A Heuristic for the P-time Reconstruction of Unique 3-Uniform Hypergraphs from their Degree Sequences*

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Abstract
One of the main problems in the wide area of graph theory is the so called *reconstruction problem*, that is the reconstruction of a (hyper)graph from its degree sequence. The problem remained open for many years, until in 2018 Deza et al. proved its NP hardness even for the simplest case of 3-uniform hypergraphs. As a consequence, the definition of classes of instances that allow a polynomial time reconstruction acquired relevance in order to restrict the NP-complete core of the problem. In this paper, we consider the class of instances $\mathcal{D}^{ext}$ defined by Ascolese et al. in 2021, and we provide some structural properties of the related 3-uniform hypergraphs. Then, we move the spotlight on its subclass $\mathcal{D}^{ext-}$ including only those elements that are unique, i.e., two non-isomorphic 3-uniform hypergraphs sharing a degree sequence do not exist in $\mathcal{D}^{ext-}$. This property suggests the possibility of a polynomial time strategy for the reconstruction of its elements. We define an algorithm that allows a fast reconstruction of some instances in $\mathcal{D}^{ext-}$, and we further provide a heuristic to solve the same problem on the entire class. The heuristic relies on the uniqueness of the elements in $\mathcal{D}^{ext-}$ and on geometric and algebraic features of the related 3-hypergraphs. Finally, statistics on the performance of the heuristic are provided.

Keywords
Uniform hypergraph, Degree sequences, Reconstruction problem

1. Introduction

The characterization of graphs and hypergraphs from their degree sequences has been one of the most challenging problems in these last decades. In the simplest case of graphs, deciding if an integer sequence is the degree sequence of a graph was solved in 1960 by Erdős and Gallai [11]. Subsequently, a number of algorithms have been developed to provide constructive proofs of this result ([15, 18, 24]). Moving to hypergraphs, the same decision problem has been widely studied (see [7, 10, 12, 20, 21]). Recently, starting from a general and non constructive characterization theorem in [8], some relevant subclasses of degree sequences have been considered (see [3]), whose elements allow a polynomial time algorithm to compute, say reconstruct, the related hypergraphs (see [1, 5, 13, 14]). In 2018 Deza et al. proved that deciding if an integer sequence

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is the degree sequence of a 3-uniform hypergraph is NP-complete [9]. As a consequence, the definition of classes of degree sequences that can be reconstructed in polynomial time acquires relevance in order to limit the NP-complete core of the problem.

In their proof Deza et al. defined as a *gadget* a class of degree sequences that show remarkable geometrical and algebraic properties. Relying on this, in [2] the authors provided a class $\mathcal{D}$ of degree sequences of unique 3-hypergraphs that preserve these properties and whose incidence matrices have strong symmetrical and structural specific characteristics. In this paper (Section 3), we define a suitable poset $\mathcal{T}$ on integer triplets and we provide a quite surprising connection between 3-hypergraphs having degree sequence in $\mathcal{D}$ and a family of ideals of $\mathcal{T}$. This connection allows us to further extend the class $\mathcal{D}$ to $\mathcal{D}^{ext}$, defined as the degree sequences of the 3-hypergraphs which correspond to ideals of $\mathcal{T}$. We show that the properties of the elements in $\mathcal{D}$ move to those in $\mathcal{D}^{ext}$. Unfortunately, the elements of $\mathcal{D}^{ext}$ lose the uniqueness property. We get it back defining the subclass $\mathcal{D}^{ext-}$ such that $\mathcal{D} \subset \mathcal{D}^{ext-} \subset \mathcal{D}^{ext}$. In Section 4, we provide a polynomial time algorithm to reconstruct two subclasses of $\mathcal{D}^{ext-}$, which we call *maximal* and *minimal* instances. We then provide a heuristic to solve the reconstruction problem for the entire class, that turns out to work perfectly for small-size degree sequences, and whose performance decreases on increasing the size of the sequence. Despite the algorithm does not always provide the reconstruction of the whole incidence matrix, it is important to point out that the obtained partially reconstructed hypergraph is free from wrong edge insertions. This property can lead to a new research line, concerning the study of error affected degree sequences reconstruction, together with the possibility of providing bounds to the number of (non-isomorphic) 3-hypergraphs sharing the same degree sequence.

2. Basic notions and definitions

We recall the basic definitions concerning hypergraphs and we fix the notation we are going to use. A hypergraph H is defined as a pair of sets $H = (V, E)$ such that $V = \{v_1, \ldots, v_n\}$ is the set of vertices and $E \subset \mathcal{P}(V) \setminus \{\emptyset\}$ is the multiset of hyperedges (briefly, edges), with $\mathcal{P}(V)$ the power set of V. A hypergraph is *simple* if it does not contain either singleton or repeated edges, and it is called k*-uniform* (briefly, k-hypergraph) if every hyperedge has exactly k vertices.

Given a k-hypergraph H and one of its vertices v, the *link hypergraph* of v in H, denoted $L_H(v)$, is defined as the hypergraph obtained from H after deleting all edges not containing v, and then removing v from all remaining edges. The *residual of* v, indicated H_v^-, is defined as the k-hypergraph obtained from H after deleting all edges containing v and the vertex v itself. It is worthwhile noticing that the link hypergraph $L_H(v)$ is $(k-1)$-uniform, while the residual hypergraph H^- is k-uniform. The *degree* of a vertex v is the number of hyperedges that contain v, and the *degree sequence* of H is the list of its vertex degrees, usually arranged in non-increasing order. A common representation of a hypergraph is using its *incidence matrix*, that is a $m \times n$ binary matrix where $m = |E|$ and $a_{i,j} = 1$ if and only if the vertex v_j belongs to the edge e_i. It is clear that the column sums of the incidence matrix of a hypergraph H gives its degree sequence, while the row sums gives the sequence of the edge cardinalities. If H is k-uniform, then the row sums is the k constant vector. We observe that the property of being simple implies that all rows of the incidence matrix are different.

