Terminating Tableaux for $\mathcal{SOQ}$ with Number Restrictions on Transitive Roles

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Abstract. We show that the description logic $\mathcal{SOQ}$ with number restrictions on transitive roles is decidable by a terminating tableau calculus. The language decided by the calculus includes the universal role, which allows us to internalize TBox axioms. Termination of the system is achieved through pattern-based blocking.

1 Introduction

Efficient tableau algorithms are available for a wide range of description logics, including logics that contain both transitive roles and functional or number restrictions, like $\mathcal{SIN}$ [1], $\mathcal{SHIF}$ [2, 3], $\mathcal{SHIQ}$ [4], $\mathcal{SHOQ}$ [5], $\mathcal{SHOIQ}$ [6], and $\mathcal{SROIQ}$ [7]. In all cases, however, the language is restricted to contain no number restrictions on *complex roles*, e.g., on transitive roles, or roles containing transitive subroles. Although desirable for applications [8], number restrictions on complex roles lead to undecidability for logics extending $\mathcal{SHIN}$ [3]. In the absence of inverse roles ($\mathcal{I}$), however, the limitation of number restrictions to simple roles can be significantly relaxed [8]. In particular, the result in [8] implies the decidability of $\mathcal{SQ}$ extended by number restrictions on transitive roles. Obtained via a small model theorem, this decidability result does not yield practical decision procedures. Nor does it seem to imply the decidability of extensions of $\mathcal{SQ}$ with nominals.

We consider the logic $\mathcal{SOQ}$ with number restrictions on transitive roles, and call it $\mathcal{SOQ}^+$. As indicated by its name, $\mathcal{SOQ}^+$ extends the basic description logic $\mathcal{ALC}$ [9] by primitive transitive roles ($\mathcal{S}$), nominals ($\mathcal{O}$), and qualified number restrictions ($\mathcal{Q}$). We show that reasoning in $\mathcal{SOQ}^+$ is decidable by giving a terminating tableau calculus for concept satisfiability in $\mathcal{SOQ}^+$ extended by the universal role. Having the universal role in the language allows us to internalize terminological axioms, reducing reasoning with respect to TBoxes to concept satisfiability [10, 11].

For termination, our calculus employs *pattern-based blocking*. Pattern-based blocking is introduced in [12, 13] for converse-free hybrid logic with global modalities. In [14], the technique is extended to graded logics subsuming $\mathcal{SOQ}$ and $\mathcal{SHOQ}$. To provide a complete treatment of number restrictions on transitive roles, we extend pattern-based blocking further, incorporating ideas [15, 16] used in tableau systems for propositional dynamic logic and propositional μ-calculus.

2 Preliminaries

Following [12–14], our formal presentation is based on simple type theory. Notationally, our presentation is based on modal syntax, but can easily be translated to the traditional DL notation following [11]. We start with two base types B and I. The interpretation of B is fixed and consists of two truth values. The interpretation of I is a nonempty set whose elements are called *individuals*. Given two types σ and τ, the *functional type* $\sigma\tau$ is interpreted as the set of all total functions from the interpretation of σ to that of τ. We write $\sigma_1\sigma_2\sigma_3$ for $\sigma_1(\sigma_2\sigma_3)$.

We employ three kinds of variables: *Nominals* x, y, z of type I (we assume there are infinitely many nominals), *propositional variables* p, q of type IB, and *role variables* r of type IIB. Since the language in question contains no role expressions other than role variables, we call role variables *roles* for short. We use the logical constants $\bot, \top : \mathrm{B},\ \neg : \mathrm{BB},\ \vee, \wedge, \rightarrow : \mathrm{BBB},\ \doteq : \mathrm{IIB},\ \exists, \forall : (\mathrm{IB})\mathrm{B}$. Terms are defined as usual. We write st for applications, $\lambda x.s$ for abstractions, and $s_1s_2s_3$ for $(s_1s_2)s_3$. We also use infix notation, e.g., $s \wedge t$ for $(\wedge)st$.

Terms of type B are called *formulas*. We employ some common notational conventions: $\exists x.s$ for $\exists(\lambda x.s)$, $\forall x.s$ for $\forall(\lambda x.s)$, and $x \not\doteq y$ for $\neg(x \doteq y)$.

Let us write $\exists X.s$ for $\exists x_1 \dots x_n.s$ if $|X| = n$ and $X = \{x_1, \dots, x_n\}$. Also, given a set X of nominals, we use the following abbreviation:

$$DX \;:=\; \bigwedge_{\substack{x,y\in X\\ x\neq y}} x \not\doteq y$$

We use the following constants, which we call *modal operators*.

$$\begin{array}{rlrcl}
\dot\neg : & (\mathrm{IB})\mathrm{IB} & \dot\neg p & = & \lambda x.\neg px \\
\dot\wedge : & (\mathrm{IB})(\mathrm{IB})\mathrm{IB} & p \mathbin{\dot\wedge} q & = & \lambda x.\, px \wedge qx \\
\dot\vee : & (\mathrm{IB})(\mathrm{IB})\mathrm{IB} & p \mathbin{\dot\vee} q & = & \lambda x.\, px \vee qx \\
\langle_\rangle_n : & (\mathrm{IIB})(\mathrm{IB})\mathrm{IB} & \langle r\rangle_n p & = & \lambda x.\exists Y.\, DY \wedge (\bigwedge_{y\in Y} rxy \wedge py) \\
[_]_n : & (\mathrm{IIB})(\mathrm{IB})\mathrm{IB} & [r]_n p & = & \lambda x.\forall Y.\, (\bigwedge_{y\in Y} rxy) \wedge DY \rightarrow \bigvee_{y\in Y} py \\
E_n : & (\mathrm{IB})\mathrm{IB} & E_n p & = & \lambda x.\exists Y.\, DY \wedge \bigwedge_{y\in Y} py \\
A_n : & (\mathrm{IB})\mathrm{IB} & A_n p & = & \lambda x.\forall Y.\, DY \rightarrow \bigvee_{y\in Y} py \\
\dot_ : & \mathrm{IIB} & \dot x & = & \lambda y.x \doteq y \\
T : & (\mathrm{IIB})\mathrm{B} & Tr & = & \forall xyz.rxy \wedge ryz \rightarrow rxz
\end{array}$$

