Implementation of Qualitative and Quantitative Methods in Ordinary Differential Equations using Maple Mathematical Software

Luis Jaime Collantes Santisteban*1*, Alberto Hananel Baigorria*1*, Samuel Collantes Santisteban*1* and Rosa Gonzales Llontop*1*

1 Pedro Ruiz Gallo National University, Av. Juan XXIII 391, Lambayeque, 14013, Peru

Abstract

The implementation of qualitative and quantitative methods in ordinary differential equations, in both its theoretical and applied parts, requires the use of mathematical software to achieve a modern approach with effectiveness in geometric and numerical analysis.

The present work was carried out with the objective of analyzing the solution of mathematical problems related to first-order ordinary differential equations, qualitatively and quantitatively. For this, the *Maple* software was used, due to its powerful mathematical machine and great symbolic capacity, with an interface that makes it easy to analyze, explore, visualize and solve mathematical problems related to qualitative and quantitative theory of ordinary differential equations.

First, specific features of *Maple* mathematical software that are useful for analyzing ordinary differential equations are identified. Then, first-order ordinary differential equations are analyzed and solved with *Maple*, considering existence, uniqueness, and stability of ordinary differential equations. A qualitative approach to the study of first-order ordinary differential equations is discussed, obtaining qualitative information about the solutions directly from the equation, without the use of a formula for the solution. In this work, worksheets have been built and developed in *Maple* that contain the solution to the problems posed in the attached data recording sheets, the same ones that appear in the literature with the names *Problem Set A*: *Practice with Maple* and *Problem Set B*: *First Order Equations*. Graphic and numerical representations are obtained that help carry out a convenient analysis and interpretation of the problems posed.

Keywords

Ordinary Differential Equations, *Maple*, Geometric and Numerical Analysis

1. Introduction

The traditional introduction of ordinary differential equations courses (Zill, 2018) has concentrated in higher education a repertoire of techniques to find solution formulas for various classes of differential equations.
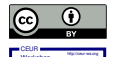
Typically, the result has been the application of formulaic techniques without a serious qualitative understanding of fundamental aspects of the topic, such as stability, asymptotic analysis, and parameter dependence. These fundamental ideas are difficult to understand because they have a lot of geometric content and involve a lot of calculations. Modern mathematical software systems can help overcome these difficulties. Collantes *et al.* (2000) obtain their results with the help of *Maple*. Belyaeva *et al.* (2021) constructed an algorithm and a program has been developed in the *Maple* environment in order to solve the fourth-order differential equations. For the use of qualitative and quantitative methods in ordinary differential equations, referring to its theory and applications, it is very important to have powerful

CEUR Workshop Proceedings (CEUR-WS.org)

mathematical software that is accessible to students and researchers. A modern system is the mathematical software *Maple*, effective in geometric and numerical analysis. Therefore, it is essential, first of all, to know its basic commands, as well as their use to perform calculations, graphs, derive and integrate functions. With this, a minimum level of competence will be achieved to allow the use of *Maple* in various ordinary differential equations topics.

In chapter 3: *Doing Mathematics with Maple*, chapter 5: *Solutions of Differential Equations* and chapter 6: *A Qualitative Approach to Differential Equations* (Coombes, K. *et al*., 1997) an introduction is presented, respectively, in the first chapter of them, to the mathematical software *Maple*, and in the following two chapters an analysis of the ordinary differential equations solutions and its qualitative approach, proposing sets of exercises and problems called *Problem Set A*: *Practice with Maple* and *Problem Set B*: *First Order Equations*, the same ones that are solved in this investigation in *Maple* worksheets.

For many ordinary differential equations, *Maple*'s **dsolve** command produces its general solution, or the specific solution to an associated initial value problem.

1.1. Symbolic solutions

Consider the first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

A solution to this equation is a function $y(x)$ of the independent variable x that satisfies $y'(x) = f(x, y(x))$. It is sometimes possible to find a formula for the solutions of (1); we call such a formula a symbolic solution. In *Maple*, the command that finds symbolic solutions is **dsolve**. To find a symbolic solution to the ordinary differential equation (1), write and execute:

   >**dsolve(diff(y(x),x)=f(x,y(x)),y(x))**.   (2)

Maple produces the answer in terms of an arbitrary constant C1. For example, consider the ordinary differential equation: $\frac{dy}{dx} = x + y$. You can solve this equation in *Maple* by writing and executing:

   >**dsolve(diff(y(x),x)=x+y(x),y(x))**.   (3)

The output of this command is $y(x) = -x - 1 + C1\_e^x$. Specific solutions can be obtained by choosing specific values for C1. For example, the solution satisfying a given initial condition can be obtained by choosing C1 appropriately. This value of C1 can be found by imposing the initial condition on the general solution and solving for C1. Alternatively, you can specify the initial condition as the ordinary differential equation when using **dsolve**.

To solve the initial value problem

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0 \tag{4}$$

it is written and executed:

   >**dsolve({diff(y(x),x)=f(x,y(x)),y(x0)=y0},y(x))**   (5)

The **dsolve** command is robust, it can solve many ordinary differential equations. In fact, it can solve most ordinary differential equations that can be solved with standard solution methods. However, there are many other equations, some of which can be solved by more advanced solution methods that *Maple* cannot solve.

The existence, uniqueness and stability of ordinary differential equations solutions are discussed below, topics that are fundamental in the theory and application of ordinary differential equations. An understanding of them helps to interpret and use the results produced by *Maple*.

1.2. Existence and uniqueness

The fundamental existence and uniqueness theorems for ordinary differential equations (Boyce & DiPrima, 2012) guarantee that every initial condition $y(x_0) = y_0$ leads to a single solution *close* to x_0, assuming that the right side of the ordinary differential equation (1) is an "appropriate" function (to be specific, assuming that f and $\partial f / \partial y$ are continuous functions).

Graphically, these theorems say that a solution curve exists through every point, and that solution curves cannot intersect. Thus, initial value problems have exactly one solution, but, since there are an infinite number of initial conditions, ordinary differential equations have an infinite number of solutions. This principle is implicit in the results obtained with **dsolve**.

When an initial condition is not specified the solution depends on an arbitrary constant; when an initial condition is specified, the solution is completely determined. It is important to remember that the existence and uniqueness theorems only guarantee the existence of a solution near the initial point x_0.

1.3. Stability

In addition to existence and uniqueness, sensitivity to the initial value of the solution of an initial value problem is a fundamental issue in the theory and application of ordinary differential equations.

Often, one is primarily interested in positive values of x, just like when x corresponds to the time in a physical problem. If an initial value problem is to predict the future of a physical system, the solution for positive x should be regularly insensitive to the initial value, this means that small changes in the initial value should lead to small changes in the solution for the positive time.

To see the importance of this principle, note that for a physical system the initial value y_0 is typically not known exactly, but it is found by measurements. When y_0 is measured, generally only an approximate value $\tilde{y}_0$ is obtained. Then if $\tilde{y}(x)$ is the solution corresponding to $\tilde{y}_0$ and the solution is very sensitive to the initial value, $\tilde{y}(x)$ will have little relation to the current state $y(x)$ of the system by increasing x.

An ordinary differential equation whose solution is regularly insensitive to the initial value when increasing x is called stable, however, when the solution is very sensitive to the initial value when increasing x, the ordinary differential equation is called unstable. As we have seen, if an equation is unstable and is used over a long-time interval, a small error in the initial value can subsequently cause a large error. Caution should be taken when using an unstable ordinary differential equation to model a physical problem.

Stability has been considered as it increases x, that is, stability to the right. Stability on the left can also be considered. There are equations that are stable both to the left and to the right and there are others that are unstable both to the left and to the right. The following theorem, which is stated without proof, is often useful in stability evaluation.

