Translating Equilibrium Description Logics into Circumscription

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Abstract

Circumscription is a powerful framework for enabling non-monotonic reasoning in Description Logics (DLs) and other knowledge representation languages based on first-order logic. It is very expressive and rather abstract, which is why it is often used as a host language for defining other non-monotonic formalisms. An alternative approach to obtain a non-monotonic version of a given first-order knowledge representation language is the Equilibrium Logic (EL). For instance, EL allows the generalization of the stable model semantics of logic programs to arbitrary propositional theories and even to arbitrary first-order theories. Recently, DLs with semantics based on EL have been proposed in the context of DL terminologies, but a deeper understanding of this formalism is still missing. The goal of this paper is to clarify the connection been DLs based on EL and DLs based on circumscription, both under the well-known *global* circumscription and the recently proposed *pointwise* circumscription. To this end, we first introduce a simple yet powerful extension of circumscribed DLs by attaching to a circumscribed KB an additional set of axioms, which filter out unintended minimal models of the first KB. As we show in this paper, such *constrained circumscription* is powerful enough to capture DLs under the EL semantics, with global and pointwise minimality. We argue that in some important cases, constrained circumscription is computationally not more expensive than its base variant. Together with the proposed translation, this provides new decidability and complexity results for DLs based on EL semantics.

1. Introduction

Description Logics (DLs), as fragments of first-order logic, are monotonic. A prominent research line on extending DLs with non-monotonic features is given by *circumscribed* DLs [1, 2, 3, 4]. At the core of *circumscription* − whether it is applied to DLs or other languages based on first-order logic − there is predicate minimization: classical models are filtered by circumscription, selecting the so-called *minimal models* where the extension of some predicates is the minimal possible. To perform such selection, a knowledge base (KB) is equipped with a *circumscription pattern* which partitions the set of predicates into *minimized*, *varying*, and *fixed* predicates and induces a preference relation among models. The freedom of selecting how to treat predicates, by selecting an appropriate circumscription pattern, makes circumscribed DLs a very expressive formalism. Circumscription is often used as a host language for defining other non-monotonic formalisms, e.g. for *defeasible inclusions* in DLs [5]. A disadvantage of circumscribed DLs is given by the high computational costs. Remarkably, the arity of minimized predicates and the interaction with varying predicates affects the decidability of basic reasoning tasks: allowing roles to be minimized makes concept satisfiability undecidable already in $\mathcal{ALC}$ [1]. To reduce the complexity of reasoning, *pointwise circumscription* has been proposed in [6]. The main difference between circumscription and pointwise circumscription is how predicate minimization is performed. In circumscription, predicates are allowed to be globally minimized across the model. For this reason, we refer to standard circumscription as *global* circumscription. While, in pointwise circumscription, the global minimization is replaced by *local* minimizations performed at the single domain elements [6].

Along the line of predicate minimization, another prominent approach is given by *Equilibrium Description Logics* (EDLs) [7] based on *Quantified Equilibrium Logic* (QEL) [8, 9]. QEL provides logical foundations to the stable model semantics of logic programs, allowing to extend it to arbitrary theories

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in first-order logic. In [7], the stable model semantics based on QEL is used for DL *terminologies*, i.e. collections of possibly recursive concept definitions, ultimately showing that for terminologies in $\mathcal{ALCI}$ the basic reasoning tasks can be performed in deterministic single exponential time. In particular, the stable model semantics based on QEL overcomes some limitations of previous works [10, 11], where terminologies are required to be syntactically monotonic. In contrast with the terminologies case, concept satisfiability w.r.t. general knowledge bases in $\mathcal{ALCI}$ is proved to be undecidable [7]. A key feature of DL terminologies is the partition of predicates into two sets: *intensional predicates*, subjected to minimization, and *extensional predicates*, that behave classically, meaning that (in the jargon of circumscription) they are fixed, and always contain the roles occurring in the terminology. Having classical roles, paired with the observation that terminologies correspond to a syntactic fragment of general knowledge bases in $\mathcal{ALCI}$, plays a crucial role when it comes to decidability.

The main contributions of this work are as follows.

∘ We study the connection between circumscribed DLs and EDLs. Although both are based on predicate minimization, circumscribed DLs and EDLs are two 'orthogonal' formalisms with different features, e.g. varying predicates of circumscribed DLs vs. *default negation* of EDLs. We show that an EDL knowledge base can be transformed into a circumscribed knowledge base such that there is a correspondence between the stable models of the first and a *restricted* set of models of the second one. Specifically, this restricted set of models is obtained by filtering out those models satisfying a further set of inclusions.

∘ To formalize such additional restrictions, we propose a *hybrid* form of circumscription where a circumscribed knowledge base is paired with a non-circumscribed one. This second knowledge base acts as a filter over the set of minimal models, allowing further refinement of the selection process of minimal models. We call this formalism *constrained circumscription*, as the non-circumscribed knowledge base constraints the model selection. A similar idea has been used in [6] where a pointwise circumscribed knowledge base is equipped with a set of constraints, mimicking concept inclusions, which are used for the normalization of complex concepts. We show that constrained circumscription enhances the expressive power of circumscription, in some cases without an additional computational cost.

∘ We define a *pointwise stable model semantics* where a pointwise minimization (in the style of pointwise circumscription) is performed. We extend the translation aforementioned. In particular, we show that a pointwise EDL KB can be translated into a pointwise circumscribed KB with constraints such that there is a correspondence between the pointwise minimal models of the second KB and stable models of the first.

