Conceptual Identification Within the Decomposition of Fuzzy Homogeneous Classes of Objects

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Abstract

Conceptual identification of fuzzy knowledge is one of the important knowledge-processing methods, which can be used for such tasks as concept matching, computation of concept similarity, re-engineering of conceptual hierarchies, etc. Since wildly used approaches to conceptual identification, which are based on the formal concept analysis and fuzzy formal concept analysis, do not consider the internal semantic dependencies among the attributes, it may lead to the construction of semantically inconsistent concepts. Therefore, in this paper, we propose a new approach to the conceptual identification of fuzzy knowledge within the decomposition of nodes of fuzzy object-oriented dynamic networks. The decomposition of fuzzy homogeneous classes of objects is considered the space for the identifying their fuzzy sub-concepts within the corresponding identification lattice. To implement the proposed approach, we developed the algorithm for identifying semantically consistent subclasses of fuzzy homogeneous classes of objects. The algorithm constructs a semantically consistent lattice of fuzzy class subclasses and discovers all subclasses and superclasses for a selected fuzzy class subclass, creating a corresponding identification lattice. In addition, we introduce a notion of a subclass neighborhood within its identification lattice, which allows the consideration of a conceptual locus of the subclass instead of the subclass itself. It makes it possible to operate with subclasses of a fuzzy class in a broader sense, calculating their similarities and differences. To explain the proposed approach, we have provided a detailed example of the conceptual identification of a particular fuzzy homogeneous class of objects, demonstrating the application of the developed algorithm.

Keywords

Fuzzy knowledge identification, fuzzy class identification, fuzzy concept identification, fuzzy class decomposition

1. Introduction

Nowadays, conceptual (class) hierarchies are the most common complex knowledge representation structure within modern object-oriented knowledge-based systems and programming languages. It provides an opportunity to formalize a particular domain via constructing a corresponding hierarchy of concepts that encapsulates the representation of concepts themselves and the relations among them. The knowledge-based systems can use such hierarchies for conceptual knowledge processing, including *representation*, *analysis*,

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classification, *integration*, *identification*, *retrieval*, *inferring*, and *transferring*. Concept identification is one of the important tasks related to knowledge analysis, integration, retrieval, and inferring since it allows the system to detect a place of particular concepts and how they are connected with other concepts in the hierarchy. Consequently, a concept can be considered not only as a single node from the hierarchy but also as its neighborhood or sub-hierarchy, which includes some number of adjacent nodes and relations among them. Using such a representation, the system can operate by concepts in the broader meaning, for example, for computation of similarity or difference of certain concepts from the same hierarchy or a few different hierarchies. In addition, a hierarchical neighborhood of a concept can be used to reduce a hierarchy representation and consequently the search space during the concept retrieval or inferring, as well as for detecting the best place for integrating a new concept into a hierarchy.

Conceptual identification has a few interpretations depending on the specifics of a certain hierarchy, the nature of the concepts, and the relations among them. Since sometimes concepts themselves, as well as the relations among them, can be vague and imprecise, knowledge-based systems should be able to perform the identification of fuzzy concepts. Therefore, in this paper, we study the identification of fuzzy concepts in fuzzy object-oriented dynamic networks, considering the decomposition of fuzzy homogeneous classes of objects, which are nodes of the networks, as spaces for the identification of fuzzy sub-concepts. As a result, we propose a new approach to identifying semantically consistent fuzzy sub-concepts of fuzzy homogeneous classes of objects.

The paper has the following structure. Section 2 contains the analysis of the main approaches to the conceptual identification of fuzzy knowledge. Section 3 presents a morphological analysis of a particular fuzzy homogeneous class of objects. Section 4 provides an approach to reducing the space for identifying fuzzy subclasses via the semantically consistent decomposition of the fuzzy homogeneous class of objects. Section 5 presents the algorithm for identifying fuzzy sub-concepts and an example of its application to identifying semantically consistent subclasses of the fuzzy homogeneous class of objects. The conclusions and acknowledgments sections finish the paper.

2. Conceptual Identification

Nowadays, one of the most common approaches to formal representation of concept hierarchies is a *formal concept analysis (FCA)* proposed by Ganter and Wille in [5]. It is a powerful framework that focuses on representing concept hierarchies in terms of so-called, *concept lattices*. Such representation of the essences from a chosen domain consists of three main stages. Firstly, to define a *formal context* for the domain by constructing the cross-table with a set of attributes of domain entities as columns and the set of objects, which model their particular instances, as rows. The cells of such a table usually contain a Boolean value meaning that a particular object has or does not have a corresponding attribute. Secondly, to define *formal concepts* within the formal context by constructing a collection of pairs of appropriate *extents* and *intents*, where the extent is a set of common attributes for the particular set of objects, while the intent is a set of objects with common attributes. And finally, to build a corresponding *concept lattice* by constructing a complete lattice of objects and a complete lattice of attributes, which are isomorphic to each other. The lattice structure assumes that a set of formal concepts is a partially ordered set with defined sub-concept-super-concept relations. Later, the FCA framework was extended for the representation of fuzzy domains (FFCA), consequently, notions of *fuzzy formal context*, *fuzzy formal concepts*, and *fuzzy concept lattice* were introduced [3, 11]. In contrast to the crisp formal concepts, the fuzzy ones are defined using the *confidence threshold*, which allows modeling the membership measure of a particular attribute for a certain object, and a certain object for the set of objects. Constructing a concept lattice or fuzzy concept lattice for a particular domain creates its lattice-based formal model that can be used for conceptual identification.

According to [20], conceptual identification is the detection of the taxonomic position of a particular object within a certain classification. In the case of FCA/FFCA, the concept lattice is used as such classification, therefore identification of a concept transforms into the detection of sub-concepts and super-concepts within the lattice. One of the commonly used approaches to conceptual identification is *rule-based identification*. The main idea of the approach is to define a system of implication rules extracting them from the defined formal context and corresponding concept lattice. In general, an implication rule can be defined in the form $P \rightarrow Q$, where P and Q are subsets of attributes of the set of all tributes used to determine a formal context or a fuzzy formal context. In the case of FCA, an implication rule can be interpreted in the following way – if an object has all attributes from the set P, it also has all attributes from the set Q. This approach was used to identify: a set of professional competencies that can help people successfully take a new position when professional retraining or changing jobs [15]; conservative access patterns, minimum behavior patterns, and canonical access patterns in two-mode social networks [13]. In the case of FFCA, an implication rule can be interpreted as follows – if a fuzzy object has all fuzzy attributes from the set P to the corresponding degree, then it also has all fuzzy attributes from the set Q to the corresponding degree. This version of the approach was used to identify: differential diagnoses for patients by a conversational recommender system [2]; causes and consequences of customer complaints within customer relationship management in financial services helping managers to accommodate the required dynamic changes according to customer expectations [14]; exceptional or suspicious cases specific to the event logs, NTFS file system, the Windows operating system, or a type of anomaly, to provide warnings for the security analysts [16-17]. However, the approach assumes the rules extraction from the formal context and corresponding concept lattice analyzing sub-concept and super-concept relations. In the case of big formal contexts, this task becomes more complicated from the computational perspective. Moreover, to identify specific concepts within a concept lattice, the system must discover in the set of rules those rules that are associated with these concepts, including all transitive rules.

Another approach to conceptual identification is the *multi-stage intersection identification* of formal concepts. The approach involves extracting new concepts via the sequence-based intersection of formal concepts within a constructed formal context. The discovered hidden concepts are identified and then integrated into the classification, where identification of the concepts means retrieval of their sub-concepts and super-concepts.

Such integration extends the initial formal context enriching it with previously non-obvious or hidden concepts. The approach was used to detect missing or hidden concepts and improve the completeness of concept coverage in biomedical terminologies NCI Thesaurus and SNOMED CT [23-24]. However, the larger the size of the formal context, the more difficult identification becomes due to the increasing number of intersections being calculated.

One more approach to conceptual identification is the *criterion-based identification* of a group of concepts. The main idea of the approach is to identify a group of concepts within a formal context or fuzzy formal context, which satisfies particular identification criteria. The FCA-version of the approach was used to identify: key nodes in massive networks using cross-face scalable centrality measure [9]; key nodes in a two-mode network, using bi-face bipartite centrality measure [10]; diversified top-k maximal clique in a social Internet of things [8]; dynamic maximal clique in online social networks [21]; user-friendly communities in signed social networks and γ-quasi-cliques for closely related users within them [22]; key structures from social networks [6]. The FFCA-version of the approach was used for the identification of location-based and content-based communities of users in social networks [4]; skyline (λ, k)-cliques in a fuzzy attributed social network [7]; cloud services in collaborative filtering-based recommendation system [12].

Each of the considered approaches implements a specific strategy for solving the problem of conceptual identification based on FCA/FFCA. In all the mentioned FCA/FFCA applications, the formal context and the fuzzy formal context were constructed using a set of objects and a set of attributes. However, this approach has one major drawback: if we consider objects as instances of a particular class of objects, this means, that they are encapsulated containers for storing data, and we do not see how their attributes are defined relative to each other. It is known that the attributes of all class objects are defined at the class level, and, as was shown in [18-19], some attributes (properties and methods) of a class may have internal semantic dependencies on other attributes. It is crucial for the semantic consistency of formal concepts within the constructed concept lattice or fuzzy concept lattice because the construction of new formal concepts is based on the set-theoretical intersection of extents ignoring internal semantic dependencies among the attributes [19]. Consequently, some constructed formal concepts may be semantically inconsistent and, therefore, physically impossible or unrealistic in a modeled domain. Another feature of FCA/FFCA is the fact that attributes are treated only as properties of objects, not as methods defined within an object class, and can be executed on all objects to change their state and attribute values. Therefore, we propose an alternative lattice-based approach to the conceptual identification of fuzzy knowledge, based on the analysis of internal semantic dependencies between the attributes (fuzzy properties and fuzzy methods) of fuzzy homogeneous classes of objects. In addition, we introduce the notion of concept' neighborhood, which allows the consideration of some subclass and superclass locus within a concept lattice instead of the single concept.

3. Fuzzy Concepts Morphology

To explain our approach to conceptual identification of fuzzy knowledge, we use the fuzzy homogeneous classes of objects which are nodes of fuzzy object-oriented dynamic networks. The formalization of internal semantic dependencies among the attributes (properties and methods) of fuzzy homogeneous classes of objects was introduced in [18]. For this purpose, the abstract model of chemical atoms and molecules was used, according to which, atoms are indivisible particles and molecules are the union of atoms and (or) smaller molecules. This model can be interpreted by attributes defined independently of other fuzzy class attributes (*fuzzy atoms*) and attributes defined based on them (*fuzzy molecules*).

