

Modification of Gaussian Elimination for the Complex System of Seismic Observations

Tatyana A. Smaglichenko

*Institute of Oil and Gas Problems
Russian Academy of Science
Gubkina str.3, 119333, Moscow, Russia
krutas39@yahoo.com*

A method for solving large and sparse systems with complex seismic observations has been developed by revising the classical Gaussian elimination. The observation matrix is divided into a set of smaller cells that can be determined based on clusters of seismic activity. Reliable inversion solutions with minimum error rating are selected within each cell; solutions that are unstable with respect to the same unknown are rejected. After the final sifting, the found unknowns are eliminated from the initial system, subsequently decreasing its dimension. A numerical example is provided to demonstrate how the method can be applied to a real dataset (a case study of the Nagano fault area in central Japan). The result of processing a huge volume of data showed that the P-wave velocity structure is much more complex than the synthetic model, which has been used to check the method effectiveness. The importance of the results for the Earthquake Early Warning (EEW) system is briefly discussed.

1. Introduction

Dense seismic networks register thousands of seismic wave arrivals in active regions. The system of observations permits us to create the system of integral equations, which can define parameters of the Earth's interior structure. The first seismological studies [1, 2] tell us that the crust and the mantle are divided into layers and the propagation velocity of seismic waves strongly increases with depth. At the same time, the observed time curves reflect the presence of "jumps," which indicate heterogeneity of the medium [3]. By using these facts and the concept of the Earth's block structure [4], researchers divided the medium into several layers and each layer into many blocks. Seismic velocities are considered deviations from the simple initial model that linearly depends on depth [5, 6]. Thus, the problem is reduced to the system of linear equations, where observations are presented as time residuals that are differences between the observed and calculated values for the simple case.

However, decades of practice reveal that there are very large variations in time residuals that cause extra complexity in geophysical models. Geological rocks are hierarchically inhomogeneous. The geophysical medium is nonlinear [7, 8]. Geophysical fields (e.g., gravitational, electromagnetic, and fields of deformations) vary in space and in time on a large range of scales. As a consequence, inhomogeneous seismic structures of different sizes can be chaotically distributed in the medium.

Small structures can be successfully retrieved using inversion of travel time residuals if the Lanczos method will be applied. The reliable reconstruction of separated large inclusions can be performed by the consecutive subtraction of selected anomalies method, also known as the method of feasible directions for mathematical problems [9]. This conclusion has been based on simplified tests with artificial data. However, in reality seismic observations are complex measurements with many sources of error. Picking and phase errors can appear while reading seismic wave arrivals on seismograms [10]. Restriction caused by the station's configuration leads to a parametrization error. Because of the complexity of the unknown structure of the Earth, similar measurements can correspond to quite different properties of the seismic medium. This leads to the appearance of equations, for which the principle of superposition does not work. Thus, the arising system of linear equations becomes quite complex. This makes the standard numerical algorithms difficult to apply.

Normally, systems of linear equations, which describe parameters of complicated seismic subjects, are large, sparse, and simultaneously inconsistent due to observation errors. The author of this paper made a modernization of the Gauss scheme [11] in order to protect the system from various errors, which can yield non-uniqueness of inversion solutions and distortion in the seismic images. The problem of round-off errors, which is typical for large and sparse systems, may be overcome via fragmentation of the initial matrix into cells. This useful idea comes from fundamental mathematical work [12]. The author applied it for seismic tomography and developed a new approach called the differentiated approach (DA) [13]. In this paper, the DA is described as an algebraic method, which can be helpful for solving linear systems with approximately known constant terms.

The following sections show that a subsystem consistent with the observations can be constructed within each cell via selection of independent pieces of the seismic information. The analysis of all trustworthy solutions makes it possible to select the steady result and to avoid observation errors.

