Goal-minimally k-diametric graphs with not so small diameters

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Abstract

An undirected graph Γ with diameter k is said to be goal-minimally k-diametric if for every edge uv of Γ the distance $d_{\Gamma-uv}(x, y) > k$ if and only if $\{x, y\} = \{u, v\}$. It is rather difficult to construct such graphs. Before our research, they were known only for diameters up to 14, except of the case $k = 11$. In this paper we construct such graphs of larger diameters using Cayley graphs with generators obtained by linear fractional transformations on the set of elements of a finite field GF(q) extended by an element ∞.

Keywords

Computer algebra, Distance, Diameter, Edge deletion, Goal-minimal, Cayley graphs

1. Introduction

Minimal graphs with respect to diameter were studied by many authors, for example see [1], [3], [7], [9], [11], [12], [17] and [18]. A special subclass of this class of graphs are so-called goal-minimal graphs with respect to diameter which were introduced by Kyš in [16] and studied by Gliviak and Plesník in [10], [19] and by Gyürki in [13] and [14].

A graph Γ with diameter k is called a *minimal graph of diameter k* if $\mathrm{diam}(\Gamma - e) > k$ for every edge $e \in E(\Gamma)$. A graph Γ is said to be *goal-minimal of diameter k* or *goal-minimally k-diametric* (k-GMD for short), if the diameter of Γ is equal to k, and for every edge $uv \in E(\Gamma)$ the inequality $d_{\Gamma-uv}(x, y) > k$ holds if and only if $\{u, v\} = \{x, y\}$.

For an example of a GMD graph with diameter 3, see Figure 1.

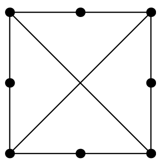

Figure 1: A 3-GMD graph on 8 vertices.

Kyš [16] conjectured that for every positive integer k there exists a k-GMD graph. He discovered such graphs only for $k = 1, 2, 3, 4, 6$. Moreover, for $k = 1, 2, 4$ he gave infinite families of k-GMD graphs. In [19] Plesník showed the first examples of k-GMD graphs for diameters $k = 5, 7, 8, 10, 12, 14$, and constructed the first infinite family of 6-GMD graphs. Gyürki [14] constructed by lifts the first known 9-GMD and 13-GMD graphs, moreover, he found an infinite family of 5-GMD graphs. Thus, before our research such graphs have been known only for values $k \leq 14$, except of the case $k = 11$.

In this paper we construct k-GMD graphs as Cayley graphs with generating set obtained from linear fractional transformations on GF(q) $\cup \{\infty\}$, having larger diameters.

The most important properties of k-GMD graphs are collected in the next theorem.

Theorem 1. *[14]*
Let k be a positive integer. A graph Γ with order at least 3 is k-GMD if and only if it has diameter k, girth $k + 2$ and for any two non-adjacent vertices u and v there exist two internally-disjoint u–v paths of length not exceeding k.

Many of the k-GMD graphs have been discovered among the graphs belonging to the family of symmetric cubic graphs and among the cages.

The symmetric cubic graphs are those cubic graphs, which are vertex-transitive and edge-transitive too. These graphs are collected into a catalogue, which can be found on the web site ([5]) of Marston Conder. We have found thirty-six k-GMD graphs in this catalogue which are shown in Table 1.

Conder ([4]) has constructed some cubic Cayley graphs in order to find minimal cubic graphs with prescribed girth. Among them, we found two which fulfill the relation $g = k + 2$ from Theorem 1, where g is the girth and k the diameter. It turns out that the first one is a 16-GMD graph and the second one is a 20-GMD graph. Conder in his paper did not specify the details of how to obtain the generators of these Cayley graphs, but fortunately, his method was described by Biggs ([2]).

The construction of Cayley graphs in this paper is a slight generalization of the Conder's method.