Definition 1. *A fuzzy atom of a fuzzy homogeneous class of objects* $T/M(T)$ *is a singleton collection*

$$A_i(T/M(T)) = \{T.x_i / \mu(T.x_i)\},$$

where $T.x_i / \mu(T.x_i) \in P(T)/M(P(T)) \cup F(T)/M(F(T))$ *is a crisp or fuzzy property or a method defined without using any other properties and (or) methods of the fuzzy class* $T/M(T)$, *where* $P(T)/M(P(T))$ *and* $F(T)/M(F(T))$ *are collections of its properties and methods, respectively.*

Definition 2. *A fuzzy molecule of a fuzzy homogeneous class of objects* $T/M(T)$ *is a collection*

$$M_i(T/M(T)) = \{T.x_i / \mu(T.x_i), T.y_{j_1} / \mu(T.y_{j_1}), \ldots, T.y_{j_n} / \mu(T.y_{j_n})\},$$

where $T.x_i / \mu(T.x_i) \in P(T)/M(P(T)) \cup F(T)/M(F(T))$, *and* $1 \le i \le |T/M(T)|$ *is a crisp or fuzzy property or a method defined based on the other methods and (or) properties* $T.y_{j_1} / \mu(T.y_{j_1}), \ldots, T.y_{j_n} / \mu(T.y_{j_n}) \in P(T)/M(P(T)) \cup F(T)/M(F(T))$ *which are crisp or fuzzy atoms and (or) parts of smaller fuzzy molecules of the fuzzy class* $T/M(T)$, *where* $1 \le j_1 \le \ldots \le j_n \le |T/M(T)|$, *where* $P(T)/M(P(T))$ *and* $F(T)/M(F(T))$ *are collections of properties and methods of the fuzzy class* $T/M(T)$, *respectively.*

Fuzzy atoms and fuzzy molecules of a fuzzy homogeneous class of objects, together determine its internal semantic dependencies.

Definition 3. *Internal semantic dependencies of a fuzzy homogeneous class of objects* $T/M(T)$ *is a set of its atoms and molecules, i.e.*

$$ISD(T/M(T)) = \{A_1(T/M(T)), \ldots, A_n(T/M(T)),$$
$$M_1(T/M(T)), \ldots, M_m(T/M(T))\}$$

where $A_i(T/M(T))$, $i = \overline{1,n}$ *are fuzzy atoms of the fuzzy class* $T/M(T)$, *while* $M_j(T/M(T))$, $j = \overline{1,m}$ *are its fuzzy molecules, respectively.*

Now, let us consider an example of a fuzzy homogeneous class of objects $Gp/0.9$, which defines the concept of a geographic place and has the following structure:

$$Gp\big(p_1 = (latitude, (\phi, \mathbb{R}))/0.91,\ p_2 = (longitude, (\lambda, \mathbb{R}))/0.91,$$
$$p_3 = (name, (n \in V_n, str))/0.71,\ p_4 = (region, (r \in V_r, str))/0.78,$$
$$f_1 = get\_latitude(gp, \mathbb{R})/0.95,\ f_2 = get\_longitude(gp, \mathbb{R})/0.95,$$
$$f_3 = get\_name(gp, str)/1,\ f_4 = get\_region(gp, str)/1\big)/0.9$$

where $Gp.p_1/0.91$ and $Gp.p_2/0.91$ are fuzzy quantitative properties, which mean the latitude and longitude of the geographic place $gp/\mu(gp)$ defined in degrees, i.e.

$$gp.latitude = \phi/\mu(\phi),\ gp.longitude = \lambda/\mu(\lambda);$$

$Gp.p_3/0.71$ is a fuzzy quantitative property which means the name of the geographic place $gp/\mu(gp)$, and is defined by the following fuzzy set

$$V_n = \{official\_name/0.95,\ historical\_name/0.74,\ regional\_name/0.62,$$
$$local\_name/0.55\},$$

where elements of the fuzzy set V_n are corresponding names of the geographic place $gp/\mu(gp)$, i.e.

$$gp.name = n/\mu(n) \in V_n;$$

$Gp.p_4/0.78$ is a fuzzy quantitative property, which means the region name where the geographic place $gp/\mu(gp)$ is located, and is defined by the following fuzzy set

$$V_r = \{official\_name/0.95,\ historical\_name/0.73,\ local\_name/0.56\},$$

where elements of the fuzzy set V_r are corresponding names of the region where the geographic place $gp/\mu(gp)$ is located, i.e.

$$gp.region = r/\mu(r) \in V_r;$$

$Gp.f_1/0.95$, $Gp.f_2/0.95$, $Gp.f_3/1$, and $Gp.f_4/1$ are fuzzy methods, which return values of corresponding properties of the geographic place $gp/\mu(gp)$, i.e.

$$gp.get\_latitude() \to round(gp.latitude,\ 1),$$
$$gp.get\_longitude() \to round(gp.longitude,\ 1),$$
$$gp.get\_name() \to gp.name,$$
$$gp.get\_region() \to gp.region.$$

Let us use the fuzzy homogeneous class of objects $Gp/0.9$ to define another fuzzy homogeneous class of objects $Tr/0.95$, which determines the concept of transfer from one geographic place to another and has the following structure:

$$\begin{aligned}Tr\Big(&p_1=\big(place_1,\ (gp_1,\ Gp)\big)/1,\ \ p_2=\big(place_2,\ (gp_2,\ Gp)\big)/1,\\ &p_3=\big(distance,\ (dst,\ km)\big)/0.94,\ \ p_4=\big(transport,\ (t\in V_t,\ str)\big)/0.81,\\ &p_5=\big(duration,\ (dr,\ h)\big)/0.87,\ \ p_6=\big(price,\ (p\in V_p,\ UAH)\big)/0.91,\\ &f_1=get\_place_1(tr,\ GP)/1,\ \ f_2=get\_place_2(tr,\ GP)/1,\\ &f_3=get\_distance(tr,\ km)/0.98,\ \ f_4=get\_transport(tr,\ str)/1,\\ &f_5=get\_duration(tr,\ h)/0.94,\ \ f_6=get\_price(tr,\ UAH)/0.96\Big)/0.95\end{aligned}$$

where $Tr.p_1/1$ and $Tr.p_2/1$ are fuzzy quantitative properties, which define two different geographic places for the transfer $tr/\mu(tr)$ between them, i.e.

$$tr.place_1=gp_1/\mu(gp_1),\ \ tr.place_2=gp_2/\mu(gp_2);$$

$Tr.p_3/0.94$ is a fuzzy quantitative property, which means a distance between two geographic places $Tr.p_1/1$ and $Tr.p_2/1$, and is defined in the following way

$$tr.distance=\sqrt{d_1+d_2},$$
$$d_1=\left(tr.place_1.get\_latitude()-tr.place_2.get\_latitude()\right)^2,$$
$$d_1=\left(tr.place_1.get\_longitude()-tr.place_2.get\_longitude()\right)^2;$$

$Tr.p_4/0.81$ is a fuzzy quantitative property, which means a kind of transport for a transfer between two geographic places and is defined as the following fuzzy set

$$V_t=\{bus/0.82,\ train/0.67,\ plane/0.93\},$$

where elements of the fuzzy set V_t are possible kinds of transport for a transfer, i.e.

$$tr.transport=t/\mu(t)\in V_t;$$

$Tr.p_5 / 0.87$ is a fuzzy quantitative property, which means duration of the transfer between two geographic places and is defined in the following way

$$tr.duration = \begin{cases} D_b,\ if\ tr.transport = bus / 0.82, \\ D_t,\ if\ tr.transport = train / 0.67, \\ D_p,\ if\ tr.transport = plane / 0.93, \end{cases}$$

where $D_b = \left\{ x_i^- / \mu\left(x_i^-\right),\ duration_b / 1,\ x_i^+ / \mu\left(x_i^+\right) \right\},\ i = \overline{1,\ldots}$ is a fuzzy set, such that

$$\frac{tr.distance}{120} < duration_b < \frac{tr.distance}{80},$$

$$x_i^- = duration_b - 4 \cdot i,\ \frac{tr.distance}{120} < duration_b - 4 \cdot i < duration_b,$$

$$\mu\left(x_i^-\right) = \frac{x_i^- \cdot 120 - tr.distance}{duration_b \cdot 120 - tr.distance} - \delta_i^-,\ \ \delta_i^- = 1 - \mu\left(x_i^-\right) - \nu\left(x_i^-\right),\ \ \nu\left(x_i^-\right) = 1 - \mu\left(x_i^-\right),$$

$$x_i^+ = duration_b + 4 \cdot i,\ duration_b < duration_b + 4 \cdot i < \frac{tr.distance}{80},$$

$$\mu\left(x_i^+\right) = \frac{tr.distance - x_i^+ \cdot 80}{tr.distance - duration_b \cdot 80} - \delta_i^+,\ \ \delta_i^+ = 1 - \mu\left(x_i^+\right) - \nu\left(x_i^+\right),\ \ \nu\left(x_i^+\right) = 1 - \mu\left(x_i^+\right),$$

$D_t = \left\{ x_i^- / \mu\left(x_i^-\right),\ duration_t / 1,\ x_i^+ / \mu\left(x_i^+\right) \right\},\ i = \overline{1,\ldots}$ is a fuzzy set, such that

$$\frac{tr.distance}{80} < duration_t < \frac{tr.distance}{50},$$

$$x_i^- = duration_t - 3 \cdot i,\ \frac{tr.distance}{80} < duration_t - 3 \cdot i < duration_t,$$

$$\mu\left(x_i^-\right) = \frac{x_i^- \cdot 80 - tr.distance}{duration_t \cdot 80 - tr.distance} - \delta_i^-,\ \ \delta_i^- = 1 - \mu\left(x_i^-\right) - \nu\left(x_i^-\right),\ \ \nu\left(x_i^-\right) = 1 - \mu\left(x_i^-\right),$$

$$x_i^+ = duration_t + 3 \cdot i,\ duration_t < duration_t + 3 \cdot i < \frac{tr.distance}{50},$$

$$\mu\left(x_i^+\right) = \frac{tr.distance - x_i^+ \cdot 50}{tr.distance - duration_t \cdot 50} - \delta_i^+,\ \ \delta_i^+ = 1 - \mu\left(x_i^+\right) - \nu\left(x_i^+\right),\ \ \nu\left(x_i^+\right) = 1 - \mu\left(x_i^+\right),$$

and $D_p = \left\{ x_i^- / \mu\left(x_i^-\right),\ duration_p / 1,\ x_i^+ / \mu\left(x_i^+\right) \right\},\ i = \overline{1,\ldots}$ is a fuzzy set, such that

$$\frac{tr.distance}{500} < duration_p < \frac{tr.distance}{450},$$

$$x_i^- = duration_p - 5 \cdot i,\ \frac{tr.distance}{500} < duration_p - 5 \cdot i < duration_p,$$

$$\mu\left(x_i^-\right)=\frac{x_i^-\cdot 500-tr.distance}{duration_p\cdot 500-tr.distance}-\delta_i^-,\ \ \delta_i^-=1-\mu\left(x_i^-\right)-\nu\left(x_i^-\right),\ \ \nu\left(x_i^-\right)=1-\mu\left(x_i^-\right),$$

$$x_i^+=duration_p+5\cdot i,\ \ duration_p<duration_p+5\cdot i<\frac{tr.distance}{450},$$

$$\mu\left(x_i^+\right)=\frac{tr.distance-x_i^+\cdot 450}{tr.distance-duration_p\cdot 450}-\delta_i^+,\ \ \delta_i^+=1-\mu\left(x_i^+\right)-\nu\left(x_i^+\right),\ \ \nu\left(x_i^+\right)=1-\mu\left(x_i^+\right);$$