The method has been applied to the aftershock area of the Nagano-ken Seibu Earthquake (magnitude 6.8 on September 14, 1984) in western Nagano (central Japan). This region was the most active in Japan before the 2011 Tohoku earthquake (magnitude 9.0 on March 11) that caused a huge tsunami. The data used in the study was recorded much earlier, from October 1995 to May 1998. Before inver-

sion of real data, the method was tested using a synthetic model of the western Nagano area. A fault zone appeared in this region after the Nagano-ken Seibu Earthquake. The crystal structure of rocks strongly changed under pressure of the forces, and seismic velocity within the fault plane became too low. This factor has been used to construct the synthetic model of the Nagano fault. This paper describes the simulation details. Reconstruction of an arbitrary cross-section is presented in order to illustrate the effectiveness of the new inversion technique. Next, the inversion algorithm is explained for a particular case using real data. Finally, the seismological illustration with all actual data is provided.

2. Modification of the Gauss Method for the Modern Problem of Complex Data

The classical Gauss method is the most simple and convenient for small systems of equations. However, rapid technical progress produces more and more diverse measurements—up to a few million. On the other hand, nature is not under human control and therefore not all data can be effectively used to get accurate information about the unknown parameters of the geophysical medium. Moreover, the complexity of the subject often leads to double interpretation of the observations. The author of this study reconsiders the Gauss scheme in order to solve large and sparse systems, which are generated from a huge amount of experimental data.

Let us consider a system of linear equations

$$\mathbf{Ax} = \mathbf{b}, \quad (1)$$

where \mathbf{A} is a rectangular matrix of dimension $m \times n$ whose elements are determined by the distribution of m seismic rays in n blocks of medium; \mathbf{x} is an unknown vector whose n components correspond to values of seismic parameter (velocity anomaly or attenuation parameter) in n blocks of medium; and \mathbf{b} is the vector with m components known from observations (travel time residuals or amplitude ratios). Due to the nature of a seismic experiment, not all seismic blocks are penetrated by the seismic rays; hence, matrix \mathbf{A} has many zero values and becomes very sparse. Nonzero elements of the matrix are often located on different co-diagonals while zero values may be on the main diagonal, which makes the condition number too large. Another problem is error accumulation, caused when recurrence formulas are used in the solution. Consequently, numerical errors that occur while processing such matrices can lead to inaccurate results. These problems can be avoided by dividing the initial matrix into cells of smaller matrices with nonzero entries [12].

As a rule, the seismic network consists of clusters of shocks. Seismic rays propagate through the medium from cluster to surface.

Thus, blocks between the shocks of cluster and station-receiver are filled and nonzero value rows of matrix A correspond to these blocks. We will use this fact in developing the method.

We can suppose without loss of generality that an earthquake network contains two clusters of seismic activity. Divide an initial matrix into four cells. Let C and D be over-determined submatrices of dimension $m_1 \times n_1$ and $m_2 \times n_2$ that are formed by means of nonzero rows of matrix A . The other two cells contain zero matrices $0_{m_1, n-n_1}$ and $0_{m_2, n-n_2}$. Figure 1 schematically shows the form of matrix A .

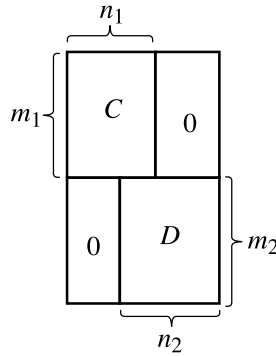


Figure 1. Form of the initial matrix after division into four cells. C and D denote matrices of dimensions $m_1 \times n_1$ and $m_2 \times n_2$, which have only nonzero entries. 0 denotes zero matrix $0_{m_1, n-n_1}$ or $0_{m_2, n-n_2}$.

Continuing in the same way, the unknown vector x with components $\{x_1 \dots x_n\}$ is divided into two smaller vectors y and z : $y = \{x_1 \dots x_{n_1}\}$ and $z = \{x_{(n+1)-n_2} \dots x_n\}$. Consequently, the observation vector b with components $\{b_1 \dots b_m\}$ is divided into two smaller vectors a and c : $a = \{b_1 \dots b_{m_1}\}$ and $c = \{b_{m_1+1} \dots b_m\}$. Thus, equation (1) becomes equivalent to this system of two equations:

$$\begin{cases} Cy = a \\ Dz = c \end{cases} \quad (2)$$

This implies that instead of the large and sparse system in equation (1), we should solve two smaller subsystems with matrices that contain only nonzero elements. Each subsystem is solved by selecting the rows that form the basis matrix (Gauss–Markov theorem, [14]). Unknown vectors y and z are estimated from inversion of the basis matrix by the Gauss–Jordan method.