3. The class $\mathcal{D}^{ext}$

We recall that the problem of reconstructing k-hypergraphs from their degree sequences is, in general, NP-hard. In [9], the authors provided a proof that relies on a reduction involving the NP-complete problem *3-partition*. In an intermediary step, they defined a class of 3-hypergraphs used as *gadget* in the reduction. A generalization of this class, denoted $\mathcal{D}$, shows interesting combinatorial properties illustrated in [2], such as the uniqueness of its elements. These remarks supported the idea that the elements of $\mathcal{D}$ can be reconstructed in polynomial time from their degree sequences.
The class $\mathcal{D}$ is defined starting from a weakly decreasing integer sequence $s = (s_1, \dots, s_n)$, with $n \geq 3$ and whose elements belong to $\mathbb{Z}$. We define a 3-hypergraph $H(s)$ with n vertices $v_1, \dots, v_n$ and whose hyperedges are the triplets (v_i, v_j, v_k) such that $s_i + s_j + s_k > 0$ (see Fig. 1 for an example). The class $\mathcal{D}_n$ is the set of all degree sequences of hypergraphs generated from a sequence s of length n, and the class $\mathcal{D}$ is the union of the classes $\mathcal{D}_n$ for each $n \geq 3$. It is known that for each sequence $\pi \in \mathcal{D}$ there exists exactly one 3-hypergraph (up to isomorphism) that realizes it (see [2]), so we will equivalently refer to the elements of $\mathcal{D}$ either as 3-hypergraphs or as degree sequences. The following property directly follows from the construction of H.

Property 1. *Given π the degree sequence obtained from $s = (s_1, \dots, s_n)$ and H the related hypergraph, if (v_i, v_j, v_k) is an edge of H and $j < k' < k$, then $(v_i, v_j, v_{k'})$ is also an edge of H.*

The way of generating the elements of $\mathcal{D}$ suggests their representation as ideals of a partially ordered set (poset). The reader is addressed to the book of D. West [25] for the definitions, main properties and notation related to the algebraic structures introduced in the sequel. So, let us define the following poset: for each positive integer n, Ω_n is the set of triplets (a_1, a_2, a_3) where $a_i \in \{1, \dots, n\}$ and $1 \leq a_1 < a_2 < a_3 \leq n$. The triplets in Ω_n can be regarded as the hyperedges of the complete 3-uniform hypergraph defined on n vertices, $v_1, \dots, v_n$. Then, we define the following linear extension of the $\leq$ order on the elements in Ω_n:

$$(a_1, a_2, a_3) \preceq (b_1, b_2, b_3) \text{ if and only if } a_i \leq b_i \text{ with } i \in \{1, 2, 3\}.$$

Let $\mathcal{T}_n = (\Omega_n, \preceq)$ be the partially ordered set thus obtained. Given $x \in \mathcal{T}_n$, the *principal ideal* $\downarrow \{x\} = \{y \in \mathcal{T}_n \text{ s.t. } y \preceq x\}$ is the intersection of all ideals that contain the element x. Since $\mathcal{T}_n$ is a finite poset, the union of ideals is finite and it is still an ideal and, furthermore, an ideal $I \subseteq \mathcal{T}_n$ can always be obtained as the finite union of principal ideals, $I = \downarrow \{x_1, \dots, x_m\} = \downarrow \{x_1\} \cup \dots \cup \downarrow \{x_m\}$ for some $x_1, \dots, x_m \in \mathcal{T}_n$.
An element $m \in I$ is *maximal* if there is no $b \in I$ such that $m \prec b$. The maximal elements of an ideal I form an *antichain*, that is a subset $A \subseteq \mathcal{T}_n$ in which no two distinct elements are comparable. It is known that the antichain of its maximal elements generates an ideal, i.e., $I = \downarrow \{A\}$, and every ideal of the poset is generated by the antichain of its maximal elements [25]. So, there is a bijective correspondence between ideals and antichains of $\mathcal{T}_n$.

Proposition 1. *Let H be a hypergraph in $\mathcal{D}_n$ and E its edge set. Then, E is an ideal in $\mathcal{T}_n$.*

Proof. Let $H \in \mathcal{D}_n$ be generated by the non-increasing integer sequence s, and (i, j, k) is one of its edges; then $s_i + s_j + s_k > 0$ by definition. We have that $s_{i'} + s_{j'} + s_{k'} > 0$ for all $(i', j', k') \preceq (i, j, k)$ and, consequently, (i', j', k') is an edge of H. Then, the edges of H are union of ideals in $\mathcal{T}_n$. □

Example 1. *The degree sequence $\pi = (8, 7, 7, 6, 6, 2)$ is in $\mathcal{D}_6$, generated by the integer sequence $s = (2, 1, 1, 0, 0, -2)$. The incidence matrix of the 3-hypergraph that realizes it is reported in Fig. 1. We note that its edges, considered as elements in $\mathcal{T}_6$, form the ideal $I_\pi =\downarrow \{(1, 3, 6), (3, 4, 5)\}$.*

However, not all ideals of $\mathcal{T}_n$, regarded as hypergraphs, are in $\mathcal{D}_n$. A counterexample can be found in [2]. Starting from this observation, we define the class $\mathcal{D}_n^{ext}$ as the class of the n-length degree sequences of 3-hypergraphs whose hyperedges form an ideal in $\mathcal{T}_n$. Obviously, $\mathcal{D}_n^{ext}$ is a proper superset of $\mathcal{D}_n$, as well as $\mathcal{D}^{ext} = \bigcup_n \mathcal{D}_n^{ext}$ is a proper superset of $\mathcal{D}$. In general, when we pass to the class $\mathcal{D}^{ext}$ we can lose the property of uniqueness, as shown in the next example.