$Tr.p_6\,/\,0.91$ is a fuzzy quantitative property, which means a price of a transfer between two geographic places and is defined as follows

$$tr.price=\begin{cases}P_b,\ if\ tr.transport=bus\,/\,0.82,\\ P_t,\ if\ tr.transport=train\,/\,0.67,\\ P_p,\ if\ tr.transport=plane\,/\,0.93;\end{cases}$$

where $P_b=\left\{x_i^-/\mu\left(x_i^-\right),\ price_b/1,\ x_i^+/\mu\left(x_i^+\right)\right\}$, $i=\overline{1,\ldots}$ is a fuzzy set, such that

$$tr.distance\cdot 20<price_b<tr.distance\cdot 60,$$

$$x_i^-=price_b-4\cdot i,\ \ tr.distance\cdot 20<price_b-4\cdot i<price_b,$$

$$\mu\left(x_i^-\right)=\frac{x_i^--tr.distance\cdot 20}{price_b-tr.distance\cdot 20}-\delta_i^-,\ \ \delta_i^-=1-\mu\left(x_i^-\right)-\nu\left(x_i^-\right),\ \ \nu\left(x_i^-\right)=1-\mu\left(x_i^-\right),$$

$$x_i^+=price_b+4\cdot i,\ \ price_b<price_b+4\cdot i<tr.distance\cdot 60,$$

$$\mu\left(x_i^+\right)=\frac{tr.distance\cdot 60-x_i^+}{tr.distance\cdot 60-price_b}-\delta_i^+,\ \ \delta_i^+=1-\mu\left(x_i^+\right)-\nu\left(x_i^+\right),\ \ \nu\left(x_i^+\right)=1-\mu\left(x_i^+\right),$$

$P_t=\left\{x_i^-/\mu\left(x_i^-\right),\ price_t/1,\ x_i^+/\mu\left(x_i^+\right)\right\}$, $i=\overline{1,\ldots}$ is a fuzzy set, such that

$$tr.distance\cdot 15<price_t<tr.distance\cdot 35,$$

$$x_i^-=price_t-2\cdot i,\ \ tr.distance\cdot 15<price_t-2\cdot i<price_t,$$

$$\mu\left(x_i^-\right)=\frac{x_i^--tr.distance\cdot 15}{price_t-tr.distance\cdot 15}-\delta_i^-,\ \ \delta_i^-=1-\mu\left(x_i^-\right)-\nu\left(x_i^-\right),\ \ \nu\left(x_i^-\right)=1-\mu\left(x_i^-\right),$$

$$x_i^+=price_t+2\cdot i,\ \ price_t<price_t+2\cdot i<tr.distance\cdot 35,$$

$$\mu\left(x_i^+\right)=\frac{tr.distance\cdot 35-x_i^+}{tr.distance\cdot 35-price_t}-\delta_i^+,\ \ \delta_i^+=1-\mu\left(x_i^+\right)-\nu\left(x_i^+\right),\ \ \nu\left(x_i^+\right)=1-\mu\left(x_i^+\right),$$

and $P_p=\left\{x_i^-/\mu\left(x_i^-\right),\ price_p/1,\ x_i^+/\mu\left(x_i^+\right)\right\}$, $i=\overline{1,\ldots}$ is a fuzzy set, such that

$$tr.distance\cdot 55<price_p<tr.distance\cdot 85,$$

$$x_i^- = price_p - 3\cdot i,\ \ tr.distance\cdot 55 < price_p - 3\cdot i < price_p,$$

$$\mu\left(x_i^-\right) = \frac{x_i^- - tr.distance\cdot 55}{price_p - tr.distance\cdot 55} - \delta_i^-,\ \ \delta_i^- = 1 - \mu\left(x_i^-\right) - \nu\left(x_i^-\right),\ \ \nu\left(x_i^-\right) = 1 - \mu\left(x_i^-\right),$$

$$x_i^+ = price_p + 3\cdot i,\ \ price_p < price_p + 3\cdot i < tr.distance\cdot 85,$$

$$\mu\left(x_i^+\right) = \frac{tr.distance\cdot 85 - x_i^+}{tr.distance\cdot 85 - price_p} - \delta_i^+,\ \ \delta_i^+ = 1 - \mu\left(x_i^+\right) - \nu\left(x_i^+\right),\ \ \nu\left(x_i^+\right) = 1 - \mu\left(x_i^+\right);$$

$Tr.f_1/1$, $Tr.f_2/1$, $Tr.f_3/0.98$, $Tr.f_4/1$, $Tr.f_5/0.94$, and $Tr.f_6/0.96$ are fuzzy methods, which return values of corresponding properties of the transfer $tr/\mu(tr)$, i.e.

$$tr.get\_place_1() \to tr.place_1,$$
$$tr.get\_place_2() \to tr.place_2,$$
$$tr.get\_distance() \to round\left(tr.distance,\ 0\right),$$
$$tr.get\_transport() \to tr.transport,$$
$$tr.get\_duration() \to tr.duration,$$
$$tr.get\_price() \to tr.price.$$

To enrich the example, let us define a fuzzy homogeneous class of objects $Jrn/0.87$, which determines the concept of a journey through the sequence of geographic places and has the following structure:

$$\begin{aligned} Jrn\Big(& p_1 = \left(transfers,\ \left(\left(tr_1,\ Tr\right),\ldots,\left(tr_n,\ Tr\right)\right)\right)/0.93,\ \ p_2 = \left(distannce,\ \left(dst,\ km\right)\right)/0.88 \\ & p_3 = \left(duration,\ \left(dr,\ h\right)\right)/0.79,\ \ p_4 = \left(price,\ \left(p,\ UAH\right)\right)/0.84, \\ & f_1 = get\_transfer\left(jrn,\ i,\ Tr\right)/1,\ \ f_2 = get\_distance\left(jrn,\ km\right)/0.91, \\ & f_3 = get\_duration\left(jrn,\ h\right)/0.89,\ \ f_4 = get\_price\left(jrn,\ UAH\right)/0.82, \\ & f_5 = compute\_discount\left(jrn,\ UAH\right)/0.78\Big)/0.87 \end{aligned}$$

where $Jrn.p_1/0.93$ is a fuzzy quantitative property, defining a sequence of transfers between different geographic places in the scope of the journey $jrn/\mu(jrn)$, i.e.

$$jrn.transfers = \left(tr_1/\mu\left(tr_1\right),\ldots,tr_n/\mu\left(tr_n\right)\right),$$

such that $tr_i.place_2 = tr_{i+1}.place_1$, $i = \overline{1, n-1}$; $Jrn.p_2/0.88$ is a fuzzy quantitative property, meaning the total distance of the transfer, during the journey $jrn/\mu(jrn)$, and is defined in the following way

$$jrn.distance = \sum_{i=1}^{n} jrn.transfers[i].get\_distance();$$

$Jrn.p_3/0.79$ is a fuzzy quantitative property, meaning the total duration of the transfer, during the journey $jrn/\mu(jrn)$, and is defined in the following way

$$jrn.duration = \sum_{i=1}^{n} \frac{a}{b},$$

$$a = \sum_{j=1}^{m} \mu(jrn.transfers[i].get\_duration()[j]) \cdot jrn.transfers[i].get\_duration()[j],$$

$$b = \sum_{j=1}^{m} \mu(jrn.transfers[i].get\_durtion()[j]),$$

where $n = |jrn.transfers|$, $m = |jrn.transfers[i].duration|$; $Jrn.p_4/0.84$ is a fuzzy quantitative property, meaning the total price of the transfer, during the journey $jrn/\mu(jrn)$, and is defined in the following way

$$jrn.price = \sum_{i=1}^{n} \left(\frac{a}{b}\right) - jrn.compute\_diccount(),$$

$$a = \sum_{j=1}^{m} \mu(jrn.transfers[i].get\_price()[j]) \cdot jrn.transfers[i].get\_price()[j],$$

$$b = \sum_{j=1}^{m} \mu(jrn.transfers[i].get\_price()[j]),$$

where $n = |jrn.transfers|$, $m = |jrn.transfers[i].price|$; $Jrn.f_1/1$, $Jrn.f_2/0.91$, $Jrn.f_3/0.89$, and $Jrn.f_4/0.82$ are fuzzy methods, returning values for corresponding properties of the journey object $jrn/\mu(jrn)$, i.e.

$$jrn.get\_transfer(i) \rightarrow jrn.transfers[i],$$
$$jrn.get\_distance() \rightarrow jrn.distance,$$
$$jrn.get\_duration() \rightarrow round(jrn.duration, 0),$$
$$jrn.get\_price() \rightarrow round(jrn.price, 1);$$

$Jrn.f_5/0.87$ is a fuzzy method, calculating a discount on the price of a journey, depending on its distance, and that is defined as follows

$$jrn.compute\_discount() \rightarrow \sum_{i=1}^{n} jrn.transfers[i].price \cdot d,$$

where d is defined by the following set of rules

$$d = \begin{cases} 0.15, \ if \ \ jrn.transfer[i].distance > 1000, \\ 0.12, \ if \ \ jrn.transfer[i].distance > 800, \\ 0.09, \ if \ \ jrn.transfer[i].distance > 600, \\ 0.05, \ if \ \ jrn.transfer[i].distance > 500, \\ 0.02, \ if \ \ jrn.transfer[i].distance > 300, \\ 0.00, \ if \ \ jrn.transfer[i].distance < 300. \end{cases}$$

Let us analyze the properties and methods of the fuzzy homogeneous class of objects $Jrn/0.87$, to detect its internal semantic dependencies. As we see, fuzzy property $Jrn.p_1/0.93$ meaning transfer between two geographic places, is defined without using any other class members, therefore, it determines a corresponding fuzzy atom of the fuzzy class $Jrn/0.87$, i.e.

$$A_1(Jrn/0.87) = \{Jrn.p_1/0.93\}.$$

Fuzzy property $Jrn.p_2/0.88$, meaning the total distance of the transfer during the journey, is defined using the property $Jrn.p_1/0.93$, consequently, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_1(Jrn/0.87) = (Jrn.p_2/0.88, \{Jrn.p_1/0.93\}).$$

Fuzzy property $Jrn.p_3/0.79$, meaning the total duration of the transfer during the journey, is defined using the property $Jrn.p_1/0.93$, therefore, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_2(Jrn/0.87) = (Jrn.p_3/0.79, \{Jrn.p_1/0.93\}).$$

Fuzzy method $Jrn.f_1/1$, returning transfers list during the journey, is defined using the property $Jrn.p_1/0.93$, therefore, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_3(Jrn/0.87) = (Jrn.f_1/1, \{Jrn.p_1/0.93\}).$$

Fuzzy method $Jrn.f_5/0.78$, returning the total price of the transfer during the journey, is defined using the property $Jrn.p_1/0.93$, therefore, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_4(Jrn/0.87) = (Jrn.f_5/0.78, \{Jrn.p_1/0.93\}).$$

Fuzzy property $Jrn.p_4/0.84$, meaning the total price of the transfer during the journey, is defined using the property $Jrn.p_1/0.93$ and fuzzy method $Jrn.f_5/0.78$, consequently, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_5\left(Jrn/0.87\right)=\left(Jrn.p_4/0.84,\ \left\{Jrn.f_5/0.78,\ Jrn.p_1/0.93\right\}\right).$$

