A Single Nonexpansive, Nonperiodic Rational Direction

K. M. Madden*
Department of Mathematics and Computer Science,
Drew University,
Madison, NJ 07940

A two-dimensional cellular automaton consists of a two-dimensional lattice of sites, each of which takes on a finite number of values, and a cellular automaton map. The cellular automaton map updates the value at each site $a \in \mathbb{Z}^2$ using a translation invariant rule that only depends on the values at the sites in some finite neighborhood of a. A number of global properties of a two-dimensional cellular automata, such as the directional entropies introduced by Milnor, can be studied using the methods of dynamical systems. In this work we consider E, the expansive one-dimensional subspaces of $\mathbb{R}^2$ as defined by Boyle and Lind. Various properties of cellular automata, including Milnor's directional entropies, vary nicely within connected components of E so it is natural to ask what subsets of $\mathbb{R}^2$ may occur as expansive one-dimensional subspaces. Boyle and Lind give an almost complete answer, the single unresolved case being when E is the complement of a line with irrational slope. In this work we construct a related example with the potential to shed light on the unresolved case.

1. Introduction

Let $\mathcal{A}$ be a fixed finite alphabet with $m \geq 2$ elements. We consider the infinite two-dimensional arrays from $\mathcal{A}$. Denote by $\mathcal{A}^{\mathbb{Z}^2}$ the space of all such arrays. Two arrays in this space are "close" if they coincide on some large finite set in $\mathbb{Z}^2$ and $\mathcal{A}^{\mathbb{Z}^2}$ is a compact metric space with respect to this topology. We note that $\mathbb{Z}^2$ acts *via* translation on $\mathcal{A}^{\mathbb{Z}^2}$ as a group of commuting homeomorphisms. The compact space $\mathcal{A}^{\mathbb{Z}^2}$ together with the group $\mathbb{Z}^2$ of commuting homeomorphisms is called the *full two-dimensional m-shift*.

A cellular automaton map updates the value at each site a in an array in $\mathcal{A}^{\mathbb{Z}^2}$ and depends only on the values of the sites in the array in a finite neighborhood of a. In [1] the typical behavior of two-dimensional cellular automata evolving from specific choices of the initial array are studied by empirical means. One might also consider the global proper-

*Electronic mail address: kmadden@drew.edu.

ties of a cellular automaton, that is, those determined by evolution of all possible arrays. Because most cellular automata maps are irreversible and multiple initial arrays may evolve to the same final array, only a subset X of $\mathcal{A}^{\mathbb{Z}^2}$ will be generated with time. The properties of this set X can then be used to describe the asymptotic behavior of the cellular automaton. Unfortunately, the set of arrays that can be generated by a two-dimensional cellular automaton can be difficult to specify and the determination of many global properties is undecidable [1].

One way around these difficulties it to consider global properties in various directions, or one-dimensional subspaces, of $\mathbb{R}^2$. Milnor [2] defined one-dimensional topological and measure theoretic directional entropy functions which can be used to characterize cellular automata. The work of Boyle and Lind [3] complements and expands on Milnor's work. Boyle and Lind define expansiveness along one-dimensional subspaces in $\mathbb{R}^2$ and show that various dynamical properties, such as directional entropy as defined by Milnor, vary continuously within connected components of expansive directions. They provide a nearly complete description of the possible one-dimensional entropy functions that can occur on expansive components of $\mathbb{Z}^2$ actions.

A direction, or one dimensional subspace S, will be called expansive for $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ roughly when any two distinct arrays in X differ on the sites in a "thickened" version of S. Otherwise, S is said to be nonexpansive. If we denote by E the collection of all expansive directions, it is natural to ask which collections of subspaces of $\mathbb{R}^2$ may occur as E for some $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$. Boyle and Lind provide an answer with the single unresolved case being when E is the complement of a single line with irrational slope. If E is the complement of a single line with rational slope, examples exist of $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ for which the expansive directions are E, but in all these examples the arrays are periodic in the direction of the slope of the rational line. Boyle and Lind ask the question of whether there exists an example of a single, nonexpansive, nonperiodic rational direction; in this work we construct such an example.