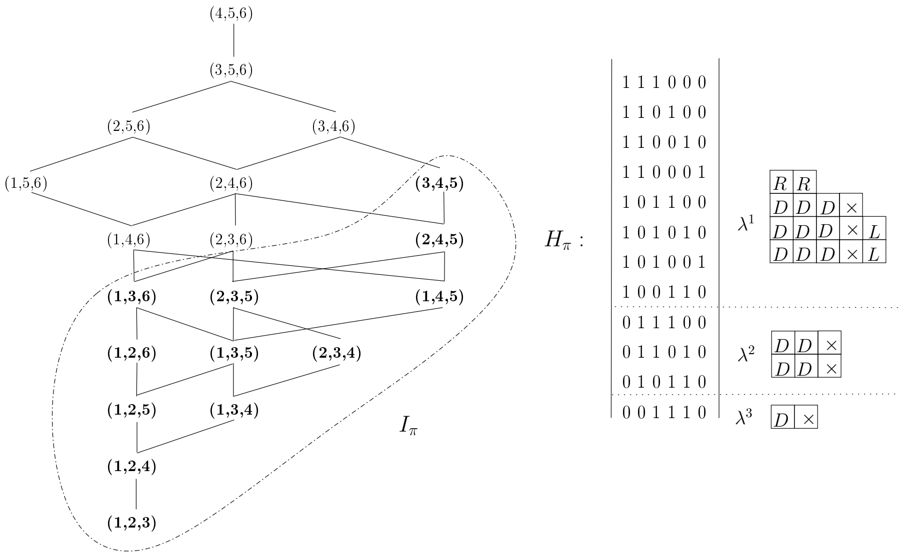

Figure 1: On the right, the (incidence matrix of the) 3-hypergraph H_π, whose degree sequence $\pi = (8, 7, 7, 6, 6, 2) \in \mathcal{D}_6$ is generated by $s = (2, 1, 1, 0, 0, -2)$. The decomposition in its three block graphs and the related Ferrers diagrams with the L, R and D labels are also provided; these notions will be introduced in the next two subsections. On the left, the complete poset $\mathcal{T}_6$, where we highlight in boldface the ideal I_π whose elements are the edges of H_π.

Example 2. *Let us consider the degree sequence $\pi^* = (25, 19, 17, 16, 12, 11, 9, 8, 6)$. It is in $\mathcal{D}^{ext}$, since it is realized by a hypergraph that is in correspondence with an ideal in the poset $\mathcal{T}_9$, that is $H_1 =\downarrow \{(1, 6, 9), (2, 3, 9), (2, 5, 7), (3, 4, 8)\}$. Moreover, there exists a second ideal in $\mathcal{T}_9$ realizing it, $H_2 =\downarrow \{(1, 5, 9), (1, 7, 8), (2, 4, 9), (3, 4, 7), (3, 5, 6)\}$. It is easy to check that the two hypergraphs are not isomorphic, so that the uniqueness property is lost for $\pi^* \in \mathcal{D}^{ext} \setminus \mathcal{D}$.*

It is interesting noticing that the class $\mathcal{D}^{ext}$ includes the non-unique degree sequences that model the 3-partition instances used in the NP-completeness proof by Deza et al. in [9]. So, the related polynomial time reconstruction algorithms and heuristics we are going to define have obviously to avoid these elements and focus on the subclass $\mathcal{D}^{ext-}$, that consists of the unique sequences of $\mathcal{D}^{ext}$ only. This class properly includes the class $\mathcal{D}$, as shown in [2], Example 2. A deeper inspection of the incidence matrix of the hypergraphs in $\mathcal{D}^{ext}$ will reveal further remarkable combinatorial properties of these objects.

The representation of hypergraphs as plane partitions
The incidence matrix of a hypergraph $H \in \mathcal{D}^{ext}_n$ can be recursively split into *block graphs*, each of them corresponding to a vertex of H: the first one is the link hypergraph related to the vertex v_1, denoted by $L_H(v_1)$; its degree sequence will be indicated as λ^1. Recursively, we define the i-th block graph of H to be the link hypergraph of the $(i-1)$-th residual hypergraph related to the vertex v_i. By abuse of notation, we indicate both the block graph and its degree sequence as λ^i. We underline that the name *block graph* is due to the fact that since H is 3-uniform, then each of its link hypergraphs is a graph. Moreover, by Property 1, we can see that the incidence matrix H is such that each residual graph has a block structure, in the sense that if the edge (j,k) is in λ^i, then (j,k') is in λ^i too, for each k' s.t. $j < k' < k$. We also note that the maximal number of block graphs is $n-2$, when H is the complete 3-hypergraph on n vertices.
For each block graph we consider the Ferrers diagram of its degree sequence regarded as an integer partition (see Fig. 1, the three Ferrers diagrams on the right). We represent these diagrams both with a sequence of bars or with the associated binary matrix. So, in each diagram λ^i the height of the j-th bar is the degree of the vertex v_{j+i} in the block graph λ^i.
Piling up the Ferrers diagrams of the block graphs λ^i, matching the columns that refer to the same vertex and starting, on each layer, from the i-th row on, we obtain a plane partition. We recall that a plane partition of the integer z of dimension $m \times n$ is a two-dimensional integer $m \times n$ matrix P such that $z = \sum_{k,j} P_{k,j}$, with $1 \leq k \leq m$ and $1 \leq j \leq n$. An overview of these combinatorial structures together with their properties can be found in [23]. Obviously, each (unique) sequence $\pi \in \mathcal{D}^{ext-}$ can be uniquely associated with a plane partition, so that the plane partition is actually an alternative representation of the hypergraph that realizes π.
A plane partition can be naturally visualized as a stack arrangement of unitary cubes of height $P_{k,j}$ lying on the point (k,j) of the plane, thus obtaining a three-dimensional object (see Fig. 2). Each Ferrers diagram λ^i is the *plane* P^i of the plane partition, that is the set of unit cubes on varying of k and j for a fixed value of height, i, whose lower left point is in position (i,i) (we enumerate rows and columns from bottom to top and from left to right, respectively). The height of the plane partition on the point (k,j) is given by the value $P_{k,j}$.
Figure 2 depicts the plane partition related to the hypergraph in Example 1. We stress that P^i corresponds to the Ferrers diagram of λ^i.
It is important to underline that P_π can be constructed up to the knowledge of the hypergraph H_π, and that the construction of P_π starting from the degree sequence π is equivalent to the reconstruction of the incidence matrix of the hypergraph H_π. It turns out that the representation of H as a plane partition allows us to uniquely detect the edges of the hypergraph H: the edge $(v_i, v_j, v_k) \in H$ is identified by the elements of the Ferrers diagram λ^i in positions $(k-i-1, j-i)$ and $(j-i, k-i)$. As an example, in Fig. 1 the edge $(1,2,6)$ of H_π is identified by the $\mathbf{R}$ box in $(4,1)$ and the bottom $\mathbf{L}$ box in $(1,5)$ of λ^1, while both boxes in λ^3 identify the edge $(3,4,5)$. The placement of the lower leftmost box of each λ^i at coordinates (i,i) is

