

Forgetting and uniform interpolation in extensions of the description logic EL

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1 Introduction

Terminologies used in life sciences, health-care, and other knowledge intensive areas are often very large and comprehensive. Examples are the medical terminology SNOMED CT (Systematised Nomenclature of Medicine Clinical Terms) [17] containing about 380 000 concept definitions and the National Cancer Institute ontology (NCI) [16] containing more than 60 000 axioms. For terminologies \mathcal{T} of this size, it is often of interest to *forget* a subvocabulary Σ of the vocabulary of \mathcal{T} ; i.e., to transform \mathcal{T} into a new terminology \mathcal{T}_Σ (called a Σ -*interpolant of \mathcal{T}*) that contains no symbols from Σ and that is indistinguishable from \mathcal{T} regarding its consequences that do not use Σ . In AI, this problem has been studied under a variety of names such as *forgetting* and *variable elimination* [14, 4, 10]. In mathematical logic, this problem has been investigated as the *uniform interpolation problem* [18]. Computing Σ -interpolants of terminologies has a number of potential applications, e.g.,

Re-use of ontologies: when using terminologies such as SNOMED CT in an application, often only a very small fraction of its vocabulary is of interest. In this case, one could use a Σ -interpolant instead of the whole terminology, where Σ is the vocabulary not of interest for the application.

Predicate hiding: a terminology developer might not want to publish a terminology completely because a certain part of its vocabulary is not intended for public use. Again, publishing Σ -interpolants, where Σ is the vocabulary to be hidden, appears to be a solution to this problem.

Exhibiting hidden relations between terms: large terminologies are difficult to maintain as small changes to its axioms can have drastic and damaging effects. To analyze possibly unwanted consequences over a certain part Γ of the vocabulary, an ontology developer can automatically generate a complete axiomatization of the relations between terms over Γ by computing a Σ -interpolant, where Σ is the complement of Γ .

Ontology versioning: to check whether two versions of a terminology have the same consequences over their common vocabulary (or a subset thereof), one can first compute their interpolants by forgetting the vocabulary not shared by the two versions and then check whether the two interpolants are logically equivalent (i.e., have the same models).

In the description of Σ -interpolants given above, we have neither specified a language in which they are axiomatized nor did we specify the language wrt. which Σ -interpolants should be indistinguishable from the original terminology. The choice of

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the latter language depends on the application: for example, if one is interested in *inclusions between concepts*, then a Σ -interpolant should imply the same concept inclusions using no symbols from Σ as the original terminology. On the other hand, if the terminology is used to query instance data using *conjunctive queries*, then a Σ -interpolant together with any instance data using no symbols from Σ should imply the same certain answers to conjunctive queries using no symbols from Σ as the original terminology.

Regarding the language \mathcal{L} in which Σ -interpolants should be axiomatized, one has to find a compromise between the following three conflicting goals:

(R) Standard reasoning problems (e.g., logical equivalence) in \mathcal{L} should not be more complex than reasoning in the language underlying the terminology.

(I) Σ -interpolants in \mathcal{L} should be uniquely determined up to logical equivalence: if T_1' and T_2' are Σ -interpolants in \mathcal{L} of terminologies T_1 and T_2 that have the same consequences not using Σ , then T_1' and T_2' should be logically equivalent.

(E) The language \mathcal{L} should be powerful enough to admit *finite* and *succinct* (ideally, polynomial size) axiomatizations of Σ -interpolants, and it should be possible to compute Σ -interpolants efficiently (ideally, in polynomial time).

For terminologies in standard description logics (DLs) such as \mathcal{EL} and any language between \mathcal{ALC} and \mathcal{SHIQO} , there do not exist languages \mathcal{L} achieving all these goals simultaneously.¹ To illustrate this point, let \mathcal{L} be second-order logic. Then \mathcal{L} trivially satisfies (E) but fails to satisfy (R) and (I), for ontologies in any standard DL.

In this paper, we consider forgetting in the lightweight description logic \mathcal{EL} underlying the designated OWL2-EL profile of the upcoming OWL Version 2 extended with role inclusions and domain and range restrictions [1]. This choice is motivated by the fact that forgetting appears to be of particular interest for large-scale and comprehensive terminologies and that many such terminologies are given in this language. We introduce three DLs for axiomatizing Σ -interpolants satisfying criteria (R) and (I) and preserving, respectively, inclusions between concepts, concept instances, and answers to conjunctive queries. These DLs do not satisfy (E), as Σ -interpolants sometimes do not exist or are of exponential size. We demonstrate that, nevertheless, Σ -interpolants typically exist and can be computed in practice for large-scale terminologies such as SNOMED CT and appropriate versions of NCI. This work has been presented in [7]; detailed proofs and additional experiments are available in a technical report [8].

2 Preliminaries

Let N_C , N_R , and N_I be countably infinite and mutually disjoint sets of concept names, role names, and individual names. \mathcal{EL} -concepts C are built according to the rule

$$C := A \mid \top \mid C \sqcap D \mid \exists r.C,$$

where $A \in N_C$, $r \in N_R$, and C, D range over \mathcal{EL} -concepts. The set of \mathcal{ELH}^r -inclusions consists of *concept inclusions* $C \sqsubseteq D$ and *concept equations* $C \equiv D$, *domain restrictions* $\text{dom}(r) \sqsubseteq C$, *range restrictions* $\text{ran}(r) \sqsubseteq C$ and *role inclusions* $r \sqsubseteq s$, where

¹ This follows from the fact that deciding whether TBoxes in these DLs imply the same concept inclusions over a signature is at least one exponential harder than deciding logical equivalence [12, 11].