where $|Y| = n + 1$ in all equations

To the right of each constant is an equation defining its semantics. Formulas of the form $[r]_n tx$ are called *box formulas* or *boxes*, and formulas $\langle r\rangle_n tx$ are called *diamond formulas* or *diamonds*. The semantics of boxes and diamonds is defined following [17–19]. Our language does not contain a dedicated symbol for the universal role. Instead, we use graded *global modalities* E_n and A_n, which are semantically equivalent to qualified number restrictions on the universal role. Formulas of the form Tr are called *transitivity assertions*.

We assume the application of modal operators to have a higher precedence than regular functional application. So, for instance, we write $\dot\neg\langle r\rangle_2\dot y\,\dot\vee\,p\;x$ for $((\dot\neg(\langle r\rangle_2(\dot y)))\,\dot\vee\,p)x$.

A *modal interpretation* $\mathfrak{M}$ is an interpretation of simple type theory that interprets B as the set $\{0,1\}$, $\bot$ as 0 (i.e., false), $\top$ as 1 (i.e., true), maps I to a non-empty set, gives the logical constants $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, $\forall$, $\doteq$ their usual meaning, and satisfies the equations defining the modal operators $\dot\neg$, $\dot\wedge$, $\dot\vee$, $\langle\_\rangle_n$, $[\_]_n$, E, A, $\dot{\_}$ and T. If $\mathfrak{M}t = 1$, we say that $\mathfrak{M}$ *satisfies* t. A formula is called *satisfiable* if it has a satisfying modal interpretation.

3 Branches

For the sake of simplicity, we will define our tableau calculus $\mathcal{T}$ on negation normal *modal expressions*, i.e., terms of the form:

$$t \;::=\; p \mid \dot\neg p \mid \dot x \mid \dot\neg\dot x \mid t\,\dot\wedge\,t \mid t\,\dot\vee\,t \mid \langle r\rangle_n t \mid [r]_n t \mid E_n t \mid A_n t$$

A *branch* Γ is a finite set of formulas s of the form

$$s \;::=\; tx \mid rxy \mid Tr \mid x\dot=y \mid x\dot{\neq}y \mid \bot \mid \alpha{:}[r]_n tx$$

where t is a negation normal modal expression. The new form $\alpha{:}[r]_n tx$ serves algorithmic purposes. The *label* α of such *label introductions* is taken from a countably infinite set of labels. Formulas of the form rxy are called *edges*. We use the formula $\bot$ to explicitly mark unsatisfiable branches. We call a branch Γ *closed* if $\bot \in \Gamma$. Otherwise, Γ is called *open*. An interpretation $\mathfrak{M}$ satisfies a branch Γ if $\mathfrak{M}$ satisfies all *proper* formulas on Γ, i.e., all formulas except for label introductions. The branch consisting of the initial formulas to be tested for satisfiability is called the *initial branch*. Initial branches must contain no edges or label introductions. This restriction is inessential for the expressiveness of the language since label introductions are semantically irrelevant, and edges rxy can equivalently be expressed as $\langle r\rangle_0\dot y x$.

Let Γ be a branch. With $\sim_\Gamma$ we denote the least equivalence relation $\sim$ on nominals such that $x \sim y$ for every equation $x\dot=y \in \Gamma$. We define the *equational closure* $\tilde\Gamma$ of a branch Γ as

$$\begin{aligned}\tilde\Gamma \;:=\; & \Gamma \cup \{tx \mid \exists x' :\; x' \sim_\Gamma x \,\wedge\, tx' \in \Gamma\}\\ & \cup \{rxy \mid \exists x', y' :\; x' \sim_\Gamma x \,\wedge\, y' \sim_\Gamma y \,\wedge\, rx'y' \in \Gamma\}\end{aligned}$$

4 Evidence and Pre-evidence

The proof of model existence for our calculus $\mathcal{T}$ proceeds in three stages. Applied to a satisfiable initial branch, the rules of $\mathcal{T}$ (defined in Sect. 5) construct a *quasi-evident* branch (defined in Sect. 6). We show that every quasi-evident branch can be extended to a *pre-evident* branch, which, in turn, can be extended to an *evident* branch. For evident branches, we show model existence.