Fuzzy method $Jrn.f_2/0.91$, returning the total distance of the transfer during the journey, is defined using the property $Jrn.p_2/0.88$, consequently, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_6\left(Jrn/0.87\right)=\left(Jrn.f_2/0.91,\ \left\{Jrn.p_2/0.88,\ Jrn.p_1/0.93\right\}\right).$$

Fuzzy method $Jrn.f_3/0.89$, returning the discount on the price of a journey, is defined using the property $Jrn.p_3/0.79$, therefore, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_7\left(Jrn/0.87\right)=\left(Jrn.f_3/0.89,\ \left\{Jrn.p_3/0.79,\ Jrn.p_1/0.93\right\}\right).$$

Fuzzy method $Jrn.f_4/0.82$, returning the total price of the transfer during the journey, is defined using the properties $Jrn.p_4/0.84$, $Jrn.p_1/0.93$ and fuzzy method $Jrn.f_5/0.78$, consequently, it determines a fuzzy molecule of the fuzzy class $Jrn/0.87$, i.e.

$$M_8\left(Jrn/0.87\right)=\left(Jrn.f_4/0.8,\ \left\{Jrn.p_4/0.84,\ Jrn.f_5/0.78,\ Jrn.p_1/0.93\right\}\right).$$

All atoms and molecules of the class $Jrn/0.87$, define its internal dependencies, i.e.

$$ISD\left(Jrn/0.87\right)=\left\{A_1\left(Jrn/0.87\right),\ M_1\left(Jrn/0.87\right),\dots,M_8\left(Jrn/0.87\right)\right\}.$$

Analyzing detected internal semantic dependencies of the fuzzy class $Jrn/0.87$, we can find some similarities and intersections among them. To observe the connections among different dependencies we visualized them in Fig. 1. The orange nodes depict corresponding internal semantic dependencies, while the violet ones mean the attributes of the fuzzy class. As we can see, all molecules of the fuzzy class $Jrn/0.87$ contain the atom $A_1\left(Jrn/0.87\right)$, i.e. $A_1\left(Jrn/0.87\right)\subseteq M_i\left(Jrn/0.87\right)$, where $i=\overline{1,8}$, while the bigger molecules contain some of the smaller ones, i.e.

$$M_1\left(Jrn/0.87\right)\subseteq M_6\left(Jrn/0.87\right), M_2\left(Jrn/0.87\right)\subseteq M_7\left(Jrn/0.87\right),$$
$$M_4\left(Jrn/0.87\right)\subseteq M_5\left(Jrn/0.87\right), M_4\left(Jrn/0.87\right)\subseteq M_8\left(Jrn/0.87\right),$$
$$M_5\left(Jrn/0.87\right)\subseteq M_8\left(Jrn/0.87\right).$$

The directed arrows mean the dependencies between a pair of attributes.

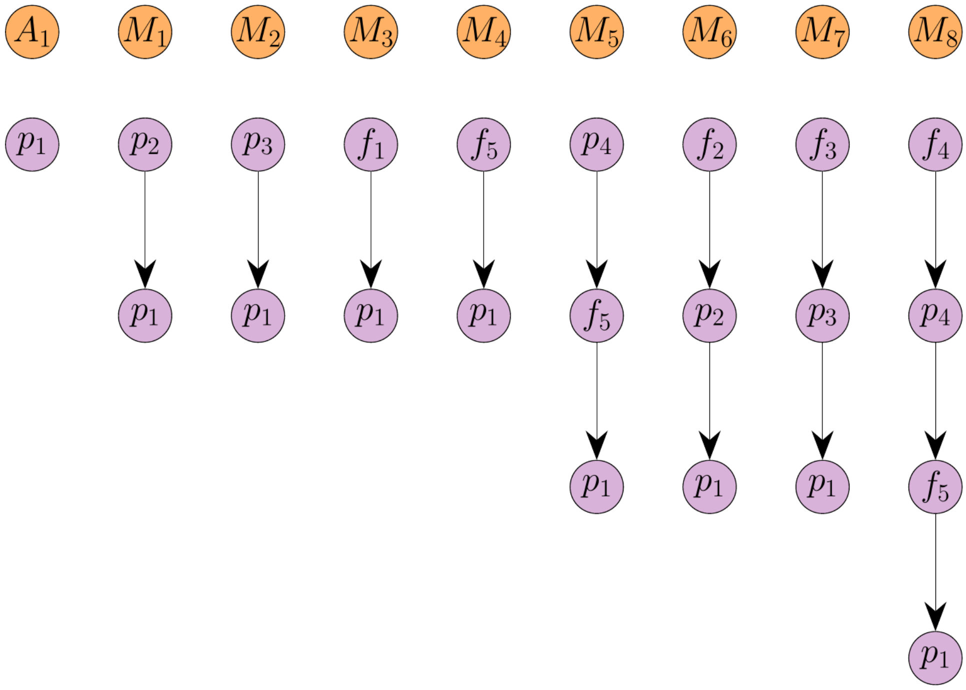

Figure 1. Internal semantic dependencies of the fuzzy homogeneous class of objects $Jrn/0.87$.

Analyzing the structure of each internal semantic dependency, illustrated in Fig. 1, we can construct a dependency graph combining all of them, i.e.

$$G(Jrn/0.87) = (A(Jrn/0.87),\ DL(Jrn/0.87)),$$

where $A(Jrn/0.87)$ is a set of attributes of the fuzzy class $Jrn/0.87$, i.e.

$$A(Jrn/0.87) = \{Jrn.p_1/0.93,\ Jrn.p_2/0.88,\ Jrn.p_3/0.79,\ Jrn.p_4/0.84,\ Jrn.f_1/1,\ Jrn.f_2/0.91,\ Jrn.f_3/0.89,\ Jrn.f_4/0.82,\ Jrn.f_5/078\},$$

and $DL(Jrn/0.87)$ is a set of dependency links among the attributes of the fuzzy class $Jrn/0.87$, i.e.

$$DL(Jrn/0.87) = \{Jrn.p_1/0.93 \perp, Jrn.p_2/0.88 \Rightarrow Jrn.p_1/0.93,\ Jrn.p_3/0.79 \Rightarrow Jrn.p_1/0.93,\ Jrn.f_1/1 \Rightarrow Jrn.p_1/0.93,\ Jrn.f_5/0.78 \Rightarrow Jrn.p_1/0.93,\ Jrn.p_4/0.84 \Rightarrow Jrn.f_5/0.78,\ Jrn.f_2/0.91 \Rightarrow Jrn.p_2/0.88,\ Jrn.f_3/0.89 \Rightarrow Jrn.p_3/0.79,\ Jrn.f_4/0.82 \Rightarrow Jrn.p_4/0.84\}.$$

The graph of internal semantic dependencies $G(Jrn/0.87)$ is represented in Fig. 2. Violet nodes represent attributes of a fuzzy class $Jrn/0.87$, edges depict dependency relations among the attributes, and edge titles mean the numbers of the molecules, which contain corresponding dependencies. To simplify the graph, we denote its nodes using only attribute identifiers.

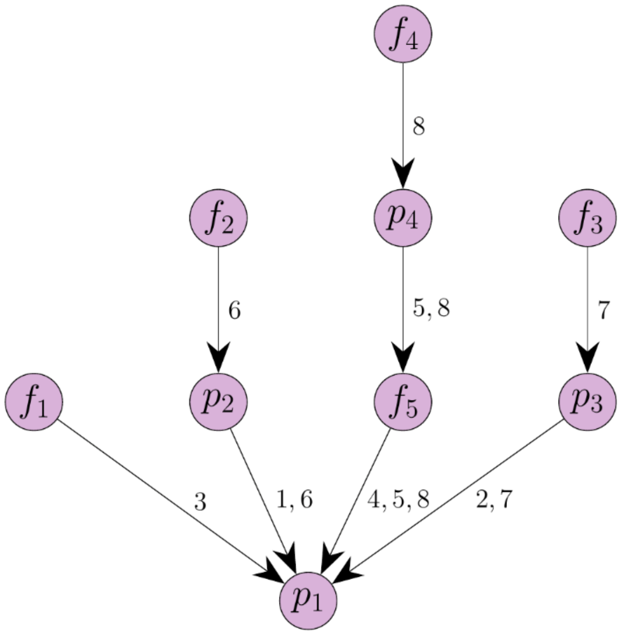

Figure 2. Graph of internal semantic dependencies of the fuzzy homogeneous class of objects $Jrn/0.87$.

Such representation of the internal semantic dependencies of the class $Jrn/0.87$ allows us to use graph-based interpretation of its subclasses. Therefore, each subgraph of graph $G(Jrn/0.87)$ defines an appropriate subclass of the fuzzy class $Jrn/0.87$. However, as we mentioned above, not all subclasses are semantically consistent, i.e. do not contradict the internal semantic dependencies of a fuzzy class. Therefore, let us define the satisfiability of internal semantic dependencies for the fuzzy homogeneous class of objects.

Definition 4. *Any subclass* $\forall SC(T)/M(SC)$ *of a fuzzy homogeneous class of objects* $T/M(T)$ *is semantically consistent if only if its graph of internal semantic dependencies*

$$G(SC(T)/M(SC)) = (A(SC(T)/M(SC)), DL(A(SC(T)/M(SC)))),$$

where $A(SC(T)/M(SC)) \subseteq A(T/M(T))$, $DL(SC(T)/M(SC)) \subseteq DL(T/M(T))$, *satisfies the following conditions:*

$$\forall (u, v) \in DL(T/M(T)) \mid (u, v) \in DL(SC(T)/M(SC)) \rightarrow$$
$$\rightarrow u \in A(SC(T)/M(SC)) \wedge v \in A(SC(T)/M(SC)),$$
$$\forall v \in A(T/M(T)) \mid v \in A(SC(T)/M(SC)) \wedge \exists u \in A(T/M(T)) \wedge$$
$$\wedge (u,v) \in DL(T/M(T)) \rightarrow (u,v) \in DL(SC(T)/M(SC)).$$

Those subclasses whose internal semantic dependency graphs do not satisfy these conditions are semantically inconsistent.