Concept C	Translation C^\sharp	Concept C	Translation C^\sharp
\top	$x = x$	$\text{dom}(r)$	$\exists y (r(x, y))$
A	$A(x)$	$\text{ran}(r)$	$\exists y (r(y, x))$
$C \sqcap D$	$C^\sharp(x) \wedge D^\sharp(x)$	$\exists u.C$	$(x = x) \wedge \exists y C^\sharp(y)$
$\exists r.C$	$\exists y (r(x, y) \wedge C^\sharp(y))$	$\exists r_1 \sqcap \dots \sqcap r_n.C$	$\exists y (r_1(x, y) \wedge \dots \wedge r_n(x, y) \wedge C^\sharp(y))$

Fig. 1. Standard translation \cdot^\sharp

C, D are \mathcal{EL} -concepts and $r, s \in \mathbb{N}_R$. An \mathcal{ELH}^r -TBox \mathcal{T} is a finite set of \mathcal{ELH}^r -inclusions. An \mathcal{ELH}^r -TBox is called \mathcal{ELH}^r -terminology if all its concept inclusions and equations are of the form $A \sqsubseteq C$ and $A \equiv C$ and no concept name occurs more than once on the left hand side. In what follows, we use $A \bowtie C$ to denote expressions of the form $A \sqsubseteq C$ and $A \equiv C$.

Assertions of the form $A(a)$ and $r(a, b)$, where $a, b \in \mathbb{N}_I$, $A \in \mathbb{N}_C$, and $r \in \mathbb{N}_R$, are called ABox-assertions. An ABox is a finite set of ABox-assertions. By $\text{obj}(\mathcal{A})$ we denote the set of individual names in \mathcal{A} . A knowledge base (KB) is a pair $(\mathcal{T}, \mathcal{A})$ consisting of a TBox \mathcal{T} and an ABox \mathcal{A} . Assertions of the form $C(a)$ and $r(a, b)$, where $a, b \in \mathbb{N}_I$, C a \mathcal{EL} -concept, and $r \in \mathbb{N}_R$, are called instance assertions. To define the semantics of DLs considered in this paper we make use of the fact that DL-expressions can be regarded as formulas in FO, where FO denotes the set of first-order predicate logic formulas with equality using unary predicates in \mathbb{N}_C , binary predicates in \mathbb{N}_R , and constants from \mathbb{N}_I ; see Figure 1 (in which the DL-constructors not considered so far are defined later). In what follows, we will not distinguish between DL-expressions and their translation into FO and regard TBoxes, ABoxes and KBs as finite subsets of FO. Thus, we use $\mathcal{T} \models \varphi$ to denote that φ follows from \mathcal{T} in first-order logic even if φ is an \mathcal{ELH}^r -inclusion and \mathcal{T} a subset of FO and similar conventions apply to DLs introduced later in this paper. FO (and, therefore, \mathcal{ELH}^r) is interpreted in models $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the *domain* $\Delta^{\mathcal{I}}$ is a non-empty set, and $\cdot^{\mathcal{I}}$ is a function mapping each concept name A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, each role name r to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and each individual name a to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

The most important ways of querying \mathcal{ELH}^r -TBoxes and KBs are subsumption (check whether $\mathcal{T} \models \alpha$ for an \mathcal{ELH}^r -inclusion α), instance checking (check whether $(\mathcal{T}, \mathcal{A}) \models \alpha$ for an instance assertion α), and conjunctive query answering. To define the latter, call a first-order formula $q(\mathbf{x})$ a *conjunctive query* if it is of the form $\exists \mathbf{y} \psi(\mathbf{x}, \mathbf{y})$, where ψ is a conjunction of expressions $A(t)$ and $r(t_1, t_2)$ with t, t_1, t_2 drawn from \mathbb{N}_I and sequences of variables \mathbf{x} and \mathbf{y} . If \mathbf{x} has length k , then a sequence \mathbf{a} of elements of $\text{obj}(\mathcal{A})$ of length k is called a *certain answer* to $q(\mathbf{x})$ of a KB $(\mathcal{T}, \mathcal{A})$ if $(\mathcal{T}, \mathcal{A}) \models q(\mathbf{a})$.

3 Forgetting

A signature Σ is a subset of $\mathbb{N}_C \cup \mathbb{N}_R^2$. Given a signature Σ , we set $\overline{\Sigma} = (\mathbb{N}_C \cup \mathbb{N}_R) \setminus \Sigma$. Given a concept, role, concept inclusion, TBox, ABox, FO-sentence, set of

² We investigate forgetting for TBoxes for DLs without nominals; thus we do not include individual names into the signature.

FO-sentences E , we denote by $\text{sig}(E)$ the signature of E , that is, the set of concept and role names occurring in it. We use the term \mathcal{ELH}_Σ^r -inclusion (Σ -ABox, Σ -query, \mathcal{L}_Σ -sentence, etc.) to denote \mathcal{ELH}^r -inclusions (ABoxes, queries, \mathcal{L} -sentences, etc.) whose signature is contained in Σ .

To define forgetting, we first formalize the notion of inseparability between TBoxes wrt. a signature. Intuitively, two TBoxes \mathcal{T}_1 and \mathcal{T}_2 are inseparable wrt. a signature Σ if they have the same Σ -consequences, where the set of Σ -consequences considered can either reflect subsumption queries, instance queries, or conjunctive queries, depending on the application. We give the definitions for sets of FO-sentences because we later require these notions for a variety of DLs.

Definition 1. Let \mathcal{T}_1 and \mathcal{T}_2 be sets of FO-sentences and Σ a signature.

– \mathcal{T}_1 and \mathcal{T}_2 are concept Σ -inseparable, in symbols $\mathcal{T}_1 \equiv_\Sigma^c \mathcal{T}_2$, if for all \mathcal{ELH}_Σ^r -inclusions α : $\mathcal{T}_1 \models \alpha \Leftrightarrow \mathcal{T}_2 \models \alpha$.

– \mathcal{T}_1 and \mathcal{T}_2 are instance Σ -inseparable, in symbols $\mathcal{T}_1 \equiv_\Sigma^i \mathcal{T}_2$, if for all Σ -ABoxes \mathcal{A} and Σ -instance assertions α using individual names from $\text{obj}(\mathcal{A})$: $(\mathcal{T}_1, \mathcal{A}) \models \alpha \Leftrightarrow (\mathcal{T}_2, \mathcal{A}) \models \alpha$.

