Reasoning About Typicality in Preferential Description Logics: Preferential vs Rational Entailment

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Abstract. Extensions of Description Logics (DLs) to reason about typicality and defeasible inheritance have been largely investigated. In this paper, we consider two such extensions, namely (i) the extension of DLs with a typicality operator **T**, having the properties of Preferential nonmonotonic entailment **P**, and (ii) its variant with a typicality operator having the properties of the stronger Rational entailment **R**. The first one has been proposed in [1, 2]. Here, we investigate the second one and we show, by a representation theorem, that it is equivalent to the approach to preferential subsumption proposed in [3]. We compare the two extensions, preferential and rational, and argue that the first one is more suitable than the second one to reason about typicality, as the latter leads to unintuitive inferences.

1 Introduction

Description logics (DLs) represents one of the most important formalisms of knowledge representation. Their success can be explained by two key advantages characterizing them. On the one hand, DLs have a well-defined semantics based on first-order logic; on the other hand, they offer a good trade-off between expressivity and complexity. DLs have been successfully implemented by a range of systems and they are at the base of languages for the semantic web such as OWL.

In a DL framework, a knowledge base (KB) comprises two components: an intensional part, called the TBox, containing the definition of concepts (and possibly roles) as well as a specification of inclusion relations among them, and an extensional part, called the ABox, containing instances of concepts and roles. Since the very objective of the TBox is to build a taxonomy of concepts, the need of representing prototypical properties and of reasoning about defeasible inheritance of such properties naturally arises.

The traditional approach is to handle defeasible inheritance by integrating some kind of nonmonotonic reasoning mechanism. This has led to study nonmonotonic extensions of DLs [4–9]. However, as the same authors have pointed out, all these proposals present some difficulties, and finding a suitable nonmonotonic extension for inheritance with exceptions is far from obvious. To give a brief account, [4] proposes the extension of DL with Reiter's default logic. However, the same authors have pointed out that this integration may lead to both semantical and computational difficulties. Furthermore, Reiter's default logic does not provide a direct way of modeling inheritance with exceptions. This has motivated the study of extensions of DLs with prioritized defaults [9, 5]. A more general approach is undertaken in [7], where an extension of DL is proposed with two epistemic operators. This extension, called $\mathcal{ALCK}_{\mathcal{NF}}$, allows one to encode Reiter's default logic as well as to express epistemic concepts and procedural rules. However, this extension has a rather complicated modal semantics, so that the integration with the existing systems requires significant changes to the standard semantics of DLs. [10] extends the work in [7] by providing a translation of an $\mathcal{ALCK}_{\mathcal{NF}}$ KB to an equivalent *flat* KB and by defining a simplified tableau algorithm for flat KBs, which includes an optimized minimality check. In [6] an extension of DL with circumscription is proposed to express prototypical properties with exceptions, by introducing "abnormality" predicates whose extension is minimized. The authors provide algorithms for checking satisfiability, subsumption and instance checking which are proved to have an optimal complexity, but are based on massive nondeterministic guessing. A calculus for circumscription in DL has not been developed yet. Moreover, the use of circumscription to model inheritance with exceptions is not that straightforward. We refer to Section 5.1 in [1, 2] for a broader discussion on the above mentioned nonmonotonic extensions of DLs.

Here, we consider an alternative approach, based on nonmonotonic entailment as defined by Kraus, Lehmann and Magidor (KLM) in [11, 12]. This approach is adopted by [2, 1] and [3]. The main advantage of this approach over previous ones is that the semantics of the resulting description logics is very simple and close to standard semantics for DLs. Furthermore, at least for what concerns [1, 2] there is a calculus for the proposed logic, and the logic can be extended in order to deal with inheritance with exceptions.

We start by considering the logic $\mathcal{ALC} + \mathbf{T}$ proposed in [2, 1], that extends the well-known description logic $\mathcal{ALC}$ by a typicality operator $\mathbf{T}$. The intended meaning of the operator $\mathbf{T}$, for any concept C, is that $\mathbf{T}(C)$ singles out the instances of C that are considered as "typical". Thus, an assertion like

"typical writers are brillant"

is represented by

$$\mathbf{T}(\mathit{Writer}) \sqsubseteq \mathit{Brillant}.$$

An $\mathcal{ALC} + \mathbf{T}$ TBox can consistently contain the above inclusion together with

$$\mathbf{T}(\mathit{Writer} \sqcap \mathit{Depressed}) \sqsubseteq \neg \mathit{Brillant}$$

(typical depressed writers are not brillant). It is worth noticing that, if the same properties were expressed by ordinary inclusions, such as $\mathit{Writer} \sqsubseteq \mathit{Brillant}$, we would simply get that there are not depressed writers, thus the KB would collapse. This collapse is avoided in $\mathcal{ALC} + \mathbf{T}$, as it is not assumed that $\mathbf{T}$ is monotonic, that is to say $C \sqsubseteq D$ does not imply $\mathbf{T}(C) \sqsubseteq \mathbf{T}(D)$.

The semantics of the $\mathbf{T}$ operator is defined by a set of postulates that are essentially a restatement of axioms and rules of nonmonotonic entailment in preferential logic **P**, as defined by KLM. The logic **P** introduces a nonmonotonic entailment $\mid\!\sim$ in order to formalize conditional assertions of the form $A \mid\!\sim B$, whose intuitive meaning is that "normally the As are Bs". The semantics of $\mathbf{T}$ is given by means of a preference relation $<$ on individuals, so that typical instances of a concept C can be defined as the instances of C that are minimal with respect to $<$. In this modal logic, $<$ works as an accessibility relation R with $R(x, y) \equiv y < x$, so that $\mathbf{T}(C)$ can be defined as $C \sqcap \Box\neg C$. The preference relation $<$ does not have infinite descending chains as the so-called Smoothness condition is assumed. As a consequence, the corresponding modal operator $\Box$ has the same properties as in Gödel-Löb modal logic G of arithmetic provability.

The family of KLM logics contains other interesting members, notably the stronger logic **R**, known as Rational Preferential Logic [12]. The axiomatization of the logic **R** is obtained from the axiomatization of **P** by adding the following rule, known as *rational monotonicity*:

$$\text{RM. } ((A \mid\!\sim B) \wedge \neg(A \mid\!\sim \neg C)) \rightarrow ((A \wedge C) \mid\!\sim B)$$

The intuitive meaning of rational monotonicity is as follows: if $A \mid\!\sim B$ and $\neg(A \mid\!\sim \neg C)$ hold, then one can infer $A \wedge C \mid\!\sim B$. This rule allows a conditional to be inferred from a set of conditionals in absence of other information. More precisely, "it says that an agent should not have to retract any previous defeasible conclusion when learning about a new fact the negation of which was not previously derivable" [12].