We write $D_\Gamma X$ as an abbreviation for $\forall x, y \in X : \; x \neq y \;\Rightarrow\; x \dot{\neq} y \in \Gamma$. A branch Γ is called *evident* if it satisfies all of the following *evidence conditions*:

$$
\begin{aligned}
(t_1 \dot\wedge t_2)x \in \Gamma \;&\Rightarrow\; t_1 x \in \tilde\Gamma \;\wedge\; t_2 x \in \tilde\Gamma \\
(t_1 \dot\vee t_2)x \in \Gamma \;&\Rightarrow\; t_1 x \in \tilde\Gamma \;\vee\; t_2 x \in \tilde\Gamma \\
\langle r\rangle_n tx \in \Gamma \;&\Rightarrow\; \exists Y\colon\; |Y| = n+1 \;\wedge\; D_\Gamma Y \;\wedge\; \{rxy, ty \mid y \in Y\} \subseteq \tilde\Gamma \\
[r]_n tx \in \Gamma \;&\Rightarrow\; |\{y \mid rxy \in \tilde\Gamma,\; ty \notin \tilde\Gamma\}/_{\sim_\Gamma}| \leq n \\
E_n tx \in \Gamma \;&\Rightarrow\; \exists Y\colon\; |Y| = n+1 \;\wedge\; D_\Gamma Y \;\wedge\; \{ty \mid y \in Y\} \subseteq \tilde\Gamma \\
A_n tx \in \Gamma \;&\Rightarrow\; |\{y \mid ty \notin \tilde\Gamma\}/_{\sim_\Gamma}| \leq n \\
\dot{x}y \in \Gamma \;&\Rightarrow\; x \sim_\Gamma y \\
\dot\neg\dot{x}y \in \Gamma \;&\Rightarrow\; x \not\sim_\Gamma y \\
x \dot= y \in \Gamma \;&\Rightarrow\; x \sim_\Gamma y \\
x \dot{\neq} y \in \Gamma \;&\Rightarrow\; x \not\sim_\Gamma y \\
\neg px \in \Gamma \;&\Rightarrow\; px \notin \tilde\Gamma \\
Tr \in \Gamma \;&\Rightarrow\; \forall x, y, z :\; rxy \in \tilde\Gamma \wedge ryz \in \tilde\Gamma \Rightarrow rxz \in \tilde\Gamma
\end{aligned}
$$

A formula s is called *evident on Γ* if Γ satisfies the right-hand side of the evidence condition corresponding to s. For instance, $(t_1 \dot\wedge t_2)x$ is evident on Γ if and only if $\{t_1 x, t_2 x\} \subseteq \tilde\Gamma$.

Theorem 4.1 (Model Existence). *Evident branches are satisfiable.*

Proof. Omitted for space reasons. Proceeds similarly to the argument in [14]. □

To simplify the treatment of transitivity, we introduce the notion of pre-evidence. We define the relation $\rhd^r_\Gamma$ as the least relation such that:

$$
\begin{aligned}
rxy \in \tilde\Gamma \;&\Rightarrow\; x \rhd^r_\Gamma y \\
x \rhd^r_\Gamma y \wedge y \rhd^r_\Gamma z \wedge Tr \in \Gamma \;&\Rightarrow\; x \rhd^r_\Gamma z
\end{aligned}
$$

We write $x \unrhd^r_\Gamma y$ iff $x \sim_\Gamma y$ or $x \rhd^r_\Gamma y$.

The *pre-evidence conditions* are obtained from the evidence conditions by omitting the condition for transitivity assertions and replacing the condition for boxes as follows:

$$[r]_n tx \in \Gamma \;\Rightarrow\; |\{y \mid x \rhd^r_\Gamma y,\; ty \notin \tilde\Gamma\}/_{\sim_\Gamma}| \leq n$$

Pre-evidence of individual formulas is defined analogously to the corresponding evidence condition. Note that for all formulas but boxes and transitivity assertions, the notions of evidence and pre-evidence coincide.

We now show that every pre-evident branch can be extended to an evident branch. Let the *evidence closure* $\hat\Gamma$ of a branch Γ be defined as $\Gamma \cup \{rxy \mid x \rhd^r_\Gamma y\}$.

Proposition 4.1. $rxy \in \hat\Gamma \iff rxy \in \tilde{\hat\Gamma} \iff x \rhd^r_\Gamma y$

Theorem 4.2 (Evidence Completion). *Γ pre-evident $\Longrightarrow$ $\hat\Gamma$ evident*

Proof. Omitted for space reasons. Uses Proposition 4.1. □

5 Tableau Rules

The tableau rules of our calculus $\mathcal{T}$ are defined in Fig. 1. In the rules, we write $\exists x \in X : \Gamma(x)$ for $\Gamma(x_1) \mid \ldots \mid \Gamma(x_n)$, where $X = \{x_1, \ldots, x_n\}$ and $\Gamma(x)$ is a set of formulas parametrized by x. In case $X = \emptyset$, the notation translates to $\perp$. Dually, we write $\forall x \in X : \Gamma(x)$ for $\Gamma(x_1), \ldots, \Gamma(x_n)$ $(X = \{x_1, \ldots, x_n\})$.

Given a term t, we write $\mathcal{N}t$ for the set of nominals that occur in t. The notation is extended to sets of terms in the natural way: $\mathcal{N}\Gamma := \bigcup\{\mathcal{N}t \mid t \in \Gamma\}$.

The side condition of $\mathcal{R}_\Diamond$ uses the notion of quasi-evidence, which we will introduce in Sect. 6. For now, assume the rule is formulated with the restriction "$\langle r \rangle_n tx$ not evident on Γ".

Given a branch Γ, we define $x{:}_\Gamma \alpha \ :\Leftrightarrow\ \exists y, t :\ \alpha{:}ty \in \Gamma \wedge y \rhd^r_\Gamma x$. A box formula tx is *subsumed* on Γ if there is a nominal y and a label α such that $y \trianglerighteq^r_\Gamma x$ and $\alpha{:}ty \in \Gamma$. The rule $\mathcal{R}_T$ is constrained to be applicable only to boxes that are not subsumed on Γ. This ensures, in particular, that $\mathcal{R}_T$ is applied at most once to each individual box formula on the branch.