It is important to know that the components $\{x_{(n+1)-n_2} \dots x_{n_1}\}$ are twice determined as solutions of both subsystems. Next, we compare both results according to the same component and select only those

values that are near equal. After finding all of the stable components, we eliminate them from the system in equation (1) that leads to size reduction of the initial system and to a new form of subsystem. The process is repeated until we have reliable results. Such a procedure permits us to avoid participating in the solution of the rows, which have errors in constant terms.

3. Testing Model

More than 130 000 P-wave arrivals from 8665 aftershocks of the 1984 Nagano-ken Seibu Earthquake were selected via the real-time processing system [15]. In order to increase the resolution of the system of linear equations, hypocenters were chosen to be uniformly located in a volume of the medium. The origin of a Cartesian coordinate system was shifted to a point close to the regional network of seismic stations and to the area of a dense distribution of quakes (Figure 2). Next, it was rotated so that the OX axis was parallel to projections of the fault plane. Thus, the coordinate system allows blocks of 100 m thickness to be placed in the fault plane.

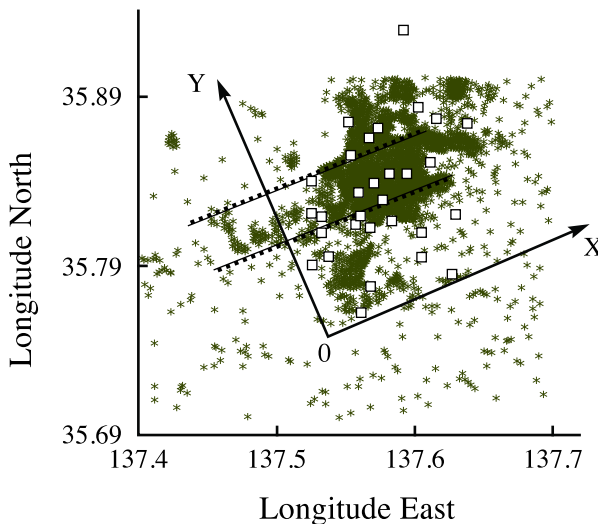


Figure 2. Earthquake locations based on a one-dimensional regional model (denoted by stars). Open squares show seismic stations. The origin 0 corresponds to geographical coordinates 35.75N and 137.53E. Dotted lines indicate projections of the fault plane. Coordinates for the end points of the lines are (35.7851N, 137.4493E, 0.3 km); (35.8365N, 137.6050E, 0.3 km); (35.8641N, 137.5900E, 9.9 km); and (35.8127N, 137.4343E, 9.9 km).

The fault plane plunged into the Earth's interior in the North direction. The block size was selected taking into account the plane geometry. Elements of matrix A given by equation (1) were obtained by ray tracing (Figure 3) through the initial simple model, which describes velocity as a linear function of depth.

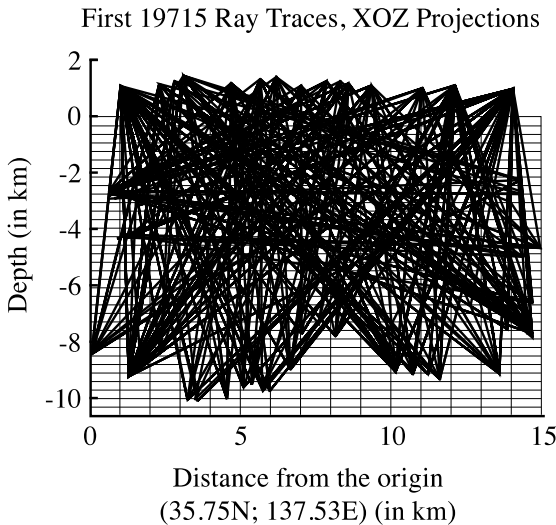


Figure 3. Ray trace diagram for XOZ projections. Axis OZ corresponds to the depths. The solid lines show the block configuration.

