Formulae for the solution of an analogical equation between Booleans using the Sheffer stroke (NAND) or the Pierce arrow (NOR)

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Abstract

This paper gives a formula for the solution of an analogical equation between Booleans using the Sheffer stroke (NAND). Naturally, a counterpart using the Pierce arrow (NOR) is also given. Although not so intuitive, these formulae are somewhat elegant. The formulae are obtained in the following way: a rapid review on analogies between sets is given. The result on sets is transposed to Booleans. This result is rewritten using solely the operators mentioned above and simplified.

Keywords

Boolean analogies, Analogies on sets, Sheffer stroke (NAND), Pierce arrow (NOR)

1. Introduction

An axiomatic approach (Section 2) that postulates reflexivity ($A : B :: A : B$) and symmetry ($C : D :: A : B$) of conformity (::), in addition to the exchange of the means ($A : C :: B : D$), for any analogy $A : B :: C : D$, allows to define analogy on commutative magmas and commutative monoids (Section 3). The additional postulate of contiguity (the same analogy should hold on the inverse of objects) allows to define analogies on commutative groups (Section 4). Adding the postulate of similarity (all features in A should appear in B or C) is used to determine the solution of analogical equations between sets in [1] (Section 5). With all the above, the analogy induced by (a) the structure of the commutative groups $(\mathcal{P}(E), \triangle)$ or $(\mathcal{P}(E), \triangledown)$[1] is the same as the analogy induced by (b) the two monoids $(\mathcal{P}(E), \cup)$ and $\mathcal{P}(E), \cap)$ holding at the same time, under the condition

$$A \subset B \cup C \quad \wedge \quad B \cap C \subset A \tag{1.1}$$

(Section 5). This condition eliminates two cases of discrepancy between the analogies induced by (a) and (b). The solution D of an analogy between sets $A : B :: C : D$ is then:

$$D = ((B \cup C) \setminus A) \cup (B \cap C). \tag{1.2}$$

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[1] $\triangle$ for symmetrical difference on sets (corresponding to XOR on Booleans), and $\triangledown$ for its counterpart corresponding to logical equivalence on Booleans.

		General	Magma	Group
	Definition		$A \star D = C \star B$	$A \star B^{-1} = C \star D^{-1}$
(o)	Reflexivity of conformity	$A : B :: A : B$	$A \star B = A \star B$	$A \star B^{-1} = A \star B^{-1}$
For any $A : B :: C : D$			$A \star D = C \star B$	$A \star B^{-1} = C \star D^{-1}$
(i)	Symmetry of conformity	$C : D :: A : B$	$C \star B = A \star D$	$C \star D^{-1} = A \star B^{-1}$
(ii)	Inversion of ratios	$B : A :: D : C$	$B \star C = D \star A$	$B \star A^{-1} = D \star C^{-1}$
(iii)	Inversion of objects (contiguity)	$A^{-1} : B^{-1} :: C^{-1} : D^{-1}$	undefined	$A^{-1} \star B = C^{-1} \star D$
(iv)	Distribution in objects (similarity)	any feature in A must appear in either B or C or both.	undefined	undefined
(v)	Exchange of the extremes	$D : B :: C : A$	$D \star A = C \star B$	$D \star B^{-1} = C \star A^{-1}$
(vi)	Exchange of the means	$A : C :: B : D$	$A \star D = B \star C$	$A \star C^{-1} = B \star D^{-1}$

Table 1
Postulates for analogy. The last two columns transcribe the definitions to the analogy naturally induced by the structures of a magma and a group.

The purpose of this paper is to transcribe (1.2) to analogy between Booleans (Section 6). As the Sheffer stroke (Section 7) is known to be functionally complete, the formulation uses only this operator (Section 10). The same is done with the Pierce arrow (Section 8).

2. Postulates for analogy

The classical way of writing down an analogy with $A : B :: C : D$ involves two basic articulations denoted by the signs : for ratio and :: that we choose to call conformity[2]. The four terms are traditionally divided into the means B and C, and the extremes A and D. Studies in the notion of analogy in its technical sense (not in its vernacular sense of mere similarity or comparison, as in analogical reasoning) extract two underlying notions, those of similarity and contiguity.

