

Measuring Mutual Information in Random Boolean Networks

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During the past few years an area of active research in the field of complex systems is that of their information storing and processing abilities. Common opinion has it that the most interesting behavior of these systems is found “at the edge of chaos,” which would seem to suggest that complex systems may have inherently nontrivial information processing abilities in the vicinity of sharp phase transitions. A comprehensive, quantitative understanding of why this is the case is however still lacking. Indeed, even “experimental” (i.e., often numerical) evidence that this is so has been questioned for a number of systems. In this paper we will investigate, both numerically and analytically, the behavior of random boolean networks (RBNs) as they undergo their order–disorder phase transition. We will use a simple mean field approximation to treat the problem, and without lack of generality we will concentrate on a particular value for the connectivity of the system. In spite of the simplicity of our arguments, we will be able to reproduce analytically the amount of mutual information contained in the system as measured from numerical simulations.

1. Introduction

The amount of information that a system is able to process (and/or store) plays an essential role when one tries to quantify the level of “complexity” [1] of a system, and indeed often the mutual information [2] stored in the system (or a concept derived from it, such as past–future mutual information) is used as a measure of its statistical complexity [3].

Over the past decade a number of authors have carried out work towards understanding under what conditions can we expect to maximize the information processing capabilities of different types of complex systems. For instance, Langton and others [4, 5] investigated the behavior of cellular automata (CA), while Crutchfield, Young, and others [6] have been concerned mainly with iterated function systems and computational complexity in this area. The definitions used for complexity were rather problem dependent, and not surprisingly two main approaches to measuring statistical complexity have been developed over the years,

as well as a large number of other *ad hoc* methods for describing structure. The first line of work uses information theory [7–10], whereas the second approach defines complexity using computation theoretic tools [6, 11].

In spite of this model dependence, the common picture that seemed to emerge from this work was that complex systems were able to show a maximally varied and self-organized behavior (i.e., maximally complex behavior) in the vicinity of sharp phase transitions [12]. Since these transitions often belonged to the class commonly known in statistical mechanics as order–disorder phase transitions, this naturally led to the notion that maximally interesting behavior of complex systems takes place “at the edge of chaos,” in an expression first coined by Packard [13]. (Note however that the disordered phase does not necessarily need to be chaotic in the strict sense of the word, i.e., ergodic.) The underlying reason was simple and appealing enough, neither very ordered systems with static structures, nor disordered systems in which information cannot be persistently stored are capable of complex information processing tasks.

The actual verification of the fact that the mutual information (or definitions of statistical complexity based on other approaches) had a maximum in the vicinity of the relevant phase transitions were a trickier business though. Early results by Langton for CAs [4, 5] and by Crutchfield [6, 11] for iterated dynamics showing sharp peaks in complexity as a function of the degree of order in the system at what appeared to be phase transitions were subsequently shown to be critically dependent on the particular measure of order chosen [3]. After this, Arnold [14] showed numerically that the two-dimensional Ising model indeed had a maximum of statistical complexity (defined through past–future mutual information) at its order–disorder transition.

Without wanting to go into the debate of what exactly constitutes a good measure of complexity—a debate often riddled with the specifics of the particular problem at hand—it would seem clear though that complexity and information must bear a close relationship. We will thus concern ourselves in this paper with the mutual information contained in random boolean networks (RBNs) [15] and its behavior as the networks undergo their order–disorder phase transition (for a computational viewpoint see [16]). By using a mean field approximation and assuming markovian behavior of the automata, we will show both numerically and analytically that the mutual information stored in the network indeed has a maximum at the transition point.