Components of vector \mathbf{b} represent travel time residuals, which are differences between the observed and calculated travel times. In other words, they are the difference in travel time for two media with real velocity distribution and velocity that increases linearly with depth. Figure 4(a) illustrates travel time residuals, which were obtained from actual observations in the western Nagano area. Synthetic data was calculated by substituting values of the synthetic model into vector \mathbf{x} in equation (1). The synthetic model has been determined as the vector with components that have the meaning of velocities selected for blocks. Extremely low velocity (4.6 km/s) used in geological study was assigned to blocks within the fault plane. Due to a lack of prior information for other blocks, their numerical values were chosen so that the observed and synthetic residuals were distributed similarly (Figure 4).

Thus, the synthetic model represented a large velocity anomaly surrounded by two uniform zones having velocity perturbations $+0.02\%$ and -0.01% within the depth ranges 0 to 2.1 km and 2.1 to 10 km, respectively (Figure 5).

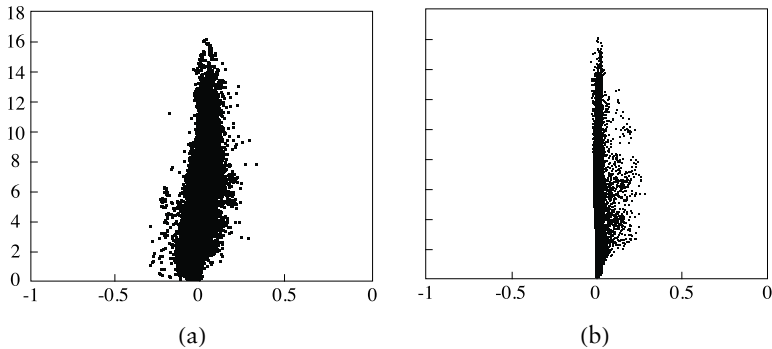


Figure 4. Travel time residuals plotted as a function of epicentral distance. Horizontal (vertical) axis corresponds to travel time residuals (epicentral distance). (a) The observed residuals. (b) Residuals calculated relative to the synthetic model.

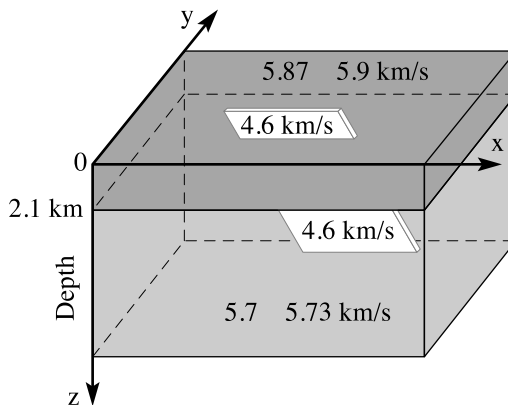


Figure 5. Synthetic velocity model used for computation of synthetic data. The lowest velocity (white) is surrounded by a high velocity zone (dark gray) down to 2.1 km and by a low velocity zone (light gray) at a greater depth.

Synthetic data was inverted using the Gauss modification. The whole model was completely reconstructed in the areas with a high distribution density of seismic rays [16]. Figure 6 illustrates the section obtained in the depth range 1.8 to 2.1 km. The retrieved image is identical to the input model, which is shown in Figure 5.

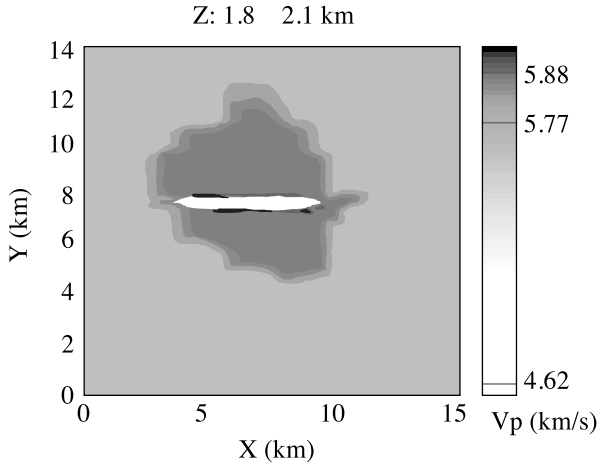


Figure 6. XOY projection of the inverted model. The projection of the inclined fault plane is denoted by a white band. The high velocity in the zone surrounding the fault is dark gray. The gray background shading corresponds to the velocity related to the initial linear model.

