Non-Gödel Negation Makes Unwitnessed Consistency Undecidable*

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Abstract. Recent results show that ontology consistency is undecidable for a wide variety of fuzzy Description Logics (DLs). Most notably, undecidability arises for a family of inexpressive fuzzy DLs using only conjunction, existential restrictions, and residual negation, even if the ontology itself is crisp. All those results depend on restricting reasoning to witnessed models. In this paper, we show that ontology consistency for inexpressive fuzzy DLs using any t-norm starting with the Łukasiewicz t-norm is also undecidable w.r.t. general models.

1 Introduction

Fuzzy Description Logics (DLs) extend the semantics of classical DLs by allowing a set of values, typically the real interval $[0, 1]$, to serve as truth degrees. They were introduced for representing and reasoning with vague or imprecise knowledge in a formal way [14,15]. While several decision procedures for fuzzy DLs exist, they usually rely on a restriction of the expressivity: either by allowing only finitely-valued semantics [5,7] or by limiting the terminological knowledge to be acyclic or unfoldable [11,9,4]. Indeed, the only algorithms for fuzzy $\mathcal{ALC}$ with general ontologies are based on the very simple Gödel semantics [13,14,15].

It was recently shown that decision procedures for more expressive fuzzy DLs cannot exist since consistency of fuzzy $\mathcal{ALC}$ ontologies becomes undecidable in the presence of GCIs whenever any continuous t-norm different from the Gödel t-norm is used [1,2,3,10,8]. Not all the constructors from $\mathcal{ALC}$ are needed for proving undecidability. In particular, it was shown in [8] that for a family of t-norms—those t-norms "starting" with the Łukasiewicz t-norm—consistency is undecidable in the logic $\mathfrak{N}\mathcal{EL}$, which allows only the constructors conjunction, existential restriction, and (residual) negation, even if all the axioms are crisp.

The crux of these undecidability results is that they all depend on restricting reasoning to the class of witnessed models. Since the existing reasoning algorithms are based also on witnessed models,[1] this semantics was a natural choice. However, it could be the case that witnessed models are the cause of undecidability and consistency w.r.t. general models is decidable.

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[1] In fact, witnessed models were introduced in [11] to correct the methods from [15].

Although possible, such a scenario would be very surprising since for first-order fuzzy logic, reasoning w.r.t. general models is typically harder than reasoning w.r.t. witnessed models [12]. In this paper we show that this is also the case for the logic $\mathfrak{NEL}$ over any t-norm starting with the Łukasiewicz t-norm: consistency of crisp ontologies w.r.t. general models is undecidable.

2 Preliminaries

We briefly introduce the family of t-norms starting with the Łukasiewicz t-norm, which will provide the semantics for the fuzzy DLs Ł$^{(0,b)}$-$\mathfrak{NEL}$.

2.1 The Łukasiewicz t-norm

Mathematical fuzzy logic generalizes the classical logical operators to deal with truth degrees from the interval $[0,1]$. A *t-norm* is an associative and commutative binary operator on $[0,1]$ that has 1 as its unit and is monotonic in both arguments. If a t-norm $\otimes$ is continuous, then there is a unique binary operator $\Rightarrow$, called the *residuum*, that satisfies $z \leq x \Rightarrow y$ iff $x \otimes z \leq y$ for all $x, y, z \in [0,1]$. In that case, for all $x, y \in [0,1]$ we have (i) $x \Rightarrow y = 1$ iff $x \leq y$ and (ii) $1 \Rightarrow y = y$. Another useful operator is the unary *residual negation* $\ominus x := x \Rightarrow 0$.

We focus on the family of t-norms *starting with* the Łukasiewicz t-norm. The Łukasiewicz t-norm, defined by $x \otimes y = \max\{0, x+y-1\}$, has the residuum $x \Rightarrow y = \min\{1, 1-x+y\}$ and residual negation $\ominus x = 1-x$. A t-norm $\otimes$ *starts with* the Łukasiewicz t-norm if there is a $b \in (0,1]$ such that for all $x, y \in [0,b]$ we have $x \otimes y = \max\{0, x+y-b\}$. Intuitively, such a t-norm behaves like a scaled-down version of the Łukasiewicz t-norm in the interval $[0,b]$.

All continuous t-norms that do not start with the Łukasiewicz t-norm have one characteristic in common: their residual negation is always the *Gödel negation* that simply maps 0 to 1 and every other truth degree to 0. In contrast, those that do start with the Łukasiewicz t-norm do not have the Gödel negation, but rather a scaled-down version of the Łukasiewicz negation, also known as the involutive negation. To be more precise, the residuum and residual negation of a t-norm starting with the Łukasiewicz t-norm satisfy $x \Rightarrow y = b-x+y$ whenever $0 \leq y < x \leq b$, and

$$\ominus x = \begin{cases} 1 & \text{if } x = 0 \\ b - x & \text{if } 0 < x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

For the rest of this paper, we assume that $\otimes$ is a t-norm starting with the Łukasiewicz t-norm in the interval $[0,b]$, for some arbitrary but fixed $b \in (0,1]$, and $\Rightarrow$, $\ominus$ are its residuum and residual negation, respectively.