A branch Γ is called a *proper extension* of a branch Δ if $\Gamma \supseteq \Delta$ and $\tilde{\Gamma} \supsetneq \tilde{\Delta}$. Note that if Γ is a proper extension of Δ, then in particular it holds $\Gamma \supsetneq \Delta$. We implicitly restrict the applicability of the tableau rules such that a rule $\mathcal{R}$ is only applicable to a formula $s \in \Gamma$ if all of the alternative branches $\Delta_1, \ldots, \Delta_n$ resulting from this application are proper extensions of Γ.

Proposition 5.1 (Soundness). *Let $\Delta_1, \ldots, \Delta_n$ be the branches obtained from a branch Γ by a rule of $\mathcal{T}$. Then Γ is satisfiable if and only if there is some $i \in \{1, \ldots, n\}$ such that Δ_i is satisfiable.*

6 Blocking Conditions and Quasi-evidence

The restrictions on the applicability of the tableau rules given by the pre-evidence conditions are not sufficient for termination. To obtain a terminating calculus, we restrain the rule $\mathcal{R}_\Diamond$ by weakening the notion of pre-evidence for diamond formulas. The weaker notion, called quasi-evidence, is then used in the side condition of $\mathcal{R}_\Diamond$ in place of pre-evidence. Quasi-evidence must be weak enough to guarantee termination but strong enough to preserve completeness.

The *edge graph* of a branch Γ is a labelled graph with the nodes $\mathcal{N}\Gamma$ and edges $\{(x, y) \mid \exists r :\ rxy \in \Gamma\}$, where a node x is labelled with all expressions t such that $tx \in \Gamma$, and an edge (x, y) is labelled with all roles r such that $rxy \in \Gamma$. A branch can always be represented graphically through its edge graph.

In [14], the notion of quasi-evidence is based on the following observation. Let Γ be a branch and x, y be nominals such that

- *x has no r-successor on Γ*, i.e., there is no z such that $rxz \in \tilde{\Gamma}$,
- for every r-diamond or r-box $tx \in \tilde{\Gamma}$, it holds $ty \in \tilde{\Gamma}$, and
- all r-diamonds and r-boxes $sy \in \tilde{\Gamma}$ are evident on Γ.

$$\mathcal{R}_{\dot\wedge}\ \frac{(s \mathbin{\dot\wedge} t)x}{sx,\ tx} \qquad\qquad \mathcal{R}_{\dot\vee}\ \frac{(s \mathbin{\dot\vee} t)x}{sx \ \mid\ tx}$$

$$\mathcal{R}_{\Diamond}\ \frac{\langle r\rangle_n tx}{\forall y,z \in Y,\ y \neq z :\ y \dot{\neq} z,\ rxy,\ ty}\ Y \text{ fresh},\ |Y| = n+1,\ \langle r\rangle_n tx \text{ not quasi-evident on } \Gamma$$

$$\mathcal{R}_{\Box}\ \frac{[r]_n tx}{\exists y,z \in Y,\ y \neq z :\ y \dot{=} z \ \mid\ \exists y \in Y :\ ty}\ Y \subseteq \{y \mid x \rhd^r_\Gamma y\},\ |Y| = |Y/{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_{T}\ \frac{Tr,\ rxy}{\alpha{:}[r]_n tx}\ \alpha \text{ fresh},\ [r]_n tx \in \tilde\Gamma,\ [r]_n tx \text{ not subsumed on } \Gamma$$

$$\mathcal{R}_{E}\ \frac{E_n tx}{\forall y,z \in Y,\ y \neq z :\ y \dot{\neq} z,\ ty}\ Y \text{ fresh},\ |Y| = n+1,\ E_n tx \text{ not evident on } \Gamma$$

$$\mathcal{R}_{A}\ \frac{A_n tx}{\exists y,z \in Y,\ y \neq z :\ y \dot{=} z \ \mid\ \exists y \in Y :\ ty}\ Y \subseteq \mathcal{N}\Gamma,\ |Y| = |Y/{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_{N}\ \frac{\dot x y}{x \dot{=} y} \qquad \mathcal{R}_{\bar N}\ \frac{\dot\neg \dot x y}{x \dot{\neq} y} \qquad \mathcal{R}^{\perp}_{\dot\neg}\ \frac{\dot\neg px}{\perp}\ px \in \tilde\Gamma \qquad \mathcal{R}^{\perp}_{\dot\neq}\ \frac{x \dot{\neq} y}{\perp}\ x \sim_\Gamma y$$

Γ is the branch to which a rule is applied. "Y fresh" stands for $Y \cap \mathcal{N}\Gamma = \emptyset$.
"α fresh" stands for $\nexists t, x :\ \alpha{:}tx \in \Gamma$

Fig. 1. Tableau rules for $\mathcal{T}$

Then all r-diamonds and r-boxes $sx \in \tilde\Gamma$ can be made evident by extending Γ with $\{rxz \mid ryz \in \tilde\Gamma\}$. As an example, consider the edge graph in Fig. 2(a). There, the formula $\langle r\rangle_0 px$ can be made evident by adding the edge rxz (represented by the dashed arrow) to the branch. In the presence of transitivity, extending a branch Γ by an edge rxz may destroy the evidence of r-boxes tu such that $u \rhd^r_\Gamma x$ (Fig. 2(b)). Note, however, that adding an edge rxz cannot destroy the evidence of a box tu such that $u \rhd^r_\Gamma x$ if we already have $u \rhd^r_\Gamma z$ (Fig. 2(c)).