2. Random boolean networks

RBNs [15] are systems composed of a number N of automata ($i = 1, \dots, N$) with only two states available (say, 0 and 1, for instance), each having associated with it a boolean function f_i of K boolean argu-

ments that will be used to update the automaton state at each time step. Each automaton i will then have associated with it K other automata i_1, i_2, \dots, i_k (the inputs or vicinity of i), whose states $(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ will be the entries of f_i . That is, the automaton i will change its state x_i at each time step according to the rule

$$x_i(t+1) = f_i(x_{i_1}(t), x_{i_2}(t), \dots, x_{i_k}(t)). \quad (1)$$

Both f_i and the identity of its K inputs are initially assigned to the automaton i at random. (In particular, the N f s are created by randomly generating outputs of value 1 with a probability p , and of value 0 with a probability $1 - p$, where p is called the *bias* of the network.) This initial random assignment of neighbors and functions will be maintained through the evolution of the system, so we will be dealing with a quenched system. Even keeping this assignment fixed, the number of possible networks that we can form for given values of N and K is extraordinarily high (a total of $(2^{2^K} N^K)^N$ possible networks). Thus, if we want to study general characteristics of RBN systems we are inevitably led to a statistical approach.

One fact that can be observed for all RBN is that although the number of available states for a network of size N grows by 2^N , the dynamics of the net separates the possible states into disjoint sets, or attractor basins. Each basin will lead the system to a different attractor. However, since the number of states available is finite and the quenched system is fully deterministic, we can be sure that the system will at some point retrace its steps in the form of periodic cycles. Thus attractors will necessarily be periodic sets of states. Since after a transient any initial state will end up in one attractor or another, their period (or rather their average period) will set the typical time scale characterizing an RBN.

It has been known for some time now [15] that RBNs show two different phases separated, for a given value of p , by a critical value of K , K_c .

1. An ordered phase for $K < K_c$ in which the networks freeze in a pattern after a short transient. In this phase almost all of the automata remain in a completely frozen state and the median period T of the attractors scale with N as a power law.
2. A disordered phase for $K > K_c$. All patterns are lost and the automata appear to be in a completely disordered state, switching from one state to another seemingly at random. The period of the attractors become unobservable in practice because the median period T grows exponentially with N , thereby rendering the system free of any time scale [14].

This behavior naturally induced the conjecture that at K_c the RBNs undergo a second order phase transition. This conjecture has been proven correct and some more information about the transition has

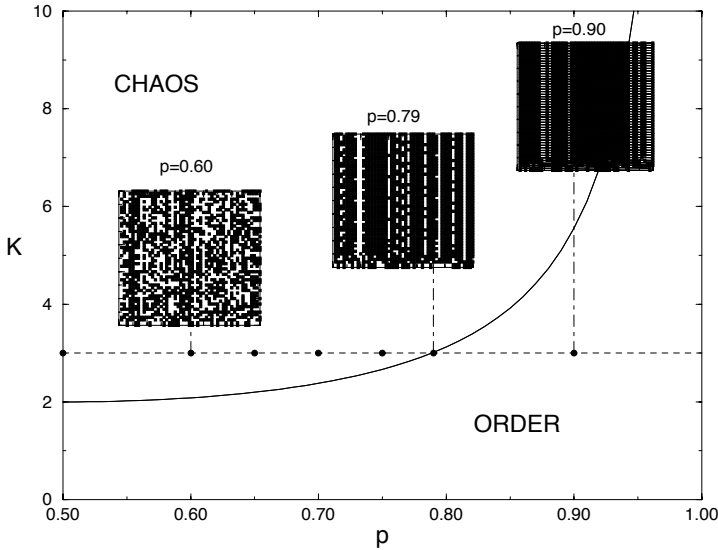


Figure 1. The boundary between the chaotic and the ordered phase is shown in a K - p phase diagram. For a constant value of K , $K = 3$, three set examples of a $N = 50$ network are shown for $p = 0.60$ (disordered phase), $p = 0.79$ (over the critical line), and $p = 0.90$ (ordered phase). Each run contains 50 consecutive states, time increasing upwards along the vertical axis.

been gained [18]. For instance, as we change the value of p the critical value K_c at which the transition takes place also changes and a “critical line” appears, as shown in Figure 1. As was shown in [19] this line corresponds to

$$K = \frac{1}{2p(1-p)}. \quad (2)$$

In the insets of Figure 1, three sets of states of a network with $N = 50$ and $K = 3$ are also shown as we move from the disordered state to the ordered one by changing p , showing a typical order–disorder transition. Each set of states contains 50 consecutive states, time running upwards along the vertical axis.

