a contraction mapping. Indeed, for $x, y \in X$, we have

$$\left|(Ux)(t) - (Uy)(t)\right| = \left|p(t)\left[x\left(g(T_1)\right) - y\left(g(T_1)\right)\right]\right| \leqslant p_1 \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \left|x(t) - y(t)\right|$$

for $t \in [T_0, T_1]_{\mathbb{T}}$ and

$$\left|(Ux)(t) - (Uy)(t)\right| = \left|p(t)\left[x\left(g(t)\right) - y\left(g(t)\right)\right]\right| \leqslant p_1 \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \left|x(t) - y(t)\right|$$

for $t \in [T_1, \infty)_{\mathbb{T}}$. Therefore, we have

$$\|Ux - Uy\| \leqslant p_1 \|x - y\|$$

for any $x, y \in X$. Hence, U is a contraction mapping.

(iii) We will prove that S is a completely continuous mapping. First, by (15), (17) and (19), we see that $(Sx)(t) \geqslant K/2$ and $(Sx)(t) \leqslant 3K/4 + (1 - p_1)K/8 \leqslant K$ for $t \in [T_0, \infty)_{\mathbb{T}}$. That is, S maps X into X.

Second, we consider the continuity of S. Let $x_n \in X$ and $\|x_n - x\| \to 0$ as $n \to \infty$, then $x \in X$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [T_0, \infty)_{\mathbb{T}}$. Consequently, for any $t \in [T_1, \infty)_{\mathbb{T}}$ we have

$$\left|f\left(t, x_n\left(h(t)\right)\right) - f\left(t, x\left(h(t)\right)\right)\right| \to 0 \quad \text{as } n \to \infty. \tag{20}$$

From (17), we obtain that

$$\left|f\left(t, x_n\left(h(t)\right)\right) - f\left(t, x\left(h(t)\right)\right)\right| \leqslant 2f(t, K). \tag{21}$$

On the other hand, from (19) we have

$$\left|(Sx_n)(t) - (Sx)(t)\right| \leqslant \int_{T_1}^{\infty} \left|f\left(s, x_n\left(h(s)\right)\right) - f\left(s, x\left(h(s)\right)\right)\right| \Delta s \tag{22}$$

for $t \in [T_0, T_1]_{\mathbb{T}}$ and

$$\left|(Sx_n)(t) - (Sx)(t)\right| \leqslant \int_{t}^{\infty} \left|f\left(s, x_n\left(h(s)\right)\right) - f\left(s, x\left(h(s)\right)\right)\right| \Delta s \tag{23}$$

for $t \in [T_1, \infty)_{\mathbb{T}}$. Therefore, from (22) and (23), we have

$$\|Sx_n - Sx\| \leqslant \int_{T_1}^{\infty} \left|f\left(s, x_n\left(h(s)\right)\right) - f\left(s, x\left(h(s)\right)\right)\right| \Delta s. \tag{24}$$

Referring to Chapter 5 in [3], we see that the Lebesgue dominated convergence theorem holds for the integral on time scales. Then, from (20) and (21), (24) yields

$$\left\|(Sx_n)(t) - (Sx)(t)\right\| \to 0 \quad (n \to \infty),$$

which proves that S is continuous on X.

Finally, we prove that SX is relatively compact. It is sufficient to verify that SX satisfies all conditions in Lemma 4. By the definition of X, we see that SX is bounded. For any $\varepsilon > 0$, take $T_2 \in [T_1, \infty)_{\mathbb{T}}$ so that

$$\int_{T_2}^{\infty} f(s, K)\,\Delta s < \varepsilon.$$

For any $x \in X$ and $t_1, t_2 \in [T_2, \infty)_{\mathbb{T}}$, we have

$$\bigl|(Sx)(t_1) - (Sx)(t_2)\bigr| < 2\varepsilon.$$

Thus, SX is uniformly Cauchy.

The remainder is to consider the equi-continuity on $[T_0, T_2]_{\mathbb{T}}$ for any $T_2 \in [T_0, \infty)_{\mathbb{T}}$. Without loss of generality, we set $T_1 < T_2$. For any $x \in X$, we have $|(Sx)(t_1) - (Sx)(t_2)| \equiv 0$ for $t_1, t_2 \in [T_0, T_1]_{\mathbb{T}}$ and

$$\begin{aligned}\bigl|(Sx)(t_1) - (Sx)(t_2)\bigr| &= \left|\int_{t_1}^{\infty} f\bigl(s, x\bigl(h(s)\bigr)\bigr)\,\Delta s - \int_{t_2}^{\infty} f\bigl(s, x\bigl(h(t)\bigr)\bigr)\,\Delta s\right| \\ &\leqslant \left|\int_{t_1}^{t_2} f(s, K)\,\Delta s\right|\end{aligned}$$

for $t_1, t_2 \in [T_1, T_2]_{\mathbb{T}}$.