∘ We extend the undecidability results in [7] showing that concept satisfiability w.r.t. EDL KBs in $\mathcal{ALC}$ and pointwise EDL KBs in $\mathcal{ALCI}$ is undecidable.

2. Preliminaries

We focus on $\mathcal{ALCHIO}$ for its expressiveness and recall here the basic notions needed in the following. We call N_C, N_R, and N_I the mutually disjoint sets of *concept names*, *role names*, and *individuals*. The expression r^- is the *inverse role* of a role name $r \in N_R$. The syntax of complex concepts in $\mathcal{ALCHIO}$ is defined by the grammar $C := \top \mid \bot \mid A \mid \{a\} \mid \neg C \mid C \sqcup C \mid C \sqcap C \mid \exists r.C \mid \forall r.C$, with $A \in N_C$, $r \in N_R$ or $r = s^-$ with $s \in N_R$, and $a \in N_I$. A *TBox* $\mathcal{T}$ is a finite set of *concept inclusions* and *role inclusions*, which are expressions of the form $C \sqsubseteq D$ and $r \sqsubseteq s$, respectively, with C and D concepts, and r and s role names or inverse roles. An expression of the form $A(a)$, where $A \in N_C$ and $a \in N_I$, is a *concept assertion*, while an expression of the form $r(a, b)$ is a *role assertion*, where $r \in N_R$. An *ABox* $\mathcal{A}$ is a collection of assertions. A *knowledge base (KB)* is a pair $\mathcal{K} = (\mathcal{A}, \mathcal{T})$, where $\mathcal{A}$ is an ABox and $\mathcal{T}$ is a TBox. Given a KB $\mathcal{K}$, we denote with $sig(\mathcal{K})$ the set of concept names, role names, and individual names occurring in $\mathcal{K}$. The classical semantics of $\mathcal{ALCHIO}$ is defined by means of *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is the *domain* and $\cdot^{\mathcal{I}}$ the *interpretation function* associating to each $a \in N_I$ a unique element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, to each $A \in N_C$ a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and to each $r \in N_R$ a set $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The extension of remaining concept and role expressions in $\mathcal{ALCIO}$ is defined as usual [12]. In this work,

we adopt the unique name assumption. The notion of a (classical) *model* of a KB, TBox, or ABox is also standard (as usual, we use '$\models$' for this). We refer to [12] for further details and fragments considered in the following.

3. Global and Pointwise Circumscription

We use [1] and [6] as main references respectively for circumscribed and pointwise circumscribed DLs. Given a KB, when we apply circumscription, beforehand we declare a *circumscription pattern* $\mathcal{P} = (M, V, F)$ defining which predicates are *minimized, varying* or *fixed*. Circumscription then simply selects the models fulfilling the requirement of the circumscription pattern, i.e. those models where the extension of the minimized predicates cannot be further reduced without violating some axioms. Thus, at the core of circumscription, there is the idea of comparing structures and discriminating which one we have to *prefer*.

Definition 1. *Given a set $F \subseteq N_C \cup N_R$ and two interpretations $\mathcal{I}$ and $\mathcal{J}$, we write $\mathcal{I} \sim_F \mathcal{J}$ if*

(i) $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$,

(ii) $a^{\mathcal{I}} = a^{\mathcal{J}}$, for all individuals $a \in N_I$,

(iii) $q^{\mathcal{I}} = q^{\mathcal{J}}$, for all $q \in F$.

The above definition simply groups interpretations sharing the same domain, interpreting individuals in the same way, and agreeing on the set of predicates in F. Note that F can be empty. In this case, we simply write $\mathcal{I} \sim \mathcal{J}$.

We now formalize the preference relation naturally induced by circumscription patterns. Formally, a circumscription pattern is a triple $\mathcal{P} = (M, V, F)$ where M, V and F are mutually disjoint sets of minimized, varying and fixed predicates.

Definition 2 (Preference relation). *Given a circumscription pattern $\mathcal{P} = (M, V, F)$ and two interpretations $\mathcal{I}, \mathcal{J}$ such that $\mathcal{I} \sim_F \mathcal{J}$, we write:*

- $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$ if $p^{\mathcal{I}} \subseteq p^{\mathcal{J}}$, for all $p \in M$;

- $\mathcal{I} \prec_{\mathcal{P}} \mathcal{J}$, if $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$ and $p^{\mathcal{I}} \subset p^{\mathcal{J}}$ for some $p \in M$.

If the circumscription pattern $\mathcal{P} = (M, V, F)$ is such that $V = \emptyset$, we denote the relations $\preceq_{\mathcal{P}}$ and $\prec_{\mathcal{P}}$ with the symbols $\subseteq_F$ and $\subset_F$, respectively.

Given a KB $\mathcal{K}$, a circumscription pattern $\mathcal{P} = (M, V, F)$ for $\mathcal{K}$ is a partition of the predicates occurring in $\mathcal{K}$. A KB $\mathcal{K}$ equipped with a circumscription pattern $\mathcal{P}$ is called *circumscribed* and we denote it with $\text{Circ}_{\mathcal{P}}(\mathcal{K})$. Given a circumscribed KB, we aim to filter the set of all possible models selecting those where the extension of minimized predicates is the smallest possible.