Conformity can be postulated to be reflexive and symmetric.[3]. The ratios can be thought to be inversible[4]. From the Greek antiquity, it is considered that analogy (in its strict technical meaning) cannot go without the exchange of the means[5]. All this leads to the postulates given in Table 1.

[2]The character : (U+2236) is named ratio in the ISO 10646 standard (Unicode) and :: (U+2237) is named proportion.
[3]I.e., a dependency relation. An equivalence relation requires transitivity in addition.
[4]*Invertendo* in the Latin tradition.
[5]*Permutando* or *alternando* in the Latin tradition.

Analogy		Corners of the square		
Transformation	Equivalent form	Transformation		D_8
identity	$A : B :: C : D$	identity	$\begin{matrix} A & B \\ C & D \end{matrix}$	e
counter-clockwise rotation	$B : D :: A : C$	rotation by $\pi/2$	$\begin{matrix} B & D \\ A & C \end{matrix}$	a
inverse of reading	$D : C :: B : A$	rotation by $2\pi/2 = \pi$	$\begin{matrix} D & C \\ B & A \end{matrix}$	a^2
clockwise rotation	$C : A :: D : B$	rotation by $3\pi/2 = -\pi/2$	$\begin{matrix} C & A \\ D & B \end{matrix}$	a^3
exchange of the means	$A : C :: B : D$	symmetry first diagonal	$\begin{matrix} A & C \\ B & D \end{matrix}$	x
inversion of ratios	$B : A :: D : C$	symmetry vertical axis	$\begin{matrix} B & A \\ D & C \end{matrix}$	ax
exchange of the extremes	$D : B :: C : A$	symmetry second diagonal	$\begin{matrix} D & B \\ C & A \end{matrix}$	a^2x
symmetry of conformity	$C : D :: A : B$	symmetry horizontal axis	$\begin{matrix} C & D \\ A & B \end{matrix}$	a^3x

Table 2
Bijection between the eight equivalent forms of an analogy and the eight elements of the dihedral group D_8, i.e., the transformations of the corners of the square.

Consecutive applications of (i), (ii), (v) or (vi) in any number and in any order lead to only eight equivalent forms of the same analogy [2] which correspond to the eight possible transformations of the corners of a square, known as the dihedral group D_8 where the internal operation is composition. This bijection is given in Table 3. In the dihedral group, the choice of the two distinguished elements, a and x among the seven non-identity elements, is not totally free. The possible choices, expressed for analogy, are visualized in Figure 1. (ii) Inversion of ratios and (vi) Exchange of the means is a possible choice. (i) Symmetry of conformity and (vi) Exchange of the means is another possible choice. For this last choice, it means that the postulates (ii) and (v) are indeed dispensable.

3. Analogy induced on commutative magmas and monoids

Let $(\mathcal{E}, \star)$ be a magma, i.e., a set equipped with an internal law, without any specific property. To define analogy on such a structure, the only device offered is its internal operation. Drawing a parallel with numbers, where, for arithmetic and geometric analogies, one has $a + d = b + c$ and $a \times d = b \times c$, it is natural to posit the following equivalence to induce analogy from a

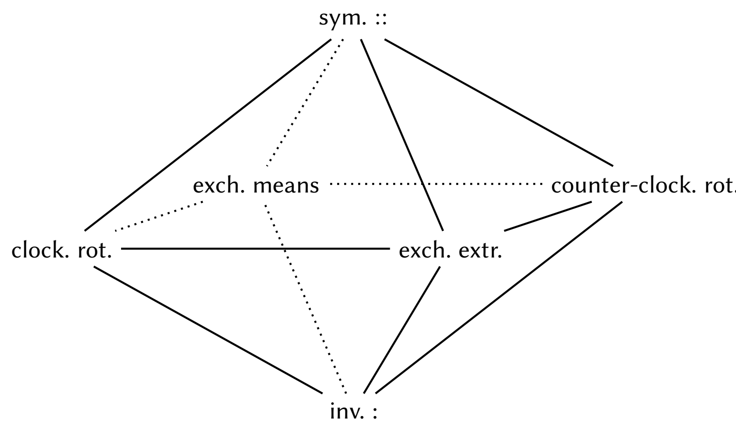

Figure 1: Any edge in the picture is a possible choice for a pair of transformations of analogy from which to get the eight equivalent forms from any analogy $A : B :: C : D$. This corresponds to selecting the elements usually denoted by a and x in the dihedral group D_8.