2.2 The Fuzzy Description Logic Ł$^{(0,b)}$-$\mathfrak{NEL}$

Syntactically, Ł$^{(0,b)}$-$\mathfrak{NEL}$ extends the DL $\mathcal{EL}$ with the unary *residual negation* constructor $\boxminus$. More precisely, concepts are built using the syntactic rule

$C ::= A \mid \top \mid C \sqcap D \mid \boxminus C \mid \exists r.C$. The semantics of $\text{Ł}^{(0,b)}$-$\mathfrak{NEL}$ is based on *interpretations* $\mathcal{I} = (\mathcal{D}^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a *domain* $\mathcal{D}^{\mathcal{I}}$ and an interpretation function that maps each concept name A to a function $A^{\mathcal{I}} : \mathcal{D}^{\mathcal{I}} \to [0,1]$, each role name r to $r^{\mathcal{I}} : \mathcal{D}^{\mathcal{I}} \times \mathcal{D}^{\mathcal{I}} \to [0,1]$, and each individual name a to an element $a^{\mathcal{I}}$ of $\mathcal{D}^{\mathcal{I}}$. This function is extended to complex concepts as follows: $\top^{\mathcal{I}}(x) = 1$, $(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$, $(\boxminus C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)$, and $(\exists r.C)^{\mathcal{I}}(x) = \sup_{y \in \mathcal{D}^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$ for all $x \in \mathcal{D}^{\mathcal{I}}$.

In contrast to traditional semantics based on witnessed models, it is possible to have an interpretation $\mathcal{I}$ such that $(\exists r.\top)^{\mathcal{I}}(x) = 1$ but where there is no $y \in \mathcal{D}^{\mathcal{I}}$ with $r^{\mathcal{I}}(x,y) = 1$. It can only be guaranteed that x has r-successors with degrees arbitrarily close to 1. This fact will produce some technical difficulties for our undecidability proof, since we it is not possible to guarantee a specific degree of membership for an r-successor of a given node (see Section 3.5).

We will use the following abbreviations: $C^0 := \top$, and $C^{k+1} := C^k \sqcap C$ for any $k \in \mathbb{N}$; $C \to D := \boxminus(C \sqcap \boxminus D)$; and $C \boxtimes D := (C \sqcap D) \to D$. It can easily be verified that, whenever $C^{\mathcal{I}}(x), D^{\mathcal{I}}(x) \in [0,b]$ for an interpretation $\mathcal{I}$ and $x \in \mathcal{D}^{\mathcal{I}}$, then we have $(C^k)^{\mathcal{I}}(x) = \max\{0, k(C^{\mathcal{I}}(x) - b) + b\}$ for every $k > 0$, $(C \to D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$, and $(C \boxtimes D)^{\mathcal{I}}(x) = \min\{C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\}$.

An *ontology* is a finite set of assertions of the form $a : C$ or $(a,b) : r$ and GCIs of the form $C \sqsubseteq D$, where a, b are individual names, and C, D are concepts. An interpretation $\mathcal{I}$ *satisfies* the assertion $a : C$ (resp. $(a,b) : r$) if $C^{\mathcal{I}}(a^{\mathcal{I}}) = 1$ (resp. $r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1$); it *satisfies* the GCI $C \sqsubseteq D$ if $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$ for every $x \in \mathcal{D}^{\mathcal{I}}$. The expression $C \equiv D$ abbreviates the two GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$. It follows that an interpretation $\mathcal{I}$ satisfies $C \equiv D$ iff $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(x)$ for every $x \in \mathcal{D}^{\mathcal{I}}$. It is a *model* of an ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$. An ontology is called *consistent* if it has a model.

Notice that we consider only *crisp* axioms; i.e. they contain no associated fuzzy degree. We will show that reasoning with this restricted form of ontologies, which is the same used for classical (crisp) DLs, is already undecidable.

To show the undecidability of ontology consistency in $\text{Ł}^{(0,b)}$-$\mathfrak{NEL}$, we will use a reduction from the Post Correspondence Problem (PCP). We first show how to encode an instance of the PCP into an $\text{Ł}^{(0,b)}$-$\mathfrak{NEL}$ ontology, and then describe how to use ontology consistency to solve the decision problem.

3 Encoding the PCP

We reduce a variant of the undecidable PCP to the consistency problem in $\text{Ł}^{(0,b)}$-$\mathfrak{NEL}$, using an idea that has been applied before to show undecidability of fuzzy DLs. The goal is to construct an ontology $\mathcal{O}_{\mathcal{P}}$ whose models contain the complete search tree for a solution to a given instance $\mathcal{P}$ of the PCP.

Definition 1 (PCP). *An instance $\mathcal{P}$ of the* Post Correspondence Problem *is a finite set of pairs $\{(v_1, w_1), \ldots, (v_n, w_n)\}$ of words over an alphabet Σ. A* solution *for $\mathcal{P}$ is a sequence $i_1 \cdots i_k \in \mathcal{N}^*$ such that $v_1 v_{i_1} \cdots v_{i_k} = w_1 w_{i_1} \cdots w_{i_k}$.*

We assume w.l.o.g. that the alphabet Σ is of the form $\{1,\dots,s\}$ for some natural number $s \geq 4$, and $\mathcal{N}$ denotes the index set $\{1,\dots,n\}$. For $\nu = i_1 \cdots i_k \in \mathcal{N}^*$, we denote $v_1 v_{i_1} \cdots v_{i_k}$ by v_ν and $w_1 w_{i_1} \cdots w_{i_k}$ by w_ν.

The *search tree* $\mathcal{T}$ of an instance $\mathcal{P}$ of the PCP is a tree over the domain $\mathcal{N}^*$, in which each node $\nu \in \mathcal{N}^*$ is labeled by v_ν and w_ν. In order to represent this search tree in an Ł$^{(0,b)}$-$\mathfrak{N}\mathcal{EL}$-interpretation, we need an appropriate encoding of words from Σ^* into the interval $[0,1]$. For technical reasons caused by the use of general semantics, we cannot guarantee that a specific real number will be used at a given node of the tree, but have to allow for a bounded error. Thus, a word can be encoded by any number from a given interval, as described next.