$$P_\pi = \begin{matrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

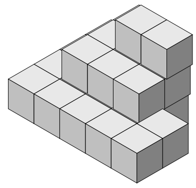

Figure 2: The plane partition associated with the degree sequence $\pi = (8, 7, 7, 6, 6, 2) \in \mathcal{D}_6^{ext-}$, represented as a matrix (on the left) and a stacking of unit cubes (on the right).

essential to maintain the correspondence between the vertices of H and the coordinates of the plane partition.

Properties of block graphs and plane partitions in $\mathcal{D}^{ext}$
Inspecting the Ferrers diagram of an integer partition one can immediately detect if it is the degree sequence of a graph. Indeed, a Ferrers diagram F_λ associated with an integer partition λ can always be decomposed into three (possibly empty) partitions $F_\lambda = D(\lambda) + R(\lambda) + L(\lambda)$, where $D(\lambda)$ is the largest square entirely contained in F_λ, called *Durfee square*, $R(\lambda)$ is the set of cells over $D(\lambda)$ and $L(\lambda)$ is the set of cells placed on the right of $D(\lambda)$ after removing the first cell of every row [22], see Fig. 3. It is known that λ is graphical if and only if $R(\lambda) \leq_d L(\lambda)'$ holds (see [17, 22]), where $L(\lambda)'$ is the conjugate partition of $L(\lambda)$, i.e., the partition obtained from $L(\lambda)$ by exchanging rows and columns, and $\leq_d$ is the dominance order [6]. We recall that provided two integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$, $\lambda \leq_d \mu$ holds if and only if $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$ for any $1 \leq k \leq n$. If $R(\lambda) = L(\lambda)'$ holds, the partition λ is a *maximal graphical partition* (see Fig. 3).

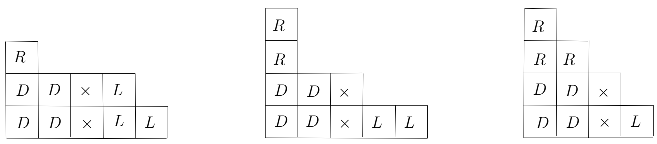

Figure 3: The Ferrers diagrams of three different partitions of $n = 10$, together with their decomposition. From left to right, a graphical partition, a maximal graphical partition and a partition that is not graphical.

Proposition 2. *Given $H \in \mathcal{D}_n^{ext}$ and its block graphs, for all $1 \leq i \leq n-2$ the integer sequence λ^i is a maximal graphical partition.*

Proof. Let us proceed by contradiction assuming that there exists an index i such that λ^i is not maximal. Since λ^i is graphical, $R(\lambda^i) <_d L(\lambda^i)'$ holds. By definition of dominance order, the smallest point (j, k) where they differ is such that $(j, k) \in L(\lambda^i)$ and $(k-1, j) \notin R(\lambda^i)$ (keeping the rows and the columns indexing bottom-up and left-right, respectively). This means that the edge $(i, j+i, k+i) \notin H$. Furthermore, $(i, j', k+i) \in H$ for each $i < j' < j+i$, by the minimality of (j, k), so there exists an index $j'' > j+i$ such that $(i, j'', k+i) \in H$. This leads to a contradiction since, by definition of $\mathcal{D}^{ext}$, the edges of H form an ideal of $\mathcal{T}$. $\square$

Again, Figure 1 clarifies the above proof and provides a representation of the sequences λ^i obtained from $\pi = (8, 7, 7, 6, 6, 2) \in \mathcal{D}_6^{ext}$ as Ferrers diagrams. A visual inspection shows that they are all maximal graphical partitions ($R(\lambda^i) = L(\lambda^i)'$ for each $i = 1, 2, 3$).
The complete 3-hypergraph on n vertices $\mathbb{H}_n$ has its generic block graph $\lambda^i = (n-i-1, \dots, n-i-1)$ of length $n-i$, for all $i = 1, \dots, n-2$, and its Ferrers diagram is a $(n-i) \times (n-i-1)$ rectangle. We stack each λ^i diagram placing its lower leftmost box in position (i, i), and we get the plane partition

$$\mathbb{P}_n = \begin{array}{ccccccc} 1 & 2 & 3 & \dots & n-3 & n-2 & n-2 \\ 1 & 2 & 3 & \dots & n-3 & n-3 & n-3 \\ \vdots & \vdots & & & & & \vdots \\ 1 & 2 & 3 & \dots & \dots & \dots & 3 \\ 1 & 2 & 2 & \dots & \dots & \dots & 2 \\ 1 & 1 & 1 & \dots & \dots & \dots & 1 \end{array}$$

Proposition 3. *Let P_π be a plane partition related to a sequence $\pi \in \mathcal{D}_n^{ext}$. There exists a submatrix P_{π^*} of P_π which is the plane partition of the complete 3-hypergraph on c vertices, for some $3 \leq c \leq n$.*

Proof. Let t be the number of planes of P_π, i.e. the maximum entry of the matrix. By construction, the submatrix composed by the last t rows and first $t+1$ columns is the plane partition related to the complete 3-hypergraph on $c = t+2$ vertices. $\square$