magma. However, observe that there are two possibilities, because the internal operation could be non-commutative.

$$\forall (A, B, C, D) \in \mathcal{E}^4, \quad A : B :: C : D \quad \overset{\text{def}}{\iff} \quad A \star D = B \star C \tag{3.1}$$

$$\text{or} \quad \forall (A, B, C, D) \in \mathcal{E}^4, \quad A : B :: C : D \quad \overset{\text{def}}{\iff} \quad A \star D = C \star B \tag{3.2}$$

With (3.1), the axiom of reflexivity of conformity would impose immediately that $\star$ be commutative because

$$\forall (A, B) \in \mathcal{E}^2, \qquad A : B :: A : B \quad \Leftrightarrow \quad \forall (A, B) \in \mathcal{E}^2, \ A \star B = B \star A.$$

For (3.2), the expression of each postulate is shown in Table 1. (o) and (i) hold because, equality being an equivalence relation, it is a dependency relation. The inverse of objects and the distribution in objects are left undefined. For all other axioms, a sufficient condition for them to be met is that $\star$ be commutative.

To summarize, to naturally induce analogy from the structure of a magma, it suffices for the internal operation to be commutative. The two definitions (3.1) and (3.2) are then the same. The axioms of object inversion and distribution within objects can be left unspecified. Observe that neither conformity nor ratio are directly defined. Finally, nothing can be said in the general case for the problem of solving an analogical equation on a commutative magma: given a triplet $(A, B, C) \in \mathcal{E}^3$, find D such that $A : B :: C : D$, i.e., find D such that $A \star D = C \star B$,

On a commutative monoid, i.e., a magma with associativity of the internal operation and a neutral element, analogy can be naturally induced in the same way as for a commutative magma.

4. Analogy induced on commutative groups

Let $(\mathcal{E}, \star)$ be a group. Let a^{-1} denote the inverse element of a.

- Ratios can be defined directly:

$$\forall (A, B) \in \mathcal{E}^2, \quad A : B \stackrel{\text{def}}{=} A \star B^{-1}. \tag{4.1}$$

  Note that this definition of the ratio is very specific: the ratio between two elements of $\mathcal{E}$ is an element of $\mathcal{E}$. This is very different from the situation with magmas in which, generally speaking, we do not know what a ratio is.
- Conformity can be defined as equality.
- The definition of analogy can then be as follows:

$$\forall (A, B, C, D) \in \mathcal{E}^4, \qquad A : B :: C : D \quad \stackrel{\text{def}}{\iff} \quad A \star B^{-1} = C \star D^{-1}. \tag{4.2}$$

The column marked Group in Table 1 gives the expression of each of the postulates using (4.2). Similarly as for magmas, conformity being equality, reflexivity and symmetry hold. Postulating the axiom of inversion of objects, i.e.,

$$\forall (A, B, C, D) \in \mathcal{E}^4, \qquad A : B :: C : D \quad \Leftrightarrow \quad A^{-1} : B^{-1} :: C^{-1} : D^{-1}, \tag{4.3}$$

has the consequence that A can be expressed in two ways in function of the other terms.

$$\begin{aligned} & A : B :: C : D \\ \Leftrightarrow \ & A \star B^{-1} = C \star D^{-1} \\ \Leftrightarrow \ & A = C \star D^{-1} \star B \end{aligned} \qquad\qquad \begin{aligned} & A^{-1} : B^{-1} :: C^{-1} : D^{-1} \\ \Leftrightarrow \ & A^{-1} \star (B^{-1})^{-1} = C^{-1} \star (D^{-1})^{-1} \\ \Leftrightarrow \ & A^{-1} \star B = C^{-1} \star D \\ \Leftrightarrow \ & B^{-1} \star A = D^{-1} \star C \\ \Leftrightarrow \ & A = B \star D^{-1} \star C \end{aligned}$$

Commutativity on the entire group is sufficient to ensure the equality

$$A = B \star D^{-1} \star C = C \star D^{-1} \star B. \tag{4.4}$$

Hence, provided the group is commutative, the group structure entails all the axioms listed in Table 1 with the exception of the axiom of distribution in objects.