2. Definitions

A homeomorphism T of a compact metric space (X, ρ) is said to be *expansive* if every pair of distinct points in X is "separated" by some iterate of T. That is, there exists an $\epsilon > 0$ such that $\rho(T^n x, T^n y) \leq \epsilon$ for all $n \in \mathbb{Z}$ implies $x = y$. In one-dimensional dynamical systems theory the expansive homeomorphisms are an important class of transformations; they play a role in the study of topological entropy and in the exploitation of hyperbolicity in smooth dynamical systems. See, for example, [4] or [5].

In higher dimensions, the notion of expansiveness can be generalized in the obvious way to the case of n commuting homeomorphisms acting

on a compact metric space (X, ρ) [3]. We may also consider whether the continuous $\mathbb{Z}^k$ actions on X that we obtain by restricting ourselves to k-dimensional subspaces of $\mathbb{Z}^d$ are expansive. In fact, as discussed in [3], given a general subspace $F \subseteq \mathbb{R}^d$, it is useful to restrict ourselves to elements of $\mathbb{Z}^d$ within a bounded distance of F and investigate the "expansiveness" of this set of homeomorphisms. We will say that the subspace F is expansive roughly if there exists $t > 0$ so that the lattice points in F thickened by t separate elements of X. The following formal definitions can be found in [3].

Definition 1.

1. Let $F \subseteq \mathbb{R}^d$ and let $t > 0$. By F^t we will mean the set of lattice points in F thickened by t. That is,

$$F^t = \{n \in \mathbb{Z}^d : d(n, F) \leq t\}$$

   where as usual $d(n, F) = \inf\{|n - w| : w \in F\}$.

2. For a subset $S \subseteq \mathbb{Z}^d$ and $x, y \in X$, let

$$\rho^S(x, y) = \sup\{\rho(ax, ay) : a \in S\}.$$

   If $S = \emptyset$, let $\rho^S(x, y) = 0$.

We will say that a subspace $F \subseteq \mathbb{R}^d$ is *expansive* if there exists $\epsilon > 0$ and a $t > 0$ such that $\rho^{F^t}(x, y) \leq \epsilon$ implies that $x = y$. If F is not expansive, we will say it is nonexpansive.

We will restrict our attention to the case when X is a closed, translation invariant subset of $\mathcal{A}^{\mathbb{Z}^2}$ with $\mathbb{Z}^2$ acting on X *via* translation: for $n = (n_1, n_2) \in \mathbb{Z}^2$ and $x \in X$, $nx(i, j) = x(i + n_1, j + n_2)$. The metric giving the topology on X is as follows: For $a, a' \in \mathcal{A}$ we let $\nu(a, a') = 1$ when $a \neq a'$ and 0 otherwise and set

$$\rho(x, y) = \sum_{(i,j) \in \mathbb{Z}^2} 2^{-(|i|+|j|)} \nu(x(i, j), y(i, j)).$$

Because $\rho(x, y) < 1$ only if $x(0, 0) = y(0, 0)$, it follows easily from the definitions that a subspace F of $\mathbb{R}^2$ is expansive if it has one of the following equivalent properties.

1. There exists a $t > 0$ so that if $x, y \in X$ agree on F^t, then $x = y$.

2. There exists $t > 0$ such that restricting $x \in X$ to F^t completely determines x.

If $X = \mathcal{A}^{\mathbb{Z}^2}$, then $\mathbb{Z}^2$ is expansive but all proper subspaces of $\mathbb{R}^2$ are nonexpansive. The interested reader is directed to [3] for further examples.

The collection of expansive (or nonexpansive) subspaces of $\mathbb{R}^2$ can be used to characterize X. In addition, Boyle and Lind show that other global properties, such as Milnor's directional entropies, vary nicely over connected components of expansive subspaces. Thus, it is natural to ask which collections of subspaces of $\mathbb{R}^2$ can occur as the expansive directions for some $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$.

If we let G denote the Grassman manifold of all one-dimensional subspaces of $\mathbb{R}^2$ with the distance between two subspaces determined by the Hausdorff metric distance between the intersections of the subspaces with the unit ball in $\mathbb{R}^2$, then the collection of nonexpansive subspaces of $\mathbb{R}^2$ is a nonempty compact set in G [3]. Furthermore, we have the following theorem.

Theorem 1. [3] Let $\mathcal{L}$ be a compact set in G that is not a singleton containing just one irrational line. Then there exists a closed, invariant subset of $\mathcal{A}^{\mathbb{Z}^2}$ whose collection of one-dimensional nonexpansive subspaces is $\mathcal{L}$.