3. Self-overlap in random boolean networks

Since RBNs appear to undergo an order–disorder phase transition, a useful way to characterize the state of the system will be its *self-overlap*, a . This is simply defined to be one minus the Hamming distance between an automaton at time t and itself at time $t+1$, averaged over all automata and times. Let us expand on this.

Suppose that we generate an RBN with bias p , and a random initial condition. We let the system evolve until the transient dies out and we are inside an attractor cycle, and then compute the states of the system for a number of time steps equal to the number of automata in the system (That is, from $t = 1$ to $t = 10,000$ for the $N = 10,000$ network that we have used. Each experimental computer point in all figures is the average of 100 different networks with random initial conditions.) Let us suppose that we count the number of times that an automata is in the state 1 both at time t and $t + 1$, and average over all automata and time steps. This will give us the *1 state self-overlap*, a_{11} . Repeating this procedure with the 0 state will then obviously give us the *0 state self-overlap*, a_{00} . Then, a will simply be given by

$$a = a_{11} + a_{00}. \quad (3)$$

On the other hand, we can analogously define a_{10} and a_{01} . Note that by symmetry we have to have $a_{10} = a_{01}$ even with $p \neq 1/2$, since a_{10} and a_{01} are the joint probability distributions, not the conditional probabilities of transitioning from 1 to 0 or *vice versa*.

It is then fairly easy to find the equation that describes the evolution of a . If we define P to be

$$P = p^2 + (1 - p)^2, \quad (4)$$

then it is not difficult to convince oneself that in a mean field approximation we must have

$$a_{t+1} = a_t^K + P(1 - a_t^K), \quad (5)$$

where K is the connectivity of the net. This equation forces a to evolve towards fixed points $a_t \rightarrow a^*$ that will depend on K and p . The stability analysis of equation (5) for $a^* = 1$ gives the critical line (equation (2)) separating the ordered phase ($a^* = 1$) from the disordered phase ($a^* < 1$). This is shown in Figure 2, where the evolution of a given by equation (5) (solid line) is plotted against the results of the numerical simulations (dots). The evolution lasts for as long as it takes the transient to die out, and once the system is in the attractor cycle a takes on its fixed point value (from now on we drop the star and designate the fixed point value simply by a).

Let us now obtain analytical expressions for the $a_{\alpha\beta}$ from our knowledge of a , the normalization conditions, and the fact that $a_{10} = a_{01}$. By equation (3) and by normalization

$$a_{00} + a_{01} + a_{10} + a_{11} = 1, \quad (6)$$

so that

$$a_{01} + a_{10} = 1 - a. \quad (7)$$

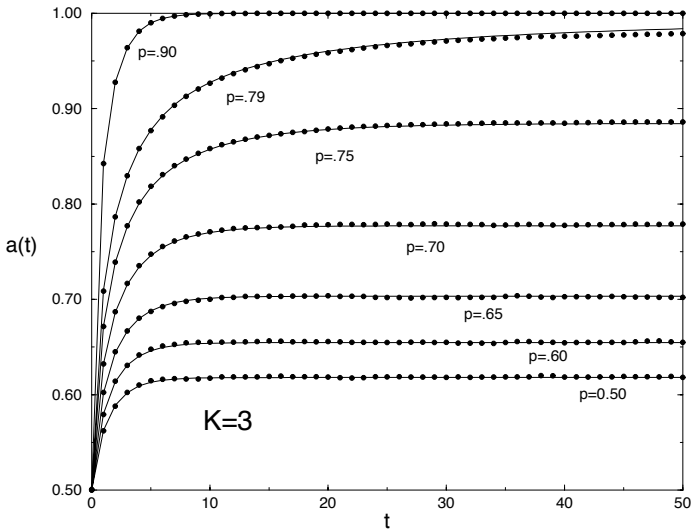


Figure 2. Plot of $a(t)$ versus t showing the numerical results from the simulations (dots) and the values predicted by the mean field approximation (equation (5)), for $K = 3$. Different values of the bias p , from $p = 0.5$ to $p = 0.9$ with $p_c = 0.79$ are shown.