5. Analogy between sets

Let $\mathcal{E}$ be a set. The set of all subsets of $\mathcal{E}$ is noted $\mathcal{P}(\mathcal{E})$. Equipped with union, $(\mathcal{P}(\mathcal{E}), \cup)$ is a commutative monoid. Union is an internal operation in $\mathcal{P}(\mathcal{E})$ that is associative and commutative. The neutral element is $\emptyset$ ($\emptyset \cup A = A \cup \emptyset = A$). However there is no inverse element in general, i.e., for any set A in $\mathcal{P}(\mathcal{E})$, there is no set B such that $A \cup B = \emptyset$. Similarly, $(\mathcal{P}(\mathcal{E}), \cap)$ is a commutative monoid. The neutral element is $\mathcal{E}$.

The symmetrical difference on sets (noted $\triangle$ and corresponding to XOR on Booleans), and another operation noted $\triangledown$ (the counterpart of logical equivalence on Booleans) are defined as follows.

$$\forall (A, B) \in \mathcal{P}(\mathcal{E})^2, \quad A \triangle B = (A \cup B) \setminus (A \cap B) \tag{5.1}$$

$$A \triangledown B = \mathcal{E} \setminus (A \triangle B) \tag{5.2}$$

$(\mathcal{P}(\mathcal{E}), \triangle)$ is a commutative group. Symmetrical difference is an internal operation in $\mathcal{P}(\mathcal{E})$ that is associative and commutative. The neutral element is $\emptyset$ ($\emptyset \triangle A = A \triangle \emptyset = A$). The inverse element of any set A in $\mathcal{P}(\mathcal{E})$ is itself: $A \triangle A = \emptyset$. Similarly, $(\mathcal{P}(\mathcal{E}), \triangledown)$ is a commutative group, with $\mathcal{E}$ as the neutral element, and each element is its one inverse.

For any quadruple of sets in a power set $\mathcal{P}(\mathcal{E})$, if the two analogies induced by the two structures of commutative monoids $(\mathcal{P}(\mathcal{E}), \cup)$ and $(\mathcal{P}(\mathcal{E}), \cap)$ hold at the same time, then, the analogy induced by the structure of commutative group $(\mathcal{P}(\mathcal{E}), \triangle)$ holds too (and similarly for $(\mathcal{P}(\mathcal{E}), \triangledown)$).

$$\begin{aligned}
A : B \overset{\cap}{::} C : D \ \wedge \ A : B \overset{\cup}{::} C : D \Leftrightarrow & \ (A \cap D) = (C \cap B) \ \wedge \ (A \cup D) = (C \cup B) \\
\Rightarrow & \ \ (A \setminus B) \cup (B \setminus A) = (C \setminus D) \cup (D \setminus C) \\
\Leftrightarrow & \ A \triangle B = C \triangle D \Leftrightarrow A : B \overset{\triangle}{::} C : D \\
\Leftrightarrow & \ A \triangledown B = C \triangledown D \Leftrightarrow A : B \overset{\triangledown}{::} C : D
\end{aligned}$$

The second line above is only an implication. Now, the analogy induced by the structure of a commutative group of $(\mathcal{P}(\mathcal{E}), \triangle)$ (or, similarly, $(\mathcal{P}(\mathcal{E}), \triangledown)$) is the same as when the two analogies induced by the two commutative monoids $(\mathcal{P}(\mathcal{E}), \cup)$ and $(\mathcal{P}(\mathcal{E}), \cap)$ hold at the same time, under the condition $A \subset B \cup C \ \wedge \ B \cap C \subset A$. This is (1.1) given in the introduction. $A \subset B \cup C$ transcribes the postulate of distribution in objects (iv) for sets with the features being the elements. $B \cap C \subset A$ is obtained by taking the set complements, i.e., using the postulate of inversion of objects (iii).

$$\begin{aligned}
& A : B \overset{\triangle}{::} C : D \Leftrightarrow \\
& A : B \overset{\triangledown}{::} C : D \Leftrightarrow A \triangle B = C \triangle D \\
& \qquad \Leftrightarrow (A \setminus B) \cup (B \setminus A) = (C \setminus D) \cup (C \setminus D) \\
& \qquad \Leftrightarrow (A \cap D) = (C \cap B) \wedge (A \cup D) = (C \cup B) \\
& \qquad \Leftrightarrow A : B \overset{\cap}{::} C : D \ \wedge \ A : B \overset{\cup}{::} C : D
\end{aligned}$$

Table 3 gives the explicit development of this correspondence.