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graph	result
C016.1	4-GMD
C018.1	4-GMD
C040.1	6-GMD
C048.1	6-GMD
C080.1	8-GMD
C090.1	8-GMD
C102.1	7-GMD
C108.1	7-GMD
C128.2	8-GMD
C144.2	8-GMD
C224.3	10-GMD
C360.2	10-GMD
C364.3	10-GMD
C384.2	10-GMD
C384.3	10-GMD
C440.3	10-GMD
C480.3	10-GMD
C512.1	12-GMD
C624.2	12-GMD
C672.7	12-GMD
C768.3	12-GMD
C880.3	12-GMD
C912.2	12-GMD
C960.1	12-GMD
C960.3	12-GMD
C1008.2	12-GMD
C1024.1	12-GMD
C1092.3	12-GMD
C1140.3	12-GMD
C1140.10	12-GMD
C1344.5	12-GMD
C1344.6	12-GMD
C1632.7	14-GMD
C1792.8	14-GMD
C2016.5	14-GMD
C2048.17	14-GMD

Table 1
k-GMD graphs from Conder's catalogue.

2. Cayley graphs and finite fields

Let G be a finite group and X be a subset of G not containing the group identity and having the property that if $h \in X$ then $h^{-1} \in X$. Then the Cayley graph of G with generating set X is the graph $C(G, X)$ with vertex set G and vertices x and y are adjacent if and only if $xy^{-1} \in X$.

Let GF(q) be the finite field with q elements. It is a well-known fact that such field exists if and only if q is a power of a prime and the multiplicative group $\mathrm{GF}^{\times}(q)$ is cyclic, so there exists a so-called primitive element (generator) ω of GF(q) such that

$$\mathrm{GF}(q) = \{0, 1, \omega, \omega^2, \ldots, \omega^{q-2}\}.$$

For our aims it is sufficient to identify the finite field of prime order p with the $\mathbb{Z}_p$. Finite fields of order p^k, where p is a prime and $k \geq 2$, we can obtain by factoring the ring of polynomial over $\mathbb{Z}_p$ by an ideal generated by an irreducible polynomial $P(x)$ of degree k.

Let us consider the set $T = \mathrm{GF}(q) \cup \{\infty\}$. For each element $g \in \mathrm{GF}(q) \setminus \{0, 1\}$ define the mapping $\varphi_g : T \to T$ by linear fractional transformation

$$\varphi_g : x \mapsto \frac{x-1}{gx-1}$$

for $x \notin \{g^{-1}, \infty\}$. Further, $\varphi_g(\infty) = g^{-1}$ and $\varphi_g(g^{-1}) = \infty$. It is easy to see that φ_g is a permutation of T. Moreover, it is an involution, i.e. $\varphi_g = \varphi_g^{-1}$. From φ_g one can derive many other permutations by the following. For every integer $1 \leq n \leq q-2$ define $\Phi_{g,n} : T \to T$ as a conjugation of φ_g under the permutation $x \mapsto \omega^n x$ in the group Sym(T). Hence, we can use these mappings (involutions) to generate some undirected Cayley graphs.

In fact, such permutations generate either the full group $PSL(2, q)$ or some of its subgroup.

3. The construction

Conder constructed his graphs as cubic Cayley graphs with generating set

$$X = \{\varphi_g, \Phi_{g,n}, \Phi_{g,2n}\} \tag{1}$$

in the group $G = \langle X \rangle$, where $n = (2^{2m} - 1)/3$ for possible $m \in \mathbb{N}$ and $g \in \mathrm{GF}(q) \setminus \{0, 1\}$.

We have performed an exhaustive computer search for arbitrary integers $1 \leq a < b \leq q-2$ in the fields with up to 49 element for generating sets X of the form

$$X = \{\varphi_g, \Phi_{g,a}, \Phi_{g,b}\} \tag{2}$$

for every $g \in \mathrm{GF}(q) \setminus \{0, 1\}$. Further, we explored the generating sets X of the form (1) in the fields GF(q) for every possible $q \leq 103$, where $q \equiv 1 \pmod 3$.