It directly follows that the incidence matrix of H_π always contains a submatrix H^* (called the *core* of H_π) which is the matrix of a complete 3-hypergraph on its first $c \leq n$ vertices. We underline that the plane partition representing H_π is composed of $c-2$ planes, and that the i-th plane represents all those hyperedges whose first element is the vertex v_i.
For each sequence $\pi \in \mathcal{D}^{ext}$, we get the following standard decomposition of the related plane partition

$$P_\pi = P_{\pi^*} + R_\pi + L_\pi,$$

with P_{π^*} the core, R_π given by the set of rows placed over the submatrix P_{π^*} and L_π given by the set of columns on the right side of the submatrix of the core adding the row immediately over. By abuse of notation, we define $L'_\pi = \bigcup_{i=1}^{c-2} L_\pi(\lambda^i)'$.

Property 2. *For each sequence $\pi \in \mathcal{D}^{ext}$, $R_\pi = L'_\pi$ holds.*

This property directly follows from the maximality of block graphs in H_π and the symmetry properties of their Ferrers diagrams.

4. The reconstruction problem in the class $\mathcal{D}^{ext-}$

In this section we introduce the sets of maximal and minimal degree sequences in $\mathcal{D}^{ext-}$, and we provide a polynomial time algorithm for their reconstruction. Then, we move the spotlight to the whole class and we provide a heuristic to determine the 3-hypergraphs related to its elements.

4.1. The reconstruction of maximal and minimal instances

We first define the algorithm *GColRec* (Greedy Column Reconstruction), that reconstructs a subset of degree sequences of $\mathcal{D}^{ext-}$ called *maximal instances*. The inputs of *GColRec* are an element $\pi = (\pi_1, \ldots, \pi_n)$ of $\mathcal{D}^{ext-}$ and its supposed core dimension c, while its output is either a 3-uniform hypergraph H consistent with π or failure.

Algorithm 1. *GColRec(π,c)*

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initialize H to the complete 3-hypergraph on the first c vertices and let π_c be its degree sequence;
update $\pi = \pi - \pi_c$;
for $i = 1 : c - 2$ **do**
  for $j = i + 1 : n - 1$ **do**
    for $k = j + 1 : n$ **do**
      if $\pi_i > 0$ *and* $\pi_j > 0$ *and* $\pi_k > 0$ **then**
        insert (i, j, k) in H and update the three elements $\pi_i = \pi_i - 1$, $\pi_j = \pi_j - 1$, $\pi_k = \pi_k - 1$;
      else
        break;
if $\pi \neq 0$ **then**
  return failure;
Output: H

Since the dimension of the core is not known in advance, we consider $n-2$ parallel computations, one for each possible value of c. The subset of the maximal instances of $\mathcal{D}^{ext-}$ is indicated by $\mathcal{M}ax$.

Proposition 4. *Let $\pi \in \mathcal{D}_n^{ext-}$ and let $R(\lambda^i)$ be an integer partition of the number z_i, for $i = 1, \ldots, c-2$. If $\pi \in \mathcal{M}ax$, then the partition $R(\lambda^i)$ is the maximum, w.r.t. dominance order, among all integer partitions of z_i in which each part is less or equal than $n - c$.*

Proof. Immediate from the definition of the reconstruction strategy *GColRec*, that maximizes the height of the bars of each block graph, from left to right. According to the length of the sequence and the dimension of its core, it is clear that each value in $R(\lambda^i)$ can not exceed $n - c$. □

We underline that the class $\mathcal{M}ax$ properly includes the class of maximal instances defined in [2], where the core computation is not present.

Example 3. *The degree sequence $\pi = (15, 15, 15, 13, 10, 10, 6, 3)$ is a maximal instance. Its block graphs, considered as integer partitions, are the maximum according to π with $c = 6$,*

$$\lambda^1 = (6, 6, 5, 4, 4, 3, 2) \qquad \lambda^2 = (5, 4, 3, 3, 2, 1) \qquad \lambda^3 = (3, 2, 2, 1) \qquad \lambda^4 = (1, 1).$$

A second greedy strategy called *GRowRec* can be defined from *GColRec*, performing the construction of each $R(\lambda^i)$ row-by-row, from λ^1 to λ^{c-2}. We define the class of *minimal instances*, $\mathcal{M}in$, as the set of the sequences in $\mathcal{D}^{ext-}$ that are correctly reconstructed by *GRowRec*. The notion of minimality follows from the greedy choices of the strategy, similarly to Proposition 4.

Theorem 1. *GColRec (respectively, GRowRec) performs in $O(n^4)$ time.*

The proof is immediate. A simple check shows that the union of the subclasses $\mathcal{M}ax$ and $\mathcal{M}in$ is strictly included in $\mathcal{D}^{ext-}$. As a matter of fact, the sequence $\pi = (21, 20, 18, 15, 12, 11, 10, 7, 3)$ is in $\mathcal{D}^{ext-} \setminus (\mathcal{M}ax \cup \mathcal{M}in)$.
We stress that the failures of the algorithms *GColRec* and *GRowRec* are due to the insertion of extra edges that do not belong to the (unique) final solution. Furthermore, wrong insertions may determine a reconstruction failure some steps ahead in the computation, so that, in general, they cannot be immediately detected, requiring an unbounded backtrack process to fix the errors. In the next section we describe a heuristic that avoids wrong edge insertions, but sometimes produces only a partial reconstruction of a 3-hypergraph in $\mathcal{D}^{ext-}$.