Definition 3. *(Minimal model) An interpretation $\mathcal{I}$ is a* minimal model *of $\text{Circ}_{\mathcal{P}}(\mathcal{K})$, in symbols $\mathcal{I} \models \text{Circ}_{\mathcal{P}}(\mathcal{K})$, if $\mathcal{I} \models \mathcal{K}$ and there is no interpretation $\mathcal{J}$ s.t. $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J} \prec_{\mathcal{P}} \mathcal{I}$. We use $MM(\mathcal{K}, \mathcal{P})$ to denote the set of minimal models of $\text{Circ}_{\mathcal{P}}(\mathcal{K})$.*

Observe that Definition 2 does not restrict how the extension of p may differ in $\mathcal{I}$ and $\mathcal{J}$ and varying predicates do not occur in any of the conditions. Thus, when searching for minimal models, circumscription allows for the reconfiguration of the predicates across the entire model.

In [6] an approximation of circumscription based on *pointwise circumscription* [13] has been introduced. Differently from (global) circumscription, in pointwise circumscription [13] predicates can be minimized only locally at a given tuple of domain elements. In [6] the minimization can only affect a domain element and the roles it participates in, leaving the rest of the structure unmodified. By repeatedly applying such local minimizations, we can often replicate the minimization across the entire model of

$$a^{(\mathcal{I},\mathcal{J})} = a^{\mathcal{I}} \qquad A^{(\mathcal{I},\mathcal{J})} = A^{\mathcal{I}} \qquad r^{(\mathcal{I},\mathcal{J})} = r^{\mathcal{I}} \qquad \top^{(\mathcal{I},\mathcal{J})} = \Delta^{\mathcal{I}} \qquad \bot^{(\mathcal{I},\mathcal{J})} = \emptyset$$

$$(r^-)^{(\mathcal{I},\mathcal{J})} = \{(e,e') \mid (e',e) \in r^{\mathcal{I}}\} \qquad\qquad (\neg C)^{(\mathcal{I},\mathcal{J})} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{J}}$$

$$(C_1 \sqcap C_2)^{(\mathcal{I},\mathcal{J})} = C_1^{(\mathcal{I},\mathcal{J})} \cap C_2^{(\mathcal{I},\mathcal{J})} \qquad\qquad (C_1 \sqcup C_2)^{(\mathcal{I},\mathcal{J})} = C_1^{(\mathcal{I},\mathcal{J})} \cup C_2^{(\mathcal{I},\mathcal{J})}$$

$$(\exists R.C)^{(\mathcal{I},\mathcal{J})} = \{e \in \Delta^{\mathcal{I}} \mid \exists e' : (e,e') \in R^{(\mathcal{I},\mathcal{J})} \wedge e' \in C^{(\mathcal{I},\mathcal{J})}\}$$

$$(\forall R.C)^{(\mathcal{I},\mathcal{J})} = \left\{ e \in \Delta^{\mathcal{I}} \mid \forall e' : \begin{array}{l} (e,e') \in R^{(\mathcal{I},\mathcal{J})} \text{ implies } e' \in C^{(\mathcal{I},\mathcal{J})} \text{ and} \\ (e,e') \in R^{\mathcal{J}} \text{ implies } e' \in C^{\mathcal{J}} \end{array} \right\}$$

Figure 1: HT semantics for DLs, with $a \in N_I$, $A \in N_C$ and $r \in N_R$.

classical circumscription, but not always: pointwise circumscription is not able to minimize predicate participating in a cycle and treats varying predicates differently. See [6, 3] for detailed comments on the differences between the two formalisms for DLs.

Definition 4. *Given two interpretations $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \sim^\bullet \mathcal{J}$ if there exists $e \in \Delta^{\mathcal{I}}$ s.t.:*

- *for all $A \in N_C$, $A^{\mathcal{I}} \setminus \{e\} = A^{\mathcal{J}} \setminus \{e\}$, and*

- *for all $r \in N_R$, $r^{\mathcal{I}} \cap (\Delta \times \Delta) = r^{\mathcal{J}} \cap (\Delta \times \Delta)$, with $\Delta = \Delta^{\mathcal{I}} \setminus \{e\}$.*

Thus, two interpretations in the relation $\sim^\bullet$ may differ in terms of concept and role names involving one domain element. We now use the relation in Definition 4 to give a *pointwise* flavor to the preference relation between interpretations of Definition 2.

Definition 5. *Assume a circumscription pattern $\mathcal{P} = (M, V, F)$ and two interpretations $\mathcal{I}, \mathcal{J}$ such that $\mathcal{I} \sim_F \mathcal{J}$. We write*

- *$\mathcal{I} \preceq_{\mathcal{P}}^\bullet \mathcal{J}$, if $\mathcal{I} \sim^\bullet \mathcal{J}$ and $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$;*

- *$\mathcal{I} \prec_{\mathcal{P}}^\bullet \mathcal{J}$, if $\mathcal{I} \preceq_{\mathcal{P}}^\bullet \mathcal{J}$ and $p^{\mathcal{I}} \subset p^{\mathcal{J}}$ for some $p \in M$.*

Given a circumscription pattern $\mathcal{P} = (M, V, F)$ such that $V = \emptyset$, we denote $\preceq_{\mathcal{P}}^\bullet$ and $\prec_{\mathcal{P}}^\bullet$ with $\subseteq_F^\bullet$ and $\subset_F^\bullet$, respectively.