4. Application to Actual Data

The initial matrix had 103 148 rows and 19 399 columns that corresponded to the number of seismic rays and blocks, respectively. We take a small part of this matrix and show how the method works on a real situation. Let \mathbf{A} be the 9×4 matrix in equation (1), where $m = 9$, $n = 4$, \mathbf{x} is the unknown 4×1 column, and \mathbf{b} is the 9×1 column:

$$\mathbf{A} = \begin{bmatrix} 0.0217 & 0.0569 & 0.0013 & 0.0 \\ 0.0714 & 0.0036 & 0.0004 & 0.0 \\ 0.0006 & 0.0003 & 0.0476 & 0.0 \\ 0.0006 & 0.0004 & 0.0483 & 0.0 \\ 0.0007 & 0.0004 & 0.0504 & 0.0 \\ 0.0 & 0.0 & 0.0215 & 0.0092 \\ 0.0 & 0.0 & 0.0222 & 0.0086 \\ 0.0 & 0.0 & 0.0222 & 0.0084 \\ 0.0 & 0.0 & 0.0050 & 0.0267 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} -0.0424 \\ -0.0450 \\ -0.0645 \\ -0.0223 \\ -0.0359 \\ -0.0154 \\ -0.2719 \\ 0.0140 \\ -0.0208 \end{bmatrix}.$$

It is readily seen that in equation (2), \mathbf{C} is the 5×3 matrix, \mathbf{a} is the 5×1 column, \mathbf{D} is the 4×2 matrix, and \mathbf{c} is the 4×1 column, where $m_1 = 5$, $n_1 = 3$, $m_2 = 4$, and $n_2 = 2$:

$$\mathbf{C} = \begin{bmatrix} 0.0217 & 0.0569 & 0.0013 \\ 0.0714 & 0.0036 & 0.0004 \\ 0.0006 & 0.0003 & 0.0476 \\ 0.0006 & 0.0004 & 0.0483 \\ 0.0007 & 0.0004 & 0.0504 \end{bmatrix},$$

$$\mathbf{a} = \begin{bmatrix} -0.0424 \\ -0.0450 \\ -0.0645 \\ -0.0223 \\ -0.0359 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 0.0215 & 0.0092 \\ 0.0222 & 0.0086 \\ 0.0222 & 0.0084 \\ 0.0050 & 0.0267 \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} -0.0154 \\ -0.2719 \\ 0.0140 \\ -0.0208 \end{bmatrix}.$$

Both submatrices \mathbf{C} and \mathbf{D} are characterized by the presence of rows, which are linearly dependent due to data clusters. We select the basis rows to make consistent subsystems. This means that unknown vectors \mathbf{y} and \mathbf{x} are estimated as:

$$y = \begin{bmatrix} 0.0217 & 0.0569 & 0.0013 \\ 0.0714 & 0.0036 & 0.0004 \\ 0.0006 & 0.0004 & 0.0483 \end{bmatrix}^{-1} \begin{bmatrix} -0.0424 \\ -0.0450 \\ -0.0223 \end{bmatrix} = \begin{bmatrix} -0.6016 \\ -0.5051 \\ -0.4499 \end{bmatrix},$$

$$z = \begin{bmatrix} 0.0215 & 0.0092 \\ 0.0050 & 0.0267 \end{bmatrix}^{-1} \begin{bmatrix} -0.0154 \\ -0.0208 \end{bmatrix} = \begin{bmatrix} -0.4149 \\ -0.7009 \end{bmatrix}.$$

An unknown component x_3 of vector \mathbf{x} is the common unknown for both subsystems, which has been found as the third component of vector \mathbf{y} and the first component of vector \mathbf{z} . Both solutions -0.4499 and -0.4149 are near equal, therefore they are pretenders to the solution of x_3 . These values have been confirmed by the solutions of an additional four subsystems. The solution with minimal error was selected as the reliable solution to component x_3 .