– \mathcal{T}_1 and \mathcal{T}_2 are query Σ -inseparable, in symbols $\mathcal{T}_1 \equiv_\Sigma^q \mathcal{T}_2$, if for all Σ -ABoxes \mathcal{A} , conjunctive Σ -queries $q(\mathbf{x})$, and vectors \mathbf{a} of the same length as \mathbf{x} of individual names in $\text{obj}(\mathcal{A})$: $(\mathcal{T}_1, \mathcal{A}) \models q(\mathbf{a}) \Leftrightarrow (\mathcal{T}_2, \mathcal{A}) \models q(\mathbf{a})$.

The definition of forgetting (Σ -interpolants) is now straightforward.

Definition 2 (Σ -interpolant). Let \mathcal{T} be an \mathcal{ELH}^r -TBox, Σ a finite signature, and \mathcal{L} a set of FO-sentences. If \mathcal{T}_Σ is a finite set of \mathcal{L}_Σ -sentences such that $\mathcal{T} \models \varphi$ for all $\varphi \in \mathcal{T}_\Sigma$, then \mathcal{T}_Σ is

- a concept Σ -interpolant of \mathcal{T} in \mathcal{L} if $\mathcal{T} \equiv_\Sigma^c \mathcal{T}_\Sigma$;
- an instance Σ -interpolant of \mathcal{T} in \mathcal{L} if $\mathcal{T} \equiv_\Sigma^i \mathcal{T}_\Sigma$;
- a query Σ -interpolant of \mathcal{T} in \mathcal{L} if $\mathcal{T} \equiv_\Sigma^q \mathcal{T}_\Sigma$.

One can show that every query Σ -interpolant is an instance Σ -interpolant and every instance Σ -interpolant is a concept Σ -interpolant. The converse implications do not hold, even for \mathcal{ELH}^r -terminologies:

Example 1. Let $\mathcal{T} = \{\text{ran}(r) \sqsubseteq A_1, \text{ran}(s) \sqsubseteq A_2, B \equiv A_1 \sqcap A_2\}$ and $\Sigma = \{A_1, A_2\}$. One can show that the empty TBox is a concept Σ -interpolant of \mathcal{T} . However, the empty TBox is not an instance Σ -interpolant of \mathcal{T} . To show this, consider the $\bar{\Sigma}$ -ABox $\mathcal{A} = \{r(a_0, b), s(a_1, b)\}$. Then $(\mathcal{T}, \mathcal{A}) \models B(b)$ but $(\emptyset, \mathcal{A}) \not\models B(b)$. Observe that no \mathcal{ELH}^r -TBox (and even no \mathcal{SHQ} -TBox) is an instance Σ -interpolant of \mathcal{T} because it is impossible to capture the ABox \mathcal{A} in a DL in which one cannot refer to the range of distinct roles in one concept. On the other hand, the TBox $\mathcal{T}' = \{\text{ran}(r) \sqcap \text{ran}(s) \sqsubseteq B\}$ given in an extension of \mathcal{ELH}^r is an instance Σ -interpolant of \mathcal{T} .

Example 2. Let $\mathcal{T} = \{A \sqsubseteq \exists s.\top, s \sqsubseteq r_1, s \sqsubseteq r_2\}$ and $\Sigma = \{s\}$. Then $\mathcal{T}' = \{A \sqsubseteq \exists r_1.\top \sqcap \exists r_2.\top\}$ is an instance Σ -interpolant of \mathcal{T} , but \mathcal{T}' is not a query Σ -interpolant of \mathcal{T} . To show the latter, let $\mathcal{A} = \{A(a)\}$ and let $q = \exists x(r_1(a, x) \wedge r_2(a, x))$. Then $(\mathcal{T}, \mathcal{A}) \models q$ but $(\mathcal{T}', \mathcal{A}) \not\models q$. Again, no \mathcal{ELH}^r -TBox (and even no TBox in \mathcal{SHIQ}) is

a query Σ -interpolant of \mathcal{T} . On the other hand, the TBox $\mathcal{T}'' = \{A \sqsubseteq \exists r_1 \sqcap r_2. \top\}$ given in an extension of \mathcal{ELH}^r with conjunctions of roles names is a query Σ -interpolant of \mathcal{T} .

Besides of exhibiting examples where concept, instance, and query Σ -interpolants are distinct, Example 1 and 2 also show that even in extremely simple cases \mathcal{ELH}^r and a variety of more expressive DLs are not sufficiently powerful to express instance and query Σ -interpolants of \mathcal{ELH}^r -terminologies. Rather surprisingly, there also exist simple examples in which \mathcal{ELH}^r -TBoxes are not sufficiently expressive to axiomatize concept Σ -interpolants of \mathcal{ELH}^r -terminologies.

Example 3. Let $\Sigma = \{\text{Research_Inst}, \text{Education_Inst}\}$ and \mathcal{T} be

$$\begin{aligned} \text{University} &\equiv \text{Research_Inst} \sqcap \text{Education_Inst} \\ \text{School} &\sqsubseteq \text{Education_Inst} \\ \text{ran}(\text{PhD_from}) &\sqsubseteq \text{Research_Inst} \end{aligned}$$

Then there does not exist an \mathcal{ELH}^r -TBox that is a concept Σ -interpolant of \mathcal{T} . Intuitively, the reason is that there is no \mathcal{ELH}_{Σ}^r -TBox which follows from \mathcal{T} and has the following infinite set of $\bar{\Sigma}$ -consequences (which are consequences of \mathcal{T}):

$$\exists \text{PhD_from}.(\text{School} \sqcap A) \sqsubseteq \exists \text{PhD_from}.(\text{University} \sqcap A),$$

where $A \in \bar{\Sigma}$. On the other hand, the TBox $\mathcal{T}' = \{\text{ran}(\text{PhD_from}) \sqcap \text{School} \sqsubseteq \text{University}\}$ given in an extension of \mathcal{ELH}^r is a concept Σ -interpolant of \mathcal{T} .