To deal with non-local constraints introduced by number restrictions on transitive roles, we refine the notion of a pattern and the quasi-evidence conditions from [14]. Similarly to the usage of "history variables" in [16] in order to track the propagation of eventuality constraints, we use labels to represent constraints on the successors of a nominal x that are introduced by r-boxes at x if r is transitive.

Given a role r, an *r-pattern* is a set consisting of modal expressions of the form μt, where $\mu \in \{\langle r\rangle_n, [r]_n \mid n \in \mathbb{N}\}$, and labels α, such that, for some n, t, x: $\alpha{:}[r]_n tx \in \Gamma$. We write $P^r_\Gamma x$ for the largest r-pattern P such that $P \subseteq \{\mu t \mid \mu tx \in \tilde\Gamma\} \cup \{\alpha \mid x{:}_\Gamma \alpha\}$. We call $P^r_\Gamma x$ the r-pattern of x on Γ. An r-pattern P is *expanded on Γ* if there are nominals x, y such that $rxy \in \tilde\Gamma$ and $P \subseteq P^r_\Gamma x$. In this case, we say that the nominal x *expands P on Γ*.

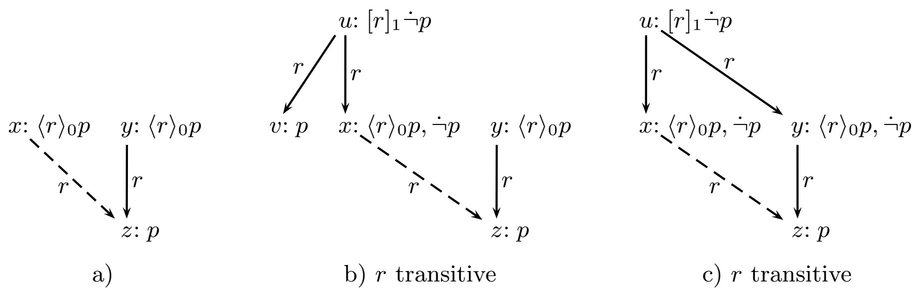

Fig. 2. Number restrictions and transitivity

A diamond $\langle r\rangle_n sx \in \Gamma$ is *quasi-evident on* Γ if it is either evident on Γ or x has no r-successor on Γ and $P^r_\Gamma x$ is expanded on Γ. The rule $\mathcal{R}_\Diamond$ can only be applied to diamonds that are not quasi-evident. Note that whenever $\langle r\rangle_n sx \in \Gamma$ is quasi-evident but not evident (on Γ), there is a nominal y that expands $P^r_\Gamma x$.

The *quasi-evidence conditions* are obtained from the pre-evidence conditions by replacing the condition for diamond formulas and adding a condition for transitivity assertions and label introductions as follows:

$$\langle r\rangle_n tx \in \Gamma \;\Rightarrow\; \langle r\rangle_n tx \text{ is quasi-evident on } \Gamma$$
$$Tr \in \Gamma \;\Rightarrow\; \forall n,t,x:\; [r]_n tx \in \tilde\Gamma \;\Rightarrow\; \exists z,\alpha:\; z \trianglerighteq^r_\Gamma x \wedge \alpha{:}[r]_n tz \in \Gamma$$
$$\alpha{:}tx \in \Gamma \;\Rightarrow\; tx \in \Gamma \wedge \forall y:\; x \triangleright^r_\Gamma y \Leftrightarrow y{:}_\Gamma\alpha$$

Lemma 6.1. *Let Γ be a branch. Let $\{[r]_n tx, [r]_n ty\} \subseteq \tilde\Gamma$ such that $Tr \in \Gamma$ and $x \trianglerighteq^r_\Gamma y$. Then:* $[r]_n tx$ is pre-evident on $\Gamma \implies [r]_n ty$ is pre-evident on Γ

Lemma 6.2. *Let Γ be a quasi-evident branch. Let $\langle r\rangle_n sx \in \Gamma$ be not evident on Γ, y be a nominal that expands $P^r_\Gamma x$ on Γ, and $\Delta := \Gamma \cup \{rxz \mid ryz \in \tilde\Gamma\}$. Then:*

1. *$\forall z:\; rxz \in \tilde\Delta \iff ryz \in \tilde\Gamma$ and $x \triangleright^r_\Delta z \iff y \triangleright^r_\Gamma z$,*
2. *$\forall m,t:\; \langle r\rangle_m t \in P^r_\Gamma x \implies \langle r\rangle_m tx$ is evident on Δ,*
3. *$\langle r\rangle_n sx$ is evident on Δ,*
4. *$\forall r',m,t,z:\; \langle r'\rangle_m tz$ is evident on $\Gamma \implies \langle r'\rangle_m tz$ is evident on Δ,*
5. *Δ is quasi-evident.*

Proof. Omitted for space reasons. Uses Lemma 6.1. □

For an illustration of Lemma 6.2, let the edge graph in Fig. 2(a) (without the dashed arrow) represent Γ. Then $\langle r\rangle_0 px$ is quasi-evident but not evident on Γ, and y expands $P^r_\Gamma x$. The graph with the dashed arrow added corresponds to the branch Δ in the lemma. The five claims for Γ and Δ are easy to verify.

Theorem 6.1 (Pre-evidence Completion). *For every quasi-evident branch Γ there is a pre-evident branch Δ such that $\Gamma \subseteq \Delta$.*

Proof. Analogous to the corresponding proof in [14], using Lemma 6.2(3-5). □

A branch is called *maximal* if it cannot be extended by any tableau rule.