In [1], it was shown that, under the condition (1.1), the solution of an analogical equation $A : B :: C : D$ of unknown D between sets is given by (1.2).

A	B	C	D	(a) $A \cap D = C \cap B$	(b) $A \cup D = C \cup B$	analogy induced by both monoids: (a) $\wedge$ (b)	(c) $A \triangle B = \mathcal{E} \setminus (A \triangledown B)$	(d) $C \triangle D = \mathcal{E} \setminus (C \triangledown D)$	analogy induced by group: (c) = (d)
F	F	F	F	T	T	T	F	F	T
F	F	F	T	T	F	F	F	T	F
F	F	T	F	T	F	F	F	T	F
F	F	T	T	T	T	T	F	F	T
F	T	F	F	T	F	F	T	F	F
F	T	F	T	T	T	T	T	T	T
F	T	T	F	F	F	F	T	T	T
F	T	T	T	F	T	F	T	F	F
T	F	F	F	T	F	F	T	F	F
T	F	F	T	F	F	F	T	T	T
T	F	T	F	T	T	T	T	T	T
T	F	T	T	F	T	F	T	F	F
T	T	F	F	T	T	T	F	F	T
T	T	F	T	F	T	F	F	T	F
T	T	T	F	F	T	F	F	T	F
T	T	T	T	T	T	T	F	F	T

Table 3
Correspondence, on sets, between the two analogies induced by the commutative monoids $(\mathcal{P}(\mathcal{E}), \cup)$ and $(\mathcal{P}(\mathcal{E}), \cap)$ holding at the same time and each of the analogies induced by the commutative groups $(\mathcal{P}(\mathcal{E}), \triangledown)$ or $(\mathcal{P}(\mathcal{E}), \triangledown)$.

6. Analogies between Booleans

There exists a correspondence between operations on sets and operations on Booleans. Here we use the correspondence between *union* and *or*, *intersection* and *and*, and the fact that the complement of a set in another one corresponds to taking the conjunction with the negation: $A \setminus B$ corresponds to $a \wedge \neg b$. With this, the solution of an analogy between Booleans, *a* : *b* :: *c* : *d*, transcribed from the solution of an analogy between sets, under the condition (transcribed from the condition on sets) that

$$a \Rightarrow b \vee c \quad \wedge \quad b \wedge c \Rightarrow a, \tag{6.1}$$

is:

$$d = ((b \vee c) \wedge \neg a) \vee (b \wedge c). \tag{6.2}$$

The condition corresponds to the cases in conflict in [3] and [4], and identified in [1], i.e., the problem of accepting or not T : F :: F : T and F : T :: T : F as valid analogies. Transposed on sets, this is tantamount to ask whether $\{e_1, e_2\}$: $\{e_2\}$ :: $\{e_3\}$: $\{e_1, e_3\}$ is a valid analogy. The condition given above for sets rejects this analogy by keeping the natural interpretation of sets as containers.

7. The Sheffer stroke

The Sheffer stroke (usually noted $|$, but noted $\uparrow$ here[6]) denotes the NAND operator. For two Boolean variables p and q,

$$p\uparrow q = \neg(p \wedge q). \tag{7.1}$$

It is known that the singleton containing the Sheffer stroke as sole Boolean operator is functionally complete. This means that any Boolean expression can be rewritten using solely the Sheffer stroke. For instance,

$$p \wedge q = (p\uparrow q)\uparrow(p\uparrow q), \tag{7.2}$$
$$p \vee q = (p\uparrow p)\uparrow(q\uparrow q), \tag{7.3}$$
$$\neg p = p\uparrow p. \tag{7.4}$$