For $v = \alpha_1 \cdots \alpha_m$ with $\alpha_1,\dots,\alpha_m \in \Sigma$, $\overleftarrow{v}$ denotes $\alpha_m \cdots \alpha_1$. The *encoding* $\mathsf{enc}(v)$ of v is the number $0.\overleftarrow{v}$, in base $\beta := s+2$. In particular, $\mathsf{enc}(\varepsilon) := 0$. For $p \in [0,1)$, the interval $\mathsf{Err}_n(p)$ contains all numbers of the form $b(p+e) \in [0,b)$ with an *error term* e with $|e| < \beta^{-n}$. The set $\mathsf{Enc}(v)$ of *bounded-error encodings* of a word $v \in \Sigma^+$ is defined as the intersection of $\mathsf{Err}_{|v|+2}(\mathsf{enc}(v))$ with $[0,b)$. The idea of these bounded-error encodings is to keep the error small enough to avoid overlapping. This way, we can decide whether two different real numbers are just different encoding of the same word; this is the case whenever they belong to the same error-bounded encoding.

Lemma 2. *Let $v,w \in \Sigma^+$, $p \in \mathsf{Enc}(v)$, $q \in \mathsf{Enc}(w)$, $g \in \mathsf{Err}_{|v|+4}(\beta^{-(|v|+2)})$, and $h \in \mathsf{Err}_{|w|+4}(\beta^{-(|w|+2)})$. Then $v = w$ iff $|p-q| < 3\max\{g,h\}$.*

Proof. We can assume w.l.o.g. that $|v| = l \leq m = |w|$. It then follows that $3\max\{g,h\} = 3b\beta^{-(l+2)} + e$ with $|e| < 3b\beta^{-(l+4)} < b\beta^{-(l+3)}$. Additionally, $p = b\mathsf{enc}(v) + e_1$ and $q = b\mathsf{enc}(w) + e_2$ with $|e_1| < b\beta^{-(l+2)}$ and $|e_2| < b\beta^{-(m+2)}$, and thus $|e_1 - e_2| < 2b\beta^{-(l+2)}$. If $v = w$, then

$$|p-q| = |e_1 - e_2| < 2b\beta^{-(l+2)} < 3b\beta^{-(l+2)} + e = 3\max\{g,h\}.$$

If $v \neq w$, then $\mathsf{enc}(v)$ and $\mathsf{enc}(w)$ must differ at least in the $(l+1)$-th digit. Thus, $|\mathsf{enc}(v) - \mathsf{enc}(w)| \geq \beta^{-(l+1)}$. This implies that

$$|p-q| \geq b|\mathsf{enc}(v) - \mathsf{enc}(w)| - |e_1 - e_2| > (\beta-2)b\beta^{-(l+2)} \geq 4b\beta^{-(l+2)},$$

and hence $|p-q| > 3b\beta^{-(l+2)} + e = 3\max\{g,h\}$. □

We now construct an ontology $\mathcal{O}_\mathcal{P}$ whose models encode the search tree of $\mathcal{P}$. More precisely, it ensures that the concept names V and W will have values in $\mathsf{Enc}(v_\nu)$ and $\mathsf{Enc}(w_\nu)$, respectively, at every domain element that encodes the node $\nu \in \mathcal{N}^*$ of the search tree $\mathcal{T}$. The concept name G will have a value in $\mathsf{Err}_{|v_\nu|+4}(1 - \beta^{-(|v_\nu|+2)})$, and similarly for H. These values provide a bound on the difference of the encoding of two words of a given length, and will be used to detect whether V and W encode the same word (see Lemma 2).

The search tree is encoded in the following way. First, we ensure that at every element of the domain the concepts V_i and W_i are interpreted as the encodings of the words v_i, w_i, respectively, without any error terms (Section 3.1).

They will be used to generate the encodings of the successor words $v_{\nu i} = v_\nu v_i$ from an encoding of v_ν, for every $\nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$. We also use similar constants to update the values of G and H, which encode the error bounds introduced by the lack of witnessed models, from each node to its successor. Afterwards, we initialize the interpretation of all the relevant concepts at the root of the tree, identified by the individual name a_0 (Section 3.2). We have to ensure, e.g. that $V^\mathcal{I}(a_0^\mathcal{I})$ is an encoding of the word $v_\varepsilon = v_1$. We then describe the concatenation of encoded constant words to given encodings (Section 3.3), the creation of successors with large enough role degrees (Section 3.4), and the transfer of the concatenated encodings to these successors (Section 3.5).

3.1 Encoding Global Constants

We use some auxiliary values that will be constant along the whole search tree; these are helpful for producing the error-bounded encodings of the words v_ν and w_ν. First, we store the encoding of the words $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ using the concept names $V_1, \ldots, V_n, W_1, \ldots, W_n$, respectively. More precisely, every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ should satisfy $V_i^\mathcal{I}(x) = b\mathsf{enc}(v_i)$ and $W_i^\mathcal{I}(x) = b\mathsf{enc}(w_i)$ for all $x \in \mathcal{D}^\mathcal{I}$ and $i \in \mathcal{N}$.