Theorem 1. Suppose that $f(x,y)$ has continuous first-order partial derivatives in the vertical region $R = \{(x,y): x_0 \leq x \leq x_1, \ -\infty < y < \infty\}$, and suppose that there exist numbers K and L such that $K \leq \dfrac{\partial f}{\partial y}(x,y) \leq L, \ \ \forall (x,y) \in R.$

If $y(x)$ and $\tilde{y}(x)$ are solutions of $dy/dx = f(x,y)$ over the interval $x_0 \le x \le x_1$ with initial values $y(x_0) = y_0$ and $\tilde{y}(x_0) = \tilde{y}_0$, respectively, then

$$|y_0 - \tilde{y}_0|e^{K(x-x_0)} \le |y(x) - \tilde{y}(x)| \le |y_0 - \tilde{y}_0|e^{L(x-x_0)}, \qquad \forall x_0 \le x \le x_1. \tag{6}$$

If $L \le 0$, then the right-hand inequality in (6) shows that

$$|y(x) - \tilde{y}(x)| \le |y_0 - \tilde{y}_0|, \ \forall x_0 \le x \le x_1. \tag{7}$$

In this way, the solutions differ by no more than the difference in the initial values, and the differential equation is stable. Furthermore, if $L > 0$, but not so big, and $x_1 - x_0$ is not so big, then

$$|y(x) - \tilde{y}(x)| \le M |y_0 - \tilde{y}_0|, \ \forall x_0 \le x \le x_1, \tag{8}$$

where $M = e^{L(x_1 - x_0)}$ is a constant of moderate size. In this way, the equation is only slightly sensitive to changes in the initial value, and the equation is only slightly unstable. On the other hand, if $K > 0$, then the left-hand inequality in (6) shows that the solution is sensitive to changes in the initial value, especially over long intervals.

We can briefly summarize these results by saying that if

$$\partial f / \partial y \le 0, \tag{9}$$

then the differential equation is stable, but if

$$\partial f / \partial y > 0, \tag{10}$$

then the equation is unstable.

The right-hand inequality in (6) of theorem is an example of a *continuous dependence* result; which shows that the solution depends continuously on the initial value.

1.4. Qualitative approach to an Ordinary Differential Equation: Direction Field

Consider the general first-order ordinary differential equation

$$\frac{dx}{dt} = f(t,x). \tag{11}$$

Qualitative information can be obtained about the solutions $x(t)$ observing (11) geometrically. Specifically, this information can be obtained from the direction field of (11). The direction field is obtained by drawing through each point (t,x) in the plane (t,x) a small line segment with a slope $f(t,x)$. Solutions, or integral curves of (11) have the property that at each of their points they are tangent to the direction field at that point, and in that way the general qualitative nature of the solutions can be determined from the direction field. Direction fields can be drawn by hand for some simple ordinary differential equations, but *Maple* can draw them for any first-order equation. The *Maple* command to draw direction fields is **dfieldplot**, in the **DEtools** package (Ortigoza, G.M., 2007[a,b]).

1.5. Autonomous Equations

Equations of the form

$$\frac{dx}{dt} = f(x), \tag{12}$$

where the function f does not depend on t, are called autonomous equations. Such equations represent physical systems whose rules of evolution do not change over time and are particularly susceptible to qualitative analysis.

2. Materials and Methods

The methodological design includes the performance of numerical simulations in *Maple*, based on the construction and development of worksheets in *Maple* that contain the solution to the problems posed in the attached data recording sheets, which correspond to exercises and problems contained in the text by Boyce & DiPrima (2012) and that have specifically been taken from Coombes *et al.* (1997) with the names *Problem Set A*: *Practice with Maple* and *Problem Set B*: *First Order Equations*. The first sheet contains exercises for using basic *Maple* commands, while the second sheet contains problems concerning first-order ordinary differential equations.

The design in *Maple* seeks to obtain graphic and numerical representations that help to carry out a convenient analysis and interpretation of the problems posed.
The developed methodology follows the following sequence:
  a) Statement of the problem.
  b) Mathematical analysis of the problem.
  c) Solution in a *Maple* worksheet of the problem algebraically, numerically and graphically.
  d) Exploration, description, analysis and interpretation of the model through the *Maple* interface.

3. Results and discussion

The results of the *Maple* worksheet corresponding to the solution of *Problem Set B. First Order Equations* are presented.

3.1. Finding the solution to the initial value problem:

$$x\frac{dy}{dx} + 2y = \sin x, \quad y(\pi/2) = 1$$

> $sol1 := dsolve\left(\left\{x \cdot diff(y(x), x) + 2 \cdot y(x) = \sin(x), y\left(\frac{\pi}{2}\right) = 1\right\}, \right.$
> $\left. y(x)\right);$

$$sol1 := y(x) = \frac{\sin(x) - \cos(x)\,x + \frac{1}{4}\pi^2 - 1}{x^2}$$

Defining this function in *Maple*:

> $y1 := unapply(rhs(sol1), x);$

$$y1 := x \rightarrow \frac{\sin(x) - \cos(x)\,x + \frac{1}{4}\pi^2 - 1}{x^2}$$
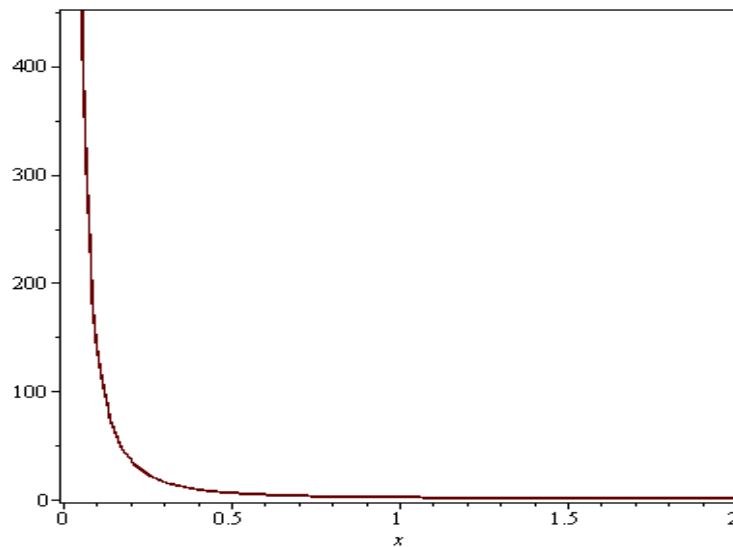
Making a list of the values of $y(x)$ in $x = 0.5, 1, 1.5, 2, \ldots, 5$.

> $evalf\left(\left\{y1\left(\dfrac{c}{2}\right)\$\ c = 1 .. 10\right\}\right);$

$\{-0.03639336405, 0.07103467055, 0.2078233180, 0.3587115330,$
$\quad 0.5087220665, 0.6509971655, 0.8022480500, 1.048351238,$
$\quad 1.768569780, 6.03214143\}$

Figure 1 shows the solution $y(x)$ on the interval $0 < x \leq 2$, with which it can be stated that the solution close to $x = 0$ is unbounded and tends to $+\infty$.
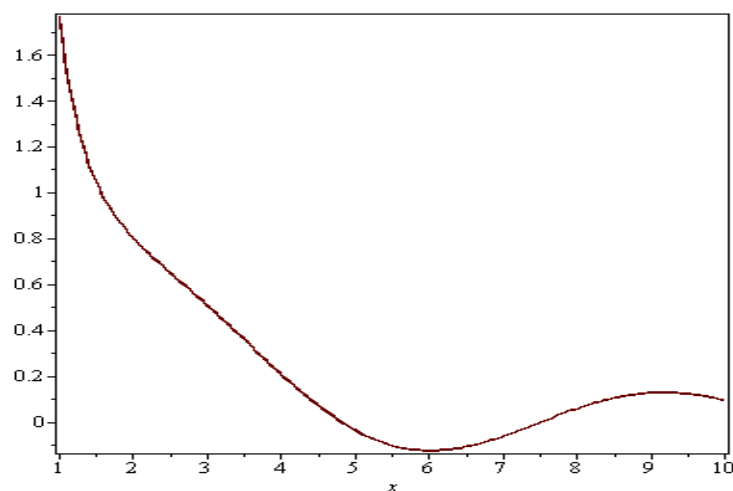
> $plot(y1(x), x = 0 .. 2);$

Figure 1: Solution to Exercise B.1(b) on the interval $\left[0, 2\right]$

Figure 2 shows the solution $y(x)$ on the interval $1 \leq x \leq 10$.