Boyle and Lind use this result to, among other things, provide an almost complete description of the possible one-dimensional entropy functions that can occur when the expansive components are proper cones or have a single rational line as their complement. Despite a great deal of effort, whether or not there exists a $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ with a single irrational nonexpansive direction remains an open question. Boyle and Lind raise a related question: Suppose L is a line with rational slope passing through $n \in \mathbb{Z}^2$. Theorem 1 guarantees the existence of $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ whose only nonexpansive direction is L, however all previously known such examples are "periodic in the n direction." That is, there exists $p \in Q$ for which $pn \in \mathbb{Z}^2$ is the identity homeomorphism on X. The fact that there can be no such periodicity in an irrational direction is a fundamental difference between rational and irrational directions. In the hope of gaining insight into the unresolved case of the theorem, Boyle and Lind ask whether there exists an example of a $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ without periodic directions whose only nonexpansive subspace is a line with rational slope. In the next section we construct such an example.

3. The example

Let $\mathcal{A} = \{0, 1, 2, 3\}$. In this section we will construct a closed, invariant subset X of $\mathcal{A}^{\mathbb{Z}^2}$ which has the following properties. (1) It is expansive in every direction except the rational direction $\pi/2$. That is, given any line L_θ with slope θ other than $\pi/2$, there exists a $t > 0$ such that restricting an array $x \in X$ to L_θ^t completely determines x. (2) It is nonexpansive in the rational direction $\pi/2$; for any $t > 0$, there are distinct arrays x,

$y \in X$ that agree on $L^t_{\pi/2}$. (3) The arrays in X need not be periodic in the $\pi/2$ direction.

Construct the two-dimensional arrays in $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ as follows. The symbols 2 and 3 occur in complete vertical sequences in the sense that $x(i,j) = 2$ (or 3) if and only if $x(i,k) = 2$ (or 3) for all $k \in \mathbb{Z}$. These vertical sequences of 2s and 3s alternate and are found in every fifth vertical column. That is, if $x(i,j) = 2$ then $x(i\pm5, j) = 3$ and $x(i\pm10, j) = 2$ and so on. We call the vertical sequences of 2s and 3s of an array its *2-3 skeleton*. Note that for any $0 \le \theta < \pi$, for t large enough, L^t_θ will determine the 2-3 skeleton of any array x.

The remaining vertical columns consist of 0s and 1s satisfying the following rules. If the configuration

$$\begin{array}{cc} x(i,j+1) & \\ x(i,j) & x(i+1,j) \end{array}$$

lies with a vertical sequence of 3s to the left and a vertical sequence of 2s to the right, then it sums to zero mod 2. For the strips bounded on the left by 2s and on the right by 3s, we change the shape. There configurations of the form

$$\begin{array}{cc} x(i,j) & x(i+1,j) \\ x(i,j-1) & \end{array}$$

sum to zero. We refer to these configurations as T_1 and T_2 respectively.

If the configurations below span a vertical sequence of 3s, or 2s respectively, then they sum to zero mod 2:

$$\begin{array}{cc} x(i,j+1) & \\ x(i,j) & x(i+2,j) \end{array} \qquad \begin{array}{cc} x(i,j) & x(i+2,j) \\ x(i,j-1). & \end{array}$$

Theorem 2. Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be all two-dimensional arrays as described and let $\mathbb{Z}^2$ act on X by translation. Then the only nonexpansive subspace of $\mathbb{R}^2$ is $L_{\pi/2}$ and the $\mathbb{Z}^2$ action is not periodic in the $\pi/2$ direction.

Proof. Clearly X is a closed, shift invariant subset of $\mathcal{A}^{\mathbb{Z}^2}$ and $\mathbb{Z}^2$ is an expansive action on the space X.

In constructing a two-dimensional array $x \in X$, each vertical 0-1 line $\mathcal{Z}$ determines a vertical 0-1 line on its right, by one of two coding rules, depending on the location of $\mathcal{Z}$ relative to the 2s and 3s. These two coding rules are two-to-one. Given a 2-3 skeleton, we can arbitrarily fill in a 0-1 line $\mathcal{Z}$ with any vertical sequence z of 0s and 1s and apply the coding rules to legally fill in to the right. Choosing inverse images going to the left gives us an uncountable number of legal points compatible with that choice for $\mathcal{Z}$; thus for any $t > 0$, there are infinitely many points in X that agree on $L^t_{\pi/2}$ and the vertical direction dynamics are nonexpansive. Because the vertical 0-1 line $\mathcal{Z}$ can be filled in arbitrarily, the vertical direction dynamics are nonperiodic.