Lemma 6.3. *Let Γ be a branch that is obtained from an initial branch. Then Γ satisfies the quasi-evidence condition for label introductions.*

Proof. Let $\Gamma_0 \to \ldots \to \Gamma_n$ be a derivation such that Γ_0 is an initial branch and $\Gamma_n = \Gamma$. The claim is shown by induction on n. $\square$

Theorem 6.2 (Quasi-evidence). *Every open and maximal branch obtained in $\mathcal{T}$ from an initial branch is quasi-evident.*

Proof. Let Γ be an open and maximal branch obtained from an initial branch. We show that every $s \in \Gamma$ that is not of the form px or rxy is either pre-evident or quasi-evident on Γ by induction on the size of s. The quasi-evidence of formulas $\alpha{:}tx$ follows by Lemma 6.3. $\square$

7 Termination

We will now show that every tableau derivation is finite. Since the tableau rules are all finitely branching, by König's lemma it suffices to show that the construction of every individual branch terminates. Since rule application always produces proper extensions of branches, it then suffices to show that the size (i.e., cardinality) of an individual branch is bounded. First, we show that the size of a branch Γ is bounded by a function in the number of nominals on Γ. Then, we show that this number itself is bounded, completing the termination proof.

We write $\Gamma \xrightarrow{\mathcal{R}} \Delta$ to denote that Δ is obtained from Γ by a single application of the rule $\mathcal{R}$. We write $\Gamma \to \Delta$ if there is some $\mathcal{R}$ such that $\Gamma \xrightarrow{\mathcal{R}} \Delta$.

We write $\mathcal{S}\Gamma$ for the set of all modal expressions occurring on Γ, possibly as subterms of other expressions, and $\mathrm{Rel}\,\Gamma$ for the set of all roles that occur on Γ. Crucial for the termination argument is the fact the the tableau rules cannot introduce any modal expressions that do not already occur on the initial branch.

Proposition 7.1. *If Γ, Δ are branches such that Δ is obtained from Γ by any rule of $\mathcal{T}$, then $\mathcal{S}\Delta = \mathcal{S}\Gamma$.*

For every pair of nominals x, y and every role r, a branch Γ may contain an edge rxy, an equation $x \doteq y$ or a disequation $x \dot{\neq} y$. For every expression $s \in \mathcal{S}\Gamma$, Γ may contain a formula sx. Tableau rule application can introduce at most one formula $\alpha{:}[r]_n tx$ for each box expression $[r]_n t$ and each nominal x. Finally, a branch may contain $\bot$. So, since the initial branch Γ_0 contains no formulas of the form $\alpha{:}tx$, the size of Γ derived from Γ_0 is bounded by $|\mathrm{Rel}\,\Gamma| \cdot |\mathcal{N}\Gamma|^2 + 2|\mathcal{N}\Gamma|^2 + 2|\mathcal{S}\Gamma| \cdot |\mathcal{N}\Gamma| + 1$. By Proposition 7.1, we know that $|\mathcal{S}\Gamma|$ and $|\mathrm{Rel}\,\Gamma|$ depend only on the initial branch.

By the above, it suffices to show that $|\mathcal{N}\Gamma|$ is bounded in the sum of the sizes of the initial formulas. We do so by giving a bound on the number of applications of $\mathcal{R}_\Diamond$ and $\mathcal{R}_E$ that can occur in the derivation of a branch, which suffices since the two rules are the only ones that can introduce new nominals.

For $\mathcal{R}_E$, we do so by defining $\psi_E\Gamma := \{E_n s \in \mathcal{S}\Gamma \mid \exists x \in \mathcal{N}\Gamma :\ E_n sx$ not evident on $\Gamma\}$ and showing that $|\psi_E\Gamma|$ decreases with every application of $\mathcal{R}_E$ (and is non-increasing otherwise, which is obvious).

Proposition 7.2. $\Gamma \xrightarrow{\mathcal{R}_E} \Delta \implies |\psi_E\Gamma| > |\psi_E\Delta|$

The proof proceeds analogously to the corresponding arguments in [12, 13].

Now we show that $\mathcal{R}_\Diamond$ can be applied at most finitely often in a derivation. Since there are only finitely many roles, it suffices to show that $\mathcal{R}_\Diamond$ can be applied at most finitely often for each role. Since $\mathcal{R}_\Diamond$ is only applicable to diamond formulas that are not quasi-evident, the following holds:

Proposition 7.3. *If $\mathcal{R}_\Diamond$ is applicable to a formula $\langle r\rangle_n sx \in \Gamma$, then either*

1. *x has an r-successor on Γ, or*
2. *$P^r_\Gamma x$ is not expanded on Γ.*

Let Δ be a branch obtained from some Γ by applying $\mathcal{R}_\Diamond$ to a formula $\langle r\rangle_n sx \in \Gamma$ such that $P^r_\Gamma x$ is not expanded on Γ. Clearly, $P^r_\Delta x$ must be expanded on Δ.