> $plot(y1(x), x = 1 .. 10);$

Figure 2: Solution to Exercise B.1(b) on the interval $\left[1, 10\right]$

Figure 3 shows the solution $y(x)$ on the interval $10 \le x \le 100$, with which it can be stated that the solution for large values of x tends to 0.

> $plot(y1(x), x = 10 .. 100);$
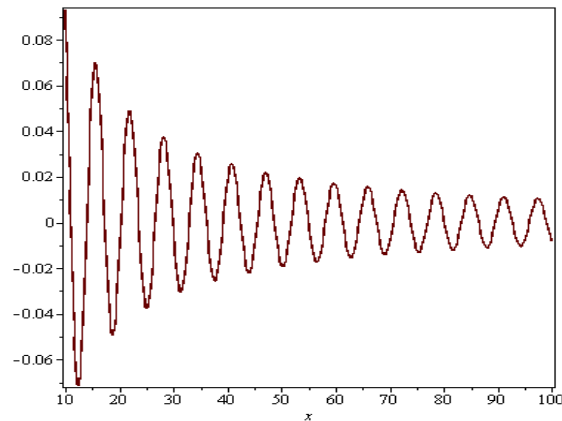
Figure 3: Solution to Exercise B.1(b) on the interval $[10,100]$

Finding the solutions $y_j(x)$ of the differential equation corresponding to the initial conditions

$$y_j\left(\frac{\pi}{2}\right) = 0.2j, \quad j = 1,\ldots,5:$$

> $sol11 := dsolve\left(\left\{x \cdot diff(y(x), x) + 2 \cdot y(x) = \sin(x), y\left(\frac{\pi}{2}\right) = a\right\},\right.$
> $\left. y(x)\right);$

$$sol11 := y(x) = \frac{\sin(x) - \cos(x) x + \frac{1}{4} a \pi^2 - 1}{x^2}$$

> $y11 := unapply(rhs(sol11), x, a);$

$$y11 := (x, a) \rightarrow \frac{\sin(x) - \cos(x) x + \frac{1}{4} a \pi^2 - 1}{x^2}$$

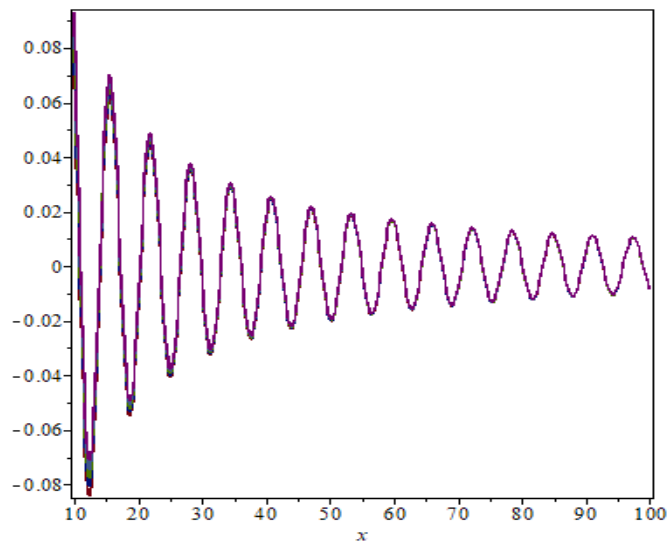Figure 4 shows, in the same graph, the solutions $y_j(x)$, $j = 1,\ldots,5$, on the interval $[0,10]$.

> $plot(\{y11(x, 0.2 \cdot j)\$ \, j = 1 .. 5\}, x = 0.2 .. 10);$
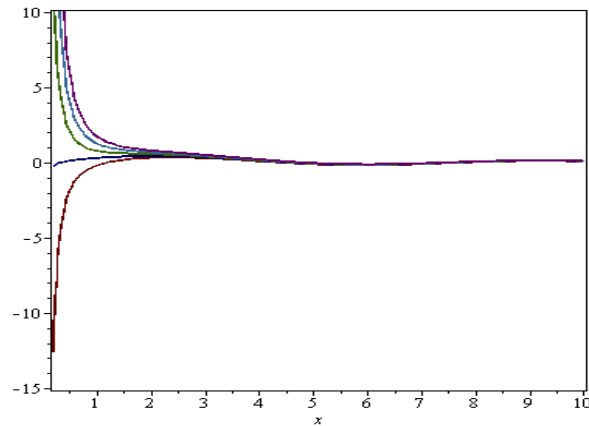
Figure 4: Solution to Exercise B.1(b) on the interval $[0,10]$

Figure 5 shows, in the same graph, the solutions $y_j(x)$, $j=1,\ldots,5$, on the interval $[10,100]$.

> $plot(\{y11(x, 0.2\cdot j)\$\ j=1..5\}, x=10..100);$

Figure 5: Solution to Exercise B.1(b) on the interval $[10,100]$

According to the graphs, it can be stated that the solutions, except one, near $x=0$, are unbounded and tend to $+\infty$ and to $-\infty$; however when $x \to +\infty$ they approach zero. Figure 6 shows a solution, with its graph, of the ordinary differential equation without singularity in $x=0$.

> $sol12 := dsolve(\{x\cdot diff(y(x),x) + 2\cdot y(x) = \sin(x), y(0)=0\}, y(x));$

$$sol12 := y(x) = \frac{\sin(x) - \cos(x)\,x}{x^2}$$

> $y12 := unapply(rhs(sol12), x);$

$$y12 := x \to \frac{\sin(x) - \cos(x)\,x}{x^2}$$

> $plot(y12(x), x=0..10);$
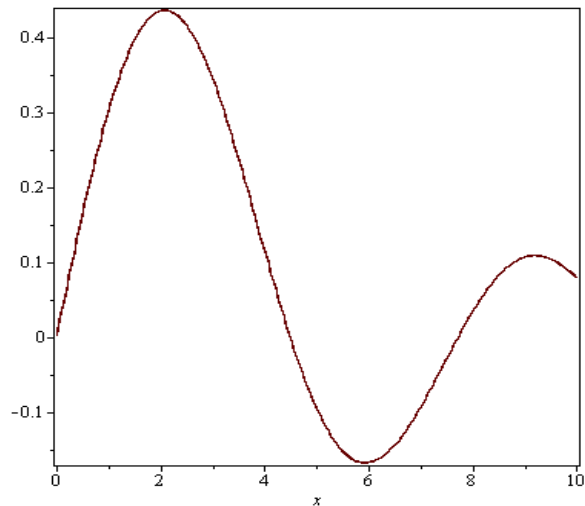
Figure 6: Solution to Exercise B.1(e) on the interval $[0,10]$, without singularity in $x = 0$

3.2. Finding the solution to the initial value problem:

$$\begin{cases} x\dfrac{dy}{dx} + y = 2x \\ \quad y(1) = c \end{cases}$$

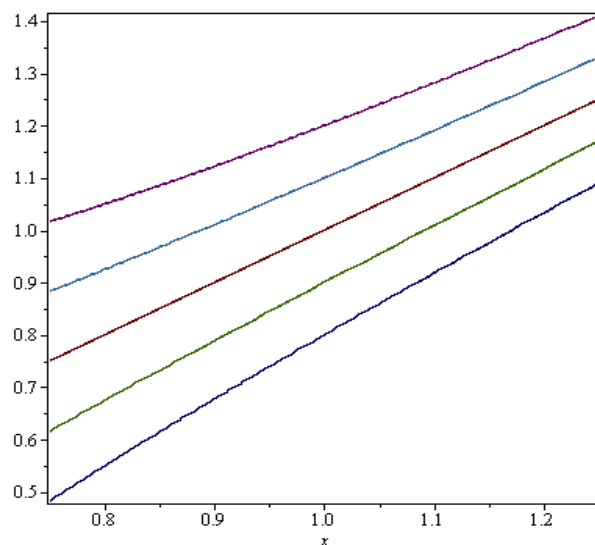> $sol2 := dsolve(\{x \cdot diff(y(x), x) + y(x) = 2 \cdot x, y(1) = c\}, y(x));$

$$sol2 := y(x) = x + \frac{c - 1}{x}$$

> $y2 := unapply(rhs(sol2), x, c);$

$$y2 := (x, c) \rightarrow x + \frac{c - 1}{x}$$

Figure 7 shows on $(0.75, \ 1.25)$ solutions for $c = 0.8, \ 0.9, \ 1, \ 1.1, \ 1.2$.