Now, we see that for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $t_1, t_2 \in [T_0, T_2]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$,

$$\bigl|(Sx)(t_1) - (Sx)(t_2)\bigr| < \varepsilon \quad \text{for all } x \in X.$$

This means that SX is equi-continuous on $[T_0, T_2]_{\mathbb{T}}$ for any $T_2 \in [T_0, \infty)_{\mathbb{T}}$.

By means of Lemma 4, SX is relatively compact. From the above, we have proved that S is a completely continuous mapping.

By Lemma 5, there exists $x \in X$ such that $(U + S)x = x$. Therefore, we have

$$x(t) = \frac{3(1 + p_1)K}{4} - p(t)x\bigl(g(t)\bigr) + \int_t^{\infty} f\bigl(s, x\bigl(h(s)\bigr)\bigr)\,\Delta s, \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{25}$$

This equation means that $x(t)$ is a solution of (1) and $\lim_{t\to\infty} z(t) = 3(1 + p_1)K/4$. Further, by the limit of $z(t)$, we have $\lim_{t\to\infty} x(t) = 3(1 + p_1)K/(4 + 4p_0)$. Note that $x \in X$, $x(t)$ is eventually positive, the sufficiency holds when $0 \leqslant p_0 < 1$.

In case $-1 < p_0 < 0$, take p_1 so that $-p_0 < p_1 < (1 - 4p_0)/5 < 1$, then $p_0 < (1 - 5p_1)/4$. Since $\lim_{t\to\infty} p(t) = p_0$ and (11) holds, we can choose $T_0 \in \mathbb{T}$ large enough such that (15) holds and

$$\frac{5p_1 - 1}{4} \leqslant -p(t) \leqslant p_1 < 1, \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{26}$$

Take $T_1 \in \mathbb{T}$ with $T_1 > T_0$ so that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Similarly, we introduce the Banach space $BC[T_0, \infty)_{\mathbb{T}}$ and its subset X as above. Define operator S as in (19) and operator U on X as follows:

$$(Ux)(t) = \begin{cases} -\frac{3Kp_1}{4} - p(t)x(g(T_1)), & t \in [T_0, T_1]_{\mathbb{T}}, \\ -\frac{3Kp_1}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Next, we show that $Ux + Sy \in X$ for any $x, y \in X$. Indeed, for any $x, y \in X$ and $t \in [T_1, \infty)_{\mathbb{T}}$, by means of (26) and (15), we have

$$(Ux)(t) + (Sy)(t) = \frac{3(1-p_1)K}{4} - p(t)x\big(g(t)\big) + \int_t^\infty f\big(s, y\big(h(s)\big)\big)\,\Delta s$$
$$\geqslant \frac{3(1-p_1)K}{4} + \frac{5p_1 - 1}{4} \times \frac{K}{2} = \frac{(5-p_1)K}{8} \geqslant \frac{K}{2}$$

and

$$(Ux)(t) + (Sy)(t) \leqslant \frac{3(1-p_1)K}{4} - p(t)K + \frac{(1-p_1)K}{8} \leqslant K.$$

That is, $Ux + Sy \in X$ for any $x, y \in X$.

The following proof is similar to that of case $0 \leqslant p_0 < 1$ and omitted. By Lemma 5, there exists $x \in X$ such that

$$x(t) = \frac{3(1-p_1)K}{4} - p(t)x\big(g(t)\big) + \int_t^\infty f\big(s, x\big(h(s)\big)\big)\,\Delta s, \quad t \in [T_1, \infty)_{\mathbb{T}},$$

which means that $x(t)$ is a solution of (1) and eventually positive. Moreover, from $\lim_{t\to\infty} z(t) = 3(1-p_1)K/4$, we have $\lim_{t\to\infty} x(t) = 3(1-p_1)K/(4+4p_0)$.

The proof is complete. □

Theorem 9. *If there exists $T_0 \in \mathbb{T}$ with $T_0 > 0$ such that*

$$p(t)e^{-g(t)} \leqslant -e^{-t}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{27}$$

and

$$\int_t^\infty f\left(s, \frac{1}{h(s)}\right)\Delta s \leqslant \frac{1}{t} + \frac{p(t)}{g(t)}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{28}$$

then Eq. (1) *has an eventually positive solution $x(t)$ with* $\lim_{t\to\infty} x(t) = 0$.