The following three Lemmas show that all other directions are expansive. ■

Lemma 1. Let $0 \le \theta < \pi$ with $\theta \notin \{0, \pi/4, \pi/2, 3\pi/4\}$. Then L_θ is an expansive subspace of $\mathbb{R}^2$.

Proof. Notice that our exceptions for θ come from lines that are parallel to sides of the configurations T_1 and T_2 used in defining our coding rules. The argument we will use is similar to the one used in Example 2.8 of [3] where it is shown that the Leddrapier three dot example is expansive in all directions θ except $\{0, \pi/2, 3\pi/4\}$. (The Leddrapier example consists of all arrays in $\{0,1\}^{\mathbb{Z}^2}$ for which any configuration of the form T_1 sums to zero.) We will show that for any θ as described, there exists a $t > 0$ such that L_θ^t completely determines any array in X.

Consider configurations S_1 and S_2 determined by vertices $\{0, e_1, e_2, 2e_1, e_1 + e_2\}$ and $\{0, e_1, e_2, e_1 + e_2, 2e_1 + e_2\}$ respectively.

There is an $s > 0$ with the following property: For large enough t, each lattice point $x(i,j)$ in $L_\theta^{t+s} \setminus L_\theta^t$ is in a translate of S_1 whose other vertices lie in L_θ^t and in a translate of S_2 whose other vertices lie in L_θ^t. Either the translate of S_1 or the translate of S_2 (depending on the location of $x(i,j)$ relative to the 2-3 skeleton) determines $x(i,j)$.

Restricting $x \in X$ to L_θ^t determines the 2-3 skeleton of x and determines x restricted to L_θ^{t+s} which in turn determines L_θ^{t+2s} and so on, and thus, L_θ is an expansive subspace of $\mathbb{R}^2$. ■

It remains to show that the directions 0, $\pi/4$, and $3\pi/4$ are expansive directions.

Lemma 2. The horizontal direction is expansive for $(X, \mathbb{Z}^2)$.

Proof. In this proof we will show that for any $t > 0$, if arrays $x, y \in X$ agree on L_0^t, then they are identical.

Suppose that $x(i,j) = x'(i,j)$ for all $i \in \mathbb{Z}$. (Note that x and x' must have the same 2-3 skeleton.) We will show that this implies that $x(i, j+1) = x'(i, j+1)$ for all $i \in \mathbb{Z}$.

Case 1. It is clear that if the ith vertical column of x lies with vertical sequences of 3s to the left and 2s at least two columns to the right, then $x(i, j+1) = x'(i, j+1)$ since

$$\begin{aligned} &x(i, j+1) + x(i, j) + x(i+1, j) \\ &\quad = x'(i, j+1) + x'(i, j) + x'(i+1, j) \mod 2. \end{aligned}$$

Case 2. A similar argument holds if the ith vertical column of x lies with a vertical sequence of 3s directly to the right.

Case 3. If the ith vertical column of x lies with vertical sequences of 2s to the left and 3s at least two columns to the right then $x(i, j+1) \neq x'(i, j+1)$

implies $x(i+1,j+1) \neq x'(i+1,j+1)$ since

$$x(i,j) + x(i,j+1) + x(i+1,j+1) \\ = x'(i,j)x'(i,j+1) + x'(i+1,j+1) \mod 2.$$

If column $(i+1)$ falls into Case 2 we have a contradiction. Otherwise, $x(i+1,j+1) \neq x'(i+1,j+1)$ implies $x(i+2,j+1) \neq x'(i+2,j+1)$ and so on until we have a contradiction.

Case 4. If the ith vertical column of x lies with a vertical sequence of 2s directly to the right, then $x(i,j+1) \neq x'(i,j+1)$ implies $x(i+2,j+1) \neq x'(i+2,j+1)$ and the $(i+2)$ vertical column falls into Case 3 giving us a contradiction.