We now introduce three extensions of \mathcal{ELH}^r which we propose to axiomatize concept, instance, and query Σ -interpolants.

Definition 3 ($\mathcal{EL}^{\text{ran},0}$, $\mathcal{EL}^{\text{ran}}$, $\mathcal{EL}^{\text{ran},\sqcap,u}$). $\mathcal{C}^{\text{ran},0}$ -concepts are constructed using the following syntax rule

$$C := D \mid \text{ran}(r) \mid \text{ran}(r) \sqcap D,$$

where D ranges over \mathcal{EL} -concepts and $r \in \mathbb{N}_R$. The set of $\mathcal{EL}^{\text{ran},0}$ -inclusions consists of concept inclusions $C \sqsubseteq D$ and role inclusions $r \sqsubseteq s$, where C is a $\mathcal{C}^{\text{ran},0}$ -concept, D an \mathcal{EL} -concept, and $r, s \in \mathbb{N}_R$.

\mathcal{C}^{ran} -concepts are constructed using the following syntax rule

$$C := A \mid \text{ran}(r) \mid C \sqcap D \mid \exists r.C,$$

where $A \in \mathbb{N}_C$, C, D range over \mathcal{C}^{ran} -concepts and $r \in \mathbb{N}_R$. The set of $\mathcal{EL}^{\text{ran}}$ -inclusions consists of all concept inclusions $C \sqsubseteq D$ and role inclusions $r \sqsubseteq s$, where C is a \mathcal{C}^{ran} -concept, D an \mathcal{EL} -concept, and $r, s \in \mathbb{N}_R$.

Let u (the universal role) be a fresh logical symbol. $\mathcal{C}^{\sqcap,u}$ -concepts are constructed using the following syntax rule

$$C := A \mid C \sqcap D \mid \exists R.C \mid \exists u.C,$$

where $A \in \mathbb{N}_C$, C, D range over $\mathcal{C}^{\sqcap,u}$ -concepts and $R = r_1 \sqcap \dots \sqcap r_n$ with $r_1, \dots, r_n \in \mathbb{N}_R$. The set of $\mathcal{EL}^{\text{ran},\sqcap,u}$ -inclusions consists of concept inclusions $C \sqsubseteq D$ and role inclusions $r \sqsubseteq s$, where C is a \mathcal{C}^{ran} -concept, D a $\mathcal{C}^{\sqcap,u}$ -concept, and $r, s \in \mathbb{N}_R$.

An X -TBox is a finite set of X -inclusions, where X ranges over $\mathcal{EL}^{\text{ran}}$, $\mathcal{EL}^{\text{ran},0}$, and $\mathcal{EL}^{\text{ran},\sqcap,u}$.

We have the following inclusions:

$$\mathcal{EL}\mathcal{H}^r \triangleleft \mathcal{EL}^{\text{ran},0} \triangleleft \mathcal{EL}^{\text{ran}} \triangleleft \mathcal{EL}^{\text{ran},\square,u}$$

where $\mathcal{L}_1 \triangleleft \mathcal{L}_2$ means that every \mathcal{L}_1 -TBox is logically equivalent to some \mathcal{L}_2 -TBox. The semantics of the additional constructors is straightforward and given in Figure 1. We regard the universal role u as a logical symbol (i.e., $u \notin \mathbb{N}_R$). This interpretation reflects the fact that the signature of the first-order translation of $\exists u.C$ coincides with the signature of C . Observe that the TBox given as a concept Σ -interpolant in Example 3 is an $\mathcal{EL}^{\text{ran},0}$ -TBox; the instance Σ -interpolant given in Example 1 is an $\mathcal{EL}^{\text{ran}}$ -TBox, and the query Σ -interpolant in Example 2 is an $\mathcal{EL}^{\text{ran},\square,u}$ -TBox. The need for the universal role is illustrated by considering $\mathcal{T} = \{A \sqsubseteq \exists r.B\}$ and $\Sigma = \{r\}$. The empty TBox is a concept and an instance Σ -interpolant of \mathcal{T} . A query Σ -interpolant is given by $\mathcal{T}' = \{A \sqsubseteq \exists u.B\}$ and reflects the fact that $(\mathcal{T}, \mathcal{A}) \models \exists x B(x)$ for $\mathcal{A} = \{A(a)\}$.

We show that the languages introduced in Definition 3 satisfy criteria (R) and (I) from the introduction. (R) is a consequence of the following result.

Theorem 1. *The following problems are PTIME-complete for $\mathcal{EL}^{\text{ran},\square,u}$ -TBoxes \mathcal{T} and ABoxes \mathcal{A} : decide whether*

- $\mathcal{T} \models C \sqsubseteq D$, for $C \sqsubseteq D$ an $\mathcal{EL}^{\text{ran},\square,u}$ -inclusion;
- $(\mathcal{T}, \mathcal{A}) \models C(a)$, where C is an \mathcal{EL} -concept.

Deciding whether $(\mathcal{T}, \mathcal{A}) \models q(\mathbf{a})$, where q is a conjunctive query, is NP-complete, and deciding this problem for fixed $q(\mathbf{a})$ (knowledge base complexity) is PTIME-complete.

It follows, in particular, that logical equivalence of $\mathcal{EL}^{\text{ran},\square,u}$ -TBoxes is decidable in PTime. These complexity results are exactly the same as for $\mathcal{EL}\mathcal{H}$ -TBoxes [15]. For (I) and the computation of Σ -interpolants below, we first investigate the relationship between the distinct inseparability notions introduced in Definition 1 and inseparability wrt. the new languages. (For \mathcal{EL} this relationship is characterized in [13].) Let X range over the superscripts ‘ran, 0’, ‘ran’, and ‘ran, \square , u ’. Say that two finite sets of FO-sentences \mathcal{T}_1 and \mathcal{T}_2 are X -inseparable wrt. Σ , in symbols $\mathcal{T}_1 \equiv_{\Sigma}^X \mathcal{T}_2$, if $\mathcal{T}_1 \models \alpha \Leftrightarrow \mathcal{T}_2 \models \alpha$, for all \mathcal{EL}_{Σ}^X -inclusions α .