4. Conceptual Identification Space Reducing

Let us compute the complete decomposition $D(Jrn/0.87)$ of the fuzzy class $Jrn/0.87$, using an algorithm for decomposing fuzzy homogeneous classes of objects via constraint-based filtering, proposed in [12], with the following configuration

$$\left(T/M(T) = Jrn/0.87,\ C = ISD(Jrn/0.87),\ N = \{1,\ldots,8\},\ M = [0,1],\ \delta = 2\right),$$

where $T/M(T)$ is a fuzzy homogeneous class of objects, C is a set of constraints (internal semantic dependencies) defined by molecules of the class $T/M(T)$, N is a sequence of required subclasses cardinalities, M is an interval that defines admitted fuzziness of each subclass, and δ is an accuracy to calculate a measure of fuzziness for subclasses. As a result, the algorithm constructed 71 semantically consistent proper non-empty subclasses of the fuzzy class $Jrn/0.87$, among 510 possible ones, namely 1 subclass of cardinality 1, i.e.

$$SC_1^1(Jrn)/0.93 = (p_1/0.93),$$

4 subclasses of cardinality 2, i.e.

$$SC_1^2(Jrn)/0.91 = (p_1/0.93,\ p_2/0.88),$$
$$SC_2^2(Jrn)/0.86 = (p_1/0.93,\ p_3/0.79),$$
$$SC_7^2(Jrn)/0.97 = (p_1/0.93,\ f_1/1),$$
$$SC_{29}^2(Jrn)/0.86 = (p_1/0.93,\ f_5/0.78),$$

9 subclasses of cardinality 3, i.e.

$$SC_1^3(Jrn)/0.87 = (p_1/0.93, p_2/0.88, p_3/0.79),$$
$$SC_5^3(Jrn)/0.94 = (p_1/0.93, p_2/0.88, f_1/1),$$
$$SC_6^3(Jrn)/0.91 = (p_1/0.93, p_3/0.79, f_1/1),$$
$$SC_{11}^3(Jrn)/0.91 = (p_1/0.93, p_2/0.88, f_2/0.91),$$
$$SC_{22}^3(Jrn)/0.87 = (p_1/0.93, p_3/0.79, f_3/0.89),$$
$$SC_{57}^3(Jrn)/0.86 = (p_1/0.93, p_2/0.88, f_5/0.78),$$
$$SC_{58}^3(Jrn)/0.83 = (p_1/0.93, p_3/0.79, f_5/0.78),$$
$$SC_{60}^3(Jrn)/0.85 = (p_1/0.93, p_4/0.84, f_5/0.78),$$
$$SC_{63}^3(Jrn)/0.9 = (p_1/0.93, f_1/1, f_5/0.78),$$

14 subclasses of cardinality 4. i.e.

$$SC_2^4(Jrn)/0.9=(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1),$$
$$SC_6^4(Jrn)/0.88=(p_1/0.93, p_2/0.88, p_3/0.79, f_2/0.91),$$
$$SC_{10}^4(Jrn)/0.93=(p_1/0.93, p_2/0.88, f_1/1, f_2/0.91),$$
$$SC_{16}^4(Jrn)/0.87=(p_1/0.93, p_2/0.88, p_3/0.79, f_3/0.89),$$
$$SC_{21}^4(Jrn)/0.9=(p_1/0.93, p_3/0.79, f_1/1, f_3/0.89),$$
$$SC_{71}^4(Jrn)/0.85=(p_1/0.93, p_2/0.88, p_3/0.79, f_5/0.78),$$
$$SC_{72}^4(Jrn)/0.86=(p_1/0.93, p_2/0.88, p_4/0.84, f_5/0.78),$$
$$SC_{73}^4(Jrn)/0.84=(p_1/0.93, p_3/0.79, p_4/0.84, f_5/0.78),$$
$$SC_{75}^4(Jrn)/0.9=(p_1/0.93, p_2/0.88, f_1/1, f_5/0.78),$$
$$SC_{76}^4(Jrn)/0.88=(p_1/0.93, p_3/0.79, f_1/1, f_5/0.78),$$
$$SC_{78}^4(Jrn)/0.89=(p_1/0.93, p_4/0.84, f_1/1, f_5/0.78),$$
$$SC_{81}^4(Jrn)/0.88=(p_1/0.93, p_2/0.88, f_2/0.91, f_5/0.78),$$
$$SC_{92}^4(Jrn)/0.85=(p_1/0.93, p_3/0.79, f_3/0.89, f_5/0.78),$$
$$SC_{109}^4(Jrn)/0.84=(p_1/0.93, p_4/0.84, f_4/0.82, f_5/0.78),$$

16 subclasses of cardinality 5. i.e.

$$SC_3^5(Jrn)/0.9=(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1, f_2/0.91),$$
$$SC_8^5(Jrn)/0.9=(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1, f_3/0.89),$$
$$SC_{12}^5(Jrn)/0.88=(p_1/0.93, p_2/0.88, p_3/0.79, f_2/0.91, f_3/0.89),$$
$$SC_{57}^5(Jrn)/0.84=(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_5/0.78),$$
$$SC_{58}^5(Jrn)/0.88=(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1, f_5/0.78),$$
$$SC_{59}^5(Jrn)/0.89=(p_1/0.93, p_2/0.88, p_4/0.84, f_1/1, f_5/0.78),$$
$$SC_{60}^5(Jrn)/0.87=(p_1/0.93, p_3/0.79, p_4/0.84, f_1/1, f_5/0.78),$$
$$SC_{62}^5(Jrn)/0.86=(p_1/0.93, p_2/0.88, p_3/0.79, f_2/0.91, f_5/0.78),$$
$$SC_{63}^5(Jrn)/0.87=(p_1/0.93, p_2/0.88, p_4/0.84, f_2/0.91, f_5/0.78),$$
$$SC_{66}^5(Jrn)/0.9=(p_1/0.93, p_2/0.88, f_1/1, f_2/0.91, f_5/0.78),$$
$$SC_{72}^5(Jrn)/0.85=(p_1/0.93, p_2/0.88, p_3/0.79, f_3/0.89, f_5/0.78),$$
$$SC_{74}^5(Jrn)/0.85=(p_1/0.93, p_3/0.79, p_4/0.84, f_3/0.89, f_5/0.78),$$
$$SC_{77}^5(Jrn)/0.88=(p_1/0.93, p_3/0.79, f_1/1, f_3/0.89, f_5/0.78),$$
$$SC_{93}^5(Jrn)/0.85=(p_1/0.93, p_2/0.88, p_4/0.84, f_4/0.82, f_5/0.78),$$
$$SC_{94}^5(Jrn)/0.83=(p_1/0.93, p_3/0.79, p_4/0.84, f_4/0.82, f_5/0.78),$$

$$SC_{99}^{5}\left(Jrn\right)/0.87=\left(p_1/0.93, p_4/0.84, f_1/1 f_4/0.82, f_5/0.78\right),$$

14 subclasses of cardinality 6. i.e.

$$SC_{4}^{6}\left(Jrn\right)/0.9=\left(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1, f_2/0.91, f_3/0.89\right),$$
$$SC_{29}^{6}\left(Jrn\right)/0.87=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_1/1, f_5/0.78\right),$$
$$SC_{30}^{6}\left(Jrn\right)/0.86=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_2/0.91, f_5/0.78\right),$$
$$SC_{31}^{6}\left(Jrn\right)/0.88=\left(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1, f_2/0.91, f_5/0.78\right),$$
$$SC_{32}^{6}\left(Jrn\right)/0.9=\left(p_1/0.93, p_2/0.88, p_4/0.84, f_1/1, f_2/0.91, f_5/0.78\right),$$
$$SC_{35}^{6}\left(Jrn\right)/0.85=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_3/0.89, f_5/0.78\right),$$
$$SC_{36}^{6}\left(Jrn\right)/0.88=\left(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1, f_3/0.89, f_5/0.87\right),$$
$$SC_{38}^{6}\left(Jrn\right)/0.87=\left(p_1/0.93, p_3/0.79, p_4/0.84, f_1/1, f_3/0.89, f_5/0.78\right),$$
$$SC_{40}^{6}\left(Jrn\right)/0.86=\left(p_1/0.93, p_2/0.88, p_3/0.79, f_2/0.91, f_3/0.89, f_5/0.78\right),$$
$$SC_{50}^{6}\left(Jrn\right)/0.84=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_4/0.82, f_5/0.78\right),$$
$$SC_{52}^{6}\left(Jrn\right)/0.88=\left(p_1/0.93, p_2/0.88, p_4/0.84, f_1/1, f_4/0.82, f_5/0.78\right),$$
$$SC_{53}^{6}\left(Jrn\right)/0.86=\left(p_1/0.93, p_3/0.79, p_4/0.84, f_1/1, f_4/0.82, f_5/0.78\right),$$
$$SC_{56}^{6}\left(Jrn\right)/0.86=\left(p_1/0.93, p_2/0.88, p_4/0.84, f_2/0.91, f_4/0.82, f_5/0.78\right),$$
$$SC_{67}^{6}\left(Jrn\right)/0.84=\left(p_1/0.93, p_3/0.79, p_4/0.84, f_3/0.89, f_4/0.82, f_5/0.78\right),$$

9 subclasses of cardinality 7. i.e.

$$SC_{9}^{7}\left(Jrn\right)/0.88=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_1/1, f_2/0.91, f_5/0.78\right),$$
$$SC_{10}^{7}\left(Jrn\right)/0.87=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_1/1, f_3/0.89, f_5/0.78\right),$$
$$SC_{11}^{7}\left(Jrn\right)/0.86=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_2/0.91, f_3/0.89, f_5/0.78\right),$$
$$SC_{12}^{7}\left(Jrn\right)/0.88=\left(p_1/0.93, p_2/0.88, p_3/0.79, f_1/1, f_2/0.91, f_3/0.89, f_5/0.78\right),$$
$$SC_{16}^{7}\left(Jrn\right)/0.86=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_1/1, f_4/0.82, f_5/0.78\right),$$
$$SC_{17}^{7}\left(Jrn\right)/0.85=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_2/0.91, f_4/0.82, f_5/0.78\right),$$
$$SC_{19}^{7}\left(Jrn\right)/0.88=\left(p_1/0.93, p_2/0.88, p_4/0.84, f_1/1, f_2/0.91, f_4/0.82, f_5/0.78\right),$$
$$SC_{22}^{7}\left(Jrn\right)/0.85=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_3/0.89, f_4/0.82, f_5/0.78\right),$$
$$SC_{25}^{7}\left(Jrn\right)/0.86=\left(p_1/0.93, p_3/0.79, p_4/0.84, f_1/1, f_3/0.89, f_4/0.82, f_5/0.78\right),$$

and 4 subclasses of cardinality 8. i.e.

$$SC_{2}^{8}\left(Jrn\right)/0.88=\left(p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_1/1, f_2/0.91, f_3/0.89, f_5/0.78\right),$$

$$SC_3^8(Jrn)/0.87 = (p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_1/1, f_2/0.91,$$
$$f_4/0.82, f_5/0.78),$$
$$SC_4^8(Jrn)/0.87 = (p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_1/1, f_3/0.89,$$
$$f_4/0.82, f_5/0.78),$$
$$SC_5^8(Jrn)/0.86 = (p_1/0.93, p_2/0.88, p_3/0.79, p_4/0.84, f_2/0.91,$$
$$f_3/0.89, f_4/0.82\, f_5/0.78).$$

All obtained subclasses of the fuzzy homogeneous class of objects $Jrn/0.87$ are elements of its semantically consistent decomposition $D(Jrn/0.87)$. Analyzing graphs of internal semantic dependencies for each constructed subclass of fuzzy class $Jrn/0.87$, we can see, that all of them are semantically consistent, according to Def. 4.