> $plot(\{y2(x, 0.8), y2(x, 0.9), y2(x, 1), y2(x, 1.1), y2(x, 1.2)\}, x = 0.75$
$..1.25);$

Figure 7: Solution to Exercise B.2(b) on the interval $(0.75, \ 1.25)$

These solutions are evaluated in $x = 0.01,\ 0.1,\ 1,\ 10$:

> $evalf(\{y2(0.01, 0.8), y2(0.01, 0.9), y2(0.01, 1), y2(0.01, 1.1),$
  $y2(0.01, 1.2)\})$;

$$\{-19.99,\ -9.99,\ 0.01,\ 10.01,\ 20.01\}$$

> $evalf(\{y2(0.1, 0.8), y2(0.1, 0.9), y2(0.1, 1), y2(0.1, 1.1), y2(0.1,$
  $1.2)\})$;

$$\{-1.9,\ -0.9,\ 0.1,\ 1.1,\ 2.1\}$$

> $evalf(\{y2(1, 0.8), y2(1, 0.9), y2(1, 1), y2(1, 1.1), y2(1, 1.2)\})$;

$$\{0.8,\ 0.9,\ 1.,\ 1.1,\ 1.2\}$$

> $evalf(\{y2(10, 0.8), y2(10, 0.9), y2(10, 1), y2(10, 1.1), y2(10, 1.2)\})$;

$$\{9.980000000,\ 9.990000000,\ 10.,\ 10.01000000,\ 10.02000000\}$$

Figures 8 and 9 show, in the same graph, the five solutions on the intervals $[0,\ 2.5]$ and $[100,\ 500]$, respectively.

> $plot(\{y2(x, 0.8), y2(x, 0.9), y2(x, 1), y2(x, 1.1), y2(x, 1.2)\}, x = 0$
  $..2.5)$;
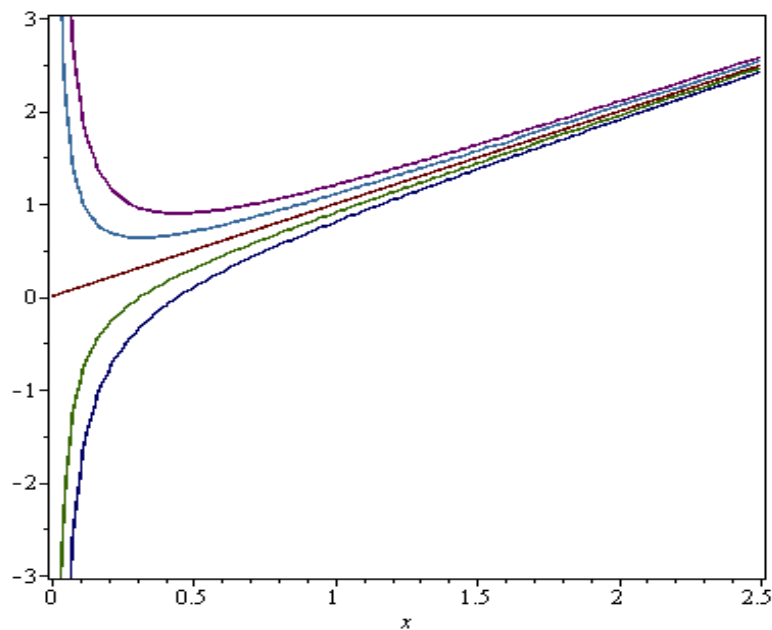
Figure 8: Solution to Exercise B.2(d) on the interval $[0,\ 2.5]$

> $plot(\{y2(x, 0.8), y2(x, 0.9), y2(x, 1), y2(x, 1.1), y2(x, 1.2)\}, x = 100$
  $..500)$;
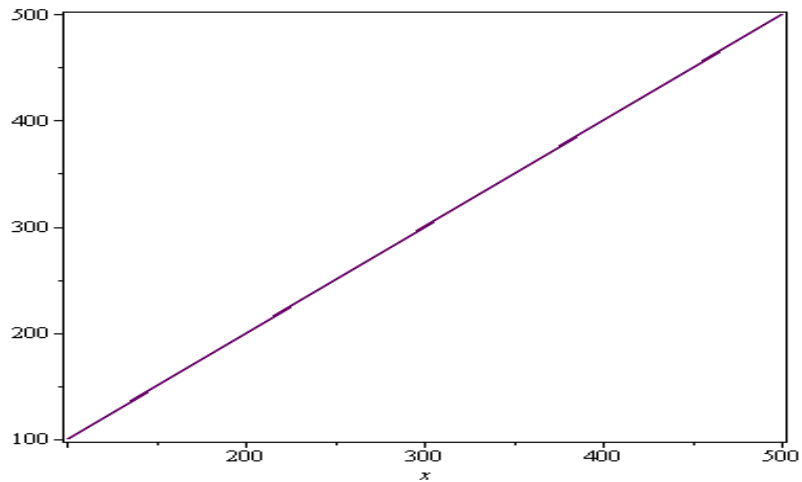
Figure 9: Solution to Exercise B.2(d) on the interval $[100,\ 500]$

Changes in the initial data affect the solution for $x \rightarrow 0^+$, but do not affect for $x \rightarrow +\infty$. The five solutions, except one, near $x = 0$ are unbounded and tend to $+\infty$ and to $-\infty$; however when $x \rightarrow +\infty$ they approach the exception: a straight line.

3.3. Finding the solution to the initial value problem:

$$\begin{cases} \dfrac{dy}{dx} - y = \cos x \\ y(0) = c \end{cases}$$

> $sol3 := dsolve(\{diff(y(x), x) - y(x) = \cos(x), y(0) = c\}, y(x));$

$$sol3 := y(x) = -\frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) + e^x\left(c + \frac{1}{2}\right)$$

> $y3 := unapply(rhs(sol3), x, c);$

$$y3 := (x, c) \rightarrow -\frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) + e^x\left(c + \frac{1}{2}\right)$$