Theorem 2. *Let \mathcal{T}_1 and \mathcal{T}_2 be $\mathcal{EL}^{\text{ran},\square,u}$ -TBoxes and Σ an infinite signature. Then the following holds:*

- $\mathcal{T}_1 \equiv_{\Sigma}^C \mathcal{T}_2$ iff $\mathcal{T}_1 \equiv_{\Sigma}^{\text{ran},0} \mathcal{T}_2$;
- $\mathcal{T}_1 \equiv_{\Sigma}^i \mathcal{T}_2$ iff $\mathcal{T}_1 \equiv_{\Sigma}^{\text{ran}} \mathcal{T}_2$;
- $\mathcal{T}_1 \equiv_{\Sigma}^q \mathcal{T}_2$ iff $\mathcal{T}_1 \equiv_{\Sigma}^{\text{ran},\square,u} \mathcal{T}_2$.

The condition that Σ is infinite is required only for the implication from right to left in Point 1. As we forget *finite* signatures, their complement is always infinite.

From Theorem 2 we immediately obtain that (I) is met for the three notions of Σ -interpolants. For example, assume that \mathcal{T}_1 and \mathcal{T}_2 are $\mathcal{EL}\mathcal{H}^r$ -TBoxes such that $\mathcal{T}_1 \equiv_{\Sigma}^q \mathcal{T}_2$ and let \mathcal{T}'_1 and \mathcal{T}'_2 be query Σ -interpolants in $\mathcal{EL}^{\text{ran},\square,u}$ of \mathcal{T}_1 and \mathcal{T}_2 , respectively. By Theorem 2, $\mathcal{T}'_1 \equiv_{\Sigma}^{\text{ran},\square,u} \mathcal{T}'_2$. But then \mathcal{T}'_1 and \mathcal{T}'_2 are logically equivalent: we have $\mathcal{T}'_1 \models \alpha$ for all $\alpha \in \mathcal{T}'_2$ because all such α are $\mathcal{EL}_{\Sigma}^{\text{ran},\square,u}$ -inclusions and $\mathcal{T}'_2 \models \alpha$. The converse direction holds for the same reason.

4 Computing Σ -interpolants

We give a recursive algorithm computing instance Σ -interpolants for \mathcal{ELH}^r -terminologies satisfying certain acyclicity conditions (similar algorithms computing concept and query Σ -interpolants are presented in the technical report [8]). In this section we assume wlog. that terminologies are *normalized* \mathcal{ELH}^r terminologies; i.e., \mathcal{ELH}^r -terminologies \mathcal{T} consisting of role inclusions and axioms of the form (here, and in what follows, we write $r \sqsubseteq_{\mathcal{T}} s$ if $\mathcal{T} \models r \sqsubseteq s$)

- $A \bowtie \exists r.B$, where $B \in \mathsf{N}_{\mathcal{C}} \cup \{\top\}$;
- $A \bowtie B_1 \sqcap \dots \sqcap B_n$, where $B_1, \dots, B_n \in \mathsf{N}_{\mathcal{C}}$;
- $\text{dom}(s) \sqsubseteq A$ and $\text{ran}(s) \sqsubseteq A$, where $A \in \mathsf{N}_{\mathcal{C}}$

such that $\text{dom}(s) \sqsubseteq A \in \mathcal{T}$ and $r \sqsubseteq_{\mathcal{T}} s$ imply $\text{dom}(r) \sqsubseteq A \in \mathcal{T}$; $\text{ran}(s) \sqsubseteq A \in \mathcal{T}$ and $r \sqsubseteq_{\mathcal{T}} s$ imply $\text{ran}(r) \sqsubseteq A \in \mathcal{T}$; and $r \sqsubseteq_{\mathcal{T}} s$ and $s \sqsubseteq_{\mathcal{T}} r$ implies $r = s$. We give the acyclicity conditions required for the algorithms to terminate. The $\overline{\Sigma}$ -cover $\mathcal{C}_{\overline{\Sigma}}^{\mathcal{T}}(r)$ of a role r wrt. a terminology \mathcal{T} consists of all $s \in \overline{\Sigma}$ such that $r \sqsubseteq_{\mathcal{T}} s$ and there does not exist $r' \in \overline{\Sigma}$ with $r' \neq s$ and $r \sqsubseteq_{\mathcal{T}} r' \sqsubseteq_{\mathcal{T}} s$.

Definition 4 (Σ -loop). Let \mathcal{T} be a normalized \mathcal{ELH}^r -terminology and Σ a signature.

Define a relation $\prec_{\Sigma} \subseteq (\mathsf{N}_{\mathcal{C}} \cap \Sigma) \times (\mathsf{N}_{\mathcal{C}} \cap \Sigma)$ as follows: $A \prec_{\Sigma} B$ if $A, B \in \Sigma$ and

- (a) $A \bowtie C \in \mathcal{T}$ for some C such that B occurs in C , or
- (b) $A \bowtie \exists r.A' \in \mathcal{T}$ for some $A' \in \mathsf{N}_{\mathcal{C}} \cup \{\top\}$ and $r \in \Sigma$ such that $\text{dom}(r) \sqsubseteq B \in \mathcal{T}$, or
- (c) $A \bowtie \exists r.A' \in \mathcal{T}$ for some $A' \in \mathsf{N}_{\mathcal{C}} \cup \{\top\}$ and r such that there exists $s \in \mathcal{C}_{\overline{\Sigma}}^{\mathcal{T}}(r)$ with $\text{ran}(r) \sqsubseteq B \in \mathcal{T}$, $\text{ran}(s) \sqsubseteq B \notin \mathcal{T}$.

We say that \mathcal{T} contains a Σ -loop if \prec_{Σ} contains a cycle.

The following example illustrates this definition and shows that the existence of Σ -loops typically entails the non-existence of Σ -interpolants, even in FO.