Since $\Gamma \to \Delta$ implies $\tilde{\Gamma} \subseteq \tilde{\Delta}$, it holds:

Proposition 7.4. *Let $s \in \Gamma$ be a diamond formula and $\Gamma \to \Delta$.*

1. *If s is evident on Γ, then s is evident on Δ.*
2. *If Δ is obtained from Γ by applying $\mathcal{R}_\Diamond$ to s, then s is evident on Δ.*

Proposition 7.5. *Let $\Gamma \to \Delta$, $x \in \mathcal{N}\Gamma$, and P be an r-pattern.*

1. *$P^r_\Gamma x \subseteq P^r_\Delta x$.*
2. *If P is expanded on Γ, then P is expanded on Δ.*

Termination of our calculus follows analogously to [14], provided we can, for each role r, bound the number of applications of $\mathcal{R}_\Diamond$. In the case of [14], this bound can be given as $|\mathrm{Pat}^{\,r}\Gamma_0|$ where Γ_0 is the initial branch and $\mathrm{Pat}^{\,r}\Gamma := \mathcal{P}(\{\langle r\rangle_n s \in \mathcal{S}\Gamma\} \cup \{[r]_n s \in \mathcal{S}\Gamma\})$. The present situation is more complex since now patterns may contain labels in addition to modal expressions. Unlike $\mathcal{S}\Gamma_0$, the set of labels introduced on the branch may grow during tableau construction. Still, we can bound the number of applications of $\mathcal{R}_\Diamond$ for every given set of labels.

A rule $\mathcal{R}$ is said to be applied to a nominal $x \in \mathcal{N}\Gamma$ if $\mathcal{R}$ is applied to a formula $tx \in \Gamma$. Given a pattern P, we define $\mathcal{A}P := \{\alpha \in P\}$.

Lemma 7.1. *Let Γ_0 be an initial branch and $\Gamma_0 \to \Gamma_1 \to \ldots$ a derivation. Let r be a role, A a set of labels, and*

$$I := \{i \mid \exists x : \Gamma_{i+1} \text{ is obtained from } \Gamma_i \text{ by applying } \mathcal{R}_\Diamond \text{ to } x \wedge \mathcal{A}(P^r_{\Gamma_i} x) = A\}$$

Then $|I| \leq |\mathrm{Pat}^{\,r}\Gamma_0| \cdot |\{\langle r\rangle_m t \in \mathcal{S}\Gamma_0\}|$.

Proof. Follows by Propositions 7.1, 7.3, 7.4 and 7.5 as Γ_0 contains no edges. An analogous argument for a similar calculus and $A = \emptyset$ is detailed in [14]. $\square$

A set of labels A is called a *pattern space* for a role r on a branch Γ if there is some $x \in \mathcal{N}\Gamma$ such that $\mathcal{A}(P^r_\Gamma x) = A$. By Lemma 7.1, it suffices to show that for each role r, the number of pattern spaces created during a derivation is bounded.

Lemma 7.2. *Let Γ_0 be an initial branch, r a role and A a set of labels. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every derivation $\Gamma_0 \to \Gamma_1 \to \ldots$:*

$$|\{x \mid \exists i, y : \; i \geq 0 \wedge \mathcal{A}(P^r_{\Gamma_i} x) = A \wedge rxy \in \Gamma_i\}| \leq f|A|$$

Proof. Let r and $\Gamma_0 \to \Gamma_1 \to \ldots$ be as required. Let $X_A := \{x \mid \exists i, y : \; i \geq 0 \wedge \mathcal{A}(P^r_{\Gamma_i} x) = A \wedge rxy \in \Gamma_i\}$. We proceed by induction on $n := |A|$. For every $x \in X_A$, let i_x be the least i such that

1. $\mathcal{A}(P^r_{\Gamma_i} x) = A$, and
2. for some y, $rxy \in \Gamma_i$.

Since Γ_0 contains no edges, $i_x \geq 1$. No single rule application can make 1 and 2 true at the same time. Hence, for every $x \in X_A$ exactly one of the following is true:

Case $\mathcal{A}(P^r_{\Gamma_{i_x-1}} x) \subsetneq A$. Then there is some y such that $rxy \in \Gamma_{i_x-1}$. So, $x \in X_B$ for some proper subset B of A. Clearly, this case is only possible if $|A| > 0$.
Case $\nexists y : \; rxy \in \Gamma_{i_x-1}$. Then $\mathcal{A}(P^r_{\Gamma_{i_x-1}} x) = A$. So, $i_x - 1$ belongs to the set I from Lemma 7.1. This is the only case possible if $|A| = 0$.

By the above, f can be defined as follows:

$$f0 := |\mathrm{Pat}^r \Gamma_0| \cdot |\{\langle r\rangle_m t \in \mathcal{S}\Gamma_0\}|$$
$$fn := |\mathrm{Pat}^r \Gamma_0| \cdot |\{\langle r\rangle_m t \in \mathcal{S}\Gamma_0\}| + \sum_{k=0}^{n-1} \binom{n}{k} fk \qquad \text{if } n > 0$$ □

We define the *level* of an r-pattern P on Γ as:

$$L_\Gamma P := |\{[r]_m t \in \mathcal{S}\Gamma \mid \exists \alpha, y : \; \alpha \in P \wedge \alpha{:}[r]_m ty \in \Gamma\}|$$

A label α is said to be *generated at level n* in a derivation $\Gamma_0 \to \Gamma_1 \to \ldots$ if there is some $i \geq 0$ such that α is generated by an application of $\mathcal{R}_T$ extending Γ_i by a formula $\alpha{:}[r]_m tx$, and $L_{\Gamma_i}(P^r_{\Gamma_i} x) = n$.