The solutions are plotted in figure 10 for $c = -0.9,\ -0.8, \ldots,\ -0.1,\ 0$. The solutions are presented on the same interval between $x = 0$ and an appropriate right endpoint $x = 4$.
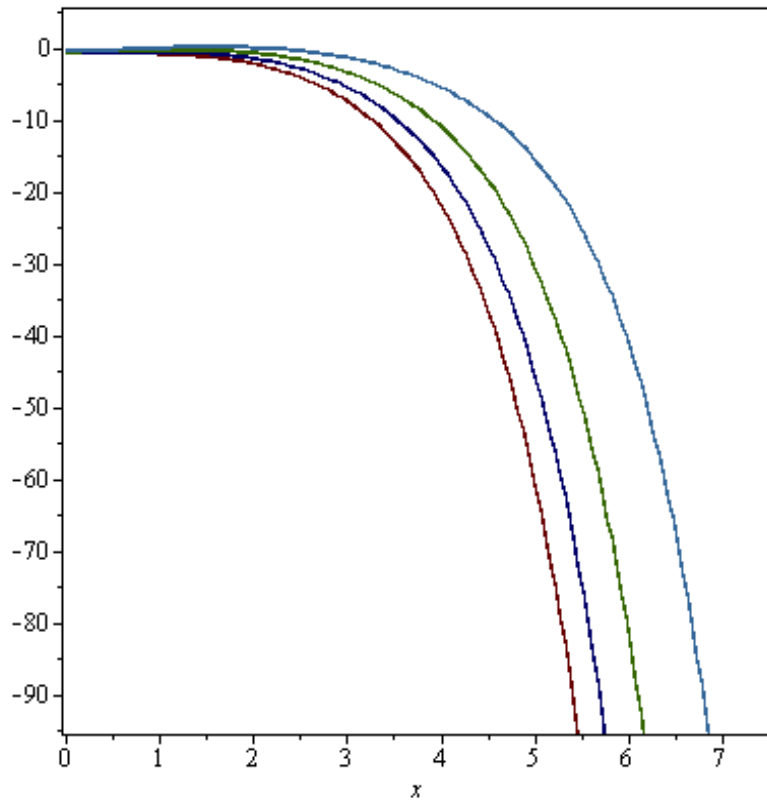
> $plot(\{y3(x, -0.9), y3(x, -0.8), y3(x, -0.7), y3(x, -0.6), y3(x, -0.5),$
$\quad y3(x, -0.4), y3(x, -0.3), y3(x, -0.2), y3(x, -0.1), y3(x, 0)\}, x = 0..4)$
$\quad ;$
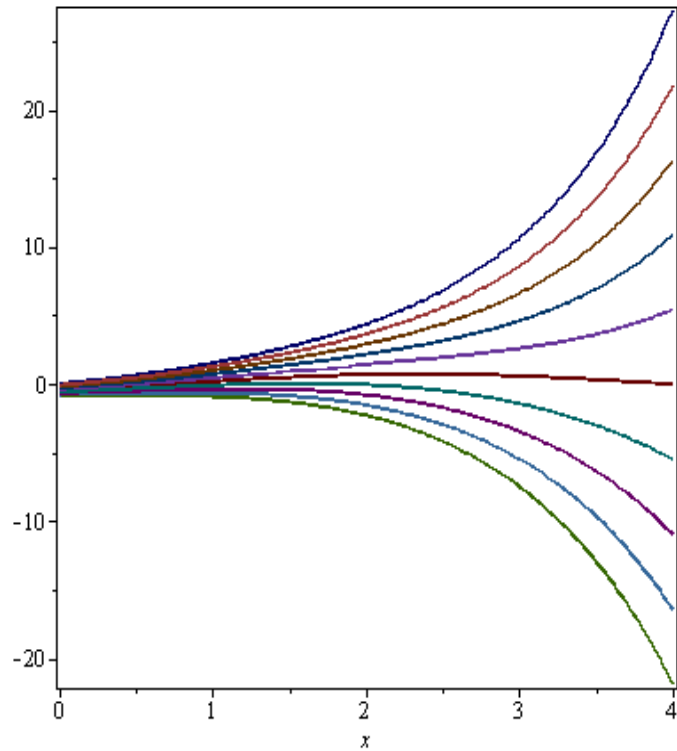
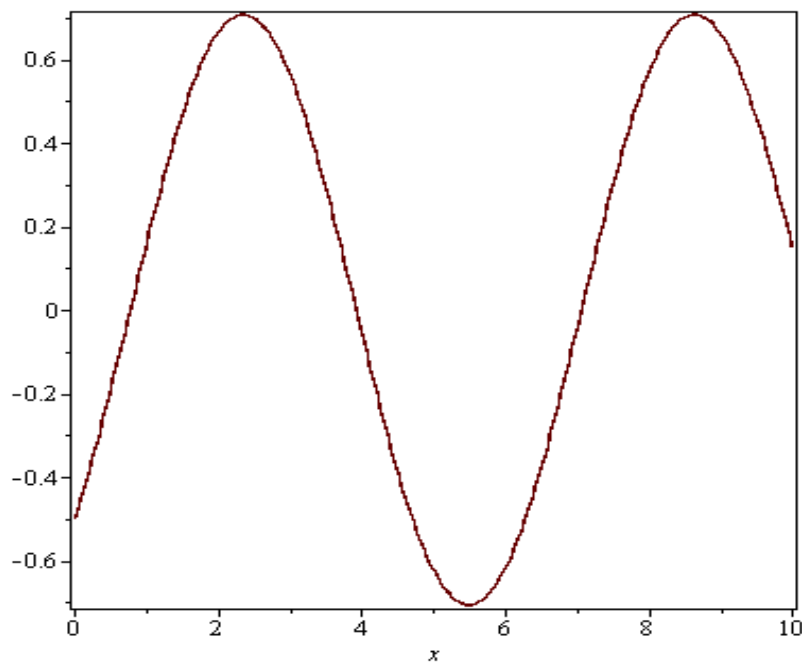**Figure 10:** Solution to Exercise B.3 on the interval $[0, \ 4]$

The aim is to explain the behavior of the solution curves for large values of x. Likewise, discuss the effect that small changes in the initial data can have on the global behavior of the solution curves.

> $plot\left(\left\{y3\left(x, -\ 0.9\right), y3\left(x, -0.8\right), y3\left(x, -0.7\right), y3\left(x, -0.6\right)\right\}, x = 0 \mathbin{..} 10\right);$
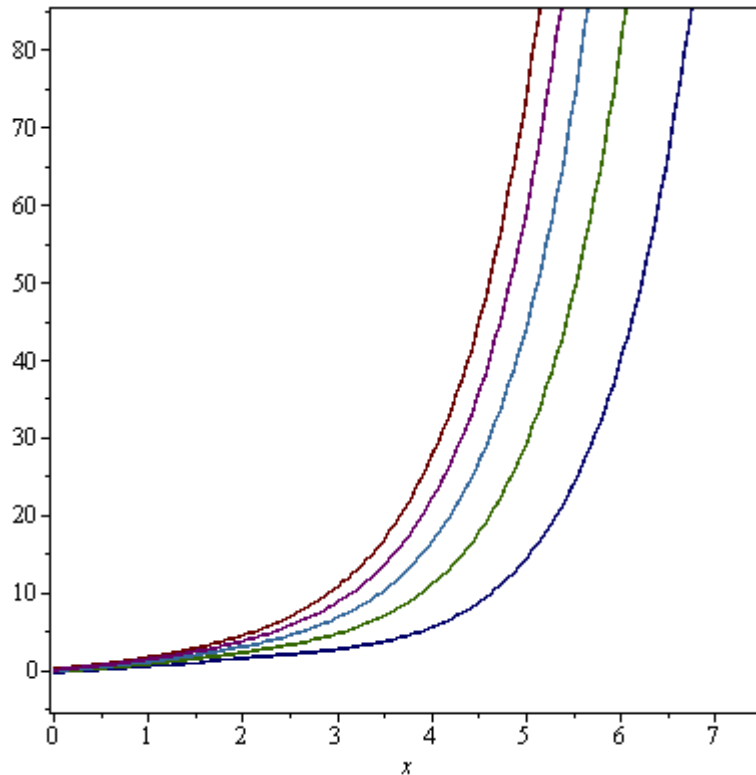
Figure 11: Solution curves (first type) of Exercise B.3 on the interval $[0,\ 10]$

> $plot(y3(x,-0.5), x = 0..10);$

Figure 12: Solution curve (second type) of Exercise B.3 on the interval $[0,\ 10]$

> $plot(\{y3(x,-0.4), y3(x,-0.3), y3(x,-0.2), y3(x,-0.1), y3(x,0)\}, x = 0$
   $..10);$

Figure 13: Solution curves (third type) of Exercise B.3 on the interval $[0, \ 10]$

According to figures 11, 12 and 13, three types of behavior are identified for the solution curves. Small changes in the initial data do not affect the solution curves for values greater than and close to $x = 0$; however, for large positive values of x, curves of the first type tend to $-\infty$, those of the second type are bounded and tend to 0, while those of the third type tend to $+\infty$.