Example 4. Consider the terminology

$$\mathcal{T} = \{\text{Elephant} \sqsubseteq \text{Mammal}, \text{Mammal} \sqsubseteq \exists \text{has_mother.Mammal}\}$$

and let $\Sigma = \{\text{Mammal}\}$. Even in FO, there exists no concept/instance/query Σ -interpolant of \mathcal{T} . To see this observe that an *infinite* axiomatization of such a Σ -interpolant is given by the inclusions

$$\{\text{Elephant} \sqsubseteq \overbrace{\exists \text{has_mother} \dots \exists \text{has_mother}}^n . \top \mid n \geq 1\}.$$

This theory cannot be finitely axiomatized in FO without additional predicates. Observe that \mathcal{T} contains the Σ -loop as $\text{Mammal} \prec_{\Sigma} \text{Mammal}$.

Examples illustrating the points (b) and (c) of Definition 4 of Σ -loops can be found in the technical report [8].

Call a concept name A primitive (pseudo-primitive) in a terminology \mathcal{T} if A does not occur on the left hand side of any axiom in \mathcal{T} (does not occur in the form $A \equiv C$ in \mathcal{T}).

For $A \in \text{sig}(\mathcal{T})$, let $\text{Pre}_\Sigma(A)$ consist of all $D = \text{ran}(r)$, $D = \exists r.\top$, and $D \in \mathbf{N}_C$ such that $\mathcal{T} \models D \sqsubseteq A$ and $\text{sig}(D) \subseteq \text{sig}(\mathcal{T}) \cap \bar{\Sigma}$; construct $P_\Sigma(A)$ as follows:

1. for A pseudo-primitive in \mathcal{T} , $P_\Sigma(A) = \text{Pre}_\Sigma(A)$;
2. if $A \equiv B_1 \sqcap \dots \sqcap B_n \in \mathcal{T}$, then

$$P_\Sigma(A) = \{C_{B_1} \sqcap \dots \sqcap C_{B_n} \mid (B_i \in \bar{\Sigma} \text{ and } C_{B_i} = B_i) \text{ or } (B_i \in \Sigma \text{ and } C_{B_i} \in P_\Sigma(B_i))\};$$

3. if $A \equiv \exists r.A' \in \mathcal{T}$, then $P_\Sigma(A)$ is the union of $\text{Pre}_\Sigma(A)$ and
 - if $A' \in \Sigma$: $\{\exists s.A' \mid s \sqsubseteq_{\mathcal{T}} r, s \in \bar{\Sigma}\}$;
 - if $A' \notin \Sigma$: $\{\exists s.D \mid s \sqsubseteq_{\mathcal{T}} r, s \in \bar{\Sigma}, D \in P_\Sigma(A')\}$.

Fig. 2. Computing $P_\Sigma(A)$

For $A \in \text{sig}(\mathcal{T})$, let $\text{Post}_\Sigma(A) = \{B \in \bar{\Sigma} \cap \text{sig}(\mathcal{T}) \mid \mathcal{T} \models A \sqsubseteq B\}$ and construct $Q_\Sigma(A)$ as follows:

1. for A primitive in \mathcal{T} , $Q_\Sigma(A) = \text{Post}_\Sigma(A)$;
2. if $A \bowtie B_1 \sqcap \dots \sqcap B_n \in \mathcal{T}$, then

$$Q_\Sigma(A) = \text{Post}_\Sigma(A) \cup \bigcup_{1 \leq i \leq n, B_i \in \Sigma} Q_\Sigma(B_i);$$

3. if $A \bowtie \exists r.A' \in \mathcal{T}$, then $Q_\Sigma(A)$ is the union of $\text{Post}_\Sigma(A)$ and

$$\{\exists s.E_s \mid s \in \mathcal{C}_{\mathcal{T}}^\Sigma(r)\} \cup \bigcup_{r \sqsubseteq_{\mathcal{T}} s} \{Q_\Sigma(B) \mid \text{dom}(s) \sqsubseteq B \in \mathcal{T}, s \in \Sigma, B \in \Sigma\},$$

where

$$E_s = \prod_{\substack{B \in \Sigma, \text{ran}(r) \sqsubseteq B \in \mathcal{T} \\ \text{ran}(s) \sqsubseteq B \notin \Sigma, D \in Q_\Sigma(B)}} D \sqcap \prod_{\substack{D \in Q_\Sigma(A') \\ A' \in \Sigma}} D \sqcap \prod_{\substack{B \in \bar{\Sigma} \\ \mathcal{T} \models \text{ran}(r) \sqcap A' \sqsubseteq B}} B.$$

Fig. 3. Computing $Q_\Sigma(A)$

The intuition behind the following algorithm for Σ -interpolants is as follows: first, one can show using Theorem 2 and a sequent-style proof system for \mathcal{ELH}^r that under the conditions of Theorem 3 there exists an instance Σ -interpolant consisting of (in addition to role inclusions and domain and range restrictions) concept inclusions of the form $C \sqsubseteq A$ and $A \sqsubseteq C$. In Figure 2, we compute the set $P_\Sigma(A)$ of C s such that $C \sqsubseteq A$ is in the interpolant by making a case distinction: in Point 1 A is pseudo-primitive; in Point 2 it is defined by a conjunction; in Point 3 it is defined as $\exists r.A'$. Points 2 and 3 are recursive as they require the sets $P_\Sigma(B)$ when B is used in the definition of A . Σ -loops describe exactly the situation in which the recursion does not terminate. In Figure 3 we compute, in a similar way, the set $Q_\Sigma(A)$ of C s such that $A \sqsubseteq C$ is in the interpolant. A sample run of the algorithm is provided in [8].

Theorem 3. Let Σ be a finite signature and \mathcal{T} a normalized \mathcal{ELH}^r -terminology without Σ -loops. Then the algorithms computing $P_\Sigma(A)$ and $Q_\Sigma(A)$ in Figures 2 and 3 terminate for all $A \in \text{sig}(\mathcal{T})$.