In this paper, we consider whether the properties of $\mathbf{T}$ in $\mathcal{ALC} + \mathbf{T}$ are the correct ones, by comparing them with the properties that would result for $\mathbf{T}$ if we adopted the stronger logic of nonmonotonic entailment **R**. We call $\mathcal{ALC} + \mathbf{T}_{\mathbf{R}}$ the resulting Description Logic. We provide some examples to show that **P** is better suited than **R** since **R** would force some inferences that we consider counterintuitive. Using **R**, for instance, we would be forced to conclude that typical writers are not brillant from the simple fact that there is a certain Mr. John who is a typical brillant person (he has, for instance, a lot of social success), who is a writer but who is not a typical writer (since he has never succeeded in publishing anything). We consider this as an unwanted inference, and therefore argue that the properties of **R** are too strong for $\mathbf{T}$, and that **P** must be preferred.

In section 4.2 we also show that the logic $\mathcal{ALC} + \mathbf{T}_{\mathbf{R}}$ is equivalent to the logic for defeasible subsumptions in DLs proposed by [3], when considered with $\mathcal{ALC}$ as the underlying DL. The idea underlying the approach by [3] is very similar to that underlying $\mathcal{ALC} + \mathbf{T}$ and $\mathcal{ALC} + \mathbf{T}_{\mathbf{R}}$: some objects in the domain are more typical than others. In the approach by [3], x is at least as typical as y if $x \geq y$. The properties of $\geq$ in [3] correspond to those of $<$ in $\mathcal{ALC} + \mathbf{T}_{\mathbf{R}}$. At a syntactic level the two logics differ, so that in [3] one finds the defeasible inclusions $C \underset{\sim}{\sqsubset} D$ instead of $\mathbf{T}(C) \sqsubseteq D$ of $\mathcal{ALC} + \mathbf{T}_{\mathbf{R}}$. But the idea is the same: in the two cases the inclusion holds if the most preferred (typical) Cs are also Ds. Indeed, it can be shown that the logic of preferential subsumption can be translated into $\mathcal{ALC} + \mathbf{T}_{\mathbf{R}}$ by replacing $C \underset{\sim}{\sqsubset} D$ with $\mathbf{T}(C) \sqsubseteq D$. The approach in [3] therefore inherits the above criticisms for extensions of DLs that use **R**.

2 The Logic $\mathcal{ALC} + \mathbf{T}$

In this section we briefly recall the description logic $\mathcal{ALC} + \mathbf{T}$ introduced in [1]. We consider an alphabet of concept names $\mathcal{C}$, of role names $\mathcal{R}$, and of individuals $\mathcal{O}$.

The language $\mathcal{L}$ of the logic $\mathcal{ALC} + \mathbf{T}$ is defined by distinguishing *concepts* and *extended concepts* as follows:

- Concepts: $A \in \mathcal{C}$ and $\top$ are *concepts* of $\mathcal{L}$; if $C, D \in \mathcal{L}$ and $R \in \mathcal{R}$, then $C \sqcap D, C \sqcup D, \neg C, \forall R.C, \exists R.C$ are *concepts* of $\mathcal{L}$;
- Extended concepts: if C is a concept, then C and $\mathbf{T}(C)$ are *extended concepts*, and all the boolean combinations of extended concepts are extended concepts of $\mathcal{L}$.

A knowledge base is a pair (TBox,ABox). TBox contains subsumptions $C \sqsubseteq D$, where $C \in \mathcal{L}$ is an extended concept of the form either C' or $\mathbf{T}(C')$, and $D \in \mathcal{L}$ is a concept. ABox contains expressions of the form $C(a)$ and aRb where $C \in \mathcal{L}$ is an extended concept, $R \in \mathcal{R}$, and $a, b \in \mathcal{O}$.

In order to provide a semantics to the operator $\mathbf{T}$, we extend the definition of a model used in "standard" terminological logic $\mathcal{ALC}$:

Definition 1 (Semantics of $\mathbf{T}$ with selection function). *A model is any structure $\langle \Delta, I, f_{\mathbf{T}} \rangle$ where:*

- *Δ is the domain;*
- *I is the extension function that maps each extended concept C to $C^I \subseteq \Delta$, and each role R to a $R^I \subseteq \Delta \times \Delta$. I assigns to each atomic concept $A \in \mathcal{C}$ a set $A^I \subseteq \Delta$ and it is extended as follows:*
  - $\top^I = \Delta$
  - $\bot^I = \emptyset$
  - $(\neg C)^I = \Delta \backslash C^I$
  - $(C \sqcap D)^I = C^I \cap D^I$
  - $(C \sqcup D)^I = C^I \cup D^I$
  - $(\forall R.C)^I = \{a \in \Delta \mid \forall b.(a, b) \in R^I \rightarrow b \in C^I\}$
  - $(\exists R.C)^I = \{a \in \Delta \mid \exists b.(a, b) \in R^I\}$
  - $(\mathbf{T}(C))^I = f_{\mathbf{T}}(C^I)$
- *Given $S \subseteq \Delta$, $f_{\mathbf{T}}$ is a function $f_{\mathbf{T}} : Pow(\Delta) \rightarrow Pow(\Delta)$ satisfying the following properties:*
  - $(f_{\mathbf{T}} - 1)$ $f_{\mathbf{T}}(S) \subseteq S$*;*
  - $(f_{\mathbf{T}} - 2)$ *if $S \neq \emptyset$, then also $f_{\mathbf{T}}(S) \neq \emptyset$;*
  - $(f_{\mathbf{T}} - 3)$ *if $f_{\mathbf{T}}(S) \subseteq R$, then $f_{\mathbf{T}}(S) = f_{\mathbf{T}}(S \cap R)$;*
  - $(f_{\mathbf{T}} - 4)$ $f_{\mathbf{T}}(\bigcup S_i) \subseteq \bigcup f_{\mathbf{T}}(S_i)$*;*
  - $(f_{\mathbf{T}} - 5)$ $\bigcap f_{\mathbf{T}}(S_i) \subseteq f_{\mathbf{T}}(\bigcup S_i)$.

Intuitively, given the extension of some concept C, $f_{\mathbf{T}}$ selects the *typical* instances of C. $(f_{\mathbf{T}} - 1)$ requests that typical elements of S belong to S. $(f_{\mathbf{T}} - 2)$ requests that if there are elements in S, then there are also *typical* such elements. The next properties constraint the behavior of $f_{\mathbf{T}}$ wrt $\cap$ and $\cup$ in such a way that they do not entail monotonicity. According to $(f_{\mathbf{T}} - 3)$, if the typical elements of S are in R, then they coincide with the typical elements of $S \cap R$, thus expressing a weak form of monotonicity (namely *cautious monotonicity*). $(f_{\mathbf{T}} - 4)$ corresponds to one direction of the equivalence $f_{\mathbf{T}}(\bigcup S_i) = \bigcup f_{\mathbf{T}}(S_i)$, so that it does not entail monotonicity. Similar considerations apply to the equation $f_{\mathbf{T}}(\bigcap S_i) = \bigcap f_{\mathbf{T}}(S_i)$, of which only the inclusion $\bigcap f_{\mathbf{T}}(S_i) \subseteq f_{\mathbf{T}}(\bigcap S_i)$ is derivable. $(f_{\mathbf{T}} - 5)$ is a further constraint on the behavior of $f_{\mathbf{T}}$ wrt arbitrary unions and intersections; it would be derivable if $f_{\mathbf{T}}$ were monotonic.