Let l be the length of the word v_i. We use the axioms

$$A_l^{\beta^l} \equiv \boxminus A_l^{\beta^l} \text{ and } \boxminus V_i \equiv A_l^{2\overleftarrow{v_i}},$$

where A_l is an auxiliary concept name. Let $\mathcal{I}$ be a model of these axioms and $x \in \mathcal{D}^\mathcal{I}$. If $l = 0$, the second axiom implies that $V_i^\mathcal{I}(x) = b\mathsf{enc}(\varepsilon) = 0$. If $l > 0$, the first axiom enforces $\beta^l(A_l^\mathcal{I}(x) - b) + b = -\beta^l(A_l^\mathcal{I}(x) - b)$; thus $A_l^\mathcal{I}(x) = b(1 - \frac{1}{2\beta^l})$. Using the second axiom, we derive that $\ominus V_i^\mathcal{I}(x) = \max\{0, b - \frac{2b\overleftarrow{v_i}}{2\beta^l}\}$. Since $\frac{\overleftarrow{v_i}}{\beta^l} = 0.\overleftarrow{v_i} < 1$, we have $V_i^\mathcal{I}(x) = b\mathsf{enc}(v_i)$.

We also encode the constant words g_i and h_i that yield the numbers $\beta^{|v_i|} - 1$ and $\beta^{|w_i|} - 1$, respectively, when read in base $\beta = s + 2$. That is, g_i is the concatenation of the symbol $(s + 1)$ with itself $|v_i|$ times, and analogously for h_i. We store them in the concept names G_i and H_i, respectively. They will be used to update the values of G and H in the search tree (see Section 3.5). The axioms $A_{|v_i|}^{\beta^{|v_i|}} \equiv \boxminus A_{|v_i|}^{\beta^{|v_i|}}$ and $\boxminus G_i \equiv A_{|v_i|}^{2g_i}$ force G_i to always be $b(1 - \beta^{-|v_i|})$; an analogous construction can be used for H_i. Using the same idea, we encode the words g_0 and h_0 represented by the numbers $\beta^{|v_1|+2} - 1$ and $\beta^{|w_1|+2} - 1$ in the concept names G_0 and H_0, respectively.

3.2 Initializing the Root

At the root of the search tree of $\mathcal{P}$, designated by the individual name a_0, the values of V and W should be $b\mathsf{enc}(v_1)$ and $b\mathsf{enc}(w_1)$, respectively. The two concept assertions $a_0 : (V \to V_1) \sqcap (V_1 \to V)$ and $a_0 : (W \to W_1) \sqcap (W_1 \to W)$ ensure this. The concept names G and H, storing the error bounds, are initialized to $b(1 - \beta^{-(|v_1|+2)})$ and $b(1 - \beta^{-(|w_1|+2)})$ using $a_0 : (G \to G_0) \sqcap (G_0 \to G)$ and $a_0 : (H \to H_0) \sqcap (H_0 \to H)$; G_0 and H_0 were defined in the previous section.

3.3 Concatenating Constants

We now describe how to express the concatenation of a given encoding with a constant word v_i. Assume that, for a given interpretation $\mathcal{I}$ and $x \in \mathcal{D}^{\mathcal{I}}$, the value $V^{\mathcal{I}}(x)$ is an element of $\mathsf{Enc}(v_\nu)$ for some $\nu \in \mathcal{N}^*$. The goal is to ensure that, for a given $i \in \mathcal{N}$, the value $V_i'^{\mathcal{I}}(x)$ is an element of $\mathsf{Enc}(v_\nu v_i) = \mathsf{Enc}(v_{\nu i})$. We introduce the axioms

$$(\boxminus V_i'')^{\beta^l} \equiv \boxminus V \text{ and } \boxminus V_i' \equiv (\boxminus V_i'') \sqcap (\boxminus V_i),$$

where $l = |v_i|$, V_i'' is a new concept name, and $V_i^{\mathcal{I}}(x) = b\mathsf{enc}(v_i)$ (see Section 3.1).

Let $\mathcal{I}$ be an interpretation that satisfies the first axiom. If $v_\nu = \varepsilon$, then $V^{\mathcal{I}}(x) = 0$, and thus $V_i''^{\mathcal{I}}(x) = 0 = \beta^{-l} V^{\mathcal{I}}(x)$. If $v_\nu \neq \varepsilon$, then $\ominus V^{\mathcal{I}}(x) \in (0, b)$, which implies that $\ominus V_i''^{\mathcal{I}}(x) \in (0, b)$ and $b - V^{\mathcal{I}}(x) = \beta^l(-V_i''^{\mathcal{I}}(x)) + b$, and thus $V_i''^{\mathcal{I}}(x) = \beta^{-l} V^{\mathcal{I}}(x)$. In both cases, $V_i''^{\mathcal{I}}(x)$ equals $V^{\mathcal{I}}(x) = b(0.\overleftarrow{v_\nu} + e)$, shifted l digits to the right. By assumption, the error term e satisfies $|e| < \beta^{-(|v_\nu|+2)}$.

Let now $\mathcal{I}$ also satisfy the second axiom. If either v_ν or v_i is ε, then $V_i'^{\mathcal{I}}(x)$ is $V_i^{\mathcal{I}}(x)$ or $V^{\mathcal{I}}(x)$, respectively. In both cases, this expresses the concatenation of v_ν and v_i. If both v_ν and v_i are non-empty words, then

$$\ominus V_i'^{\mathcal{I}}(x) = b \max\{0, 1 - \beta^{-l}(0.\overleftarrow{v_\nu} + e) - (0.\overleftarrow{v_i})\} = b \max\{0, 1 - (0.\overleftarrow{v_{\nu i}} + e')\}$$

with $|e'| = \beta^{-l}|e| < \beta^{-(|v_{\nu i}|+2)}$. Thus, $0.\overleftarrow{v_{\nu i}} + e' < 1$ and $V_i'^{\mathcal{I}}(x) \in \mathsf{Enc}(v_{\nu i})$.