4.2. A heuristic for the class $\mathcal{D}^{ext-}$

Let us consider the reconstruction of the class $\mathcal{D}_n^{ext-}$. We introduce the notion of *complementarity*: given a 3-hypergraph H on n vertices, we denote $d_M = \binom{n-1}{2}$ the maximum admitted degree of each vertex, that corresponds to the common degree in the complete 3-hypergraph on n vertices $\mathbb{H}_n$. For a given $\pi \in \mathcal{D}_n^{ext}$, we define its *complementary sequence* $\overline{\pi} = (d_M, \ldots, d_M) - \pi$. It is clear that $\overline{\pi}$ is the degree sequence of the *complementary* 3-hypergraph $\overline{H}_\pi$ with n vertices and containing all edges not in H_π. We underline that $\overline{H}_\pi$ keeps all properties of the hypergraphs of the class $\mathcal{D}_n^{ext}$, in particular the strong block-structure of the incidence matrix (and the uniqueness if we restrict to $\mathcal{D}^{ext-}$). Similarly, we define the degree sequences of the complementary block graphs in $\overline{H}_\pi$ as $\overline{\lambda}^i = (n - i - 1) - \lambda^i$ for $i = 1, \ldots, n-2$, and the related complementary plane partition as $\overline{P}_\pi = \mathbb{P}_n - P_\pi$, with $\mathbb{P}_n$ the plane partition associated to $\mathbb{H}_n$. From now on, we will omit the subscript that identifies the degree sequence π when no ambiguities may arise.
The heuristic we propose starts from a sequence $\pi \in \mathcal{D}_n^{ext-}$ and requires a parallel reconstruction of both H and its complementary $\overline{H}$ at the same time, avoiding the insertion of extra edges during the process. All edges of $\mathbb{H}_n$ are considered and assigned either to H or to $\overline{H}$, if no ambiguities occur, otherwise are kept unassigned. The hypergraphs H and $\overline{H}$ are modelled by two three-dimensional matrices that represent their plane partitions P and $\overline{P}$, respectively. Both matrices are initialized to the null matrix, and their elements are updated to the value 1 according to the detected edges in H and $\overline{H}$. The heuristic, indicated as *HeuRec*, is composed by the following three steps:

Step 1

The pseudo-code of the first step of the reconstruction is provided in Algorithm 2, *Preprocessing*, that computes from the input sequence π of length n an admitted dimension of the core c

and the complementary sequence $\overline{\pi}$. Then, it assigns to H all hyperedges of the core, i.e., all hyperedges of $\mathbb{H}_c$ on the vertices $v_1, \ldots, v_c$. After their insertion, the sequence π is updated by subtracting the common degree value $d_c = \frac{(c-2)(c-1)}{2}$ of $\mathbb{H}_c$ from the first c elements of the input sequence, obtaining the sequence π^0. Since the dimension of the core is not known in advance, we perform $n-2$ parallel computations, one for each possible value of c.

Property 3. *Let c be the dimension of the core of the degree sequence π. An edge (v_i, v_j, v_k) in H_π whose minimum vertex index i is greater than $c-2$ does not exist.*

It directly follows from the definition of core of a plane partition. As a consequence, all edges whose minimum vertex is v_i, with $i > c-2$, are assigned to $\overline{H}$, and they form the complete 3-hypergraph defined on the last $n-(c-2)$ vertices. After their insertion in $\overline{H}$, *Preprocessing* creates the sequence $\overline{\pi}^0$ by subtracting the value $\overline{d}_c = \frac{(n-c)(n-c+1)}{2}$ from the last $n-(c-2)$ elements of $\overline{\pi}$.

Theorem 2. *Preprocessing performs in $O(n^3)$ time.*

Proof. The plane partition related to the hypergraph H can be represented as a three-dimensional matrix whose dimensions are $(n-2) \times (n-1) \times (c-2)$ for a hypergraph on n vertices and with core dimension c. The same for $\overline{H}$. The insertion of the edges of the core requires to iterate through the whole matrix, procedure that runs in $O(n^3)$. □

The reconstruction of each plane of P and $\overline{P}$ proceeds in Step 2, starting from the updated sequences $\pi^0 = (\pi^0_1, \ldots, \pi^0_n)$ and $\overline{\pi}^0 = (\overline{\pi}^0_1, \ldots, \overline{\pi}^0_n)$.

Algorithm 2. *Preprocessing(π)*

Input: π

compute $\overline{\pi}$;

assumed c the dimension of the core, compute $d_c = \frac{(c-2)(c-1)}{2}$ and $\overline{d}_c = \frac{(n-c)(n-c+1)}{2}$;

insert all edges of the complete hypergraph on the first c vertices in H and compute π^0 by updating π;

insert all edges of the complete hypergraph on the last $n-(c-2)$ vertices in $\overline{H}$ and compute $\overline{\pi}^0$ by updating $\overline{\pi}$;

Output: H, $\overline{H}$, π^0, $\overline{\pi}^0$

Step 2

Now the heuristic inserts in H and $\overline{H}$ the edges that definitely belong to one of them. The insertions are performed either by checking the cardinalities of the possible positions or the values of the (supposed) degree sequences starting from π^0 and $\overline{\pi}^0$. The reconstruction process proceeds plane by plane from P^1 up to P^{c-2}, resp. from $\overline{P}^1$ up to $\overline{P}^{c-2}$. The edges' insertion in a generic plane P^i is sketched in Algorithm 3, *Insert-H*, and hereafter detailed. An analogous algorithm, *Insert-$\overline{H}$*, is also defined. A further definition is required: we call *R-area* the set of positions in P^i that are delimited by the upper row of the core, the i-th column and the (discrete) diagonal line of P^i (these two last included). In other words, the R-area is the part of P^i where the elements of $R(\lambda^i)$ may lie. Symmetrically, we can define the L-area of P^i that includes the elements of $L(\lambda^i)$, see Fig. 4.

Insert-H: starting from the sequence π^{i-1}, the algorithm computes the maximal and the minimal partition of the i-th element π_i^{i-1} of π^{i-1} included in the R-area of P^i. Let these partitions be p_{max} and p_{min}, respectively. The algorithm inserts in the R-area of P^i the elements common to p_{max} and p_{min}, i.e. the partition $p_{max} \cap p_{min}$. Accordingly, it inserts in the L-area their symmetric elements. It is worthwhile noticing that each element added to the R-area with its symmetric in the L-area form an edge whose first vertex is v_i, and which is common to all 3-hypergraphs sharing the same degree sequence π, if any. Example 4 clarifies this construction.