In $[1, 2]$ an alternative semantics for $\mathbf{T}$ based on a preference relation is provided. The idea is that there is a global preference relation among individuals and that the typical members of a concept C, i.e. selected by $f_{\mathbf{T}}(C^I)$, are the minimal elements of C wrt this preference relation. Observe that this notion is global, that is to say, it does not compare individuals with respect to a specific concept (something like y is more typical than x wrt concept C). In this framework, an object $x \in \Delta$ is a *typical instance* of some concept C, if $x \in C^I$ and there is no C-element in Δ *more typical* than x. The typicality preference relation is partial since it is not always possible to establish which object is more typical than which other.

Let us first define the concept of minimal elements of a given set $S \subseteq \Delta$ wrt a relation $<$:

Definition 2 (Minimal elements of S). *Given a relation $<$ over a domain Δ, and given any $S \subseteq \Delta$, we define:*

$$Min_<(S) = \{x : x \in S \textit{ and } \nexists y \in S \textit{ s.t. } y < x\}$$

Moreover, we say that a relation $<$ over a set Δ satisfies the *Smoothness Condition* iff for all $S \subseteq \Delta$, for all $x \in S$, either $x \in Min_<(S)$ or $\exists y \in Min_<(S)$ such that $y < x$.

In $[1, 2]$ the following Representation Theorem has been proved:

Theorem 1 ([1]). *Given any model $\langle \Delta, I, f_{\mathbf{T}} \rangle$, $f_{\mathbf{T}}$ satisfies postulates $(f_{\mathbf{T}} - 1)$ to $(f_{\mathbf{T}} - 5)$ iff there exists an irreflexive and transitive relation $<$ over Δ satisfying the Smoothness Condition, such that for all $S \subseteq \Delta$, $f_{\mathbf{T}}(S) = Min_<(S)$.*

The above representation theorem allows to use the following semantics for $\mathcal{ALC} + \mathbf{T}$, which is similar to the one of Preferential logic **P** as defined by KLM.

Definition 3 (Semantics of $\mathcal{ALC} + \mathbf{T}$). *A model $\mathcal{M}$ of $\mathcal{ALC} + \mathbf{T}$ is any structure $\langle \Delta, I, < \rangle$ where:*

- *Δ is the domain;*
- *$<$ is an irreflexive and transitive relation over Δ satisfying the Smoothness Condition (Definition 2)*
- *I is the extension function that maps each extended concept C to $C^I \subseteq \Delta$, and each role R to a $R^I \subseteq \Delta \times \Delta$. I assigns to each atomic concept $A \in \mathcal{C}$ a set $A^I \subseteq \Delta$ and it is extended as follows:*
  - $\top^I = \Delta$
  - $\bot^I = \emptyset$
  - $(\neg C)^I = \Delta \backslash C^I$
  - $(C \sqcap D)^I = C^I \cap D^I$
  - $(C \sqcup D)^I = C^I \cup D^I$
  - $(\forall R.C)^I = \{a \in \Delta \mid \forall b.(a, b) \in R^I \rightarrow b \in C^I\}$

- $(\exists R.C)^I = \{a \in \Delta \mid \exists b.(a, b) \in R^I\}$
- $(\mathbf{T}(C))^I = Min_<(C^I)$

As mentioned before, the intuitive idea is as follows: typical elements of a concept C, i.e. instances of the extended concept $\mathbf{T}(C)$ (thus, belonging to $(\mathbf{T}(C))^I$) correspond to minimal elements of C, i.e. to elements in $Min_<(C^I)$. An inclusion relation $\mathbf{T}(C) \sqsubseteq D$ is satisfied in a model $\mathcal{M}$ if $Min_<(C^I) \subseteq D^I$. This is stated in a rigorous manner by the following definition:

Definition 4 (Model satisfying a Knowledge Base). *Consider a model $\mathcal{M}$, as defined in Definition 3. We extend I so that it assigns to each individual a of $\mathcal{O}$ an element a^I of the domain Δ. Given a KB (TBox,ABox), we say that:*

– *$\mathcal{M}$ satisfies TBox if for all inclusions $C \sqsubseteq D$ in TBox, and all elements $x \in \Delta$, if $x \in C^I$ then $x \in D^I$.*
– *$\mathcal{M}$ satisfies ABox if: (i) for all $C(a)$ in ABox, we have that $a^I \in C^I$, (ii) for all aRb in ABox, we have that $(a^I, b^I) \in R^I$.*

$\mathcal{M}$ satisfies a knowledge base if it satisfies both its TBox and its ABox.

If a model does not satisfy an inclusion $C \sqsubseteq D$, we will say that $C \not\sqsubseteq D$ holds in the model.

The following equation between the typicality operator $\mathbf{T}$ and the nonmonotonic entailment operator $\mid\!\sim$ in KLM logic **P** (describing what can be *typically* derived from a given premise) holds:

$$C \mid\!\sim D \text{ iff } \mathbf{T}(C) \sqsubseteq D$$

3 Extension of $\mathcal{ALC} + \mathbf{T}$ with a modular preference relation: the logic $\mathcal{ALC} + \mathbf{T_R}$

In the Introduction we have recalled that the family of KLM logics contains other interesting members, notably the stronger logic **R**, known as Rational Preferential Logic [12]. The axiomatization of the logic **R** is obtained from the axiomatization of **P** by adding the rule of rational monotonicity:

$$\text{RM. } ((A \mid\!\sim B) \wedge \neg(A \mid\!\sim \neg C)) \rightarrow ((A \wedge C) \mid\!\sim B)$$

Let us now consider the properties that would result for $\mathbf{T}$ if we adopted the stronger logic of nonmonotonic entailment **R**. If we added to the conditions above for $f_{\mathbf{T}}$ the following condition of Rational Monotonicity:

$$(f_{\mathbf{T}} - \mathbf{R}) \text{ if } f_{\mathbf{T}}(S) \cap R \neq \emptyset, \text{ then } f_{\mathbf{T}}(S \cap R) \subseteq f_{\mathbf{T}}(S)$$

we would obtain a stronger DL based on Rational Entailment, as described in [12]. $(f_{\mathbf{T}} - \mathbf{R})$ forces again a form of monotonicity: if there is a typical S having the property R, then all typical S and Rs inherit the properties of typical Ss. We call $\mathcal{ALC} + \mathbf{T_R}$ the logic resulting from the addition of $(f_{\mathbf{T}} - \mathbf{R})$ to the properties $(f_{\mathbf{T}} - 1) - (f_{\mathbf{T}} - 5)$.

Definition 5 (Semantics of $\mathcal{ALC} + \mathbf{T_R}$ with selection function $f_{\mathbf{T}}$). *An $\mathcal{ALC} + \mathbf{T_R}$ model with selection function is any structure $\langle \Delta, I, f_{\mathbf{T}} \rangle$, defined as in Definition 1, in which $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}} - \mathbf{R})$.*

As for the logic $\mathcal{ALC} + \mathbf{T}$, the semantics of $\mathcal{ALC} + \mathbf{T_R}$ can be formulated in terms of possible world structures $\langle \Delta, I, < \rangle$ in which $<$ is *modular*, i.e. for each x, y, z, if $x < y$, then either $z < y$ or $x < z$.