The whole system of real observations has been solved using the approach described. This permitted us to make a visualization of the P-wave velocity structure in the western Nagano region. We chose the depth range for which the testing result has already been demonstrated (see Section 3). By comparing Figures 6 and 7, the synthetic image can be distinguished from the actual one. Seismic structure was partially predicted in the synthetic model. It is important to note that very low velocity within the fault plane is well determined when processing real data. This means that the Gauss modification method accurately reconstructs the main fault structure. At the same time, surrounding zones have complex patterns that likely correspond to reality. Thus, our synthetic modeling is insufficient to predict real features of the medium.

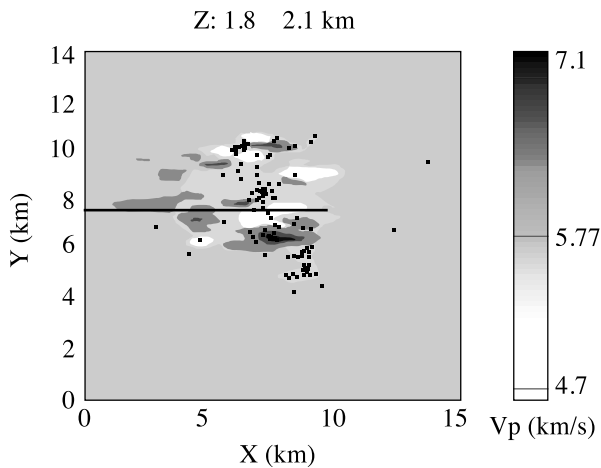


Figure 7. XOY projection of the velocity model and relocated events (dots). Low (high) velocity zone is white (dark gray). Projection of the main fault plane is denoted by a solid line.

In Figure 8, we demonstrate another cross-section (the XOZ projection, $Y = 10$ km) that shows relocated events and the velocity structure obtained at different depths.

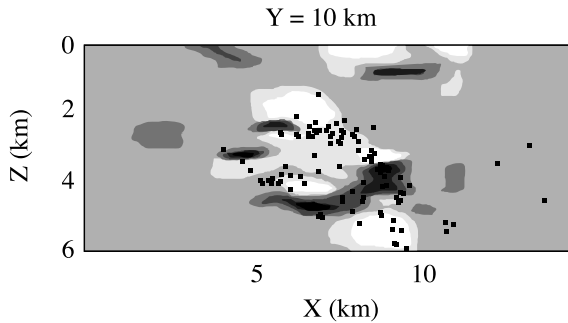


Figure 8. XOZ projection of the velocity model and relocated events. Low (high) velocity zone is white (dark gray). Hypocenters are denoted by dots. Axis Z corresponds to the depths.

Shock locations are mainly concentrated near the boundary between low and high seismic velocities. Analysis of the velocity distribution with respect to the shock location is very important in order to make the Earthquake Early Warning (EEW) system effective. EEW is a modern tool for the protection of human lives and property. It is known that strong quakes can be at different depths. Accurate location determination for the most dangerous vibrations will help to avoid false alarms in the EEW system. The range and distribution of seismic instrumentation for the EEW system can be determined on the basis of the velocity structure. This will be the object of another paper.

5. Conclusion

The modification of the Gauss method was developed to process a seismic observation system that is complex. Mathematical procedures have been described for a simple case of two clusters of seismic activity. In general, an inversion solution is estimated as steady if the found value has been confirmed for a significant number of clusters. Thus, the method is easy to implement in the general case.

The complex model was created using geological data and a series of detailed comparisons of real and synthetic datasets. The model consisted of three large structures, including the fault plane. The method reconstructed this model quite well. This suggests that the Gauss method modification can be effective to map such structures. On the other hand, the inversion of real data revealed that the synthetic model is too simplified. We conclude that the testing process requires

efforts to produce more realistic synthetic images. This can be done by introducing a wide range of prior information, such as gravity, petrology electromagnetic, and other data.

Application of the method to a huge amount of data obtained in one of the most active seismic regions in the world showed that shocks are neighbors of velocity contrasts. This result is significant and confirmed by other recent studies.

Acknowledgments

I would like to express my deep gratitude to Prof. Shigeki Horiuchi (Home Seismometer Corporation, Tsukuba, Japan) for constructive remarks that helped in developing a new inversion technique. I very much appreciate my colleagues from NIED, Japan for support and assistance during our cooperative research in 2003 and 2005. I thank the organizers of the 2011 Interdisciplinary Symposium on Complex Systems, including Ali Sanayei, and anonymous reviewers for useful recommendations for improving the quality of this paper. My sincere gratitude goes to Alexey Nikolaev and Anatoliy Dmitrievskiy, members of the Russian Academy of Sciences, for fruitful discussion of complexity of geophysical fields. The travel grant for my participation in the Symposium has been awarded by the Russian Foundation for Basic Research.