Insert-$\overline{H}$: acts analogously to compute $\overline{P}^i$ from the sequence $\overline{\pi}^{i-1}$.

Algorithm 3. *Insert-H(H,π^{i-1})*

Input: H,π^{i-1}
compute p_{max} and p_{min};
insert in the R-area of P^i the elements in $p_{max} \cap p_{min}$;
fill the L-area of P^i symmetrically with respect to the R-area;
Output: H

Example 4. *Let us consider the sequence* $\pi = (38, 32, 32, 32, 28, 21, 21, 19, 13, 12, 7) \in \mathcal{D}_{11}$, *given by* $s = (15, 8, 8, 8, 5, -2, -2, -4, -8, -11, -13)$. *We can argue from* s *the size of the core,* $c = 7$. *In the first step of HeuRec the algorithm Preprocessing gives as output the sequences* $\pi^0 = (23, 17, 17, 17, 13, 6, 6, 19, 13, 12, 7)$ *and* $\overline{\pi}^0 = (7, 13, 13, 13, 17, 14, 14, 16, 22, 23, 28)$, *together with the matrices* H *and* $\overline{H}$ *where the insertion of the edges belonging to the (respective) core has been performed. We now show the steps of the algorithm Insert-H for* $i = 1$, *whose output is depicted in Fig. 4, right. The performance of Insert-*$\overline{H}$ *is analogous. Since the length of the sequence is* $n = 11$ *and the core has size* $c = 7$, *the plane* P^1 *is a* 9×10 *rectangle in which Preprocessing already inserted the edges on the left-bottom* 5×6 *rectangle, i.e. the core (see Fig 4). In the following step, the partitions* p_{min} *and* p_{max} *of* $\pi_1^0 = 23$ *are computed. Since they must be contained in the R-area, the partitions will be* $p_{min} = (4, 4, 3, 3, 3, 3, 2, 1)$ *and* $p_{max} = (4, 4, 4, 4, 4, 3)$, *highlighted with different colors in Fig. 4, left. Insert-H performs the insertion of the elements belonging to both of them,* $p_{min} \cap p_{max}$, *pointed out in Fig. 4, left, with dashed lines. Finally, their symmetric in the L-area are also inserted, reaching the final (partial) reconstruction of* P^1, *shown in Fig 4, right.*

Theorem 3. *Insert-H (resp., Insert-*$\overline{H}$*) performs in* $O(n^2)$ *time.*

Proof. In *Insert-H* the computation of both p_{max} and p_{min} can be performed by simple arithmetic operations, resulting in a constant time complexity. The update of both $R(\lambda^i)$ and $L(\lambda^i)$ requires $O(n^2)$ time, since P^i is modelled by a matrix of dimension $O(n^2)$. □

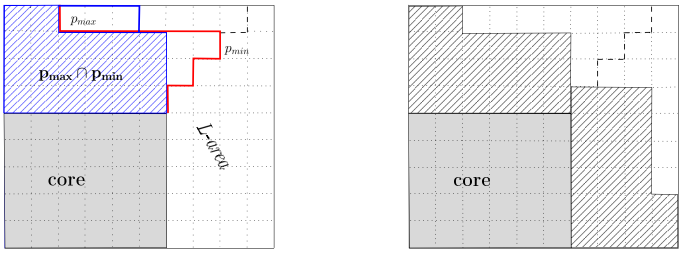

Figure 4: The figure shows the main passages of *Insert-H* on the first plane P^1 of the plane partition related to $\pi = (38, 32, 32, 32, 28, 21, 21, 19, 13, 12, 7) \in \mathcal{D}_{11}$, whose core has size $c = 7$. On the left, the computation of the elements in the *R-area* that definitively belong to the plane partition; on the right, their insertion together with the symmetric elements in the *L-area*.

Finally, the update of π^{i-1} into π^i is performed. We compute the sequence $q = (0, \ldots, 0, q_i, \ldots, q_n)$ where each q_j, with $i+1 \leq j \leq n$, counts the number of the edges involving the vertices v_i and v_j that are inside the R or L-area of H and that are not yet added in $\overline{H}$, while q_i counts the number of edges inserted in P^i. In other words, the vector q counts the edges of P^i that either belong to H or still maintain their placement ambiguity.
Then, we define $\pi^i = \pi^{i-1} - q$. In general, q_i can be different from π_i^{i-1}. Such a discrepancy is useful to keep possible all ways of inserting the edges in the P^{i+1} plane. A symmetric computation leads to the update of $\overline{\pi}^{i-1}$ to $\overline{\pi}^i$. The following property is immediate.

Property 4. *Let π be the degree sequence of a 3-hypergraph with core dimension c. If Insert-H adds the edge (i, j, k) to the plane P^i of H, with $i < j < k$, then it also belongs to all 3-hypergraphs whose degree sequence is π.*

Step 3

As a final step, the heuristic considers the set A of *ambiguous edges*, i.e. the edges that have not yet been inserted in H or in $\overline{H}$, and the updated sequences $\pi = \pi^{c-2}$ and $\overline{\pi} = \overline{\pi}^{c-2}$ obtained after the performance of Step 2. In this final step the heuristic benefits of the membership of π in $\mathcal{D}_n^{ext}$, in particular of the ideal (and filter) characterization w.r.t. the poset $\mathcal{T}_n$.
If $(i, j, k) \preceq (i', j', k')$ in $\mathcal{T}_n$ then, for the related edges e_1 and e_2, respectively, it holds that if $e_2 \in H$ then $e_1 \in H$ (symmetrically, if $e_1 \in \overline{H}$ then $e_2 \in \overline{H}$, as stated in Property 1).
The procedure described in Algorithm 4, *Poset*, is iteratively repeated on each edge $m \in A$ until no further edge insertions in H and $\overline{H}$ are possible. The heuristic produces a successful reconstruction if all edges of A are inserted in H or $\overline{H}$. The matrix H is the final output.