Definition 6 (Semantics of $\mathcal{ALC} + \mathbf{T_R}$). *An $\mathcal{ALC} + \mathbf{T_R}$ model $\mathcal{M}$ is any structure $\langle \Delta, I, < \rangle$, defined as in Definition 3, in which $<$ is further assumed to be modular.*

The equivalence between this semantics and the one formulated with $f_{\mathbf{T}}$ is proven by the representation theorem below (Theorem 6). First of all, we need to recall Lemma 2.1 in [2]:

Lemma 1 (Lemma 2.1 in [2], page 5). *If $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}} - 1) - (f_{\mathbf{T}} - 5)$, then $f_{\mathbf{T}}(S \cup R) \cap S \subseteq f_{\mathbf{T}}(S)$.*

Now we are able to prove the representation theorem:

Theorem 2 (Representation Theorem). *A KB is satisfiable in an $\mathcal{ALC} + \mathbf{T_R}$ model described in Definition 6 iff it is satisfiable in a model $\langle \Delta, I, f_{\mathbf{T}} \rangle$ where $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}} - 1) - (f_{\mathbf{T}} - 5)$ plus $(f_{\mathbf{T}} - \mathbf{R})$, and $(\mathbf{T}(C))^I = f_{\mathbf{T}}(C^I)$.*

Proof. Here we only consider the property $(f_{\mathbf{T}} - \mathbf{R})$. For the other properties, we refer to the proof of the Representation Theorem for $\mathcal{ALC} + \mathbf{T}$, as presented in [2], Theorem 2.1, page 5. The *only if* direction is trivial and left to the reader. For the *if* direction, as in [2], we define the $<$ relation as follows:

– for all $x, y \in \Delta$, we let $x < y$ if $\forall S \subseteq \Delta$, if $y \in f_{\mathbf{T}}(S)$, then (a) $x \notin S$ and (b) $\exists R \subseteq \Delta$ such that $S \subset R$ and $x \in f_{\mathbf{T}}(R)$.

Notice that given $(f_{\mathbf{T}} - \mathbf{R})$, this condition is equivalent to the simplified condition that only contains (a). Indeed, if (a) holds, it follows that also (b) holds. To be convinced, take any S such that $y \in f_{\mathbf{T}}(S)$, and $x \notin S$. We show that $x \in f_{\mathbf{T}}(S \cup \{x\})$, hence (b) holds. For a contradiction, suppose $x \notin f_{\mathbf{T}}(S \cup \{x\})$, then by $(f_{\mathbf{T}} - 1)$ and $(f_{\mathbf{T}} - 2)$, $f_{\mathbf{T}}(S \cup \{x\}) \cap S \neq \emptyset$, and by $(f_{\mathbf{T}} - \mathbf{R})$, $f_{\mathbf{T}}(S) = f_{\mathbf{T}}((S \cup \{x\}) \cap S) \subseteq f_{\mathbf{T}}(S \cup \{x\})$. Hence, $y \in f_{\mathbf{T}}(S \cup \{x\})$, which contradicts (a), given that $x \in S \cup \{x\}$. Therefore, we will consider the simplified definition of $<$:

– for all $x, y \in \Delta$, we let $x < y$ if $\forall S \subseteq \Delta$, if $y \in f_{\mathbf{T}}(S)$, then $x \notin S$.

We then show that if $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}} - \mathbf{R})$, then $<$ is modular. Let $x < y$. Consider z and suppose $z \not< y$. This means that there is R such that $y \in f_{\mathbf{T}}(R)$, and $z \in R$ We reason as follows. First, notice that by Lemma 1, $y \in f_{\mathbf{T}}(\{y, z\})$ (given that $y, z \in R$, $y \in f_{\mathbf{T}}(R \cup \{y, z\}) \cap \{y, z\}$, hence $y \in f_{\mathbf{T}}(\{y, z\})$). In order to show that $<$ is modular, we want to show that $x < z$. For a contradiction, suppose that $x \not< z$. Then there is Z such that $z \in f_{\mathbf{T}}(Z)$ and $x \in Z$. Consider $Z \cup \{y, z\}$, by $(f_{\mathbf{T}} - 1)$, $f_{\mathbf{T}}(Z \cup \{y, z\}) \subseteq Z \cup \{y, z\}$, and by $(f_{\mathbf{T}} - 2)$, $f_{\mathbf{T}}(Z \cup \{y, z\}) \neq \emptyset$. Hence, either $f_{\mathbf{T}}(Z \cup \{y, z\}) \cap Z \neq \emptyset$

or $f_{\mathbf{T}}(Z\cup\{y,z\})\cap Z=\emptyset$, and $f_{\mathbf{T}}(Z\cup\{y,z\})\cap\{y,z\}\neq\emptyset$. In the last case, $y\in f_{\mathbf{T}}(Z\cup\{y,z\})$. In the first case, by $(f_{\mathbf{T}}-\mathbf{R})$, $f_{\mathbf{T}}(Z)=f_{\mathbf{T}}((Z\cup\{y,z\})\cap Z)\subseteq f_{\mathbf{T}}(Z\cup\{y,z\})$, hence $z\in f_{\mathbf{T}}(Z\cup\{y,z\})$. From this, we derive that $f_{\mathbf{T}}(Z\cup\{y,z\})\cap\{y,z\}\neq\emptyset$, hence, by $(f_{\mathbf{T}}-\mathbf{R})$, $f_{\mathbf{T}}(\{y,z\})=f_{\mathbf{T}}((Z\cup\{y,z\})\cap\{y,z\})\subseteq f_{\mathbf{T}}(Z\cup\{y,z\})$, and $y\in f_{\mathbf{T}}(Z\cup\{y,z\})$. In both cases, we have that $y\in f_{\mathbf{T}}(Z\cup\{y,z\})$, however this is impossible, given that $x\in Z\cup\{y,z\}$ and $x<y$. We therefore conclude that if $z\not< y$, then $x<z$, hence modularity holds. ■

The following facts hold in $\mathcal{ALC}+\mathbf{T_R}$:

(R) $(\mathbf{T}(A)\sqcap B\not\sqsubseteq\bot)$ implies $\mathbf{T}(A\sqcap B)\sqsubseteq\mathbf{T}(A)$
(*) $(\mathbf{T}(A)\sqcap B\not\sqsubseteq\bot)$ implies $\mathbf{T}(B)\sqcap A\sqsubseteq\mathbf{T}(A)$