The values of the concept names G and H should also be updated. These values also have an error term, which is two orders of magnitude smaller than the encoding of the words from the search tree. Let $G^{\mathcal{I}}(x) = b(1 - \beta^{-(|v_\nu|+2)} + e)$ with $|e| < \beta^{-(|v_\nu|+4)}$, which corresponds to the word $(s+1)\cdots(s+1)$ of length $|v_\nu|+2$. We have to concatenate this to the word $(s+1)\cdots(s+1)$ of length $l := |v_i|$, stored in G_i. As above, the axioms $(\boxminus G_i'')^{\beta^l} \equiv \boxminus G$ and $\boxminus G_i' \equiv (\boxminus G_i'') \sqcap (\boxminus G_i)$ ensure that

$$G_i'^{\mathcal{I}}(x) = b - b(1 - \beta^{-l}(1 - \beta^{-(|v_\nu|+2)} + e) - (1 - \beta^{-l})) = b(1 - \beta^{-(|v_{\nu i}|+2)} + e')$$

with $|e'| < \beta^{-(|v_{\nu i}|+4)}$.

3.4 Creating Successor Nodes

The axioms $\top \sqsubseteq \exists r_i.\top$ for each $i \in \mathcal{N}$ enforce the existence of r_i-successors to each node of the search tree. This is the main difference to the proof of undecidability for $Ł^{(0,b)}$-$\mathfrak{NEL}$ w.r.t. witnessed models in [8]. In fact, in any witnessed model $\mathcal{I}$ of these axioms we have for each $x \in \mathcal{D}^{\mathcal{I}}$ a $y \in \mathcal{D}^{\mathcal{I}}$ such that $r_i^{\mathcal{I}}(x, y) = 1$. In the more general setting considered here, only the following holds.

Lemma 3. *Let $\mathcal{I}$ satisfy $\top \sqsubseteq \exists r_i.\top$. Then for each $x \in \mathcal{D}^{\mathcal{I}}$ and every error bound $e > 0$ there is a $y_e \in \mathcal{D}^{\mathcal{I}}$ such that $r_i^{\mathcal{I}}(x, y_e) > 1 - e$.*

This is the main reason why we need to allow for bounded errors in the encodings of the words. If we could always guarantee the existence of r_i-successors with degree 1, then the axioms presented in the next section could be used to transfer all values exactly. However, since only Lemma 3 holds, the transfer might lead to an additional error.

3.5 Transferring Values

In Section 3.2, we have ensured that every interpretation contains an individual that encodes the root of the search tree. We are also able to compute the values needed in the successors using the constants from Section 3.1 and the concatenation from Section 3.3. Intuitively, these values should now be transfered to the successors created in the previous section; however, in general this transfer of values from a node of the search tree $\mathcal{T}$ to a successor node will incur an error on the transferred value. Thus, we have to make sure that this error does not grow beyond the bounds of our encoding.

We consider first the case that $b = 1$. We need to transfer the value of V_i' at a node to the value of V at some r_i-successor of this node. We show that, if $V_i'^{\mathcal{I}}(x) = b(\mathsf{enc}(v_{\nu i}) + e')$ is a bounded-error encoding of $v_{\nu i}$, then the axioms

$$\exists r_i.(\boxminus V) \sqsubseteq \boxminus V_i' \text{ and } \exists r_i.V \sqsubseteq V_i'$$

ensure that the value of V is also a bounded-error encoding of this word at all r_i-successors with large enough role degree since the incurred error stays below $e := b(\beta^{-(|v_{\nu i}|+2)} - |e'|) > 0$.

Lemma 4. *Let $\mathcal{I}$ satisfy $\exists r.(\boxminus D) \sqsubseteq \boxminus C$ and $\exists r.D \sqsubseteq C$, $b = 1$, and $0 < e < \frac{1}{2}$. Then for all $x, y \in \mathcal{D}^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) > 1 - e$ we have $|C^{\mathcal{I}}(x) - D^{\mathcal{I}}(y)| < e$.*

Proof. Let x and y be as above. The axioms force $\mathcal{I}$ to satisfy

$$r^{\mathcal{I}}(x, y) \otimes \ominus D^{\mathcal{I}}(y) \le \sup_{y' \in \mathcal{D}^{\mathcal{I}}} r^{\mathcal{I}}(x, y') \otimes \ominus D^{\mathcal{I}}(y') \le \ominus C^{\mathcal{I}}(x), \text{ and}$$
$$r^{\mathcal{I}}(x, y) \otimes D^{\mathcal{I}}(y) \le \sup_{y' \in \mathcal{D}^{\mathcal{I}}} r^{\mathcal{I}}(x, y') \otimes D^{\mathcal{I}}(y') \le C^{\mathcal{I}}(x).$$

This implies that $r^{\mathcal{I}}(x, y) - D^{\mathcal{I}}(y) \le \max\{0, r^{\mathcal{I}}(x, y) - D^{\mathcal{I}}(y)\} \le 1 - C^{\mathcal{I}}(x)$ and $r^{\mathcal{I}}(x, y) + D^{\mathcal{I}}(y) - 1 \le \max\{0, r^{\mathcal{I}}(x, y) + D^{\mathcal{I}}(y) - 1\} \le C^{\mathcal{I}}(x)$. From the first inequality if follows that $C^{\mathcal{I}}(x) - D^{\mathcal{I}}(y) \le 1 - r^{\mathcal{I}}(x, y) < e$ and the second one yields $D^{\mathcal{I}}(y) - C^{\mathcal{I}}(x) \le 1 - r^{\mathcal{I}}(x, y) < e$. This shows that the absolute difference between $C^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(y)$ is always smaller than e. ☐

To transfer the value of G_i' along r_i to G, we use the axioms $\exists r_i.(\boxminus G) \sqsubseteq \boxminus G_i'$ and $\exists r_i.G \sqsubseteq G_i'$. The error bound $e := b(\beta^{-(|v_{\nu i}|+4)} - |e'|)$ ensures that the additional error to $G_i'^{\mathcal{I}}(x) = b(1 - \beta^{-(|v_{\nu i}|+2)} + e')$ is small enough.