Definition 6. *An interpretation $\mathcal{I}$ is a pointwise minimal model of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$, in symbols $\mathcal{I} \models^\bullet \mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$, if $\mathcal{I} \models \mathcal{K}$ and there is no interpretation $\mathcal{J}$ s.t. $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J} \prec_{\mathcal{P}}^\bullet \mathcal{I}$. We use $PMM(\mathcal{K}, \mathcal{P})$ to denote the set of pointwise minimal models of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$.*

The standard definitions of concept satisfiability, concept subsumption, and concept instances are adapted to (resp. pointwise) minimal models in the obvious way and can be reduced polynomial one into the other [1, 6]. In this work, we focus on concept satisfiability w.r.t. (resp. pointwise) circumscribed KBs: given a concept C and a circumscribed KB $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$, we write $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \models^{(\bullet)} C$, if $C^{\mathcal{I}} \neq \emptyset$ holds for some $\mathcal{I} \in MM(\mathcal{K}, \mathcal{P})$ (resp. $PMM(\mathcal{K}, \mathcal{P})$).

4. Equilibrium Description Logics

Following [7], we recall Equilibrium Description Logics (EDLs), i.e. DLs under the stable model semantics of Quantified Equilibrium Logic. *Equilibrium Logic (EL)* [14] is a powerful formalism that allows, e.g., extending the stable model semantics of *Answer Set Programming (ASP)* to arbitrary theories. EL is built upon the logic of *Here-and-There (HT)* with an additional minimality requirement and its first-order counterpart, *Quantified Equilibrium Logic (QEL)*, has been introduced in [9]. An interpretation in the HT logic consists of a pair of structures $(\mathcal{I}, \mathcal{J})$ sharing the same domain and interpreting individuals in the same way, where $\mathcal{I}$ is called 'here' interpretation and $\mathcal{J}$ is called 'there' interpretation. Intuitively

speaking, the 'here' interpretation corresponds to a *world* (in the sense of Kripke frames) where the truth of an atom can be seen as a *belief*. While the 'there' world corresponds to what is *assumed to be possible*. These two interpretations are related by the inclusion relation, intuitively corresponding to the fact that only possible facts can be believed. We formalize this requirement by recalling the relations $\subseteq_F$ and $\subset_F$ of Definition 2 where the set of *fixed predicates* $F \subseteq N_C \cup N_R$ specifies those predicates over which the 'here' and the 'there' interpretations must agree.

Definition 7. *A* Here-and-There (HT) interpretation *is a pair* $(\mathcal{I}, \mathcal{J})$ *of interpretations with* $\mathcal{I} \subseteq_F \mathcal{J}$, *with* $F \subseteq N_C \cup N_R$. *The* interpretation function $\cdot^{(\mathcal{I}, \mathcal{J})}$ *is defined in Figure 1.*

In general, under the HT semantics predicates do not obey the law of the excluded middle. Already in the case of propositional HT logic, the formula $x \vee \neg x$ is not a tautology [8]. Given an HT interpretation $(\mathcal{I}, \mathcal{J})$ as in Definition 7, we can refer to the predicates in the F as *classical*: indeed since the interpretations $\mathcal{I}$ and $\mathcal{J}$ agree on the extension of predicates in F, the law of the excluded middle applies for each $p \in F$.

As already pointed out in [7], the universally quantified concept of the form $\forall r.C$ can be translated in FOL as $\forall y((r(x,y) \to C(y))$. Thus the interpretation must align with the interpretation of implication in quantified HT. Indeed, in HT logic – whether quantified or not – the implication is intuitionistic [8, 9] and it is true in an HT interpretation when the following are satisfied: (1) if the antecedent is true in the HT interpretation, then the consequence is true in the HT interpretation, and (2) the 'there' interpretation is a classical model of it. As a matter of fact, concept inclusions are also affected by this double nature of implication, as they are 'explicit' implications in DLs.

Definition 8. *Assume a KB* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ *and an HT interpretation* $(\mathcal{I}, \mathcal{J})$. *We write:*

- $(\mathcal{I}, \mathcal{J}) \models C \sqsubseteq D$, *if* $C^{(\mathcal{I}, \mathcal{J})} \subseteq D^{(\mathcal{I}, \mathcal{J})}$ *and* $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$;

- $(\mathcal{I}, \mathcal{J}) \models \mathcal{K}$ *if* $\mathcal{I} \models \mathcal{A}$ *and* $(\mathcal{I}, \mathcal{J}) \models C \sqsubseteq D$ *for all* $C \sqsubseteq D \in \mathcal{T}$.

We can now define *stable models* of a DL KB.

Definition 9 (Stable models). *Given* $F \subseteq N_C \cup N_R$, *an interpretation* $\mathcal{J}$ *is a* stable model *of a KB* $\mathcal{K}$ *under fixed predicates* F, *if*

(i) *the HT interpretation* $(\mathcal{J}, \mathcal{J})$ *is a model of* $\mathcal{K}$, *and*

(ii) *there is no* $\mathcal{I}$ *s.t.* $(\mathcal{I}, \mathcal{J})$ *is a model of* $\mathcal{K}$ *and* $\mathcal{I} \subset_F \mathcal{J}$.

We denote with $SM_F(\mathcal{K})$ *the set of all stable models for* $\mathcal{K}$ *with fixed predicates* F. *If* $F = \emptyset$, *we drop the subscript* F *and write* $SM(\mathcal{K})$.