Intuitive operators are associative and commutative as is the case for $+$ or $\times$ on numbers. However, remarkably, the Sheffer stroke is commutative

$$p\uparrow q = q\uparrow p, \tag{7.5}$$

but not associative, i.e., in general

$$(p\uparrow q)\uparrow r \neq p\uparrow(q\uparrow r). \tag{7.6}$$

By virtue of $p\uparrow p = \neg p$, trivially,

$$(p\uparrow p)\uparrow(p\uparrow p) = \neg(\neg p) = p. \tag{7.7}$$

The notation p^2 for $p\uparrow p$ can be introduced, and applying it twice, reduces (7.7) to:

$$(p^2)^2 = p. \tag{7.8}$$

8. The Pierce arrow

The Pierce arrow is the NOR operator, i.e.,

$$p\downarrow q = \neg(p \vee q). \tag{8.1}$$

It has similar properties as the Sheffer stroke: it is commutative, but not associative, negation is obtained by self-application

$$\neg p = p\downarrow p, \tag{8.2}$$

[6]As in [5] and other works, we prefer $\uparrow$ over $|$ for symmetry reasons due to the use of the Pierce arrow $\downarrow$.

and any Boolean formula can be rewritten using it solely, i.e., alone, it is functionally complete. There is a kind of symmetry with the Sheffer stroke for the expression of conjunction and disjunction, due to the fact that they are dual[7]:

$$p \wedge q = (p{\downarrow}p){\downarrow}(q{\downarrow}q) \tag{8.3}$$

$$p \vee q = (p{\downarrow}q){\downarrow}(p{\downarrow}q). \tag{8.4}$$

The notation p^2 can be used with the Pierce arrow with the same meaning and same value as with the Sheffer stroke:

$$p^2 = \neg p = p{\uparrow}p = p{\downarrow}p. \tag{8.5}$$

Consequently, (7.8) also holds for the Pierce arrow.

9. Relations between the Sheffer stroke and the Pierce arrow

The following properties can easily be established by using the expression of disjunction for the two operators:

$$b^2{\uparrow}c^2 = (b{\downarrow}c)^2, \tag{9.1}$$

$$a^2{\uparrow}(b^2{\uparrow}c^2) = (a{\downarrow}(b{\downarrow}c))^2. \tag{9.2}$$

Rather than using p, q and r for variable names, we used a, b and c on purpose, to ease the reading of Section 10. The same can be done for conjunction:

$$b^2{\downarrow}c^2 = (b{\uparrow}c)^2, \tag{9.3}$$

$$a^2{\downarrow}(b^2{\downarrow}c^2) = (a{\uparrow}(b{\uparrow}c))^2. \tag{9.4}$$

10. Formulae for the solution of a Boolean analogy

The rewriting of the solution of an analogy between Booleans into an expression that involves only the Sheffer stroke can be worked out by hand from (6.2). It is safer to rely on a program to automatically perform this rewriting. We give such a program in Figure 2. It starts from a tree representation of (6.2), i.e., (6.2) in Polish notation.

The result is as follows, with spaces for clarity.

$$\begin{aligned} d = (\ (\ (\ & (((b{\uparrow}b){\uparrow}(c{\uparrow}c)){\uparrow}(a{\uparrow}a)) \uparrow (((b{\uparrow}b){\uparrow}(c{\uparrow}c)){\uparrow}(a{\uparrow}a))\ ) \uparrow \\ & (\ (((b{\uparrow}b){\uparrow}(c{\uparrow}c)){\uparrow}(a{\uparrow}a)) \uparrow (((b{\uparrow}b){\uparrow}(c{\uparrow}c)){\uparrow}(a{\uparrow}a))\ )\ ) \uparrow \\ & (((b{\uparrow}c){\uparrow}(b{\uparrow}c)){\uparrow}((b{\uparrow}c){\uparrow}(b{\uparrow}c)))\ ) \end{aligned}$$

This lengthy formula can be simplified by

- locating occurrences of (7.8), i.e., $(p{\uparrow}p){\uparrow}(p{\uparrow}p) = p$,
- introducing the p^2 notation, and

[7]The dual f^d of an operator f is defined as follows [5]: $f^d(a_1, a_2, \ldots a_n) = (f(a_1^2, a_2^2, \ldots a_n^2))^2$.

```python
def and_(p, q):
    return f'(({p}↑{q})↑({p}↑{q}))'
def or_(p, q):
    return f'(({p}↑{p})↑({q}↑{q}))'
def not_(p):
    return f'({p}↑{p})'
def a():
    return 'a'
def b():
    return 'b'
def c():
    return 'c'

# d = '((b or c) and non(a)) or (b and c)'

d = or_( and_(or_(b(), c()), not_(a())), and_(b(),
↪  c()) )

print(d)
```

Figure 2: Program for automatic generation of the solution of an analogy between Booleans using the Sheffer stroke only.