Proof. For simplicity, we consider here the semantics of $\mathcal{ALC}+\mathbf{T_R}$ with selection function (Definition 5). By the Representation Theorem above, this is equivalent to considering the semantics of $\mathcal{ALC}+\mathbf{T_R}$ with $<$ (Definition 6). It is immediate to see that **(R)** holds in an $\mathcal{ALC}+\mathbf{T_R}$ model with selection function satisfying $(f_{\mathbf{T}}-\mathbf{R})$. For **(*)**: If $(\mathbf{T}(A)\sqcap B\not\sqsubseteq\bot)$ holds in a model, then $f_{\mathbf{T}}(A^I)\cap B^I\neq\emptyset$, hence by $(f_{\mathbf{T}}-\mathbf{R})$, $f_{\mathbf{T}}((A\cap B)^I)\subseteq f_{\mathbf{T}}(A^I)$. On the other hand, from Lemma 1, we have that $f_{\mathbf{T}}(B^I)\cap A^I\subseteq f_{\mathbf{T}}((A\cap B^I)$. Hence, $f_{\mathbf{T}}(B^I)\cap A^I\subseteq f_{\mathbf{T}}((A\cap B)^I)\subseteq f_{\mathbf{T}}(A^I)$, and $\mathbf{T}(B)\sqcap A\sqsubseteq\mathbf{T}(A)$ holds. ■

Both properties allow us to draw conclusions from the simple fact that there is *one* individual that (i) is a typical instance of the concept A and that (ii) has the property B. From **(R)**, we derive that *all* typical A and Bs are typical As. From $(*)$ we derive something about typical Bs, even if A and B are unrelated properties. In particular, we derive that typical Bs that also have the property A are typical As.

From **(*)** we derive the counterintuitive example of the Introduction, where from an empty TBox and an ABox containing the following facts:

(a) $\mathbf{T}(\mathit{Brillant})(\mathit{john})$
(b) $\mathit{Writer}(\mathit{john})$
(c) $\neg\mathbf{T}(\mathit{Writer})(\mathit{john})$

we can then conclude that

(d) $\mathbf{T}(\mathit{Writer})\sqsubseteq\neg\mathit{Brillant}$

Indeed, from the ABox we can first obtain that $\mathbf{T}(\mathit{Brillant})\sqcap\mathit{Writer}\not\sqsubseteq\mathbf{T}(\mathit{Writer})$, then, by making the contrapositive of $(*)$, we get $\mathbf{T}(\mathit{Writer})\sqcap\mathit{Brillant}\sqsubseteq\bot$, from which we can immediately conclude (d) $\mathbf{T}(\mathit{Writer})\sqsubseteq\neg\mathit{Brillant}$.

As a further example, given the following ABox:

$\mathbf{T}(\mathit{Graduated})(\mathit{andras})$
$\mathit{SoccerPlayer}(\mathit{andras})$
$\mathbf{T}(\mathit{SoccerPlayer})(\mathit{lilian})$
$\mathit{Graduated}(\mathit{lilian})$

and an empty TBox, we can get that:

$\mathbf{T}(SoccerPlayer)(andras)$

which does not make sense given that $lilian$ is a different person not related to $andras$, hence we do not want to use $lilian$'s properties to make inferences about $andras$.

In our opinion, the inferences that hold in $\mathcal{ALC} + \mathbf{T_R}$ are rather arbitrary and counterintuitive. In conclusion, we believe that the logic **R** is too strong and unsuitable to reason about typicality.

4 $\mathcal{ALC} + \mathbf{T_R}$ vs Preferential Subsumption

In [3] a general preferential semantic framework for defeasible subsumption in DLs is presented. Here we recall this approach applied to the standard logic $\mathcal{ALC}$, in order to compare it with the logic $\mathcal{ALC} + \mathbf{T_R}$. We call the resulting logic $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$. The idea underlying $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$ is very similar to that underlying $\mathcal{ALC} + \mathbf{T}$ and $\mathcal{ALC} + \mathbf{T_R}$: some objects in the domain are more typical than others. In $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$, x is more typical than y if $x \geq y$. The properties of $\geq$ in $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$ correspond to those of $<$ in $\mathcal{ALC} + \mathbf{T_R}$. At a syntactic level the two logics differ, so that in $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$ one finds the defeasible inclusions $C \underset{\sim}{\sqsubset} D$ instead of $\mathbf{T}(C) \sqsubseteq D$ of $\mathcal{ALC} + \mathbf{T_R}$. But the idea is the same: in the two cases the inclusion holds if the most preferred (typical) Cs are also Ds.

We show that the logic $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$ is equivalent to the logic $\mathcal{ALC} + \mathbf{T_R}$, namely we show that the logic $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$ of preferential subsumption can be translated into $\mathcal{ALC} + \mathbf{T_R}$ by replacing $C \underset{\sim}{\sqsubset} D$ with $\mathbf{T}(C) \sqsubseteq D$. Therefore we conclude that the approach in [3] inherits the above criticisms for extensions of DLs that use **R**.

4.1 Preferential Subsumption

In [3] it is considered an alphabet of concept names $\mathcal{C}$, of role names $\mathcal{R}$, and of individuals $\mathcal{O}$. Concepts of the language $\mathcal{L}$ of the logic $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$ are defined as for standard $\mathcal{ALC}$, that is to say: $A \in \mathcal{C}$ and $\top$ are *concepts* of $\mathcal{L}$; if $C, D \in \mathcal{L}$ and $R \in \mathcal{R}$, then $C \sqcap D, C \sqcup D, \neg C, \forall R.C, \exists R.C$ are *concepts* of $\mathcal{L}$.

A knowledge base is a pair (TBox,ABox). TBox contains subsumptions $C \sqsubseteq D$ as well as preferential (or defeasible) subsumptions $C \underset{\sim}{\sqsubset} D$, where $C, D \in \mathcal{L}$. ABox contains expressions of the form $C(a)$ and aRb where $C \in \mathcal{L}$ is a concept, $R \in \mathcal{R}$, and $a, b \in \mathcal{O}$.

The basic idea of the semantics for preferential subsumption is that it is assumed that some objects of the domain Δ are viewed as more typical than others. Before giving a formal description of a model for $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$, we introduce the following definition of *maximal* elements of a given $S \subseteq \Delta$. As usual, given a partial order $\leq$, we have that $x = y$ if and only if $x \leq y$ and $y \leq x$, so that $x \neq y$ if and only if either $x \not\leq y$ or $y \not\leq x$.

Definition 7 (Maximal elements of S). *Given a relation $\leq$ over a domain Δ, and given any $S \subseteq \Delta$, we define:*

$$S^{-} = \{x : x \in S \mid \nexists y \in S \text{ s.t. } x \leq y \text{ and } x \neq y\}$$

Intuitively, given a concept C, the set C^{I^-} represents the maximally preferred (or typical) elements of C. A preferential subsumption $C \sqsubseteq\!\!\!\sim D$ is then satisfied in a model $\mathcal{M}$ if $C^{I^-} \subseteq D^I$.
Furthermore, we remember that a relation $\leq$ over a set Δ is *Noetherian* (or bounded) iff there is no infinite strictly ascending chain of objects of Δ. As the same authors have pointed out in [3], if $\leq$ is transitive, then the Noetherian property is equivalent to the following condition:

(N) for every $S \subseteq \Delta$ and $x \in S$, either: $x \in S^-$ or there is $y \in S$ such that $x \leq y$ and $y \in S^-$.