Let \mathcal{T}_Σ consist of the following inclusions, where A , r , and s range over $\text{sig}(\mathcal{T}) \cap \bar{\Sigma}$:

- $r \sqsubseteq s$, for $r \sqsubseteq_{\mathcal{T}} s$;

- $D \sqsubseteq A$, for all $D \in P_\Sigma(A)$;
- $A \sqsubseteq D$, for all $D \in Q_\Sigma(A)$;
- $\text{ran}(r) \sqsubseteq D$, for all $D \in Q_\Sigma(B)$ such that $\text{ran}(r) \sqsubseteq B \in \mathcal{T}$ and $B \in \Sigma$;
- $\text{dom}(r) \sqsubseteq D$, for all $D \in Q_\Sigma(B)$ such that $\text{dom}(r) \sqsubseteq B \in \mathcal{T}$ and $B \in \Sigma$.

Then \mathcal{T}_Σ is an instance Σ -interpolant of \mathcal{T} .

$P_\Sigma(A)$ and $Q_\Sigma(A)$ are both of exponential size, in the worst case. For $P_\Sigma(A)$, this is clear from Point 2 of the construction: let \mathcal{T} consist of $A \equiv B_1 \sqcap \dots \sqcap B_n$ and $A_i^j \sqsubseteq B_i$ ($1 \leq i, j \leq n$) and let $\Sigma = \{B_i \mid 1 \leq i \leq n\}$. Then $P_\Sigma(A)$ is of size n^n , and one can show that there does not exist a shorter Σ -interpolant in $\mathcal{EL}^{\text{ran}, \sqcap, \sqcup}$. For $Q_\Sigma(A)$ this follows from the fact that one might have to construct a complete unfolding of the terminology.

If we admit disjunctions in C in axioms $C \sqsubseteq D$ of Σ -interpolants, then we can replace, in Point 2, $P_\Sigma(A)$ for $A \equiv B_1 \sqcap \dots \sqcap B_n \in \mathcal{T}$ by the singleton set consisting of

$$\prod_{1 \leq i \leq n, B_i \in \Sigma} B_i \sqcap \prod_{1 \leq i \leq n, B_i \in \Sigma} \bigsqcup_{C_{B_i} \in P_\Sigma(B_i)} C_{B_i}.$$

We will see below that in practice this construction leads to much smaller Σ -interpolants. However, this improvement does not come for free. Consider the language $\mathcal{EL}^{\text{ran}, \sqcup}$, where the only difference to $\mathcal{EL}^{\text{ran}}$ is that \mathcal{C}^{ran} -concepts now admit ‘ \sqcup ’ as a binary concept constructor. Every $\mathcal{EL}^{\text{ran}, \sqcup}$ -TBox is logically equivalent to an (exponentially larger) $\mathcal{EL}^{\text{ran}}$ -TBox, and so $\mathcal{EL}^{\text{ran}, \sqcup}$ inherits many desirable properties from $\mathcal{EL}^{\text{ran}}$. However, one can show that, in contrast to $\mathcal{EL}^{\text{ran}}$, logical equivalence between $\mathcal{EL}^{\text{ran}, \sqcup}$ -TBoxes is coNP-hard.

5 Experiments

We have implemented a prototype called NUI that computes instance Σ -interpolants as presented in Theorem 3. We have applied NUI to a version of SNOMED CT dated 09 February 2005 (without two left-identities) and the \mathcal{ELH}^r -fragment of the release 08.08d of NCI. The first terminology has approx. 380K axioms, almost the same number of concept names, and 56 role names. The \mathcal{ELH}^r -fragment of NCI has approx. 63K axioms, approx. 65K concept names, and 123 role names. We note that the algorithms given above compute (for ease of exposition) a large number of redundant axioms and NUI implements a variety of straightforward optimizations.

$ \Sigma \cap \text{sig}(\cdot) $	SNOMED CT	$ \Sigma \cap \text{sig}(\cdot) $	NCI
2 000	93.0%	5 000	97.0%
3 000	84.5%	10 000	81.1%
4 000	67.0%	15 000	72.0%
5 000	59.5%	20 000	59.2%

Table 1. Success rate of NUI

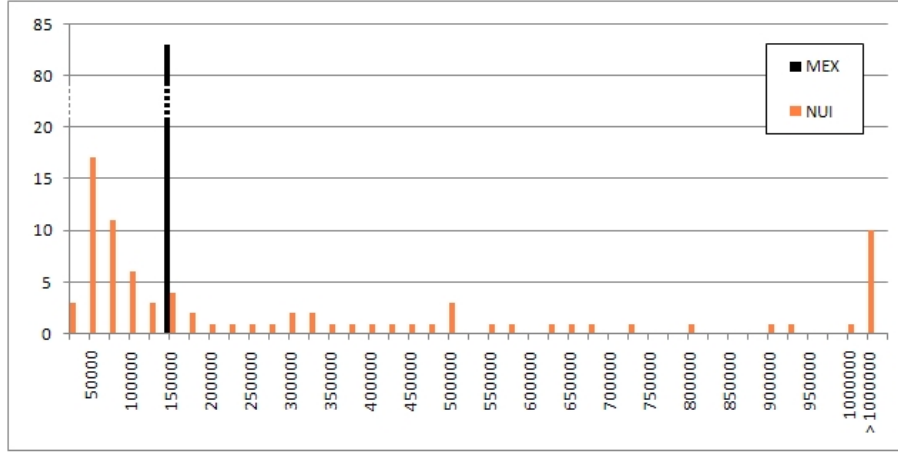


Fig. 4. Size distribution of MEX-modules and instance Σ -interpolants of SNOMED CT

First observe that neither SNOMED CT nor NCI contain any Σ -loops, for any signature Σ . Thus, Σ -interpolants always exist and can, in principle, be computed using our algorithm.