But then, by symmetry,

$$a_{10} = a_{01} = \frac{1 - a}{2}. \tag{8}$$

We still have two more normalization conditions, derived from the fact that the probability of finding a mean field automaton in the state 1 is p , and $1 - p$ for the state 0

$$a_{11} + a_{10} = p, \tag{9}$$

$$a_{00} + a_{01} = 1 - p, \tag{10}$$

whence

$$a_{00} = \frac{a}{2} - \left(p - \frac{1}{2} \right), \tag{11}$$

$$a_{11} = \frac{a}{2} + \left(p - \frac{1}{2} \right), \tag{12}$$

which satisfy equation (3). Figure 3 shows the analytical expressions for the $a_{\alpha\beta}$ (solid lines) together with the results from the numerical simulations (dots).

So far we have simply approximated the whole network by a set of mean field automata. However, since the $a_{\alpha\beta}$ are equivalent to $p_{\alpha\cap\beta}$ we

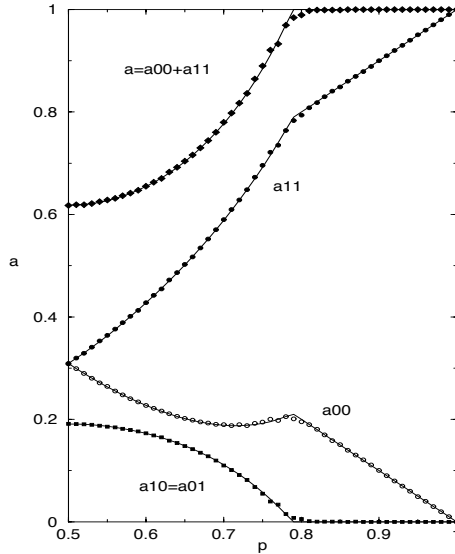


Figure 3. Analytical (continuous line) and numerical results (dots) for the $a_{\alpha\beta}$ as p ranges from 0.5 to 1.

can now calculate the conditional probabilities

$$p_{\alpha\beta} = \frac{a_{\alpha\beta}}{p_\beta}. \tag{13}$$

If we now assume that our mean field automata are markovian these conditional probabilities will completely characterize their transition probabilities [2]. Therefore, the transition matrix for the mean field markovian automaton is:

$$\mathbf{T} = \begin{pmatrix} p_{0|0} & p_{1|0} \\ p_{0|1} & p_{1|1} \end{pmatrix} = \begin{pmatrix} \frac{a+1-2p}{2(1-p)} & \frac{1-a}{2(1-p)} \\ \frac{1-a}{2p} & \frac{a-1+2p}{2p} \end{pmatrix} \tag{14}$$

which satisfy

$$\sum_{\alpha=0}^1 p_{\alpha|0} = 1, \tag{15}$$

$$\sum_{\alpha=0}^1 p_{\alpha|1} = 1, \tag{16}$$

where $p_{\alpha|0}, p_{\alpha|1}$ are the probabilities of transitioning from the states 0, 1 to the state α .

Thus, we have now reduced the whole network to a set of mean field automata evolving independently under markovian conditions, all the

effects of their interactions being encoded in a . To compute the past-future mutual information stored in the system we only have to apply information theory [3, 20]. The one-automaton entropy is simply

$$H(x_{t+1}) = -p \log p - (1-p) \log(1-p), \quad (17)$$

whereas the Shannon uncertainty associated to the markovian evolution of this automaton will be

$$H(x_{t+1} | x_t) = pH(x_{t+1} | x_t = 1) + (1-p)H(x_{t+1} | x_t = 0), \quad (18)$$

with

$$\begin{aligned} H(x_{t+1} | x_t = 1) = & -\frac{a-1+2p}{2p} \log\left(\frac{a-1+2p}{2p}\right) \\ & -\frac{1-a}{2p} \log\left(\frac{1-a}{2p}\right), \end{aligned} \quad (19)$$

and

$$\begin{aligned} H(x_{t+1} | x_t = 0) = & -\frac{a+1-2p}{2(1-p)} \log\left(\frac{a+1-2p}{2(1-p)}\right) \\ & -\frac{1-a}{2(1-p)} \log\left(\frac{1-a}{2(1-p)}\right). \end{aligned} \quad (20)$$