In order to simultaneously satisfy all the above-mentioned restrictions in the r_i-successors, we use only those $y \in \mathcal{D}^{\mathcal{I}}$ with $r_i^{\mathcal{I}}(x, y) > 1 - e$, where e is the minimum of the individual error bounds for V_i', W_i', G_i', and H_i'. This is not a problem since Lemma 3 shows that such a successor y always exists.

In the case that $b < 1$, the transfer of values is considerably easier. In fact, no error bounds are necessary in this case since encodings of words can be transferred without additional errors through any r_i-successor with degree $> b$.

Lemma 5. *Let $b < 1$ and assume that $\mathcal{I}$ satisfies $\exists r.(\boxminus D) \sqsubseteq \boxminus C$ and $\exists r.D \sqsubseteq C$. If $x, y \in \mathcal{D}^{\mathcal{I}}$ with $C^{\mathcal{I}}(x) \in [0, b)$ and $r^{\mathcal{I}}(x, y) > b$, then $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(y)$.*

Proof. We have $r^{\mathcal{I}}(x,y) \otimes \ominus D^{\mathcal{I}}(y) \leq \ominus C^{\mathcal{I}}(x)$ and $r^{\mathcal{I}}(x,y) \otimes D^{\mathcal{I}}(y) \leq C^{\mathcal{I}}(x)$. If $C^{\mathcal{I}}(x) = 0$, then $r^{\mathcal{I}}(x,y) \otimes D^{\mathcal{I}}(y) = 0$, and thus $D^{\mathcal{I}}(y) = 0$. Consider now the case that $C^{\mathcal{I}}(x) \in (0,b)$. If $D^{\mathcal{I}}(y) \geq b$, then $C^{\mathcal{I}}(x) \geq r^{\mathcal{I}}(x,y) \otimes D^{\mathcal{I}}(y) \geq b$, contradicting the assumption. Likewise, $D^{\mathcal{I}}(y) = 0$ implies $b - C^{\mathcal{I}}(x) \geq r^{\mathcal{I}}(x,y) > b$, which is also impossible. Thus, we must have $D^{\mathcal{I}}(y) \in (0,b)$, which implies $b - D^{\mathcal{I}}(y) \leq b - C^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(y) \leq C^{\mathcal{I}}(x)$, and hence $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(y)$. □

As before, Lemma 3 ensures that a successor with degree strictly greater than b always exists, which shows that we can always transfer all desired values exactly.

3.6 Canonical Model

From the constructions of Sections 3.1 to 3.5 we obtain the ontology $\mathcal{O}_{\mathcal{P}}$:

$$\begin{aligned}\mathcal{O}_{\mathcal{P}} := \mathcal{O}_0 \cup \mathcal{O}_{G_0:=g_0} \cup \mathcal{O}_{H_0:=h_0} \\ \cup \bigcup_{i=1}^{n} \mathcal{O}_{V_i:=v_i} \cup \mathcal{O}_{W_i:=w_i} \cup \mathcal{O}_{G_i:=g_i} \cup \mathcal{O}_{H_i:=h_i} \cup \mathcal{O}_{r_i} \\ \cup \mathcal{O}_{V_i'=V\circ v_i} \cup \mathcal{O}_{W_i'=W\circ w_i} \cup \mathcal{O}_{V_i' \overset{r_i}{\rightsquigarrow} V} \cup \mathcal{O}_{W_i' \overset{r_i}{\rightsquigarrow} W} \\ \cup \mathcal{O}_{G_i'=G\circ g_i} \cup \mathcal{O}_{H_i'=H\circ h_i} \cup \mathcal{O}_{G_i' \overset{r_i}{\rightsquigarrow} G} \cup \mathcal{O}_{H_i' \overset{r_i}{\rightsquigarrow} H}, \qquad \text{where}\end{aligned}$$

$$\begin{aligned}\mathcal{O}_0 &:= \{a_0 : (V \to V_1) \sqcap (V_1 \to V),\ a_0 : (W \to W_1) \sqcap (W_1 \to W)\} \cup \\ &\quad\ \{a_0 : (G \to G_0) \sqcap (G_0 \to G),\ a_0 : (H \to H_0) \sqcap (H_0 \to H)\}, \\ \mathcal{O}_{C:=u} &:= \{C_{|u|}^{\beta^{|u|}} \equiv \boxminus C_{|u|}^{\beta^{|u|}},\ \boxminus C \equiv C_{|u|}^{2\overleftarrow{u}}\}, \\ \mathcal{O}_{D'=C\circ u} &:= \{(\boxminus D'')^{\beta^{|u|}} \equiv \boxminus C,\ \boxminus D' \equiv (\boxminus D'') \sqcap (\boxminus D)\}, \\ \mathcal{O}_r &:= \{\top \sqsubseteq \exists r.\top\}, \\ \mathcal{O}_{C \overset{r}{\rightsquigarrow} D} &:= \{\exists r.(\boxminus D) \sqsubseteq \boxminus C,\ \exists r.D \sqsubseteq C\}.\end{aligned}$$