Similarly to the approach of this paper, in the stable model semantics of [15], the set of predicates is partitioned in *intentional predicates* – subjected to the minimization, and *extensional predicates* – which are *classical* and thus obey the law of the excluded middle. The latter requirement, in the setting of full first-order logic, can be easily expressed with the conjunct $p \vee \neg p$, for each extensional predicate p. However, in the setting of the DLs considered in this work, we can express fixed concept names but not fixed roles. This is not a limit, as we already parametrized the relation $\subseteq_F$ expressing the predicates that are fixed.

In the semantics given in Definition 9, the negation $\neg$ behaves as *negation as failure* or *default negation* in logic programs: the truth of a negated concept $\neg C$ at a certain domain element in a stable model intuitively means that the concept C could not be proved at such domain element. As already underlined in [7], the fact that negation is not anymore classical implies that KBs that are equivalent under the classical semantics might not be equivalent under the stable model semantics.

Example 1 (From [6]). *Assume that we want to describe the following scenario. Administrators can grant access to classified files. If a classified file is read, the permission to do so should have been granted. Consider the knowledge base $\mathcal{K}$*

$$\text{Classified\_Document}(f_1) \quad \text{User}(John) \quad \text{read}(John, f_1)$$
$$\exists \text{access\_granted\_by}.\text{Admin} \sqsubseteq \text{Has\_Read\_Permission}$$
$$\text{Classified\_Document} \sqcap \neg\forall(\text{read})^-.\text{Has\_Read\_Permission} \sqsubseteq \bot$$

Assume that Admin *and* access_granted_by *are fixed predicates. We derive that all classified documents* must *be read by someone who has permission to do so. As an effect of '¬' behaving as default negation, such permission is not 'justified' by the last inclusion above and must come from an Admin. Thus, in every stable model, we derive that John, who is reading the classified document f_1, has permission granted by an admin.*

In Definition 4 we introduce a relation comparing interpretations that differ only at a single domain element, in terms of concept names and role names involving it. We lift this *pointwise* comparison to defining *pointwise stable models* as follows. Recall the definition of $\subseteq_F^\bullet$ and $\subset_F^\bullet$.

Definition 10 (Pointwise Stable Models). *Given $F \subseteq N_C \cup N_R$, an interpretation $\mathcal{J}$ is a* pointwise stable model *of a KB $\mathcal{K}$ under fixed predicates F, if*

 (i) *the HT interpretation $(\mathcal{J}, \mathcal{J})$ is a model of $\mathcal{K}$, and*

 (ii) *there is no $\mathcal{I}$ s.t. $(\mathcal{I}, \mathcal{J})$ is a model of $\mathcal{K}$ and $\mathcal{I} \subset_F^\bullet \mathcal{J}$.*

We denote with $PSM_F(\mathcal{K})$ the set of all stable models for $\mathcal{K}$ with fixed predicates F. If $F = \emptyset$, we drop the subscript F and write $PSM(\mathcal{K})$.

Remark. *Pointwise stable models* have been already considered for full first-order logic in [15]. However, there are some key differences with the above definition. Ferraris et al. achieve the stable model semantics by means of an operator associating each first-order formula Φ to a second-order formula $SM[\Phi]$ whose models are stable. In the case of pointwise circumscription, the operator associates the formula $PSM[\Phi]$ which can be expressed at the first-order level. The performed pointwise minimization in $PSM[\Phi]$ allows only for the minimization of a single predicate. In our setting, multiple predicates with multiple arities can be minimized, as long as they 'concern' a unique domain element.

The reasoning tasks of *concept satisfiability, subsumption* and *instance checking* can be adapted in the usual way to stable models and pointwise stable models.

We strengthen the undecidability result for $\mathcal{ALCI}$ of [7] showing that concept satisfiability is undecidable in $\mathcal{ALC}$ under the stable model semantics via a reduction from the *domino problem*.

Theorem 1. *Concept satisfiability w.r.t general KBs in $\mathcal{ALC}$ under the stable model semantics is undecidable.*

In [6], concept satisfiability is proved to be undecidable w.r.t general KBs in $\mathcal{ALCIO}$ under pointwise circumscription. The result is obtained via a reduction from the domino problem and the constructed TBox is negation-free. As already proved in [7], for negation-free KBs, minimal models coincide with stable models. The same holds if we restrict to pointwise minimal models and pointwise stable models. Since nominals can be simulated in the pointwise stable model semantics using default negation as done in [7], the following result follows.

Theorem 2. *Concept satisfiability w.r.t general KBs in $\mathcal{ALCI}$ under the pointwise stable model semantics is undecidable.*

5. Constrained Circumscription

In order to characterize the connection of EDL KBs and circumscribed KBs, we extend circumscription with *constraints*. The notion of constraints has been introduced in [6] for pointwise circumscribed KBs as a tool for normalizing complex concepts without affecting minimization. A constraint in the sense of [6] is a pair (C, D) with the intuitive meaning *"if C holds, then D must hold too"*. Coupled with a circumscribed (or pointwise circumscribed) KB, constraints allow for further refinement of the set of minimal models, as the requirement expressed by constraints acts outside circumscription. As one can observe, a constraint is nothing but a GCI. In this work, we extend the notion of constraints of [6] to KBs as follows.