We also remember that a relation $\leq$ is modular iff, for each x, y, z such that $x \neq y, y \neq z, x \neq z$, if $x \leq y$, then either $z \leq y$ or $x \leq z$. In order to provide a semantics for $\mathcal{ALC}_{\sqsubseteq\!\!\!\sim}$, we extend the definition of model of $\mathcal{ALC}$ as follows:

Definition 8 (Semantics of $\mathcal{ALC}_{\sqsubseteq\!\!\!\sim}$). *A model $\mathcal{M}$ (ordered interpretation) is any structure $\langle \Delta, \leq, I \rangle$, where Δ and I are defined as in Definition 1, and $\leq$ is a Noetherian, modular, reflexive, transitive, anti-symmetric relation over Δ. For any concept C, C^I is defined in the usual way.*

Definition 9 (Model satisfying a Knowledge Base). *Consider a model $\mathcal{M}$, as defined in Definition 8. We extend I so that it assigns to each individual a of $\mathcal{O}$ an element a^I of the domain Δ. Given a KB (TBox,ABox), we say that:*

- *$\mathcal{M}$ satisfies TBox if:*
  - *for all subsumptions $C \sqsubseteq D$ in TBox, and all elements $x \in \Delta$, if $x \in C^I$ then $x \in D^I$*
  - *for all preferential subsumptions $C \sqsubseteq\!\!\!\sim D$ in TBox, and all elements $x \in \Delta$, if $x \in C^{I^-}$ then $x \in D^I$.*
- *$\mathcal{M}$ satisfies ABox if: (i) for all $C(a)$ in ABox, we have that $a^I \in C^I$, (ii) for all aRb in ABox, we have that $(a^I, b^I) \in R^I$.*

$\mathcal{M}$ satisfies a knowledge base if it satisfies both its TBox and its ABox.

4.2 Equivalence between $\mathcal{ALC} + \mathbf{T_R}$ and Preferential Subsumption

In this section we show that the semantics of $\mathcal{ALC} + \mathbf{T_R}$ proposed in Section 3 corresponds to the semantics of preferential subsumption of [3] outlined in Section 4.1. We first give a formal translation of an $\mathcal{ALC} + \mathbf{T_R}$ TBox into an $\mathcal{ALC}_{\sqsubseteq\!\!\!\sim}$ TBox, and vice-versa. Intuitively, a preferential subsumption $C \sqsubseteq\!\!\!\sim D$ corresponds to an inclusion relation $\mathbf{T}(C) \sqsubseteq D$. We then show that, on the one hand, given an $\mathcal{ALC} + \mathbf{T_R}$ TBox, say T, if it is satisfiable, so is the $\mathcal{ALC}_{\sqsubseteq\!\!\!\sim}$ TBox T' obtained by this translation; on the other hand, if an $\mathcal{ALC}_{\sqsubseteq\!\!\!\sim}$ TBox T' is satisfiable, so is the $\mathcal{ALC} + \mathbf{T_R}$ TBox T obtained by the translation.

Definition 10 (Translation of an $\mathcal{ALC} + \mathbf{T_R}$ TBox). *Given an $\mathcal{ALC} + \mathbf{T_R}$ TBox T, we define an $\mathcal{ALC}_{\sqsubseteq\!\!\!\sim}$ TBox T' as follows:*

- *for all inclusions $C \sqsubseteq D \in T$, where C is a concept, then $C \sqsubseteq D \in T'$;*

– *for all inclusions* $\mathbf{T}(C) \sqsubseteq D \in T$*, then* $C \precsim D \in T'$.

Definition 11 (Translation of an $\mathcal{ALC}_{\precsim}$ TBox). *Given an* $\mathcal{ALC}_{\precsim}$ *TBox* T'*, we define an* $\mathcal{ALC} + \mathbf{T_R}$ *TBox* T *as follows:*
– *for all subsumptions* $C \sqsubseteq D \in T'$*, then* $C \sqsubseteq D \in T$*;*
– *for all preferential subsumptions* $C \precsim D \in T'$*, then* $\mathbf{T}(C) \sqsubseteq D \in T$.

Let us first prove that:

Theorem 3. *Given an* $\mathcal{ALC} + \mathbf{T_R}$ *TBox* T*, if it satisfiable in an* $\mathcal{ALC} + \mathbf{T_R}$ *model* $\mathcal{M}$*, then the* $\mathcal{ALC}_{\precsim}$ *TBox* T'*, obtained by the translation of* T *as in Definition 10, is also satisfiable in an* $\mathcal{ALC}_{\precsim}$ *model* $\mathcal{M}'$.

Proof. Let $\mathcal{M} = \langle \Delta, <, I \rangle$ be the $\mathcal{ALC} + \mathbf{T_R}$ model satisfying T. We first define a model $\mathcal{M}^* = \langle \Delta', \leq, I' \rangle$ as follows:
– $\Delta' = \Delta$;
– we let $x \leq y$ iff $y < x$;
– we let $C^{I'} = C^I$ for any $\mathcal{ALC}$[4] concept C.

We then define a model $\mathcal{M}' = \langle \Delta', \leq', I' \rangle$, where $\leq'$ is defined as follows: $x \leq' y$ if $x \leq y$; for all $x \in \Delta$, we let $x \leq' x$.

We first show that $\mathcal{M}'$ is an $\mathcal{ALC}_{\precsim}$ model. To this aim, we just need to prove that the relation $\leq'$ is Noetherian (i.e. it satisfies the property **(N)**), modular, reflexive, transitive, and anti-symmetric.

Reflexivity follows from the definition of $\leq'$.

To show that $\leq'$ is transitive, suppose (1) $x \leq' y$ and (2) $y \leq' z$. We have to show that also $x \leq' z$. If $x = y$, then (2) $x \leq' z$ and we are done. The case in which $y = z$ is symmetric. In case $x \neq y$ and $y \neq z$, by definition of $\leq'$ we have that $x \leq y$ and $y \leq z$. By definition of $\leq$, we have that $y < x$ and $z < y$. Since $<$ is transitive, we have that $z < x$ and, by definition of $\leq$, $x \leq z$. We conclude by definition of $\leq'$ that $x \leq' z$.

The relation $\leq'$ is modular. Consider $x \leq' y$ and a given $z \in \Delta$ such that $x \neq y, z \neq y$, and $x \neq z$. We have to show that either $(*)$ $x \leq' z$ or $(**)$ $z \leq' y$. Since $x \neq y$, it can be observed that $x \leq y$, then $y < x$. Since $<$ is modular, we have that either $z < x$ or $y < z$. By definitions of $\leq$ and $\leq'$, if $z < x$ then $(*)$ $x \leq' z$, whereas if $y < z$, then $(**)$ $z \leq' y$. In both cases, we are done.