A model of this ontology is given by the interpretation $\mathcal{I}_{\mathcal{P}} = (\mathcal{N}^*, \cdot^{\mathcal{I}_{\mathcal{P}}})$, where $\cdot^{\mathcal{I}_{\mathcal{P}}}$ is defined as follows:

- $a_0^{\mathcal{I}_{\mathcal{P}}} = \varepsilon$,
- $V^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{benc}(v_\nu)$, $\quad V_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{benc}(v_i)$, $\quad V_i'^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{benc}(v_{\nu i})$,
- $W^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{benc}(w_\nu)$, $\quad W_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{benc}(w_i)$, $\quad W_i'^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{benc}(w_{\nu i})$,
- $G^{\mathcal{I}_{\mathcal{P}}}(\nu) = b(1 - \beta^{-(|v_\nu|+2)})$, $\quad H^{\mathcal{I}_{\mathcal{P}}}(\nu) = b(1 - \beta^{-(|w_\nu|+2)})$,
- $G_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = b(1 - \beta^{-|v_i|})$, $\quad H_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = b(1 - \beta^{-|w_i|})$,
- $r_i^{\mathcal{I}_{\mathcal{P}}}(\nu, \nu i) = 1$ and $r_i^{\mathcal{I}_{\mathcal{P}}}(\nu, \nu') = 0$ if $\nu' \neq \nu i$.

Notice that the values of all the auxiliary concept names used in the construction are fully determined by these concept names. Intuitively, $\mathcal{I}_{\mathcal{P}}$ is an encoding of the search tree of $\mathcal{P}$. We now show that every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$ encodes the search tree $\mathcal{T}$ of $\mathcal{P}$ under the following notion of encoding.

Definition 6. *Let $\mathcal{I}$ be an interpretation, $x \in \mathcal{D}^{\mathcal{I}}$, and $\nu \in \mathcal{N}^*$. We say that $\mathcal{I}$* at x encodes $\mathcal{T}$ at ν *if, for every $i \in \mathcal{N}$,*

a) $V^{\mathcal{I}}(x) \in \mathsf{Enc}(v_\nu)$, $\quad W^{\mathcal{I}}(x) \in \mathsf{Enc}(w_\nu)$,
b) $G^{\mathcal{I}}(x) \in \mathsf{Err}_{|v_\nu|+4}(1-\beta^{-(|v_\nu|+2)})$, $\quad H^{\mathcal{I}}(x) \in \mathsf{Err}_{|w_\nu|+4}(1-\beta^{-(|w_\nu|+2)})$.

In the following, whenever we talk about these properties, we will only consider V and G since W and H can be treated in an analogous manner.

Lemma 7. *For every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$ there is a function $f : \mathcal{N}^* \to \mathcal{D}^{\mathcal{I}}$ such that for all $\nu \in \mathcal{N}^*$, $\mathcal{I}$ at $f(\nu)$ encodes $\mathcal{T}$ at ν.*

Proof. We construct the function f by induction on $\nu \in \mathcal{N}^*$. For $\nu = \varepsilon$, observe that the values of V and G at $a_0^{\mathcal{I}}$ are forced by $\mathcal{O}_0$ to be $V_1^{\mathcal{I}}(\varepsilon) = b\mathsf{enc}(v_\varepsilon)$ and $G_0^{\mathcal{I}}(\varepsilon) = b(1-\beta^{-(|v_\varepsilon|+2)})$, respectively. Thus, for $f(\nu) := a_0^{\mathcal{I}}$ the conditions are satisfied even without any error terms.

Assume now that $\mathcal{I}$ at $f(\nu)$ encodes $\mathcal{T}$ at ν and let $i \in \mathcal{N}$. As described in Sections 3.4 and 3.5, one can find $0 < e < \frac{1}{2}$ such that the values of V_i' and G_i' are transferred to V and G, respectively, without increasing the error bounds on the encoded values. Specifically, we can choose

$$e := \min_{i=1,\dots,n} \{\tfrac{1}{3}, b(\beta^{-(|v_{\nu i}|+2)} - |e_{V_i'}|),\ b(\beta^{-(|v_{\nu i}|+4)} - |e_{G_i'}|),\ \dots\}$$

if $e_{V_i'}$ and $e_{G_i'}$ are the error terms of V_i' and G_i', respectively. Lemmata 4 and 5 show that $V^{\mathcal{I}}(y)$ is a bounded-error encoding of $v_{\nu i}$ whenever $r_i^{\mathcal{I}}(x,y) > 1-e$, and similarly for $G^{\mathcal{I}}(y)$. By Lemma 3, there is a $y_i \in \mathcal{D}^{\mathcal{I}}$ such that $\mathcal{I}$ at y_i encodes $\mathcal{T}$ at νi. We can thus define $f(\nu i) := y_i$ to satisfy the condition. $\square$

Thus, in any model of $\mathcal{O}_{\mathcal{P}}$ we can find an encoding of the search tree $\mathcal{T}$ of $\mathcal{P}$. The canonical model $\mathcal{I}_{\mathcal{P}}$ is the special case in which all error terms are 0. In the next section we use this encoding to solve the instance $\mathcal{P}$ of the PCP.

4 Finding a Solution

To decide whether $\mathcal{P}$ has a solution, we use the additional ontology $\mathcal{O}_{\neq}$, consisting of the single axiom $(V \to W) \sqcap (W \to V) \sqsubseteq (G \boxtimes H)^3$. Recall that $G \boxtimes H$ is interpreted as the minimum of the values of G and H in the interval $[0, b]$.