References

- [1] B. Gutenberg and C. F. Richter, "Materials for the Study of Deep-Focus Earthquakes," *Bulletin of the Seismological Society of America*, 26(4), 1936 pp. 341–390.
- [2] E. F. Savarensky and D. P. Kirnos, *Elements of Seismology and Seismometry*, 2nd edition (in Russian), Moscow: Nauka, 1955.
- [3] E. F. Savarensky, "To the Question of Inhomogeneous Deep Structure of the Earth Using Seismological Data" (in Russian), *Soviet Doklady of Soviet Academy of Sciences*, 27(1), 1940.
- [4] A. Wegener, "Die Entstehung der Kontinente," *Geologische Rundschau*, 3(4), 1912 pp. 276–292. doi:10.1007/BF02202896.
- [5] A. S. Alexeev, M. M. Lavrent'ev, R. S. Muhomedov, and V. G. Romanov, "Numerical Method for the Solution of the Three-Dimensional Inversion Problem of Seismology" (in Russian), *Mathematical Problems of Geophysics, Soviet Academy of Sciences, Siberian Department*, 1, 1969 pp. 170–201.
- [6] K. Aki and W. K. H. Lee, "Determination of Three-Dimensional Velocity Anomalies under a Seismic Array Using First P-Arrival Times from Local Earthquakes, 1. Homogeneous Initial Model," *Journal of Geophysical Research*, 81(23), 1976 pp. 4381–4399. doi:10.1029/JB081i023p04381.

- [7] I. A. Beresnev and A. V. Nikolaev, "Experimental Investigations of Non-linear Seismic Effects," *Physics of the Earth and Planetary Interiors*, 50(1), 1988 pp. 83–87. doi:10.1016/0031-9201(88)90096-9.
- [8] P. P. Dimitriu, "Preliminary Results of Vibrator-Aided Experiments in Non-Linear Seismology Conducted at Uetze, F.R.G.," *Physics of the Earth and Planetary Interiors*, 63(1–4), 1990 pp. 172–180. doi:10.1016/0031-9201(90)90016-Q.
- [9] T. A. Smaglichenko, A. V. Nikolaev, S. Horiuchi, and A. Hasegawa, "The Method for Consecutive Subtraction of Selected Anomalies: The Estimated Crustal Velocity Structure in the 1996 Onikobe (M = 5.9) Earthquake Area, Northeastern Japan," *Geophysical Journal International*, 153(3), 2003 pp. 627–644. doi:10.1046/j.1365-246X.2003.01918.x.
- [10] A. Douglas, D. Bowers, and J. B. Young, "On the Onset of P Seismograms," *Geophysical Journal International*, 129(3), 1997 pp. 681–690. doi:10.1111/j.1365-246X.1997.tb04503.x.
- [11] C. P. Gauss, *Beiträge zur Theorie der algebraischen Gleichungen*, Göttingen, 1849.
- [12] C. Lanczos, *Applied Analysis*, Englewood Cliffs, NJ: Prentice Hall, 1956.
- [13] T. A. Smaglichenko. "New Differentiated Approach for Seismic Tomography," presented at the 2007 European Geosciences Union Conference in Vienna, Austria.
<http://meetings.copernicus.org/www.cosis.net/abstracts/EGU2007/00474/EGU2007-J-00474.pdf>.
- [14] S. D. Silvey, *Statistical Inference*, London: Penguin, 1970.
- [15] S. Horiuchi, T. Matsuzawa, and A. Hasegawa, "A Real-Time Processing System of Seismic Wave Using Personal Computers," *Journal of Physics of the Earth*, 40, 1992 pp. 395–406.
- [16] T. A. Smaglichenko and A. V. Nikolaev, "A New Scheme for the Solution of Inverse Problems in Seismic Tomography," *Doklady Earth Sciences*, 415(6), 2007, pp. 890–894.