Proof. Take $T_1 \in \mathbb{T}$ with $T_1 > T_0$ so that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Define the Banach space $BC[T_0, \infty)_{\mathbb{T}}$ as in (3). Let

$$X = \left\{x \in BC[T_0, \infty)_{\mathbb{T}}\colon e^{-t} \leqslant x(t) \leqslant \frac{1}{t} \text{ for } t \in [T_1, \infty)_{\mathbb{T}} \text{ and } e^{-T_1} \leqslant x(t) \leqslant \frac{1}{t} \text{ for } t \in [T_0, T_1]_{\mathbb{T}}\right\},$$

then X is a bounded, convex and closed subset of $BC[T_0, \infty)_{\mathbb{T}}$. Define an operator S on X as follows:

$$(Sx)(t) = \begin{cases} -p(T_1)x(g(T_1)) + \int_{T_1}^\infty f(s, x(h(s)))\,\Delta s, & t \in [T_0, T_1]_{\mathbb{T}}, \\ -p(t)x(g(t)) + \int_t^\infty f(s, x(h(s)))\,\Delta s, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

First, we show that $Sx \in X$ for all $x \in X$. Indeed, from (27) and (28), we have for $t \in [T_1, \infty)_{\mathbb{T}}$,

$$(Sx)(t) = -p(t)x\big(g(t)\big) + \int_t^\infty f\big(s, x\big(h(s)\big)\big)\,\Delta s$$
$$\leqslant \frac{-p(t)}{g(t)} + \frac{1}{t} + \frac{p(t)}{g(t)} \leqslant \frac{1}{t}$$

and

$$(Sx)(t) \geqslant -p(t)e^{-g(t)} \geqslant e^{-t}.$$

It follows that $e^{-T_1} \leqslant (Sx)(t) \leqslant 1/t$ for $t \in [T_0, T_1]_{\mathbb{T}}$. Thus, we have proved that $Sx \in X$ for all $x \in X$. The rest of the proof is similar to that of Theorem 8 and hence omitted.

By Corollary 6, we see that there exists $x \in X$ such that

$$x(t) = -p(t)x\big(g(t)\big) + \int_t^\infty f\big(s, x\big(h(s)\big)\big)\,\Delta s, \quad t \in [T_1, \infty)_{\mathbb{T}}, \tag{29}$$

which means that $x(t)$ is an eventually positive solution of (1). Note from the definition of X, we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

The following result can be proved similar to the proof of Theorem 9 and hence omitted.

Theorem 10. *If there exist a constant $K > 0$ and $T_0 \in \mathbb{T}$ with $T_0 > 0$ such that*

$$0 \leqslant p(t) \leqslant Kg(t)e^{-t}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{30}$$

$$\int_t^\infty f\big(s, e^{-h(s)}\big)\,\Delta s \geqslant (K+1)e^{-t}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{31}$$

and

$$\int_t^\infty f\left(s, \frac{1}{h(s)}\right)\Delta s \leqslant \frac{1}{t}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{32}$$

then Eq. (1) *has an eventually positive solution $x(t)$ with* $\lim_{t\to\infty} x(t) = 0$.

Example 11. Let $q > 1$ and $\mathbb{T} = \{q^n\colon n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ is the set of nonnegative integers. Consider the following equation:

$$\left[x(t) + \frac{t+1}{2t}x\big(\rho(t)\big)\right]^\Delta + \frac{x(\sigma(t))}{t\sigma(t)} = 0, \quad t \in \mathbb{T}. \tag{33}$$

Then $p(t) = \frac{t+1}{2t}$, $g(t) = \rho(t)$, $h(t) = \sigma(t)$ and $f(t, x) = \frac{x}{t\sigma(t)}$. It is easy to see that all the conditions (H1)–(H3) are satisfied. Also, $\int_1^\infty f(s, K)\,\Delta s = K$ for any $K > 0$. By Theorem 8, Eq. (33) has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = a > 0$.

Example 12. Let $\tau > 0$ and $\mathbb{T} = \{n\tau\colon n \in \mathbb{N}_0\}$. Consider the following equation:

$$\big[x(t) - e^{-\tau}x(t-\tau)\big]^\Delta + \frac{(2t+\tau)x((t+\tau)^2)}{t^2} = 0, \quad t \in \mathbb{T}, \tag{34}$$

where $p(t) = -e^{-\tau}$, $g(t) = t - \tau$, $h(t) = (t + \tau)^2$ and $f(t, x) = (2t + \tau)x/t^2$. We can readily verify that p, g and h satisfy all the conditions (H1)–(H3). Also, $p(t)e^{-g(t)} = -e^{-t}$ and $\int_t^\infty f(s, 1/h(s))\,\Delta s = 1/t^2$. Then, we see that (28) holds eventually. By Theorem 9, Eq. (34) has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$.

Example 13. Let $\mathbb{T} = \{t \geqslant 0\colon\ t \in \mathbb{R}\}$. Consider the following equation:

$$\big[x(t) + (t-1)e^{-t}x(t-1)\big]^\Delta + e^{-t/4}x\left(\frac{t}{4}\right) = 0, \quad t \in \mathbb{T}, \tag{35}$$

where $p(t) = (t-1)e^{-t}$, $g(t) = t - 1$, $h(t) = t/4$ and $f(t, x) = e^{-t/4}x$. Then, $\int_t^\infty f(s, 1/h(s))\,\Delta s \leqslant 4e^{-t/4}$ for $t \geqslant 4$. Further, $\int_t^\infty f(s, e^{-h(s)})\,\Delta s = 2e^{-t/2}$. Taking $K = 1$, we see that (30)–(32) hold eventually. By Theorem 10, Eq. (35) has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$.

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