3.4. Finding the implicit solution to the ordinary differential equation:

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^{y}} \, y^{2}.$$

> $sol4 := dsolve\left(diff(y(x), x) = \frac{x - \exp(-x)}{y(x) + \exp(y(x))}, y(x) \right);$

$$sol4 := \frac{1}{2} x^{2} + e^{-x} - \frac{1}{2} y(x)^{2} - e^{y(x)} + \_C1 = 0$$

> $f4 := (x, y) \rightarrow \frac{1}{2} x^{2} + e^{-x} - \frac{1}{2} y^{2} - e^{y};$
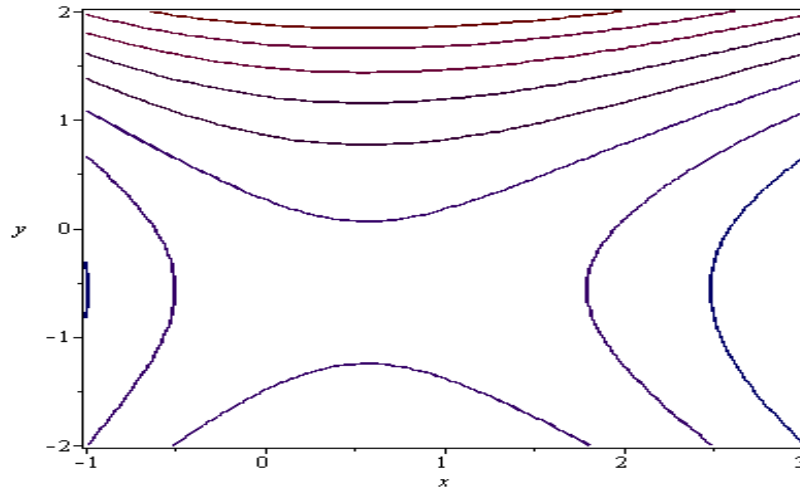
$$f4 := (x, y) \rightarrow \frac{1}{2} x^{2} + e^{-x} - \frac{1}{2} y^{2} - e^{y}$$

It is noted that the solution is implicitly given in the form $f(x, y) = c.$

In figure 14, **contourplot** has been used to see what the solution curves look like. For the ranges of x and y, it has been taken $x = -1 \ldots 3$ and $y = -2 \ldots 2.$

> $with(plots) :$
> $contourplot(f4(x, y), x = -1 .. 3, y = -2 .. 2);$

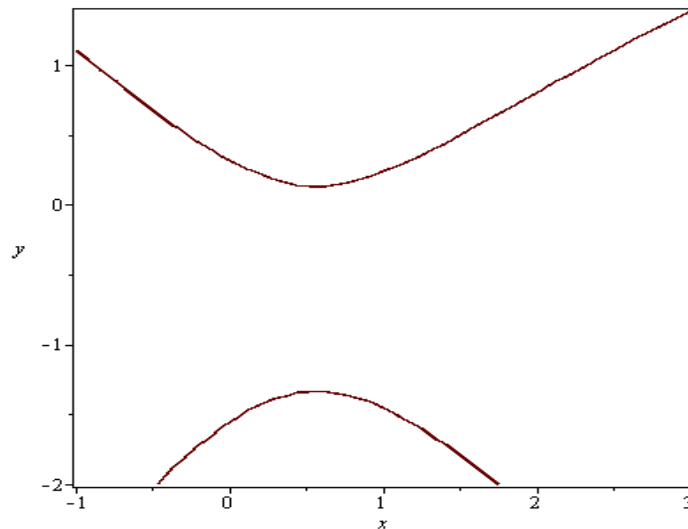**Figure 14:** Solution curves of Exercise B.4(b) with $-1 \leq x \leq 3, -2 \leq y \leq 2$, first form

Figure 15 presents a second way to use **contourplot** to see what the solution curves look like. In figure 16, **implicitplot** has been used to graph the solution satisfying the initial condition $y(1.5) = 0.5$. The graph shows two curves, one of them is the solution, which is showed in figure 17.

> $d := f4(1.5, 0.5);$
>> $d := -0.425591111$

> $g4 := (x, y) \rightarrow \dfrac{1}{2} x^2 + e^{-x} - \dfrac{1}{2} y^2 - e^y - d;$

$$g4 := (x, y) \rightarrow \dfrac{1}{2} x^2 + e^{-x} - \dfrac{1}{2} y^2 - e^y - d$$

> $g4(1.5, 0.5);$
>> $0.$

> $contourplot(g4(x, y), x = -1 .. 3, y = -2 .. 2);$
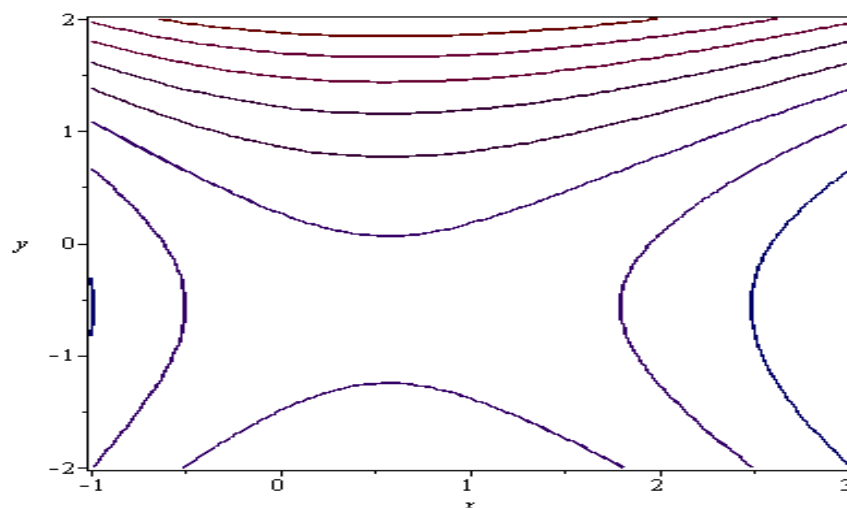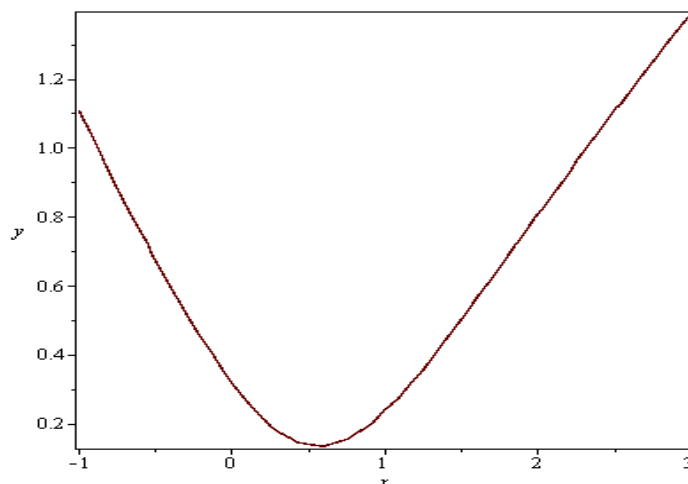

Figure 15: Solution curves of Exercise B.4(b) with $-1 \leq x \leq 3, -2 \leq y \leq 2$, second form

> $implicitplot(f4(x, y) = f4(1.5, 0.5), x = -1 .. 3, y = -2 .. 2);$

Figure 16: Probable solution curves of Exercise B.4(c) with $-1 \le x \le 3, -2 \le y \le 2$

> $implicitplot(f4(x, y) = f4(1.5, 0.5), x = -1 ..3, y = 0 ..2);$
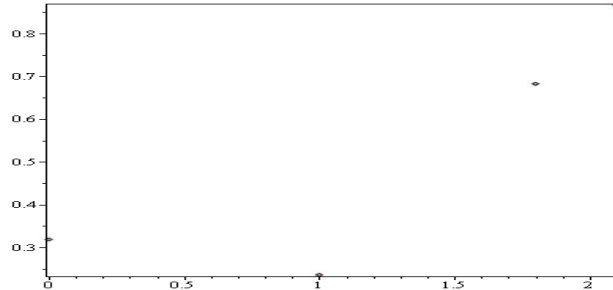

Figure 17: Solution curve of Exercise B.4(c) with $-1 \le x \le 3, 0 \le y \le 2$

Given a value x_1, to find $y(x_1)$ it is necessary to find the solution of $f(x_1, y) = f(1.5, \ 0.5)$ (viewed as an equation in y) near of $y = 0.5$. This can be done with the **fsolve** command. It's found $y(0)$, $y(1)$, $y(1.8)$, $y(2.1)$. In figure 18 these values are marked on a graph.