Lemma 8. *$\mathcal{P}$ has a solution iff $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{\neq}$ is inconsistent.*

Proof. If this ontology has no model, then in particular the canonical model $\mathcal{I}_{\mathcal{P}}$ of $\mathcal{O}_{\mathcal{P}}$ cannot satisfy $\mathcal{O}_{\neq}$. We thus have

$$(V \to W)^{\mathcal{I}_{\mathcal{P}}}(\nu) \otimes (W \to V)^{\mathcal{I}_{\mathcal{P}}}(\nu) > ((G \boxtimes H)^3)^{\mathcal{I}_{\mathcal{P}}}(\nu)$$

for some $\nu \in \mathcal{N}^*$. If $V^{\mathcal{I}_{\mathcal{P}}}(\nu) = W^{\mathcal{I}_{\mathcal{P}}}(\nu)$, then $v_\nu = w_\nu$, i.e. $\mathcal{P}$ has a solution. We now assume without loss of generality that $V^{\mathcal{I}_{\mathcal{P}}}(\nu) < W^{\mathcal{I}_{\mathcal{P}}}(\nu)$, and thus

$$b - W^{\mathcal{I}_{\mathcal{P}}}(\nu) + V^{\mathcal{I}_{\mathcal{P}}}(\nu) > ((G \boxtimes H)^3)^{\mathcal{I}_{\mathcal{P}}}(\nu) = b - 3b\max\{\beta^{-(|v_\nu|+2)}, \beta^{-(|w_\nu|+2)}\}.$$

This implies that

$$|\mathsf{enc}(w_\nu) - \mathsf{enc}(v_\nu)| < 3b \max\{\beta^{-(|v_\nu|+2)}, \beta^{-(|w_\nu|+2)}\},$$

and by Lemma 2 we again have $v_\nu = w_\nu$.

Assume now that $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{\neq}$ has a model $\mathcal{I}$ and let f be the function constructed in Lemma 7 and $\nu \in \mathcal{N}^*$. In particular, we have

$$((G \boxtimes H)^3)^\mathcal{I}(f(\nu)) = b - 3\max\{g,h\} \in (0,b)$$

with $g \in \mathsf{Err}_{|v_\nu|+4}(\beta^{-(|v_\nu|+2)})$ and $h \in \mathsf{Err}_{|w_\nu|+4}(\beta^{-(|w_\nu|+2)})$. Since $\mathcal{I}$ satisfies $\mathcal{O}_{\neq}$, it holds that

$$((V \to W) \sqcap (W \to V))^\mathcal{I}(f(\nu)) \leq ((G \boxtimes H)^3)^\mathcal{I}(f(\nu)).$$

If we assume w.l.o.g. that $V^\mathcal{I}(f(\nu)) \leq W^\mathcal{I}(f(\nu))$, then this implies

$$W^\mathcal{I}(f(\nu)) - V^\mathcal{I}(f(\nu)) \geq 3\max\{g,h\},$$

and by Lemma 2, $v_\nu \neq w_\nu$. Since this holds for all $\nu \in \mathcal{N}^*$, $\mathcal{P}$ has no solution. □

A direct consequence of this lemma is the undecidability of ontology consistency in the fuzzy DL $\text{Ł}^{(0,b)}$-$\mathfrak{N}\mathcal{EL}$.

Theorem 9. *Consistency of* $\text{Ł}^{(0,b)}$*-*$\mathfrak{N}\mathcal{EL}$*-ontologies is undecidable.*

In particular, this shows that ontology consistency in Ł-$\mathcal{ALC}$ is undecidable w.r.t. general models, even if the ontologies contain only crisp axioms.

5 Conclusions

We have shown that ontology consistency w.r.t. general models is undecidable in fuzzy DLs based on t-norms starting with the Łukasiewicz t-norm using only the constructors from $\mathcal{EL}$ and residual negation. This was achieved by a modification of the undecidability proofs for these logics w.r.t. witnessed model semantics [8]. The main problem introduced by general semantics is that it is impossible to ensure that values are transferred exactly from a node to its role successors. To simulate the search tree for an instance of the PCP, we allow for a bounded error in the represented value. The bounds are small enough to ensure that different words can still be distinguished.

Ontology consistency is a central decision problem for DLs since other reasoning tasks like concept satisfiability or subsumption can be reduced to it. However, the converse reduction does not hold. It is thus natural to ask whether these other problems are also undecidable. Since our construction uses only crisp assertions that involve only one individual name, it follows that also concept satisfiability in $\text{Ł}^{(0,b)}$-$\mathfrak{N}\mathcal{EL}$ is undecidable w.r.t. general models.

Together with the results from [6], this fully characterizes the decidability of consistency and satisfiability w.r.t. general models in all fuzzy DLs between

$\otimes$-$\mathfrak{NEL}$ and $\otimes$-$\mathcal{SHOI}^{-\forall}$:[2] it is decidable (in ExpTime) iff the t-norm $\otimes$ does not start with the Łukasiewicz t-norm, i.e. the fuzzy DL has Gödel negation.

In future work we plan to extend the study of fuzzy DLs with general semantics, aiming towards a full characterization of their complexity properties, as has been done for witnessed models. In particular, we think that the presented approach using error bounds can be applied to modify other undecidability results for fuzzy DLs w.r.t. witnessed models, e.g. for Π-$\mathcal{ALC}$ [8]. We also plan to generalize the framework presented in [8] to deal with general models.

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[2] $\otimes$-$\mathcal{SHOI}^{-\forall}$ extends $\otimes$-$\mathfrak{NEL}$ by the constructors $\sqcup$, $\to$, inverse and transitive roles, fuzzy role hierarchies, and nominals.