A similar argument shows that if $x(i,j) = x'(i,j)$ for all $i \in \mathbb{Z}$ then $x(i,j-1) = x'(i,j-1)$ for all $i \in \mathbb{Z}$. Thus, if $x(i,j) = x'(i,j)$ for all $(i,j) \in L_0^t \cap \mathbb{Z}^2$, then $x(i,j) = x'(i,j)$ for all $(i,j) \in L_0^{t+1} \cap \mathbb{Z}^2$ and so on, giving $x = x'$. ■

Lemma 3. The direction $\theta = \pi/4$ is expansive for $(X, \mathbb{Z}^2)$.

Proof. We show that if for all $i \in \mathbb{Z}$, $x(i,j) = x'(i,j)$ for $j = i-1, i, i+1$ then $x(i,i+2) = x'(i,i+2)$.

Case 1. Suppose that the ith vertical column of x lies with vertical sequences of 3s to the left and 2s at least two columns to the right. Then $x(i,i+2) = x'(i,i+2)$ since $x(i,i+2) + x(i,i+1) + x(i+1,i+1)$ and $x'(i,i+2) + x'(i,i+1) + x'(i+1,i+1)$ are equivalent mod 2.

Case 2. A similar argument holds if the ith vertical column of x lies with a vertical sequence of 3s directly to the right since $x(i,i+2) + x(i,i+1) + x(i+2,i+1)$ and $x'(i,i+2) + x'(i,i+1) + x'(i+2,i+1)$ are equivalent mod 2.

Case 3. If the ith vertical column of x lies with vertical sequences of 2s to the left and 3s at least two columns to the right. Then $x(i,i+2) = x'(i,i+2)$ since $x(i,i+2) + x(i,i+1) + x(i+1,i+2)$ and $x'(i,i+2) + x'(i,i+1) + x'(i+1,i+2)$ are equivalent mod 2.

Case 4. Suppose that the ith vertical column of x lies with a vertical sequence of 2s directly to the right. Then $x(i,i+2) = x'(i,i+2)$ since $x(i,i+2) + x(i,i+1) + x(i+2,i+2)$ and $x'(i,i+2) + x'(i,i+1) + x'(i+2,i+2)$ are equivalent mod 2.

Similarly if for all $i \in \mathbb{Z}$, $x(i,j) = x'(i,j)$ for $j = i-1, i, i+1$ then $x(i,i-2) = x'(i,i-2)$. Thus, if $x(i,j) = x'(i,j)$ for all $(i,j) \in L_0^t \cap \mathbb{Z}^2$ and $t > \sqrt{2}/2$, then $x(i,j) = x'(i,j)$ for all $(i,j) \in L_0^{t+\sqrt{2}/2} \cap \mathbb{Z}^2$ and so on, giving $x = x'$. ■

An argument similar to the one used in Lemma 3 shows that $\theta = 3\pi/4$ is an expansive direction for $(X, \mathbb{Z}^2)$ as well.

4. Conclusion

The hope for the example constructed in this work was that it would yield new insights into the following open question.

Question 1. Does there exist $X \in \mathcal{A}^{\mathbb{Z}^2}$ whose only nonexpansive direction is irrational?

Unfortunately, the fact that the nonexpansive direction in our example was rational figured heavily in its construction and the answer to Question 1 will have to come from other methods.

The resolution of Question 1 will lead to a complete understanding of the possible sets of nonexpansive subspaces for two-dimensional cellular automata. This understanding is a worthy goal in its own right and it is also necessary to extend the theory to higher dimensions. We can ask which collections of $\mathbb{R}^m$ can occur as the expansive subspaces for some $X \in \mathcal{A}^{\mathbb{Z}^m}$ for $m > 2$. The answer to this question is still quite incomplete, see [3] for what is known. Question 1 has fairly complicated implications for higher dimensions and until it is resolved, the theory cannot advance effectively.

Acknowledgment

The author would like to thank Mike Boyle for many useful suggestions.

References

[1] N. Packard and S. Wolfram, "Two Dimensional Cellular Automata," *Journal of Statistical Physics*, **38** (1985) 901–946.

[2] J. Milnor, "On the Entropy Geometry of Cellular Automata," *Complex Systems*, **2** (1988) 357–386.

[3] M. Boyle and D. Lind, "Expansive Subdynamics," *Transactions of the American Mathematical Scoiety*, **349**(1) (1997) 55–102.

[4] R. Mañé, *Ergodic Theory and Differentiable Dynamics* (Springer-Verlag, New York, 1983).

[5] P. Walters, *An Introduction to Ergodic Theory* (Springer-Verlag, New York, 1982).