Definition 11. *Assume two KBs $\mathcal{K}$ and $\mathcal{C}$ equipped with a circumscription pattern $\mathcal{P}$. Given an interpretation $\mathcal{I}$, we say that $\mathcal{I}$ is a (pointwise) minimal model of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \wedge \mathcal{C}$, in symbols $\mathcal{I} \models^{(\bullet)} \mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \wedge \mathcal{C}$, if $\mathcal{I} \models^{(\bullet)} \mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ and $\mathcal{I} \models \mathcal{C}$. We call* constraint set *the KB $\mathcal{C}$.*

Compared to [6] where constraints were restricted to mimicking concept inclusions, we not only allow assertions in the constraints but also role inclusions. Furthermore, we remark that the two KBs $\mathcal{K}$ and $\mathcal{C}$ can share the signature.

Example 2. *Recall the KB $\mathcal{K}$ of Example 1. Let us consider $\mathcal{K}' \subseteq \mathcal{K}$ given by and circumscribed with $M = \{\mathrm{Has\_Read\_permission}\}$ and keeping all the other predicates* fixed. *Consider the constraint set $\mathcal{C} = \{\mathrm{Classified\_Document} \sqsubseteq \forall(\mathrm{read})^-.\mathrm{Has\_Read\_Permission}\}$. As in Example 1, the inclusion above imposes that whenever a classified document is read, the user reading it has permission to do so. Since the predicate $\mathrm{Has\_Read\_Permission}$ is minimized, in minimal models of $\mathcal{K}'$, any permission must be granted by an Admin. Applying Definition 11, we find that in every minimal model $\mathcal{I}$ of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}') \wedge \mathcal{C}$, John got the reading permission from an Admin. Counterintuitively, if we push the constraint set $\mathcal{C}$ inside the circumscription, i.e. we look for minimal models of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}' \cup \mathcal{C})$ we can derive that John has permission to read f_1 without the approval of an administrator.*

Following [1], we can prove that the standard *filtration technique* (see [12]) can be used to show that circumscribed $\mathcal{ALCHIO}$ with constraints has the 'small' (exponential size) model property under the assumption that roles are varying. The latter result yields the following.

Theorem 3. *Concept satisfiability in circumscribed $\mathcal{ALCHIO}$ with constraints is NExpTime^{NP}-complete if all roles are varying.*

Filtration has been used in [2] for showing that circumscribed DL-Lite$_{Bool}^{\mathcal{H}}$ has the small model property under the condition that *no* varying role is subsumed by a fixed role. A KB in DL-Lite$_{Bool}^{\mathcal{H}}$ that satisfies this condition is called *role-layered*. The model construction is the same applied for circumscribed $\mathcal{ALCHIO}$ with varying roles, however, the restricted syntax of DL-Lite$_{Bool}$ allows for handling the presence of fixed roles too. The upper bound of Theorem 3 applies also if role-layered circumscribed KBs in DL-Lite$_{Bool}^{\mathcal{H}}$ are paired with constraints in $\mathcal{ALCHIO}$.

Corollary 1. *Concept satisfiability in role-layered circumscribed DL-Lite$_{Bool}^{\mathcal{H}}$ with constraints in $\mathcal{ALCHIO}$ is in NExpTime^{NP}.*

Expressing Closed Predicates. In DLs with closed predicates [16, 17], the extension of some predicates – so-called *closed* – is restricted only to named individuals whose participation in the closed predicates is 'justified' by ABox assertions. An interpretation is a model of a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ w.r.t. a set Σ of closed predicates, in symbols $\mathcal{I} \models (\mathcal{A}, \mathcal{T}, \Sigma)$, if $\mathcal{I} \models \mathcal{K}$ and for each $A, r \in \Sigma$:

- $d \in A^{\mathcal{I}}$ implies $d = a^{\mathcal{I}}$ for some individual a such that $A(a) \in \mathcal{A}$, and

- $(d, e) \in r^{\mathcal{I}}$ implies $d = a^{\mathcal{I}}$ and $e = b^{\mathcal{I}}$ with a, b such that $r(a, b) \in \mathcal{A}$.

Intuitively speaking, the extension of a closed predicate p is 'circumscribed' to instances of p given by ABox assertions. Although the underlying principle is close to predicate minimization, DLs with closed predicates and (pointwise) circumscribed DLs are two different non-monotonic formalisms and we see no obvious formal translation. On the other hand, constrained circumscription can easily capture DLs with close predicates.

Given a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ with a set of closed predicates Σ, let $\Sigma' = \{p' \mid p \in \Sigma\}$, i.e. we consider a copy of the signature in Σ. Let $\mathcal{A}' = \{p'(\vec{c}) \mid p(\vec{c}) \in \mathcal{A} \text{ and } p \in \Sigma\}$. We obtain this way a new KB $\mathcal{K}' = (\mathcal{A} \cup \mathcal{A}', \mathcal{T})$. We now define a circumscription pattern $\mathcal{P}$ for $\mathcal{K}'$ by stating that all predicates in Σ' are minimized. Since $\mathcal{T}$ does not contain any occurrence of symbols in Σ', in a minimal model $\mathcal{I}'$ we will have that each $p' \in \Sigma'$ contains only named individuals or pairs of named individuals, whose participation in $(p')^{\mathcal{I}}$ is justified by an assertion in $\mathcal{A}'$. To the circumscribed KB $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}')$, we add the set of constraints $\mathcal{C} = \{p \sqsubseteq p' \mid p \in \Sigma\}$. In every model $\mathcal{I}$ of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}') \wedge \mathcal{C}$, every element of $p^{\mathcal{I}}$ is an element of $(p')^{\mathcal{I}}$. In this way, we force p to be closed.