The relation $\leq'$ is anti-symmetric. Suppose, by absurd, that $x \leq' y$ and $y \leq' x$, but $x \neq y$. By definition of $\leq'$, this means that $x \leq y$ and $y \leq x$ and, by definition of $\leq$, that $x < y$ and $y < x$ in $\mathcal{M}$. By transitivity of $<$, we have that $x < x$, against the fact that $\mathcal{M}$ is an $\mathcal{ALC} + \mathbf{T_R}$ model and, then, $<$ is irreflexive. The relation $\leq'$ is then also anti-symmetric, since it is obtained from $\leq$ by adding only relations $x \leq' x$.

In order to prove that the relation $\leq'$ satisfies **(N)** we need the following fact:

Fact 1 *Given the models* $\mathcal{M}$ *and* $\mathcal{M}'$ *above, we have that* $S^- = Min_<(S)$.

[4] In other words, the interpretation I' corresponds to I with the exception of the extension of extended concepts $\mathbf{T}(C)$, not belonging to the language of the logic $\mathcal{ALC}_{\precsim}$.

Proof of Fact 1. First, we prove that $S^- \subseteq Min_<(S)$. Let $x \in S^-$, thus $x \in S$. This means that, for all $y \in S, y \neq x$, we have that $x \not\leq' y$. By definition of $\leq'$, also $x \not\leq y$ and, by definition of $\leq$, $y \not< x$. Therefore, $x \in S$ and, for all $y \in S$, we have $y \not< x$, that is to say $x \in Min_<(S)$.

In order to prove that $Min_<(S) \subseteq S^-$, consider $x \in Min_<(S)$, thus $x \in S$. This means that, for all $y \in S, y \neq x$, we have that $y \not< x$. Again, by definitions of $\leq$ and $\leq'$, we conclude that $x \not\leq' y$, that is to say $x \in S^-$. ($\Box$ *proof of Fact 1*)

Let us now prove that $\leq'$ satisfies the property **(N)**. Suppose, on the contrary, that there exist $S \subseteq \Delta$ and $x \in S$ such that $x \notin S^-$ and $(\star)$ $\nexists y \in S^-$ such that $x \leq' y$. Since $S \neq \emptyset$ ($x \in S$), also $Min_<(S) \neq \emptyset$ since $<$ satisfies the smoothness condition. Therefore, $(\star)$ means that, for all $y \in S^-, x \neq y$, we have $x \not\leq' y$. From Fact 1, we have that $x \notin Min_<(S)$. Moreover, for all $y \in Min_<$, we have that $y \not< x$ by definition of $\leq'$ and $\leq$. This contradicts the fact that the relation $<$ satisfies the smoothness condition.

Finally, we show that $\mathcal{M}'$ is a model for T'. For subsumptions of the form $C \sqsubseteq D \in T'$ the proof is straightforward, since $C \sqsubseteq D \in T$ and $C^{I'} \subseteq D^{I'}$ by the following facts: (i) $\mathcal{M}$ is a model of T, then $C^I \subseteq D^I$, (ii) $C^I = C^{I'}$ and (iii) $D^I = D^{I'}$ (C and D are $\mathcal{ALC}$ concepts). For preferential subsumptions of the form $C \sqsubseteq\!\!\!\!\sim D \in T'$, we observe that $\mathbf{T}(C) \sqsubseteq D \in T$ and, since $\mathcal{M}$ is a model of T, we have that $(\mathbf{T}(C))^I = Min_<(C^I) \subseteq D^I$. Moreover, since D is an $\mathcal{ALC}$ concept (not mentioning $\mathbf{T}$), we can also observe that $D^I = D^{I'}$. By Fact 1, we have that $C^{I'-} = Min_<(C^I)$. We conclude that $C^{I'-} = Min_<(C^I) \subseteq D^I = D^{I'}$, and the proof is over. ■

Let us now prove that:

Theorem 4. *Given an $\mathcal{ALC}_{\sqsubseteq\!\!\!\!\sim}$ TBox T', if it satisfiable in an $\mathcal{ALC}_{\sqsubseteq\!\!\!\!\sim}$ model $\mathcal{M}'$, then the $\mathcal{ALC} + \mathbf{T_R}$ TBox T, obtained by the translation of T' as in Definition 11, is also satisfiable in an $\mathcal{ALC} + \mathbf{T_R}$ model $\mathcal{M}$.*

Proof. Let $\mathcal{M}' = \langle \Delta', \leq', I' \rangle$ be the $\mathcal{ALC}_{\sqsubseteq\!\!\!\!\sim}$ model satisfying T'. We define a model $\mathcal{M} = \langle \Delta, <, I \rangle$ as follows:

– $\Delta = \Delta'$;
– we let $x < y$ iff $y \leq' x$ and $y \neq x$;
– we define I as follows:
  • $C^I = C^{I'}$ for all $\mathcal{ALC}$ concepts C;
  • $(\mathbf{T}(C))^I = Min_<(C^I)$.

We prove that $\mathcal{M}$ is an $\mathcal{ALC} + \mathbf{T_R}$ model satisfying T. First, we prove that $<$ is irreflexive, transitive and satisfies the smoothness condition.

By definition of $<$, $x < y$ if and only if $x \neq y$, therefore $<$ is irreflexive.

For transitivity, consider $x < y$ and $y < z$. By definition, we have that $x \neq y$, $y \neq z$, $y \leq' x$, and $z \leq' y$. Since $\mathcal{M}'$ is an $\mathcal{ALC}_{\sqsubseteq\!\!\!\!\sim}$ model, the relation $\leq'$ is transitive, therefore $z \leq' x$ and, by definition of $<$, we conclude that $x < z$.

As we have done for Fact 1, we can prove the following fact:

Fact 2 *Given the models $\mathcal{M}'$ and $\mathcal{M}$ above, we have that $S^- = Min_<(S)$.*

Let us now prove that the relation $<$ satisfies the smoothness condition. By absurd, suppose that that this does not hold, that is to say there are a set $S \subseteq \Delta$ and an element $x \in S$ such that (i) $x \notin Min_<(S)$ and (ii) $\nexists y \in Min_<(S)$ such that $y < x$. First, observe that, since $\mathcal{M}'$ is an $\mathcal{ALC}_{\precsim}$ model, the relation $\leq'$ satisfies condition **(N)**, therefore $S^- \neq \emptyset$. Therefore, (i) and (ii) imply that, for all $y \in Min_<(S)$, $y \not< x$. We can also observe that $x \not\leq' y$, since otherwise we would have $y < x$. By (i) and Fact 2, we have that $x \notin S^-$. This is in contrast with the fact that $\leq'$ satisfies condition **(N)**, since there are $S \subseteq \Delta$ and $x \in S$ such that $x \notin S^-$ and, for all $y \in S^-$ (again, by Fact 2, $y \in S^-$ if and only if $y \in Min_<(S)$), $x \neq y$, we have that $x \not\leq' y$.

Finally, we prove that the model $\mathcal{M}$ satisfies the Tbox T. First of all, by definition of $\mathcal{M}$, we have that $C^I = C^{I'}$ for all $\mathcal{ALC}$ concepts C, that is to say for all concepts not mentioning $\mathbf{T}$. Therefore, we have that $\mathcal{M}$ satisifies inclusions of the form $C \sqsubseteq D \in T$ where C is an $\mathcal{ALC}$ concept, since $C \sqsubseteq D \in T'$ and $\mathcal{M}'$ satisfies T', then $C^{I'} \subseteq D^{I'}$, thus $C^I \subseteq D^I$.