It is known that the power set of a certain set is a partially ordered set, which defines a complete bounded lattice [21]. Therefore, the power set of the set of properties and methods of the fuzzy homogeneous class of objects $Jrn/0.87$ is a poset $(PS(Jrn/0.87), \subseteq)$, which defines a complete bounded lattice of subclasses of the fuzzy class $Jrn/0.87$. i.e.

$$L(Jrn/0.87) = (PS(Jrn/0.87), \subseteq, \cup, \cap, 0, 1),$$

where $\cup$ and $\cap$ are the least upper bound (join) and the greatest lower bound (meet) operations, defined on the set $PS(Jrn/0.87)$ i.e.

$$\forall SC_1/M(SC_1) \in PS(Jrn/0.87),\ \forall SC_2/M(SC_2) \in PS(Jrn/0.87) \rightarrow$$
$$\rightarrow SC_1/M(SC_1) \cup SC_2/M(SC_2) \in PS(Jrn/0.87),$$
$$SC_1/M(SC_1) \cap SC_2/M(SC_2) \in PS(Jrn/0.87),$$

and where 0 is the least element $SC_1^0(Jrn)/0.0$, and 1 is the greatest elements $SC_1^9(Jrn)/0.87$ of the lattice, i.e.

$$\forall SC/M(SC) \in PS(Jrn/0.87) \rightarrow SC_1^0(Jrn)/0.0 \cap SC/M(SC) = SC_1^0(Jrn)/0.0,$$
$$SC_1^9(Jrn)/0.87 \cup SC/M(SC) = SC_1^9(Jrn)/0.87.$$

As we know, the cardinality of $PS(Jrn/0.87)$ is equal to $2^n = 2^9 = 512$, where $n = |Jrn/0.87|$, therefore, illustrating the Hasse diagram of such a lattice is a non-trivial task, because of its size. Therefore, let us construct and illustrate the tower of subclass lattice $L(Jrn/0.87)$, using the corresponding approach proposed in [20] (see Fig. 3). As we can see, Fig. 3 represents three objects similar to tower buildings, which consist of sections and floors of a certain capacity. The tower sections are vertical columns of floors

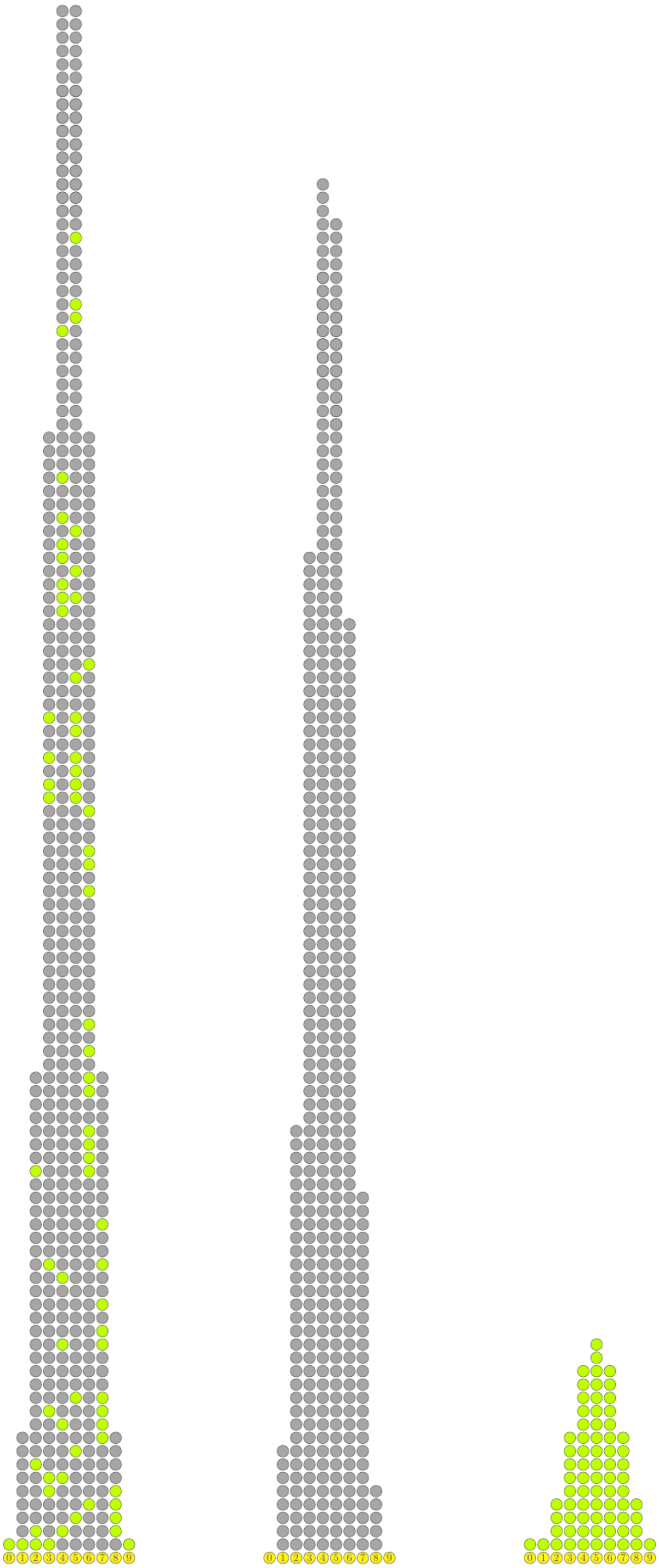

Figure 3. Tower of the complete bounded subclass lattice of the fuzzy homogeneous class of objects $Jrn/0.87$.

represented by grey and lime circles. The capacity of all floors, within the particular section, is represented by yellow circles with a corresponding number. The tower of the subclass lattice $L(Jrn/0.87)$ is depicted on the left side of Fig. 3. The circles colored in gray can be interpreted as the unlighted tower floors because they mean semantically inconsistent subclasses of the fuzzy homogeneous class of objects $Jrn/0.87$. The circles colored in lime have an opposite interpretation since they mean semantically consistent subclasses, detected by the decomposition algorithm.

The second and third towers depicted in the middle and on the right in Fig. 3 are towers of sublattices of the subclass lattice $L(Jrn/0.87)$, which contain only semantically inconsistent and only semantically consistent subclasses of the fuzzy homogeneous class of objects $Jrn/0.87$, respectively. Comparing the number of subclasses of both kinds, we can see, that there are only 71 semantically consistent proper non-empty subclasses among the 510 formally possible. In more detail, this comparison can be represented by Tab. 1. The first row of the table means the cardinality of subclasses, while the second and the third rows contain the number of all formally possible and all semantically consistent subclasses of the fuzzy homogeneous class of objects $Jrn/0.87$ of certain cardinality. The fourth row represents the ratio third row to the second row in percent. According to [20], the decomposition consistency of the fuzzy homogeneous class of objects $Jrn/0.87$ is approximately equal to 13.9%, i.e.

Table 1

Quantitative analysis of subclasses of the fuzzy homogeneous class of objects $Jrn/0.87$.

Cardinality	1	2	3	4	5	6	7	8	Total
Possible Subclasses	9	36	84	126	126	84	36	9	**510**
Consistent Subclasses	1	4	9	14	16	14	9	4	**71**
Decomposition Consistency	11%	11%	11%	11%	13%	17%	25%	44%	**14%**

$$DC(Jrn/0.87) = \frac{|D(Jrn/0.87)|}{|PS(Jrn/0.87)|-2} \cdot 100\% = \frac{71}{510} \approx 13.9\%.$$

It means that only 13.9% of all possible proper non-empty subclasses of the fuzzy homogeneous class of objects $Jrn/0.87$ are semantically consistent ones. Consequently, we can reduce the search space for the conceptual identification within the semantically consistent decomposition of a fuzzy homogeneous class of objects. To do this, let us construct the sublattice of the subclass lattice $L(Jrn/0.87)$, which contains only semantically consistent subclasses of the fuzzy class $Jrn/0.87$. According to the definition provided in [21], the sublattice of the lattice L is a subset X of L, such that $a \in X, b \in X \to a \wedge b \in X, a \vee b \in X$. It is obvious that the empty subclass $SC_1^0(Jrn)/0.0 \subseteq Jrn/0.87$, as well as the subclass $SC_1^9(Jrn)/0.87 \subseteq Jrn/0.87$, are semantically consistent subclasses of the fuzzy homogeneous class of objects $Jrn/0.87$, therefore the set of all its semantically consistent subclasses is defined as follows

$$CS\left(Jrn/0.87\right) = D\left(Jrn/0.87\right) \cup \left\{SC_1^0\left(Jrn\right)/0.0,\ SC_1^9\left(Jrn\right)/0.87\right\}.$$

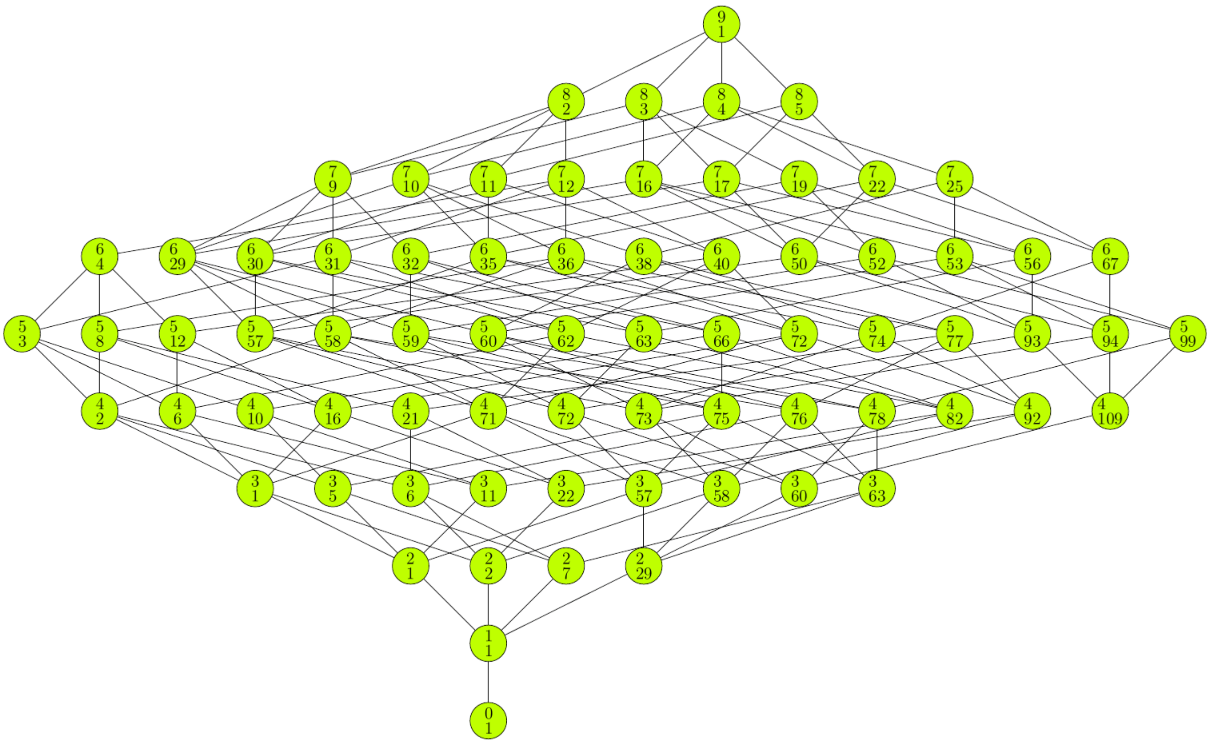

Figure 4. Hasse diagram of complete bounded semantically consistent subclass lattice of the fuzzy homogeneous class of objects $Jrn/0.87$.