Theorem 4. *Concept satisfiability w.r.t KBs with closed predicates can be polynomially reduced to concept satisfiability w.r.t. (pointwise) circumscribed KBs with constraints.*

The result above almost directly implies that nominals can be simulated using constraints. Indeed this is already possible via closed predicates: each nominal o can be simulated by replacing every occurrence of o with a closed predicate N and adding an assertion $N(o)$. A similar idea has been used in [7] to show that nominals can be simulated in EDLs.

6. From EDLs to Circumscribed DLs with Constraints

Given a KB $\mathcal{K}$ and a set F of fixed predicates, we aim to build a KB $\bar{\mathcal{K}}$ with a circumscription pattern $\mathcal{P}$ and a set of constraints $\mathcal{C}$ such that there is a correspondence between stable models of $\mathcal{K}$ and model of $\mathrm{Circ}_{\mathcal{P}}(\bar{\mathcal{K}}) \wedge \mathcal{C}$ up to the original signature. We give a sketch of the translation that we adapted from [18].

The first step of our construction is to simulate HT interpretations. We consider a copy Σ' of the signature Σ of $\mathcal{K}$, which we use to split the 'here' and the 'there' interpretations. Intuitively speaking, the interpretation of primed predicates corresponds to the interpretations of predicates in the 'there' interpretation. We construct the KB $\bar{\mathcal{K}}$ taking the union of two KBs $\mathcal{K}'$ and $\mathcal{K}^{\star}$. The KB $\mathcal{K}'$ is obtained from $\mathcal{K}$ by replacing each symbol with its primed counterpart. We use $\mathcal{K}'$ to require that the 'there' is a classical model of $\mathcal{K}$. The KB $\mathcal{K}^{\star}$ is obtained by replacing each concept C occurring negatively in $\mathcal{K}$ with C'. This second KB is used to simulate the interpretation of negated concepts in the HT semantics (see Figure 1).

To capture the stable model semantics in the setting of circumscription, we require that all predicates in the original signature of $\mathcal{K}$ are minimized while all primed predicates are fixed. To synchronize the 'here' and the 'there', ensuring that the interpretations of primed and non-primed symbols coincide, we introduce constraints of the form $P \sqsubseteq P'$ and $P' \sqsubseteq P$, for each predicate symbol P. These constraints filter out minimal models that are not stable.

A natural question is: *do we really need to rely on constrained circumscription?* Circumscription and Equilibrium Logic are two orthogonal formalisms, as already argued in [15].

Example 3. *Consider the TBox $\mathcal{T} = \{\neg A \sqsubseteq A\}$ with $F = \emptyset$. It is easy to show that $\mathcal{T}$ has no stable model. Indeed, given $\mathcal{I}$ such that $\Delta^{\mathcal{I}} = \{x\}$ and $\mathcal{I} \models \mathcal{T}$, then $A^{\mathcal{I}} = \{x\}$. The interpretation $\mathcal{J}$ such that $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ and $A^{\mathcal{J}} = \emptyset$ is such that $(\mathcal{J}, \mathcal{I}) \models \mathcal{T}$. If we apply (pointwise) circumscription, the most natural option to 'mimic' the stable model semantics is to minimize A. One can observe that the previous model $\mathcal{I}$ of $\mathcal{T}$ is minimal.*

Example 4. *Consider the TBox $\mathcal{T} = \{B \sqsubseteq A \sqcup \neg A\}$ and assume $F = \{B\}$. We can show that A is satisfiable in a stable model of $\mathcal{T}$. Indeed, the interpretation $\mathcal{I}$ such that $\Delta^{\mathcal{I}} = \{x\}$, $B^{\mathcal{I}} = A^{\mathcal{I}} = \{x\}$ is a stable model $\mathcal{T}$ such that $A^{\mathcal{I}} = \emptyset$. If one considers now (pointwise) circumscription, as before a natural*

direction is to respect the 'nature' of predicates, i.e. keep B fixed and minimize A. However, assuming that A is minimized, it is straightforward to see that there is no minimal model $\mathcal{J}$ of $\mathcal{T}$ such that $A^{\mathcal{J}} \neq \emptyset$.

The transformation. Given a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$, let Σ denote the set of all concept names and role names occurring in $\mathcal{K}$. We consider a copy of the signature Σ and we denote it with $\Sigma' = \{A' | A \in A\} \cup \{r' | r \in \Sigma\}$. Given a complex concept C we denote with C' the concept over Σ' obtained globally replacing each predicate symbol in C with its primed version. Following [18], we define an operator τ recursively as follows:

- $\tau(A) = A, \tau(r) = r, \tau(\top) = \top, \tau(\bot) = \bot, \tau(\{a\}) = \{a\},$

- $\tau(\neg C) = \neg C', \tau(\exists R.C) = \exists R.\tau(C)$ and $\tau(\forall R.C) = \forall R.\tau(C) \sqcap \forall R'.C'$

- $\tau(C \circ D) = \tau(C) \circ \tau(D)$, with $\circ \in \{\sqcap, \sqcup\}$.