For inclusions of the form $\mathbf{T}(C) \sqsubseteq D \in T$, we have that $C \precsim D \in T'$ and, since $\mathcal{M}'$ is a model of T', we have that $C^{I'-} \subseteq D^{I'}$. By Fact 2 and the fact that C and D are $\mathcal{ALC}$ concepts, we can conclude that $Min_<(C^I) = C^{I'-} \subseteq D^{I'} = D^I$, and we are done. ■

By Theorems 3 and 4 above, we can conclude that the semantics of preferential subsumption is equivalent to the one of preferential (rational) description logics, therefore it inherits the criticisms for extensions of DLs that use **R** discussed in Section 3.

From a knowledge representation point of view, it can be observed that the language of $\mathcal{ALC} + \mathbf{T}$, as well as of $\mathcal{ALC} + \mathbf{T_R}$, is more general than the one of $\mathcal{ALC}_{\precsim}$. In the logics $\mathcal{ALC} + \mathbf{T}$ and $\mathcal{ALC} + \mathbf{T_R}$, it is also possible to use the $\mathbf{T}$ operator in the ABox, in order to express that individuals are typical members of a concept. For instance, an ABox can contain the following facts:

$\mathbf{T}(\mathit{Writer})(\mathit{sophie})$
$\mathbf{T}(\mathit{Writer} \sqcap \mathit{Depressed})(\mathit{mick})$,

representing that sophie is a typical writer and that mick is a typical depressed writer, respectively. Moreover, it is possible to reason about prototypical properties of those individuals; in the above example, from the KB of the Introduction, one can infer that sophie is brillant, whereas mick is not, i.e.:

$\mathit{Brillant}(\mathit{sophie})$
$\neg \mathit{Brillant}(\mathit{mick})$

It is not obvious how such typical properties of individuals occurring in the ABox can be encoded in the logic $\mathcal{ALC}_{\precsim}$ with preferential subsumptions. Moreover, although we have considered only TBoxes containing inclusions of the form

$$\mathbf{T}(C) \sqsubseteq D$$

where C and D are concepts not mentioning $\mathbf{T}$, nothing prevents us a more general use of the $\mathbf{T}$ operator in the definition of the TBox, for instance to formalize inclusions of the form

$$\mathbf{T}(C) \sqsubseteq \mathbf{T}(D).$$

This cannot be done in $\mathcal{ALC}_{\underset{\sim}{\sqsubset}}$ of [3].

5 Conclusions

In this work we have investigated the role of rational monotonicity in the context of nonmonotonic extensions of DLs.

We have first compared two approaches based on the semantics of KLM *rational* preferential entailment, namely:

1. the logic $\mathcal{ALC} + \mathbf{T}$, extending standard $\mathcal{ALC}$ by means of a typicality operator $\mathbf{T}$, which allows to express inclusion relations of the form $\mathbf{T}(C) \sqsubseteq D$, representing the fact that "typical" elements of concept C have the property D/are also members of D; the semantics of the operator $\mathbf{T}$ is based on a set of postulates that are essentially a reformulation of the axioms and rules of KLM logic **P**;
2. the logic $\mathcal{ALC} + \mathbf{T_R}$, which is equivalent to the approach by [3]. The logic $\mathcal{ALC} + \mathbf{T_R}$ is based on the same idea of $\mathcal{ALC}+\mathbf{T}$, but the semantics of $\mathbf{T}$ refers to the strongest KLM logic **R**.

We have provided some examples to show that the former is more appropriate than the latter when reasoning about typicality. Of course, both $\mathcal{ALC} + \mathbf{T}$ and $\mathcal{ALC} + \mathbf{T_R}$ are monotonic, so they must be completed by some kind of nonmonotonic mechanism. For $\mathcal{ALC} + \mathbf{T}$, some work has been done in [13]. More in detail, the *monotonic* logic $\mathcal{ALC} + \mathbf{T}$ is not sufficient to perform some kind of defeasible reasoning. Concerning the example of the Introduction, if the KB contains:

$\mathbf{T}(\mathit{Writer}) \sqsubseteq \mathit{Brillant}$
$\mathbf{T}(\mathit{Writer} \sqcap \mathit{Depressed}) \sqsubseteq \neg \mathit{Brillant}$

we get for instance that:

KB $\cup \{\mathit{Writer}(mick), \mathit{Depressed}(mick)\} \not\models \neg \mathit{Brillant}(mick)$
KB $\not\models \mathbf{T}(\mathit{Writer} \sqcap \mathit{Fat}) \sqsubseteq \mathit{Brillant}$

In order to derive the conclusion about $mick$ we should know (or assume) that $mick$ is a typical depressed writer, but we do not dispose of this information. Similarly, in order to derive that also a typical fat writer is brillant, we must be able to infer or assume that a "typical fat writer" is also a "typical writer", since there is no reason why it should not be the case; this cannot be derived by the logic itself given the nonmonotonic nature of $\mathbf{T}$. The basic monotonic logic $\mathcal{ALC} + \mathbf{T}$ is then too weak to enforce these extra assumptions, so that an additional mechanism to perform defeasible inferences is needed.

In [13] a minimal model semantics $\mathcal{ALC} + \mathbf{T}_{min}$ is proposed. The idea is (i) to define a preference relation among models and then (ii) to define a semantic entailment, denoted with $\models^{\mathcal{L}_T}_{min}$, determined by minimal models. Intuitively, a model $\mathcal{M}$ is preferred to a model $\mathcal{N}$ is $\mathcal{M}$ admits more typical instances of concepts with respect to $\mathcal{N}$. Taking the KB of the example above, one can obtain, for instance:

KB $\cup$ $\{ Writer(mick), Depressed(mick)\} \models^{\mathcal{L}_T}_{min} \neg Brillant(mick)$
KB $\cup$ $\{\exists HasChild.(Writer \sqcap Depressed)(roy)\} \models^{\mathcal{L}_T}_{min} \exists HasChild.\neg Brillant(roy)$
KB $\models^{\mathcal{L}_T}_{min} \mathbf{T}(Writer \sqcap Fat) \sqsubseteq Brillant$

Decision procedures for checking satisfiability in $\mathcal{ALC} + \mathbf{T}$ and $\mathcal{ALC} + \mathbf{T}_{min}$, as well as complexity results, have been provided. In detail, it has been shown that satisfiability in $\mathcal{ALC} + \mathbf{T}$ is EXPTIME-complete [1, 2]; since checking satisfiability of an $\mathcal{ALC}$ KB is known to be EXPTIME-complete, this means that adding the operator $\mathbf{T}$ is essentially inexpensive. Moreover, it has been proved that checking query entailment in $\mathcal{ALC} + \mathbf{T}_{min}$ is in CO-NEXP$^{\mathrm{NP}}$ [13].

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