- reestablishing the order of appearance of a, b and c by commutativity of the Sheffer stroke.

$$
\begin{aligned}
d &= (((b{\uparrow}b){\uparrow}(c{\uparrow}c)){\uparrow}(a{\uparrow}a)){\uparrow}(b{\uparrow}c) \\
&= (\ (b^2{\uparrow}c^2) \uparrow a^2\ )\ \uparrow\ (b{\uparrow}c) \\
&= (\ a^2 \uparrow (b^2{\uparrow}c^2)\ )\ \uparrow\ (b{\uparrow}c)
\end{aligned}
\tag{10.1}
$$

For the Pierce arrow, the formula output by a similar program is as follows. Similarly, it can be simplified.

$$
\begin{aligned}
d &= (((((b{\downarrow}c){\downarrow}(b{\downarrow}c)){\downarrow}((b{\downarrow}c){\downarrow}(b{\downarrow}c)))\downarrow((a{\downarrow}a){\downarrow}(a{\downarrow}a))){\downarrow}((b{\downarrow}b){\downarrow}(c{\downarrow}c)))\ \downarrow \\
&\quad (((((b{\downarrow}c){\downarrow}(b{\downarrow}c)){\downarrow}((b{\downarrow}c){\downarrow}(b{\downarrow}c)))\downarrow((a{\downarrow}a){\downarrow}(a{\downarrow}a))){\downarrow}((b{\downarrow}b){\downarrow}(c{\downarrow}c)))) \\
&= \Big(\ (((((b{\downarrow}c){\downarrow}(b{\downarrow}c)){\downarrow}((b{\downarrow}c){\downarrow}(b{\downarrow}c)))\downarrow((a{\downarrow}a){\downarrow}(a{\downarrow}a))){\downarrow}((b{\downarrow}b){\downarrow}(c{\downarrow}c))\ \Big)^2 \\
&= \Big(\ ((((b{\downarrow}c){\downarrow}a){\downarrow}((b{\downarrow}b){\downarrow}(c{\downarrow}c)))\ \Big)^2 \\
&= \Big(\ ((b{\downarrow}c){\downarrow}a)\downarrow(b^2{\downarrow}c^2)\ \Big)^2 \\
&= \Big(\ (a{\downarrow}(b{\downarrow}c))\downarrow(b^2{\downarrow}c^2)\ \Big)^2
\end{aligned}
\tag{10.2}
$$

This second formula could have been obtained directly from (10.1) by exploiting the relations seen in Section 9, i.e., the duality between the two operators.

a	b	c	$b{\uparrow}c$	$b^2{\uparrow}c^2$	$a{\uparrow}(b{\uparrow}c)$	$a^2{\uparrow}(b^2{\uparrow}c^2)$	d in (10.1)	d in (10.3)
F	F	F	T	F	T	T	F	F
F	F	T	T	T	T	F	T	T
F	T	F	T	T	T	F	T	T
F	T	T	F	T	T	F	T	T
T	F	F	T	F	F	T	F	F
T	F	T	T	T	F	T	F	F
T	T	F	T	T	F	T	F	F
T	T	T	F	T	T	T	T	T

Table 4
True value tables of (10.1) and (10.3).

$$
\begin{aligned}
&(a^2{\uparrow}(b^2{\uparrow}c^2)){\uparrow}(b{\uparrow}c) && && (10.1)\\
&\quad = \quad (a{\downarrow}(b{\downarrow}c))^2{\uparrow}(b{\uparrow}c) && \text{by } (9.2) &&\\
&\quad = \quad (a{\downarrow}(b{\downarrow}c))^2{\uparrow}(b^2{\downarrow}c^2)^2) && \text{by } (9.3) &&\\
&\quad = \quad \big(\ (a{\downarrow}(b{\downarrow}c)){\downarrow}(b^2{\downarrow}c^2)\ \big)^2 && \text{by } (9.1) && (10.2)
\end{aligned}
$$