> $fsolve(f4(0, y) = f4(1.5, 0.5), y);$
> > 0.3183856549
> $fsolve(f4(1, y) = f4(1.5, 0.5), y);$
> > 0.2356325919
> $fsolve(f4(1.8, y) = f4(1.5, 0.5), y);$
> > 0.6821874213
> $fsolve(f4(2.1, y) = f4(1.5, 0.5), y);$
> > 0.8662118724

> $PLOT(\text{POINTS}([\,[\,0, fsolve(\,f4(0, y) = f4(1.5, 0.5), y)\,]\,], [\,1,$
  $fsolve(\,f4(1, y) = f4(1.5, 0.5), y)\,]\,], [\,1.8, fsolve(f4(1.8, y) = f4(1.5,$
  $0.5), y)\,]\,], [\,2.1, fsolve(\,f4(2.1, y) = f4(1.5, 0.5), y)\,]\,],$
  $\text{SYMBOL(DIAMOND)}));$

Figure 18: Some points of the solution curve of Exercise B.4(d)

3.5. Finding the implicit solution to the ordinary differential equation:

$$(e^x \sin y - 2y \sin x)\, dx + (e^x \cos y + 2 \cos x)\, dy = 0.$$

> $sol5 := dsolve\left(\, diff\,(y(x), x) = \dfrac{2 \cdot y(x) \cdot \sin(x) - \exp(x) \cdot \sin(y(x))}{\exp(x) \cdot \cos(y(x)) + 2 \cdot \cos(x)},\right.$
  $\left. y(x)\right);$

$\qquad sol5 := 2\, y(x)\, \cos(x) + e^x \sin(y(x)) + \_C1 = 0$

> $f5 := (x, y) \to 2 \cdot y \cdot \cos(x) + e^x \cdot \sin(y);$
  $\qquad f5 := (x, y) \to 2\, y \cos(x) + e^x \sin(y)$

It is noted that the solution is implicitly given in the form $f(x, y) = c$. In figure 19, **contourplot** has been used to see what the solution curves look like. For the ranges of x and y, it has been taken $x = -3 \dots 3$ and $y = -3 \dots 3$.
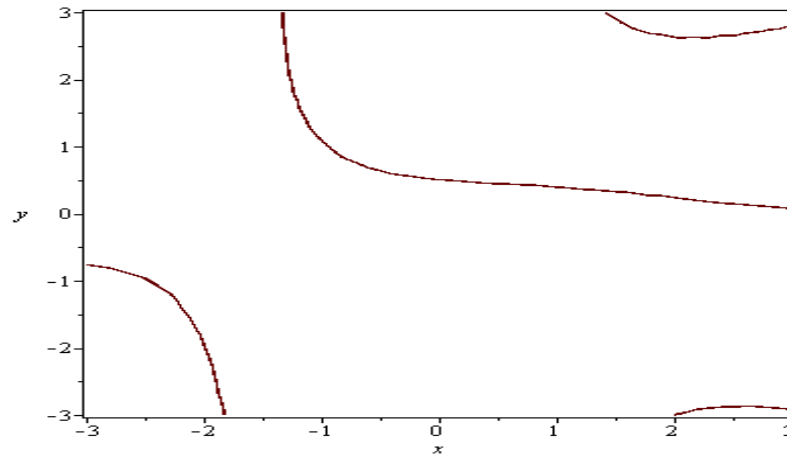
> $with(\,plots\,) :$
> $contourplot(\,f5(x, y), x = -3\,..3, y = -3\,..3);$
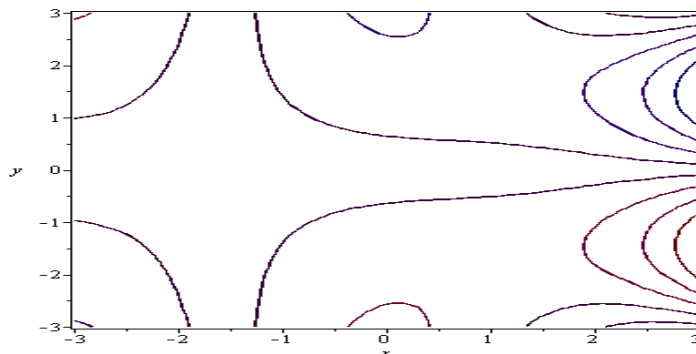
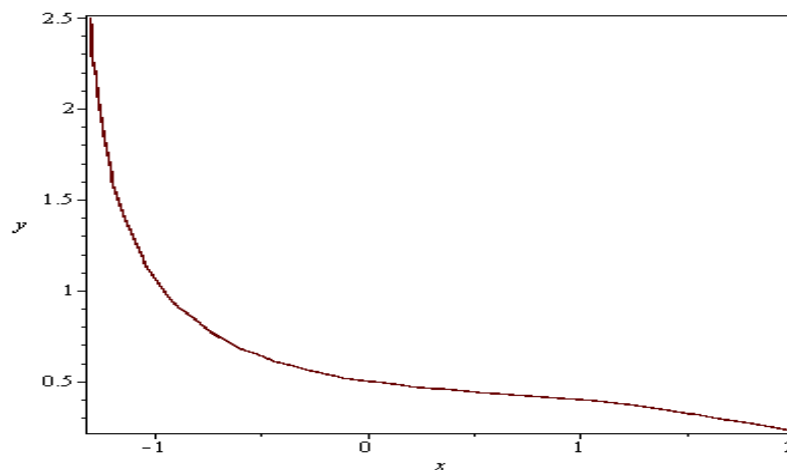**Figure 19:** Solution curves of Exercise B.5(b) with $-3 \le x \le 3,\ -3 \le y \le 3$

In figure 20, **implicitplot** has been used to graph the solution satisfying the initial condition $y(0) = 0.5$. The graph shows several curves, one of them is the solution, given in figure 21.

> $implicitplot(f5(x, y) = f5(0, 0.5), x = -3..3, y = -3..3);$

Figure 20: Probable solution curves of Exercise B.5(c) with $-3 \leq x \leq 3$, $-3 \leq y \leq 3$

> $implicitplot(f5(x, y) = f5(0, 0.5), x = -1.8..2, y = -3..2.5);$

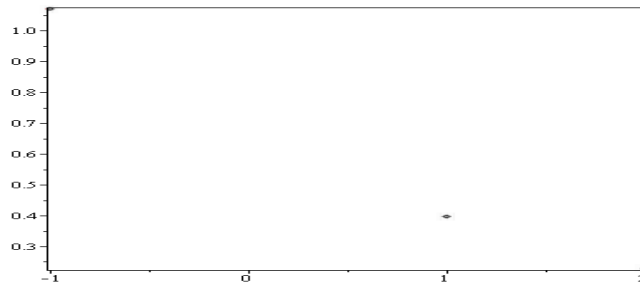**Figure 21:** Solution curve of Exercise B.5(c) with $-1.8 \leq x \leq 2$, $0 \leq y \leq 2.5$

Given a value x_1, to find $y(x_1)$ it is necessary to find the solution of $f(x_1, y) = f(0, 0.5)$ (viewed as an equation in y) near $y = 0.5$. This can be done with the **fsolve** command. It's found $y(-1)$, $y(1)$, $y(2)$. In figure 22 these values are marked on a graph.