In our experiments, we focus on the case of forgetting a large signature Σ (and keeping a “small” signature $\bar{\Sigma} \cap \text{sig}(\cdot)$), as this corresponds to many application scenarios. The experiments have been performed on a standard PC with 2.13 GHz and 3 GB of RAM.

Success rate: Table 1 shows the rate at which NUI succeeds to compute instance Σ -interpolants of SNOMED CT and NCI wrt. various signatures. All failed cases are due to memory overflow after several hours. For each table entry, 100 samples have been used. The signatures contain concept and role names randomly selected from the full signature of SNOMED CT (we never forget the role ‘roleGroup’ as this would make forgetting trivial) and NCI, respectively. $\bar{\Sigma} \cap \text{sig}(\cdot)$ always contains 20 role names. For NCI and signatures of size ≤ 4000 , NUI had a 100% success rate.

Size: We compare the size of instance Σ -interpolants of SNOMED CT and NCI computed by NUI with the size of extracted $\bar{\Sigma} \cap \text{sig}(\cdot)$ -modules; i.e., minimal *subsets* of the respective terminologies which preserve, e.g., inclusions between $\bar{\Sigma} \cap \text{sig}(\cdot)$ -concepts. We use MEX-modules [5] of SNOMED CT and, respectively, \top -local modules [2] of NCI. The size of Σ -interpolants, terminologies, and modules is measured as number of symbols rather than number of axioms as Σ -interpolants can contain large axioms.

For SNOMED CT, we generated 100 random signatures with 3 000 concept names and 20 role names in $\bar{\Sigma} \cap \text{sig}(\cdot)$. For 17 of those signatures, NUI failed to compute an instance Σ -interpolant. For the remaining 83 signatures, Figure 4 shows the number of interpolants and $\bar{\Sigma} \cap \text{sig}(\cdot)$ -modules (vertical axis) of a given size (horizontal axis). The size of the $\bar{\Sigma} \cap \text{sig}(\cdot)$ -modules lies between 125K and 150K symbols, which is about 3% of SNOMED CT. 48.19% of instance Σ -interpolants are smaller than the corresponding modules. However, 10 instance Σ -interpolants contain more than one million symbols and the largest instance Σ -interpolant is more than 11 times larger than SNOMED CT.

For NCI, we computed instance Σ -interpolants and $\bar{\Sigma} \cap \text{sig}(\cdot)$ -modules wrt. random signatures with 7 000 concept names and 20 role names in $\bar{\Sigma} \cap \text{sig}(\cdot)$. NUI succeeded

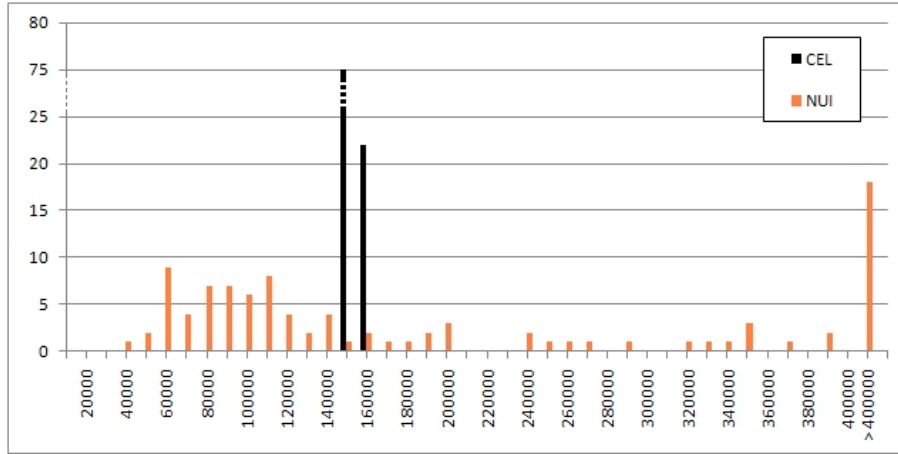


Fig. 5. Size distribution of CEL-modules and instance Σ -interpolants of NCI

to compute interpolants for 97 of 100 signatures, each within 25 min. Figure 5 shows the size distribution of the successfully computed interpolants and the corresponding modules. The size of the $\overline{\Sigma} \cap \text{sig}(\cdot)$ -modules lies between 140K and 160K symbols, which is about 22.5% of NCI. 74.47% of the instance Σ -interpolants are smaller than the corresponding modules. On the other hand, 18 instance Σ -interpolants consist of more than 400K symbols and the largest instance Σ -interpolant is more than 12 times larger than NCI.

Forgetting with disjunction: All failures in Table 1 are due to the fact that $P_{\Sigma}(A)$ is too large. Indeed, if we admit disjunction and consider $\mathcal{EL}^{\text{ran}, \sqcup}$, then NUI succeeds to compute *all* Σ -interpolants from Table 1, each within 15 min. Moreover, for NCI, no signature for which NUI fails has been detected. For SNOMED CT, however, NUI still typically fails for $|\overline{\Sigma} \cap \text{sig}(\cdot)| \geq 30\,000$.

6 Discussion

The notion of forgetting in DL ontologies has recently been investigated in a number of research papers. [9, 19] consider forgetting in DL-Lite and [3] investigate in how far forgetting in DLs can be reduced to forgetting in logic programs. [6], on which this paper is based, proposes forgetting for acyclic \mathcal{EL} -terminologies and concept inclusions.

The main novel contributions of this paper are (i) the first algorithms with experimental results indicating the practical feasibility of forgetting in DL-terminologies and (ii) the first systematic analysis of the distinct languages required to axiomatize Σ -interpolants for distinct query languages. Many open problems remain; e.g., we conjecture that Σ -interpolants of \mathcal{ELH}^r -terminologies (and possibly even TBoxes) exist in the languages introduced whenever they exist in FO. Such a result would provide further justification for those languages. Secondly, it would be of interest to prove decidability (and complexity) of the decision problem whether there exists a Σ -interpolant for a given \mathcal{ELH}^r -terminology (TBox). Note that our acyclicity conditions are sufficient but not necessary for the existence of Σ -interpolants.

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