The uncertainty is thus,

$$\begin{aligned} H(x_{t+1} | x_t) = & -\frac{a-1+2p}{2} \log\left(\frac{a-1+2p}{2p}\right) - \frac{1-a}{p} \log\left(\frac{1-a}{2p}\right) \\ & -\frac{a+1-2p}{2} \log\left(\frac{a+1-2p}{2(1-p)}\right) - \frac{1-a}{2} \log\left(\frac{1-a}{2(1-p)}\right), \end{aligned} \quad (21)$$

whence the past-future mutual information will be:

$$\begin{aligned} I = & H(x_{t+1}) - H(x_{t+1} | x_t) \\ = & -p \log p - (1-p) \log(1-p) + \frac{a-1+2p}{2} \log\left(\frac{a-1+2p}{2p}\right) \\ & + \frac{1-a}{p} \log\left(\frac{1-a}{2p}\right) + \frac{a+1-2p}{2} \log\left(\frac{a+1-2p}{2(1-p)}\right) \\ & + \frac{1-a}{2} \log\left(\frac{1-a}{2(1-p)}\right). \end{aligned} \quad (22)$$

Figure 4 shows the analytical expressions (solid lines) as well as the experimental results from the simulations for both the one-automaton

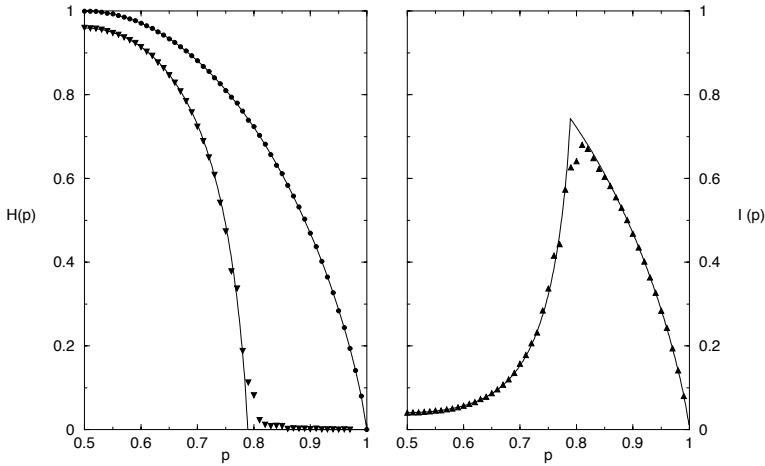


Figure 4. Numerical and analytical results for both $H(x_{t+1})$ (filled dots for the numerical results) and $H(x_{t+1} | x_t)$ (filled triangles) are shown in the left-hand side. Note how $H(x_{t+1} | x_t)$ is always smaller and decays faster than $H(x_{t+1})$, becoming 0 in the ordered phase. In the right-hand side I versus p is plotted, showing a peak at the critical value $p_c = 0.79$ as expected.

entropy (dots) and the Shannon uncertainty (triangles). Note how the uncertainty is always smaller and decays faster than the one-automaton entropy. In particular, for $p \geq p_c$ (where $p_c = 0.79$ for our net with $K = 3$), we have $a = 1$ and $H(x_{t+1} | x_t) = 0$. Thus in the ordered phase the mutual information becomes simply the one-automaton entropy. Given this discussion, it is obvious that the mutual information that can be stored in the system has to have a maximum precisely at p_c . This is shown in Figure 4, where the mutual information I is plotted against p (again, both the analytical expression above as well as the experimental results).

Finally, in Figure 5 the mutual information is plotted against the one-automaton entropy H . From $H = 0$, corresponding to $p = 1$; to $H \approx 0.75$, which corresponds to the critical value $p = p_c$, we see that I is just a straight line of slope 1. This is as it should be; since, as we just saw, $H(x_{t+1} | x_t)$ is 0 for p beyond p_c , and $I = H(x_{t+1})$ in this region. Precisely at $H(p_c)$, I reaches a maximum, and beyond this point it starts to decay nonlinearly as the Shannon uncertainty switches on.