> $fsolve(f5(-1, y) = f5(0, 0.5), y);$
$\qquad$ 1.070377493
> $fsolve(f5(1, y) = f5(0, 0.5), y);$
$\qquad$ 0.3968306836
> $fsolve(f5(2, y) = f5(0, 0.5), y);$
$\qquad$ 0.2278495202

> $PLOT(\text{POINTS}([[-1, fsolve(f5(-1, y) = f5(0, 0.5), y)], [1,$
$fsolve(f5(1, y) = f5(0, 0.5), y)], [2, fsolve(f5(2, y) = f5(0, 0.5),$
$y)]], \text{SYMBOL}(\text{DIAMOND})));$

Figure 22: Some points of the solution curve of Exercise B.5(d)

3.6. In this problem the continuous dependence of the solutions with respect to the initial data is studied. Finding the solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y}{1+x^2}, \quad y(0) = c.$$

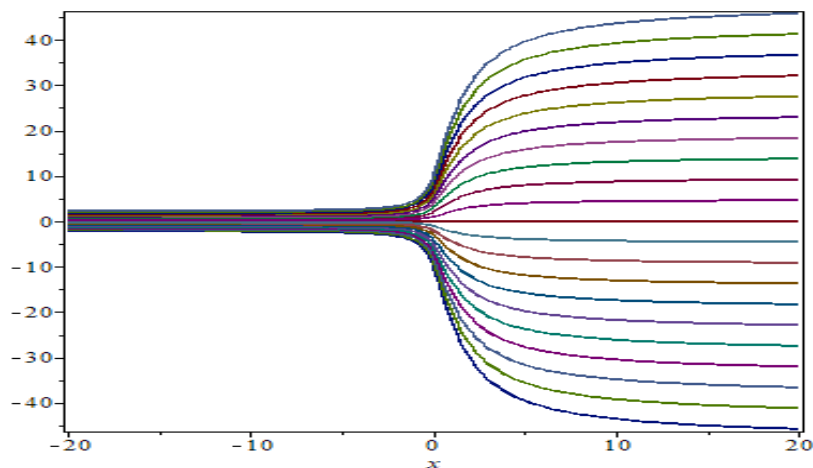> $sol6 := dsolve\left(\left\{diff(y(x), x) = \frac{y(x)}{1+x^2}, y(0) = c\right\}, y(x)\right);$

$$sol6 := y(x) = c\,e^{\arctan(x)}$$

> $y6 := unapply(rhs(sol6), x, c);$
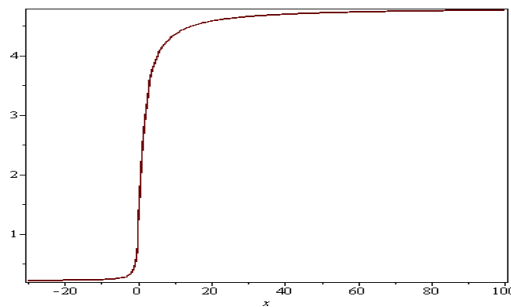
$$y6 := (x, c) \rightarrow c\,e^{\arctan(x)}$$

It is denoted as y_c the solution of the given problem. In figure 23, for $c = -10$, -9,…, -1, 0, 1, the solutions y_c are graphed together on the interval $-20 \le x \le 20$. Figure 24 shows the solution curve on the interval $[-30,\ 100]$.

> $plot(\{y6(x, c)\$ \ c = -10..10\}, x = -20..20);$

Figure 23: Solution curves of Exercise B.6(b) on the interval $\left[-20, \; 20\right]$

> $plot(\,y6(x,\,1),\,x=-30\,..100)\,;$

Figure 24: Solution curve y_1 of Exercise B.6(b) on the interval $\left[-30, \; 100\right]$

Computing $\lim_{x\to\pm\infty} y_1(x)$:

> $\lim_{x\,\to\,+\,\infty} y6(x,\,1)\,;$

$$e^{\frac{1}{2}\,\pi}$$

> $evalf\!\left(\exp\!\left(\dfrac{\pi}{2}\right)\right)\,;$

$$4.810477382$$

> $\lim_{x\,\to\,-\,\infty} y6(x,\,1)\,;$

$$e^{-\frac{1}{2}\,\pi}$$

> $evalf\!\left(\exp\!\left(-\dfrac{\pi}{2}\right)\right)\,;$

$$0.2078795763$$

For any pair of numbers a, b and $x \in \tilde{}$:

$$\left|y_a(x) - y_b(x)\right| = \left|ae^{\tan^{-1}x} - be^{\tan^{-1}x}\right| = \left|(a-b)e^{\tan^{-1}x}\right| \le L\left|a-b\right|.$$

> $plot\!\left(\exp\!\left(\tan^{-1}(x)\right),\,x=-30\,..100\right)\,;$
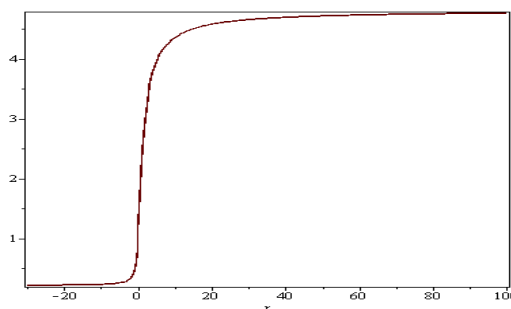
Figure 25: Graph of the function $e^{\tan^{-1}(x)}$ from Exercise B.6(d) on the interval $\left[-30, \; 100\right]$

According to figure 25 it can be taken $L = 5$, $\forall x \in \tilde{}$, and $L = 1$, $\forall x < 0$.

5. Conclusions

In this research work, the production of two worksheets in *Maple* was achieved, which combine text with the numerical, symbolic and graphic output of the mathematical software. The first sheet contains an introduction to the basic *Maple* commands, and solves the set of exercises and problems called *Problem Set A*: *Practice with Maple*, allowing to master the basic skills necessary to work with the software. With this set of problems, a minimum level of competence was reached that allowed *Maple* to be used throughout the research.

Through *Maple* it has been possible to discuss the existence, uniqueness and stability of solutions of the differential equations considered in the data record: *Problem Set B*: *First Order Equations*, topics that are fundamental in the theory and application of these equations. With the help of *Maple*, in each of the problems solved, it has been possible to explore, describe, analyze and interpret solutions of each intervening model, in view of *Maple*'s simple and interactive interface, thus prioritizing the analysis of the concepts involved and the resulting solutions.

Through *Maple*, it has been possible to implement quantitative and qualitative methods in first-order ordinary differential equations, visualized algebraically, numerically and graphically. In particular, a qualitative approach to the study of these equations has been considered. With this approach, qualitative information about the solutions is obtained directly from the equation, without the use of a formula for the solution.

Due to the great symbolic capacity of the software, it is concluded that *Maple* is very powerful for the quantitative and qualitative analysis of first-order ordinary differential equations. The *Maple* symbolic calculation software made it possible to efficiently carry out calculations and present solutions algebraically, numerically and graphically, so that the researcher can spend more time assimilating the concepts and analyzing the problems.

The use of *Maple* has contributed, according to this research work, to a better understanding of the mathematical models, given that the researcher can focus his attention on the phases of approach, formalization and "concretion" of the different problems raised. It is concluded that the use of *Maple* allows the resolution of numerous problems formulated in terms of first-order ordinary differential equations and constitutes a fundamental tool in scientific research.

References

[1] D.G. Zill, A First Course in Differential Equations with Modeling Applications, 12th. ed., Cengage Learning, New York, NY, 2018.
[2] J. Collantes, F. Concha and B. Chiné, Axial symmetric flow model for a flat bottom hydrocyclone, Chemical Engineering Journal 80 (2000) 257–265. doi:10.1016/S1383-5866(00)00099-X.
[3] I. Belyaeva, I. Kirichenko, O. Ptashnyi, N. Chekanova and T. Yarkho, Integrating linear ordinary fourth-order differential equations in the MAPLE programming environment, Eastern-European Journal of Enterprise Technologies 3(4(111)) (2021) 51-57. doi:10.15587/1729-4061.2021.233944.
[4] K.R. Coombes, B.R. Hunt, R.L. Lipsman, J.E. Osborn and G.J. Stuck, Differential Equations with Maple, 2nd. ed., Wiley, New York, NY, 1997.
[5] G.M. Ortigoza, Resolviendo ecuaciones diferenciales ordinarias con Maple y Mathematica, Revista Mexicana de Física, 53(2) (2007) 155-167.
Disponible en: <http://www.scielo.org.mx/scielo.php?script=sci_arttext&pid=S1870-35422007000200004&lng=es&nrm=iso>.
[6] G.M. Ortigoza, Animaciones en Matlab y Maple de ecuaciones diferenciales parciales de la física-matemática, Revista Mexicana de Física, E 53(1) (2007) 56-66.

Disponible en: <http://www.scielo.org.mx/scielo.php?script=sci_arttext&pid=S1870-35422007000100008&lng=es&nrm=iso>.

[7] W.E. Boyce and R.C. DiPrima, Elementary Differential Equations, 10th. ed., Wiley, New York, NY, 2012.