Theorem 4. *Poset performs in $O(n^3)$ time.*

Proof. Both the check of the maximality and minimality of an element m in A, and the related computation of I and F, require a scan of the whole set, performed in $O(n^3)$ time. The update of the sequences π^* and $\overline{\pi}^*$, together with the possible insertion of m, run in $O(n)$ time. Summing up, the whole *Poset* procedure requires $O(n^3)$ steps. □

Algorithm 4. *Poset(H,$\overline{H}$,π,$\overline{\pi}$,A,m)*

Input: H, $\overline{H}$, π, $\overline{\pi}$, A, m

if *m is a maximal element in $\mathcal{T}_n$* **then**
    compute $I = \downarrow \{m\} \cap A$;
    compute ρ the degree sequence of the 3-hypergraph I and $\pi^* = \pi - \rho$;
    if *there exists i such that $\pi^*_i < 0$* **then**
        insert m in $\overline{H}$ and update $\overline{\pi}$;
        remove m from the set A;
if *m is a minimal element in $\mathcal{T}_n$* **then**
    compute $F = \uparrow \{m\} \cap A$;
    compute ρ the degree sequence of the 3-hypergraph F and $\overline{\pi}^* = \overline{\pi} - \rho$;
    if $\overline{\pi}^*_i < 0$ *for some i* **then**
        insert m in H and update π;
        remove m from the set A;

Output: H, $\overline{H}$, π, $\overline{\pi}$, A

Remark 1. *The algorithm Poset does not perform wrong edge-insertions, since it relies on the properties of ideal and filter of H and $\overline{H}$ only, without any further assumption.*

We stress again that *Poset* terminates in a finite number of iterations, at most $|A|$, leading to the (unique) solution or to a partial reconstruction where no wrong insertions are performed.

Theorem 5. *HeuRec performs in polynomial time with a total cost of $O(n^{10})$.*

Proof. It directly follows from the previous analysis. In particular, Step 1 is performed in $O(n^3)$, see Theorem 2. In Step 2, *Insert-H* and *Insert-$\overline{H}$* run on each of the $c - 2$ planes of the partition, for a total cost of $O(n^3)$, see Theorem 3. Finally, Step 3 requires a running time of $O(n^9)$, since *Poset* can be performed $|A|$ times at most, with $|A| = O(n^3)$, see Theorem 4. Then, we have that the total cost of the three steps is $O(n^9)$. Since the three steps are required to be performed for n times at most (one for each value of c), the total running time is $O(n^{10})$. □

Example 5. *The sequences $\pi_1 = (95, 95, 52, 50, 47, 44, 30, 30, 24, 17, 16, 16, 13, 13, 9, 9, 9, 9, 9, 7)$ and $\pi_2 = (38, 32, 32, 32, 28, 21, 21, 19, 13, 12, 7)$ are two sequences of different length in $\mathcal{D}^{ext-}$ that are neither maximal nor minimal instances. $HeuRec$ correctly reconstructs the corresponding hypergraphs. On the other hand, let us consider the sequence $\pi = (51, 48, 39, 36, 32, 31, 29, 27, 24, 20, 15, 11) \in \mathcal{D}^{ext}_{12}$. HeuRec does not fully reconstruct the related 3-hypergraph, as the computation stops with a non-empty set A of ambiguous edges. Moreover, π is a maximal instance that can be reconstructed by GColRec. As a matter of fact, HeuRec does not extend GColRec.*

Statistical results

We checked the performance of *HeuRec* on randomly generated sequences $\pi \in \mathcal{D}$, varying their length. We underline that the computation of sequences in $\mathcal{D}$ instead of $\mathcal{D}^{ext-}$ allows to guarantee the uniqueness property without limiting or affecting the heuristic performance. Table 1 shows the success rate according to the different lengths of π up to $n = 30$. The statistics was obtained as the mean of several runs of $100K$ block trials for each length.

$n \leq 9$ 100%	$n = 10$ 99.998%	$n = 11$ 99.95%	$n = 12$ 99.78%	$n = 13$ 99.34%	$n = 14$ 98.43%	$n = 15$ 96.75%	$n = 16$ 93.95%	$n = 17$ 90.25%	$n = 18$ 84.87%	$n = 19$ 78.54%
$n = 20$ 71.28%	$n = 21$ 62.95%	$n = 22$ 54.79%	$n = 23$ 46.85%	$n = 24$ 39.14%	$n = 25$ 32.42%	$n = 26$ 26.07%	$n = 27$ 20.79%	$n = 28$ 16.37%	$n = 29$ 12.70%	$n = 30$ 9.70%

Table 1
Success rate of *HeuRec* on sequences of the class $\mathcal{D}_n$, up to $n = 30$. We generated all sequences of the classes until $n = 6$. From $n = 7$ to $n = 30$, we performed the algorithm on several blocks of $100K$ randomly generated sequences. What here reported is the mean of the performance of the blocks.

It is known that the cardinality of the class $\mathcal{D}_n^{ext}$ has an exponential growth rate (w.r.t. to the length n of its elements), see [2]. From the obtained data, we observe a high success rate for small degree sequences, while the performance drastically deteriorate when n increases (second line of Table 1), as expected. From Example 5, the possibility of running on the same input sequence π the three algorithms, *HeuRec*, *GColRec* and *GRowRec*, can be considered to improve the set of instances that are fast reconstructable. Unfortunately, not all sequences in $\mathcal{D}^{ext-}$ can be reconstructed by the three algorithms: as an example, consider $\pi = (70, 55, 49, 36, 33, 32, 28, 20, 20, 17, 17, 16, 10, 8, 6)$ in $\mathcal{D}$. In view of achieving a polynomial time reconstruction strategy for the class $\mathcal{D}^{ext-}$, the possibility of defining new techniques that reduce the set A of the ambiguous edges we get at the end of the run deserves deeper investigation.

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