4. Conclusions

By using a mean field approximation and a markovian ansatz for the evolution of an RBN, we have been able to show with a few, back of the envelope type of calculations, that the past–future mutual information

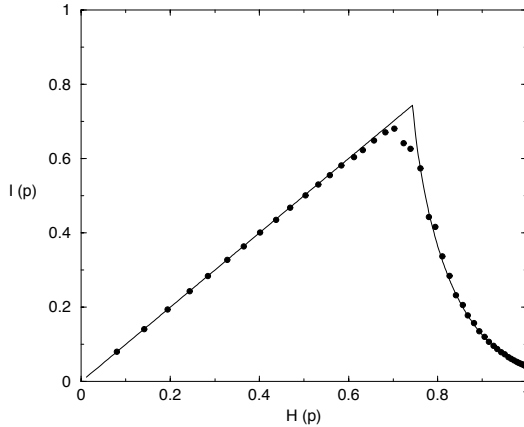


Figure 5. The past–future mutual information I versus the one-automaton entropy $H(x_{t+1})$. From $H = 0$ (corresponding to $p = 1$) to $H(p_c)$ we have $I = H(x_{t+1})$ since $H(x_{t+1} | x_t) = 0$. Therefore, in this region the mutual information simply increases linearly with the one block entropy. Beyond this point however we enter the disordered phase and $H(x_{t+1} | x_t)$ switches on, growing faster (in absolute value) than the one block entropy. Therefore, I shows a peak exactly at the transition point.

contained in a RBN reaches a maximum at the point at which this system undergoes its order–disorder phase transition. Also, in Figure 5 we can see how the mutual information as a function of the amount of disorder present in the system (the one-automaton entropy) reaches a maximum at the point that corresponds to the phase transition.

Similar results obtained in [4, 5] (for CAs) and in [6] (for symbolic dynamics of the logistic map) were explained by Li [3] on the ground that the peak was an artifact created by the particular quantity chosen to measure the disorder of the system. Thus, for instance, Li criticizes Langton arguing that since in the ordered phase we have $I = H(x_t)$, it is only natural for him to find a straight line as the boundary of his plot of complexity against disorder (as we do). Li surmises that if instead of using $H(x_t)$ as a measure of the disorder of the system one chooses to use the Shannon uncertainty of the source $H(x_{t+1} | x_t)$ ($H_{t+1|t}$ for short) the left side of the plot would no longer be a straight line, and the maximum of I would not be reached for intermediate values of the disorder. Rather, in the I versus $H_{t+1|t}$ plot the maximum of I falls over the y -axis since I reaches a maximum at 0 ($H_{t+1|t}$), and I monotonically decreases as $H_{t+1|t}$ increases. Thus, the intuitive picture of the relationship between complexity and disorder proposed by Langton and others (i.e., unimodal relationship between complexity and disorder with complexity reaching a maximum at intermediate values of the latter) would no longer seem to

be correct. This Li takes as support of his conclusion that the dependence of I on the amount of disorder in the system can take many varied forms.

We think that the argument just presented, although trivially correct, fails to capture the essence behind the idea of unimodal dependence between I and the amount of disorder in the system. We should first note that $I(H_{t+1|t} = 0)$ is not a single-valued function. Rather, since $H_{t+1|t} = 0$ for $p_c \leq p \leq 1$, at $H_{t+1|t} = 0$ I grows from 0 (corresponding to $p = 1$) to its maximum value (corresponding to $p = p_c$). That is, we have not gotten rid of the straight line in the I versus H_t graph, we have merely made it into a vertical line placed at $H_{t+1|t} = 0$. Note however that the maximum of I would still be reached at the transition point between the two phases of the system. This is, in fact, the central point of the issue at hand. The postulated unimodal dependence between I (or complexity) and disorder rests under the assumption that, as we vary the order parameter, the system goes from an ordered phase into a disordered one with I attaining its maximum value neither at one phase nor the other, but precisely at the transition point between them. If the quantity chosen as the order parameter varies over both phases then I will reach this maximum for intermediate values of the parameter. If, on the other hand, a whole phase of the system is mapped into a single value of the order parameter, then quite obviously the maximum will be at one of the edges of the graph. Thus one could say that the essence of unimodality relies not on I reaching its maximum for intermediate values of the order parameter, but on such a maximum being at the transition point between the ordered and the disordered phases.

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