Given the TBox $\mathcal{T}$, we denote with $\mathcal{T}^*$ the resulting TBox, with predicates occurring in $\Sigma \cup \Sigma'$, obtained applying τ to all GCIs in $\mathcal{T}$, i.e. $\mathcal{T}^{\star} = \{\tau(C) \sqsubseteq \tau(D) | C \sqsubseteq D \in \mathcal{T}\}$. Observe that the assertions in the ABox can be seen as inclusions in the standard way. It is easy to observe that they are not affected by the τ transformation. We $\mathcal{K}^{\star} = (\mathcal{A}, \mathcal{T}^{\star})$. We denote with $\mathcal{K}'$ the KB obtained from $\mathcal{K}$ by replacing each symbol $p \in \Sigma$ with its primed counterpart $p' \in \Sigma'$.

Given a set Σ of concept names and role names and interpretation $\mathcal{J}$, we denote with $\mathcal{J}'$ the interpretation such that $\Delta^{\mathcal{J}} = \Delta^{\mathcal{J}'}, (p')^{\mathcal{J}'} = p^{\mathcal{J}}$ for all $P \in \Sigma$. Given an HT interpretation $(\mathcal{I}, \mathcal{J})$, we denote with $\mathcal{I} \cup \mathcal{J}'$ the interpretation such that $\Delta^{\mathcal{I} \cup \mathcal{J}'} = \Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}, P^{\mathcal{I} \cup \mathcal{J}'} = P^{\mathcal{I}}$ for all $P \in \Sigma$ and $(P')^{\mathcal{I} \cup \mathcal{J}'} = P^{\mathcal{J}}$ for all $P' \in \Sigma'$.

As mentioned at the beginning of this section, we aim to simulate the HT semantics. Roughly speaking, given an HT interpretation, we copy the 'there' interpretation using the primed symbols, while unprimed symbols are interpreted as in the 'here'.

Lemma 1. *Assume a KB $\mathcal{K}$ and two interpretations $\mathcal{I}$ and $\mathcal{J}$. Let Σ be the set of predicates occurring in $\mathcal{K}$ and $F \subseteq \Sigma$ be a set of fixed predicates. Then $(\mathcal{I}, \mathcal{J})$ is a HT model of $\mathcal{K}$ with fixed predicates in F if and only if $\mathcal{I} \cup \mathcal{J}'$ is a model of $\mathcal{K}^* \cup \mathcal{K}' \cup \{p \sqsubseteq p' \in \Sigma\} \cup \{p' \sqsubseteq p | p \in F\}$.*

We now use circumscription to capture stable models. With the following theorem, we formalize the intuitive explanation given at the beginning of this section.

Theorem 5. *Assume a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and an interpretation $\mathcal{I}$. Let Σ be the set of predicates occurring in $\mathcal{K}$ and $F \subseteq \Sigma, \mathcal{I} \in SM_F(\mathcal{K})$ if and only if $\mathcal{I} \cup \mathcal{I}'$ is a model of $\mathrm{Circ}_{\mathcal{P}_F}(\mathcal{K}^* \cup \mathcal{K}') \wedge \mathcal{C}$, with $\mathcal{C} = \{p \sqsubseteq p', p' \sqsubseteq p | p \in \Sigma\}$ and $\mathcal{P}_F = (M, \emptyset, F \cup \Sigma')$ with $M = \Sigma \smallsetminus F$.*

Since $\mathcal{K}'$ contains only fixed predicates, alternatively we can add it to the set of constraints. Theorem 5 extends to the case of pointwise EDLs and pointwise circumscribed DLs with constraints.

Theorem 6. *Assume a KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and an interpretation $\mathcal{I}$. Let Σ be the set of predicates occurring in $\mathcal{K}$ and $F \subseteq \Sigma$, $\mathcal{I} \in PSM_F(\mathcal{K})$ if and only if $\mathcal{I} \cup \mathcal{I}'$ is a model of $\mathrm{Circ}_{\mathcal{P}_F}^{\bullet}(\mathcal{K}^* \cup \mathcal{K}') \wedge \mathcal{C}$, with $\mathcal{C} = \{p \sqsubseteq p', p' \sqsubseteq p | p \in \Sigma\}$ and $\mathcal{P}_F = (M, \emptyset, F \cup \Sigma')$ with $M = \Sigma \smallsetminus F$.*

7. Conclusions

In this paper, we established a translation from Equilibrium DLs (EDLs) and circumscribed DLs with *constraints*, both under global and pointwise minimization. These two formalisms are interesting on their own and understanding the complexity of one can help in understanding the complexity of the other. Therefore, our future research will follow two main lines. We plan to investigate the expressive power and the computational complexity of EDLs. In particular, we aim to focus on lightweight EDLs in the DL-Lite family. Via the translation of EDLs into circumscribed DLs with constraints, we can refine our investigation by studying the complexity of reasoning in circumscribed DLs with constraints.

The preliminary results presented in this paper involve varying predicates, that are not supported in EDLs. We are also interested in identifying syntactic restrictions under which global and pointwise circumscription (with constraints) coincide, inheriting the good computational properties of the latter for both EDLs and pointwise EDLs. Furthermore, we aim to explore the connection between EDLs and DLs based on MKNF [19], and fuzzy DLs under the (3-valued) Gödel semantics [20].

Acknowledgments

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