Therefore, a set of all semantically consistent subclasses $CS\left(Jrn/0.87\right) \subseteq PS\left(Jrn/0.87\right)$ of the fuzzy class of objects $Jrn/0.87$ defines a carrier $\left(CS\left(Jrn/0.87\right), \subseteq\right)$ of the consistent subclass lattice $CSL\left(Jrn/0.87\right)$, which is a sublattice of the subclass lattice $L\left(Jrn/0.87\right)$, since

$$\forall SC_1/M\left(SC_1\right) \in CS\left(Jrn/0.87\right),\ \forall SC_2/M\left(SC_2\right) \in CS\left(Jrn/0.87\right) \rightarrow$$
$$\rightarrow SC_1/M\left(SC_1\right) \cup SC_2/M\left(SC_2\right) \in CS\left(Jrn/0.87\right),$$
$$SC_1/M\left(SC_1\right) \cap SC_2/M\left(SC_2\right) \in CS\left(Jrn/0.87\right),$$

where $\cup$ and $\cap$ are the least upper bound (join) and the greatest lower bound (meet) operations, defined on the set $CS\left(Jrn/0.87\right)$. In addition, the carrier of the sublattice

$CSL(Jrn/0.87)$ contains the least element $SC_1^0(Jrn)/0.0$, and the greatest element $SC_1^9(Jrn)/0.87$, which makes it the a bounded one, since

$$\forall SC/M(SC) \in CS(Jrn/0.87) \rightarrow$$
$$\rightarrow SC_0^1(Jrn)/0.0 \cap SC/M(SC) = SC_0^1(Jrn)/0.0,$$
$$SC_1^9(Jrn)/0.87 \cup SC/M(SC) = SC_1^9(Jrn)/0.87.$$

Consequently, the semantically consistent sublattice of subclasses lattice of the fuzzy homogeneous class of objects $Jrn/0.87$ is defined as follows

$$CSL(Jrn/0.87) = (CS(Jrn/0.87), \subseteq, \cup, \cap, 0, 1),$$

where 0 means the least element, while 1 means the greatest element of the lattice. The Hasse diagram of the lattice $CSL(Jrn/0.87)$ is depicted in the Fig. 4. The superscript of each lattice node means the cardinality of the corresponding subclass of the fuzzy class $Jrn/0.87$, and the subscript indicates the number of the subclass among all possible subclasses of such cardinality.

Constructing the sublattice $CSL(Jrn/0.87)$ allows us to reduce the subclass identification space by $512/73 \approx 7$ times and to perform the subclass identification, analyzing only semantically consistent subclasses of the fuzzy class $Jrn/0.87$.

5. Identification of Consistent Fuzzy Knowledge

To develop the identification of consistent fuzzy knowledge within the decomposition of fuzzy homogeneous classes of objects via constraint-based filtering, we modified the corresponding decomposition algorithm, proposed in [19], adding subclasses and superclasses detection procedures (see Algorithm 1). The main idea of the algorithm is to detect sets of all semantically consistent subclasses $S^{\downarrow}(SC/M(SC))$ and superclasses $S^{\uparrow}(SC/M(SC))$ for the selected semantically consistent subclass $SC/M(SC)$ of the fuzzy homogeneous class of objects $T/M(T)$. As the input parameters, the algorithm uses the following: a fuzzy homogeneous class of objects $T/M(T)$ as a space for the fuzzy knowledge identification; a semantically consistent subclass $SC/M(SC)$ of the fuzzy class $T/M(T)$ as an identification target; a set of constraints $C = ISD(T/M(T))$ defined by molecules of the fuzzy class $T/M(T)$, to detect its semantically consistent subclasses; a required accuracy δ for computation of the measure of fuzziness of subclasses and superclasses of the class $SC/M(SC)$.

In general, the procedure of the identification of consistent fuzzy knowledge can be split into a few successive stages. In the first stage, the algorithm constructs all formally possible subclasses of the fuzzy class $T/M(T)$, which have a cardinality greater or less than the cardinality of the subclass $SC/M(SC)$. This reduces the identification space again, avoiding subclasses of the same cardinality as the $SC/M(SC)$ subclass, since such subclasses definitely are not superclass or subclass of the subclass $SC/M(SC)$.

Algorithm 1. Identification of Consistent Fuzzy Knowledge

Require: $T/M(T)$, $SC/M(SC)$, C, δ

Ensure: $S^{\uparrow}$, $S^{\downarrow}$

$t := \{\}$;
$S^{\downarrow} := \{\}$;
$S^{\uparrow} := \{\}$;
for $i = 1, \ldots, 2^{|T/M(T)|} - 1$ **do**
  if binary(i).count(1) $\neq |SC/M(SC)|$ **then**
    for $a_j/\mu(a_j) \in T/M(T)$, $j = 1, \ldots, |T/M(T)|$ **do**
      if $(i \ \& \ (1 \ll j)) > 0$ **then**
        t.add$(a_j/\mu(a_j))$;
    satisfy := **true**;
    for all $c \in C$ **do**
      if is_satisfy(t, c) = **false then**
        staisfy := **false**;
        break;
    if satisfy **then**
      $M(t) =$ compute_fuzziness(t, δ);
      if is_subclass$(SC/M(SC), t/M(t))$ **then**
        $S^{\uparrow}$.add$(t/M(t))$;
      if is_superclass$(SC/M(SC), t/M(t))$ **then**
        $S^{\downarrow}$.add$(t/M(t))$;
    $t := \{\}$;
return $S^{\uparrow}$, $S^{\downarrow}$.

In the second stage, the algorithm detects all semantically consistent subclasses of the fuzzy homogeneous class of objects $T/M(T)$, among previously generated, performing the constraint-based filtering, using Procedure 1. It verifies the satisfiability of each constraint $c \in C$, defined by molecules of the fuzzy class $T / M(T)$, for the subclass $SC / M(SC)$. In general, the subclass $SC / M(SC)$ can satisfy or not satisfy the constraint $c \in C$ as well as the constraint can be inapplicable to the subclass, therefore, the procedure can return as a value *true*, *false*, or *none*, respectively. In the third stage, the algorithm computes the fuzziness for each semantically consistent potential subclass or superclass of the class $SC / M(SC)$, using Procedure 2. After that, the algorithm verifies the subclass and (or) superclass relation between the detected semantically consistent subclasses of the fuzzy class $T / M(T)$ and subclass $SC / M(SC)$, using Procedure 3 and Procedure 4, respectively.

Procedure 1. is_satisfy(t, c)

Input: t, c
Output: satisfy $\to$ { **true**, **false**, **none** }

```
satisfy := none;
if c[0] ∈ t then
   satisfy := false;
   for i ∈ 1,...,|c| do
      for all a / μ(a) ∈ c[i] do
         if a / μ(a) ∈ t then
            satisfy := true;
         else
            satisfy := false;
            break;
      if satisfy then
         return satisfy;
return satisfy.
```

Procedure 2. compute_fuzziness(t, δ)

Input: t, δ
Output: $M(t) \to [0, 1]$

```
sum := 0;
for all a / μ(a) ∈ t do
   sum := sum + μ(a);
M(t) := round(sum / max(|t|, 1), δ);
return M(t).
```

Procedure 3. $\text{is\_subclass}\left(SC/M\left(SC\right),\ t/M\left(t\right)\right)$

Input: $SC/M\left(SC\right),\ t/M\left(t\right)$
Output: $\text{is\_subclass} \to \{\,\textbf{true}, \textbf{false}\,\}$

if $\left|SC/M\left(SC\right)\right| > \left|t/M\left(t\right)\right|$ **then**
  return false;
for all $a/\mu\left(a\right) \in SC/M\left(SC\right)$ **do**
  if $a/\mu\left(a\right) \notin t/M\left(t\right)$ **then**
    return false;
return true.

Procedure 4. $\text{is\_superclass}\left(SC/M\left(SC\right),\ t/M\left(t\right)\right)$

Input: $SC/M\left(SC\right),\ t/M\left(t\right)$
Output: $\text{is\_superclass} \to \{\,\textbf{true}, \textbf{false}\,\}$

if $\left|SC/M\left(SC\right)\right| < \left|t/M\left(t\right)\right|$ **then**
  return false;
for all $a/\mu\left(a\right) \in t/M\left(t\right)$ **do**
  if $a/\mu\left(a\right) \notin SC/M\left(SC\right)$ **then**
    return false;
return true.

As a result, the algorithm computes a set of all semantically consistent proper non-empty subclasses $S^{\downarrow}\left(SC/M\left(SC\right)\right)$ and superclasses $S^{\uparrow}\left(SC/M\left(SC\right)\right)$ for the subclass $SC/M\left(SC\right)$.

To demonstrate the conceptual identification of fuzzy knowledge using Algorithm 1, let us apply the proposed approach to the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$ of the fuzzy homogeneous class of objects $Jrn/0.87$, described in the previous section. Consequently, we obtain the following sets of proper non-empty subclasses

$$S^{\downarrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)=\left\{SC_{57}^{5}\left(Jrn\right)/0.84,\ SC_{58}^{5}\left(Jrn\right)/0.88,\ SC_{59}^{5}\left(Jrn\right)/0.89,\right.$$
$$SC_{60}^{5}\left(Jrn\right)/0.87,\ SC_{2}^{4}\left(Jrn\right)/0.9,\ SC_{71}^{4}\left(Jrn\right)/0.85,\ SC_{72}^{4}\left(Jrn\right)/0.86,\ SC_{73}^{4}\left(Jrn\right)/0.84,$$
$$SC_{75}^{4}\left(Jrn\right)/0.9,\ SC_{76}^{4}\left(Jrn\right)/0.88,\ SC_{78}^{4}\left(Jrn\right)/0.89,\ SC_{1}^{3}\left(Jrn\right)/0.87,\ SC_{5}^{3}\left(Jrn\right)/0.94,$$
$$SC_{6}^{3}\left(Jrn\right)/0.91,\ SC_{57}^{3}\left(Jrn\right)/0.86,\ SC_{58}^{3}\left(Jrn\right)/0.83,\ SC_{60}^{3}\left(Jrn\right)/0.85,\ SC_{63}^{3}\left(Jrn\right)/0.9,$$
$$\left.SC_{1}^{2}\left(Jrn\right)/0.91,\ SC_{2}^{2}\left(Jrn\right)/0.86,\ SC_{7}^{2}\left(Jrn\right)/0.97,\ SC_{29}^{2}\left(Jrn\right)/0.86,\ SC_{1}^{1}\left(Jrn\right)/0.93\right\},$$

and superclasses

$$S^{\uparrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)=\left\{SC_{9}^{7}\left(Jrn\right)/0.88,\ SC_{10}^{7}\left(Jrn\right)/0.87,\ SC_{16}^{7}\left(Jrn\right)/0.86,\right.$$
$$\left.SC_{2}^{8}\left(Jrn\right)/0.88,\ SC_{3}^{8}\left(Jrn\right)/0.87,\ SC_{4}^{8}\left(Jrn\right)/0.87\right\}.$$

As was noted above, subclasses $SC_{1}^{0}\left(Jrn\right)/0.0$, and $SC_{1}^{9}\left(Jrn\right)/0.87$ are semantically consistent subclasses of the fuzzy class $Jrn/0.87$, therefore the identification space for the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$ is defined as follows

$$IS_{29}^{6}\left(Jrn/0.87\right)=S^{\uparrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)\cup S^{\downarrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)\cup$$
$$\cup\left\{SC_{1}^{0}\left(Jrn\right)/0.0,\ SC_{1}^{9}\left(Jrn\right)/0.87\right\}.$$