Lemma 7.3. *Let $\Gamma_0 \to \Gamma_1 \to \ldots$ be a derivation where Γ_0 is initial and $Tr \in \Gamma_0$. Let $x \in \mathcal{N}\Gamma_i$. Then every $\alpha \in P^r_{\Gamma_i} x$ is generated at level strictly less than $L_{\Gamma_i}(P^r_{\Gamma_i} x)$.*

Proof. Assume, by contradiction, Γ_i, r, and x are all as required and there is some $\alpha \in P^r_{\Gamma_i} x$ such that α is generated at level $m \geq L_{\Gamma_i}(P^r_{\Gamma_i} x)$. Then there is some $j < i$ such that α is generated by an application of $\mathcal{R}_T$ to some $ryz \in \Gamma_j$ such that $L_{\Gamma_j}(P^r_{\Gamma_j} y) = m$. By the applicability restriction on $\mathcal{R}_T$ (non-subsumption), $L_{\Gamma_{j+1}}(P^r_{\Gamma_{j+1}} y) = m + 1$, and therefore $L_{\Gamma_i}(P^r_{\Gamma_i} y) > m$. Since $\alpha \in P^r_{\Gamma_i} x$, by Lemma 6.3 it holds $y \rhd^r_{\Gamma_i} x$. Since r is transitive, $\mathcal{A}(P^r_{\Gamma_i} y) \subseteq \mathcal{A}(P^r_{\Gamma_i} x)$. Consequently, $L_{\Gamma_i}(P^r_{\Gamma_i} x) \geq L_{\Gamma_i}(P^r_{\Gamma_i} y) > m \geq L_{\Gamma_i}(P^r_{\Gamma_i} x)$. Contradiction. □

By Lemma 7.3, the number of pattern spaces with level n (i.e., pattern spaces whose patterns have level n) is bounded from above by 2^m, where m is the number of labels generated at levels less than n. Clearly, the level of r-patterns in a derivation from Γ_0 is bounded by $|\{[r]_k t \in \mathcal{S}\Gamma_0\}|$. Also, by the applicability restriction on $\mathcal{R}_T$ (non-subsumption), no labels can be generated at level $|\{[r]_k t \in \mathcal{S}\Gamma_0\}|$. Hence, in order to show that the number of pattern spaces created during a derivation is bounded, it suffices to bound the number of labels generated at all levels less than $|\{[r]_k t \in \mathcal{S}\Gamma_0\}|$. A label α is called *r-label* (in a derivation $\Gamma_0 \to \Gamma_1 \to \ldots$) if there are i, n, t, x such that $\alpha{:}[r]_n tx \in \Gamma_i$.

Lemma 7.4. *Let Γ_0 be an initial branch and $Tr \in \Gamma_0$. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every derivation $\Gamma_0 \to \Gamma_1 \to \ldots$ and $0 \leq n < |\{[r]_k t \in \mathcal{S}\Gamma_0\}|$: $|\{\alpha \mid \exists m \leq n :\ \alpha$ is an r-label generated at level $m\}| \leq fn$.*

Proof. We define f by induction on n. Let $A_m := \{\alpha \mid \exists k \leq m :\ \alpha$ is an r-label generated at level $k\}$, and $A_m := \emptyset$ if $m < 0$. A new label can only be generated by an application of $\mathcal{R}_T$. Therefore, by the applicability condition of $\mathcal{R}_T$, $|A_n| \leq |\{[r]_m t \in \mathcal{S}\Gamma_0\}| \cdot |\{x \mid \exists i, y :\ i \geq 0 \wedge L_{\Gamma_i}(P^r_{\Gamma_i} x) \leq n \wedge rxy \in \Gamma_i\}|$. By Lemma 7.3, $|A_n| \leq |\{[r]_m t \in \mathcal{S}\Gamma_0\}| \cdot |\bigcup_{B \subseteq A_{n-1}} \{x \mid \exists i, y :\ i \geq 0 \wedge \mathcal{A}(P^r_{\Gamma_i} x) = B \wedge rxy \in \Gamma_i\}|$. By Lemma 7.2, there is a function g such that $|A_n| \leq |\{[r]_m t \in \mathcal{S}\Gamma_0\}| \cdot \sum_{k=0}^{|A_{n-1}|} \binom{|A_{n-1}|}{k} gk \leq |\{[r]_m t \in \mathcal{S}\Gamma_0\}| \cdot 2^{|A_{n-1}|} g|A_{n-1}|$. Hence, we can define:

$$
\begin{aligned}
f0 &:= |\{[r]_m t \in \mathcal{S}\Gamma_0\}| \cdot g0 \\
fn &:= |\{[r]_m t \in \mathcal{S}\Gamma_0\}| \cdot 2^{f(n-1)} g(f(n-1)) \qquad \text{if } n > 0
\end{aligned}
$$

□

8 Conclusion

To account for non-local constraints introduced by number restrictions on transitive roles, the notion of patterns from [14] needs to be extended. The extension is semantically intuitive and allows for a simple proof of model existence. As it comes to termination, the reasoning in [14] needs to be refined considerably.

The termination proof establishes a non-elementary complexity bound for the associated decision procedure. Presently, we do not know if this bound is strict. Still, we believe that this procedure is well-suited for efficient implementation despite its potentially high worst-case complexity. In fact, if applied to problems that do not contain number restrictions on transitive roles, the complexity of the procedure matches the NExpTime bound of [14], which is even lower than the 2-NExpTime bound established for practically successful procedures of [2–6].

While $\mathcal{SHQ}$ with number restrictions on transitive roles is, in general, undecidable [8], decidability can be regained for $\mathcal{SHQ}$-terminologies satisfying certain admissibility criteria [8]. We believe that the present algorithm can be extended to cover terminologies satisfying these criteria.

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