Remarkably, (10.1) is equivalent to the following formula, where the whole is squared and variables are squared.[8]

$$d = \big(\ (a{\uparrow}(b{\uparrow}c)) \uparrow (b^2{\uparrow}c^2)\ \big)^2 \tag{10.3}$$

The equivalence between (10.1) and (10.3) is shown by the table of truth values for the two formulae in Table 4. The grayed-out lines are the two lines corresponding to the cases where condition (6.1) is not verified. In this table, the symmetry around the central line says that the value of d is negated by taking the negation of each of the variables a, b and c. This just states that, considered as an operator on three variables, the solution of an analogy is self-dual:

$$d(a^2, b^2, c^2) = d(a, b, c)^2.$$

This follows intuition as, d being the solution of an analogy, the postulate of inversion of objects (III) should hold. For the same reason, an equivalent form to (10.2) is:

$$d = (\ a^2 \downarrow (b^2{\downarrow}c^2)\ )\ \downarrow\ (b{\downarrow}c) \tag{10.4}$$

Thus, remarkably, the formulae using the Pierce arrow (NOR) are the same as the ones using the Sheffer stroke (NAND). That is, (10.4) is the same as (10.1) and (10.2) is the same as (10.3), except for the operator.

[8]The submitted version of this paper contained a regrettable error in the justification of this equivalence. We fortunately became aware of it before the feedback of the reviewers, who, of course, spotted it. We thank one of them for suggesting a proof of this equivalence.

11. Conclusion

This paper gave formulae for the solution of an analogical equation between Booleans using solely the Sheffer stroke (NAND) or the Pierce arrow (NOR).

These formulae were obtained by transposing a formula on sets to Booleans. To justify this first formula, we reminded postulates for analogy and briefly showed how analogy can be induced from some algebraic structures (see also [6]). We then gave a rapid review on analogies between sets and stressed the fact that there is a discrepancy between analogy induced by union or intersection and analogy induced by symmetrical difference. Transposed to Booleans, this discrepancy tantamounts to ask whether $T : F :: F : T$ and $F : T :: T : F$ (by inversion of ratios (ii)) should be considered valid analogies.

Although not so intuitive, the formulae for Booleans using the Sheffer stroke or the Pierce arrow are somewhat elegant. They reflect the self-duality of the solution of a Boolean analogical equation. Any of the two operators, Sheffer stroke or Pierce arrow, can indifferently be used for them. It is an open question whether these formulae are the most economical ones in terms of number of occurrences of operators or variables, i.e., whether their efficiency is the best possible [5].

12. Acknowledgments

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References

[1] Y. Lepage, De l’analogie rendant compte de la commutation en linguistique, Mémoire d’habilitation à diriger les recherches, Université de Grenoble, 2003.

[2] J. J. Rallier des Ourmes, Proportion, in: Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers, par une société de gens de lettres, volume XIII, Chez Briasson, David, Le Breton ou Durand, Paris, 1765, pp. 466a–467b.

[3] S. Klein, Culture, mysticism and social structure and the calculation of behavior, in: Proceedings of the European Conference on Artificial Intelligence (ECAI 1982), 1982, pp. 141–146.

[4] L. Miclet, H. Prade, Handling analogical proportions in classical logic and fuzzy logics settings, in: C. Sossai, G. Chemello (Eds.), Symbolic and Quantitative Approaches to Reasoning with Uncertainty, Springer, Berlin, Heidelberg, 2009, pp. 638–650.

[5] M. Couceiro, E. Lehtonen, P. Mercuriali, R. Péchoux, On the efficiency of normal form systems for representing Boolean functions, Theoretical Computer Science 813 (2020) 341–361. URL: https://inria.hal.science/hal-02153506. doi:10.1016/j.tcs.2020.01.009.

[6] N. Stroppa, Définitions et caractérisation de modèles à base d’analogies pour l’apprentissage automatique des langues naturelles, Thèse de doctorat, École nationale supérieure des télécommunications, 2005. URL: https://hal.archives-ouvertes.fr/tel-00145147/.