Thus, the identification space $IS_{29}^{6}\left(Jrn/0.87\right)$ of the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$ defines the carrier $\left(IS_{29}^{6}\left(Jrn/0.87\right),\subseteq\right)$ of the identification space lattice $ISL_{29}^{6}\left(Jrn/0.87\right)$, which is a sublattice of the consistent subclass lattice $CSL\left(Jrn/0.87\right)$ and subclass lattice $L\left(Jrn/0.87\right)$, since $IS_{29}^{6}\left(Jrn/0.87\right)\subseteq CS\left(Jrn/0.87\right)\subseteq PS\left(Jrn/0.87\right)$ and

$$\forall SC_{1}/M\left(SC_{1}\right)\in IS_{29}^{6}\left(Jrn/0.87\right),\ \forall SC_{2}/M\left(SC_{2}\right)\in IS_{29}^{6}\left(Jrn/0.87\right)\rightarrow$$
$$\rightarrow SC_{1}/M\left(SC_{1}\right)\cup SC_{2}/M\left(SC_{2}\right)\in IS_{29}^{6}\left(Jrn/0.87\right),$$
$$SC_{1}/M\left(SC_{1}\right)\cap SC_{2}/M\left(SC_{2}\right)\in IS_{29}^{6}\left(Jrn/0.87\right),$$

where $\cup$ and $\cap$ are the least upper bound (join) and the greatest lower bound (meet) operations, defined on the set $IS\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)$. Since the carrier of the lattice $ISL_{29}^{6}\left(Jrn/0.87\right)$ contains the least element $SC_{1}^{0}\left(Jrn\right)/0.0$, and the greatest element $SC_{1}^{9}\left(Jrn\right)/0.87$, it makes the identification sublattice $ISL_{29}^{6}\left(Jrn/0.87\right)$ a bounded one, since

$$\forall SC/M\left(SC\right)\in IS_{29}^{6}\left(Jrn/0.87\right)\rightarrow$$
$$\rightarrow SC_{0}^{1}\left(Jrn\right)/0.0\cap SC/M\left(SC\right)=SC_{0}^{1}\left(Jrn\right)/0.0,$$
$$SC_{1}^{9}\left(Jrn\right)/0.87\cup SC/M\left(SC\right)=SC_{1}^{9}\left(Jrn\right)/0.87.$$

Thus, the identification lattice for the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$ of the fuzzy homogeneous class of objects $Jrn/0.87$ is defined as follows

$$ISL_{29}^{6}\left(Jrn/0.87\right)=\left(IS_{29}^{6}\left(Jrn/0.87\right),\ \subseteq,\ \cup,\ \cap,\ 0,\ 1\right),$$

where 0 means the least element, while 1 means the greatest element of the lattice. The Hasse diagram of the lattice $ISL_{29}^{6}\left(Jrn/0.87\right)$ is depicted in the Fig. 5. The yellow node means the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$, while the lime nodes represent its subclasses and superclasses.

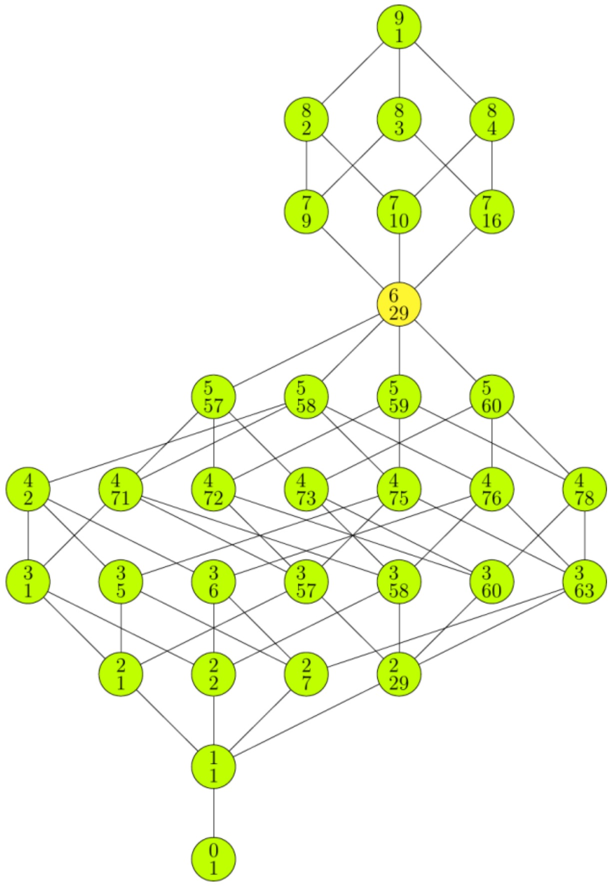

Figure 5. Hasse diagram of complete bounded identification space lattice of the subclass $SC_{29}^{6}\left(Jrn\right)/0.87 \subseteq Jrn/0.87$.

Let us define a concept of a subclass neighborhood within the identification lattice, which will provide an opportunity to consider the so-called subclass locus instead of the subclass itself.

Definition 5. *The neighborhood of the subclass* $SC/M\left(SC\right)$ *of a fuzzy homogeneous class of objects* $T/M\left(T\right)$ *is a pair* $N\left(SC/M\left(SC\right)\right)=\left(S_n^{\downarrow}, S_n^{\uparrow}\right)$, *where*

$$S_n^{\downarrow}=\left\{SC_1^{\downarrow}/M\left(SC_1^{\downarrow}\right),\dots,SC_k^{\downarrow}/M\left(SC_k^{\downarrow}\right)\right\},$$
$$S_n^{\downarrow}\subseteq IS\left(SC/M\left(SC\right)\right),$$

$$S_n^{\uparrow}=\left\{SC_1^{\uparrow}/M\left(SC_1^{\uparrow}\right),\ldots,SC_w^{\uparrow}/M\left(SC_w^{\uparrow}\right)\right\},$$
$$S_n^{\uparrow}\subseteq IS\left(SC/M\left(SC\right)\right),$$

are sets of subclasses and superclasses of the subclass $SC/M\left(SC\right)$ *such that*

$$\forall SC_i^{\downarrow}/M\left(SC_i^{\downarrow}\right)\in S_n^{\downarrow},\ i=\overline{1,k}\rightarrow SC_i^{\downarrow}/M\left(SC_i^{\downarrow}\right)\subseteq SC/M\left(SC\right),$$
$$\forall SC_j^{\uparrow}/M\left(SC_j^{\uparrow}\right)\in S_n^{\uparrow},\ j=\overline{1,w}\rightarrow SC/M\left(SC\right)\subseteq SC_j^{\uparrow}/M\left(SC_j^{\uparrow}\right),$$

and where $IS\left(SC/M\left(SC\right)\right)$ *is a carrier of the subclass identification lattice* $ISL\left(SC/M\left(SC\right)\right)$*.*

Using concept of subclass neighborhood, let us define a notion of a subclass neighborhood measure.

Definition 6. *The neighborhood measure of the subclass* $SC/M\left(SC\right)$ *of a fuzzy homogeneous class of objects* $T/M\left(T\right)$ *is a pair* $N_{\mu}\left(SC/M\left(SC\right)\right)=\left(n^{\downarrow},n^{\uparrow}\right)$*, where* $n^{\downarrow}$ *and* $n^{\uparrow}$ *are subclass and superclass neighborhood, respectively, and defined in the following way*

$$n^{\downarrow}=\frac{\left|S_n^{\downarrow}\left(SC/M\left(SC\right)\right)\right|}{\left|S^{\downarrow}\left(SC/M\left(SC\right)\right)\right|},$$
$$n^{\uparrow}=\frac{\left|S_n^{\uparrow}\left(SC/M\left(SC\right)\right)\right|}{\left|S^{\uparrow}\left(SC/M\left(SC\right)\right)\right|}.$$

Now let us consider an example of the subclass neighborhood and its measure for the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$. Suppose that neighborhood of the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$ is defined in the following way

$$N\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)=\left(S_n^{\downarrow},S_n^{\uparrow}\right),$$
$$S_n^{\downarrow}=\left\{SC_{59}^{5}\left(Jrn\right)/0.89,\ SC_{60}^{5}\left(Jrn\right)/0.87,\ SC_{78}^{4}\left(Jrn\right)/0.89\right\},$$
$$S_n^{\uparrow}=\left\{SC_{9}^{7}\left(Jrn\right)/0.88,\ SC_{16}^{7}\left(Jrn\right)/0.86,\ SC_{3}^{8}\left(Jrn\right)/0.87\right\},$$

then its neighborhood measure is $N_{\mu}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)=\left(0.13^{\downarrow},\ 0.43^{\uparrow}\right)$, where

$$n^{\downarrow}=\frac{\left|S_n^{\downarrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)\right|}{\left|S^{\downarrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)\right|}=\frac{3}{24}\approx 0.13,$$

$$n^{\uparrow} = \frac{\left|S_n^{\uparrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)\right|}{\left|S^{\uparrow}\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)\right|} = \frac{3}{7} \approx 0.43.$$

It is clear that $n^{\downarrow} \in [0,1]$ and $n^{\uparrow} \in [0,1]$, therefore if their values are closer to 0, this means that the subclass and superclass neighborhood is low, but if it is closer to 1 then their neighborhood is high. Using a neighborhood $N\left(SC_{29}^{6}\left(Jrn\right)/0.87\right)$ of the subclass $SC_{29}^{6}\left(Jrn\right)/0.87$ we can consider a subclass locus defined by its subclasses and superclasses instead of the subclass itself.

6. Conclusions

In this paper, we analyzed known approaches to the conceptual identification of fuzzy knowledge using a formal concept analysis and its fuzzy extension. Since the FCA/FFCA-based computation of formal concepts can construct semantically inconsistent concepts, which are impossible or unreal within a modeled domain, we proposed another new lattice-based approach to the conceptual identification of fuzzy knowledge. We study the conceptual identification of subclasses within the decomposition of fuzzy homogeneous classes of objects. The proposed approach allows us to identify semantically consistent subclasses within the decomposition of a fuzzy homogeneous class of objects constructing corresponding sub-class lattice. Such lattice is considered as a space for conceptual identification of any of its elements via detection of its subclasses and superclasses. Identification of a specific subclass involves the construction of a corresponding identification lattice, which is a sub-lattice of the subclass lattice. To implement the approach, we developed the corresponding identification algorithm, extending the algorithm for the decomposition of fuzzy homogeneous classes of objects via constraint-based filtering, proposed in [18]. As a result, the algorithm constructs all semantically consistent subclasses of a fuzzy homogeneous class of objects and then, verifies the subclass-superclass relation between each of them and a subclass, which needs to be identified.

To demonstrate the conceptual identification of fuzzy knowledge using the developed algorithm, we provided an example of conceptual identification of a semantically consistent subclass of a fuzzy homogeneous class of objects, which defines a fuzzy concept of a journey through the sequence of geographic places. To visualize the identification process, the corresponding identification lattice was constructed. Using this lattice, we introduced the notions of subclass neighborhood and its measure. The proposed approach can be extended for the conceptual identification of fuzzy knowledge within the fuzzy conceptual hierarchies and scaling of big concept lattices.

Acknowledgements

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