Our computer search of these cases yielded more than 90 new k-GMD graphs. They are shown in Table 2. All these graphs are mutually non-isomorphic, since they can be distinguished by the value of the total distance in a graph. As one can see, these k-GMD graphs covers diameters $k = 12, 16, 18, 19, 20, 21, 22, 23, 24$ and 26. Thus, we have found k-GMD graphs for nine new values of k. So at present there are known k-GMD graphs for 22 distinct values of k. The graphs were generated by the computer system GAP [8] and the goal-minimality property was examined by a computer program based on the algorithm described in [15].

Example 1. *The multiplicative group of* $GF(13) = (\mathbb{Z}_{13}, \oplus, \odot)$ *is generated by* $g = 2$. *If we take* $a = 1$ *and* $b = 2$, *then we get the following permutations of* $T = \mathbb{Z}_{13} \cup \{\infty\}$:

$$\begin{aligned} \varphi_2 = & \quad (0,1)(3,12)(4,8)(6,7)(9,11)(10,\infty) \\ \Phi_{2,1} = & \quad (0,2)(1,12)(3,8)(5,9)(6,11)(7,\infty) \\ \Phi_{2,2} = & \quad (0,4)(1,\infty)(2,11)(3,6)(5,10)(9,12). \end{aligned}$$

These permutations generate a subgroup Γ *of index 2 in* $PSL(2,13)$, *so* $|\Gamma| = 1092$. *The corresponding Cayley graph has diameter 12, girth 14 and it is a 12-GMD graph. This graph is displayed in Table 2 under the number 1.*

We plan to continue in this search, but it requires too much computer time. So we would like to know the answer to the next open question.

Question. How to choose $g \in \mathrm{GF}(q)$ and integers a and b in (2) in order to obtain a k-GMD graph for some integer k?

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Table 2: k-GMD graphs obtained from finite fields.

graph	q	g	a	b	order	result	|Aut|	Total distance
1	13	ω^2	1	2	1 092	12-GMD	2184	4784052
2	13	ω^7	3	6	1 092	12-GMD	6552	4795518
3	16	ω^3	5	10	4 080	16-GMD	24480	81753000
4	19	ω^5	2	4	6 840	16-GMD	41040	252765360
5	19	ω^5	2	10	6 840	16-GMD	13680	254352240
6	19	ω^5	6	12	6 840	16-GMD	41040	251773560
7	25	ω^4	8	16	7 800	16-GMD	46800	332853300
8	27	ω^2	2	14	9 828	16-GMD	19656	540692334
9	29	ω^3	2	8	12 180	16-GMD	24360	860906760
10	29	ω^3	4	8	12 180	16-GMD	24360	853263810
11	29	ω^6	1	13	12 180	16-GMD	24360	847100730
12	29	ω^6	2	9	12 180	16-GMD	12180	851241930
13	29	ω^6	4	9	12 180	16-GMD	24360	858708270
14	29	ω^8	9	18	12 180	16-GMD	24360	857313660
15	29	ω^{13}	8	16	12 180	16-GMD	73080	856521960
16	31	ω^{14}	4	14	14 880	16-GMD	14880	1311828240
17	31	ω^{17}	2	14	29 760	18-GMD	29760	5777665920
18	32	ω^3	7	16	32 736	18-GMD	32736	7024883712
19	32	ω^5	1	16	32 736	18-GMD	65472	7002459552
20	32	ω^5	2	4	32 736	18-GMD	65472	6949099872
21	32	ω^5	8	18	32 736	18-GMD	32736	7046915040
22	32	ω^7	3	12	32 736	18-GMD	32736	6986680800
23	32	ω^7	7	15	32 736	18-GMD	32736	6958511472
24	32	ω^{15}	1	14	32 736	18-GMD	32736	6939573696
25	32	ω^{15}	3	6	32 736	18-GMD	65472	7024261728
26	32	ω^{15}	3	11	32 736	18-GMD	32736	7012296720
27	32	ω^{15}	5	12	32 736	18-GMD	32736	6987941136
28	37	ω^2	10	22	25 308	18-GMD	25308	4065059538
29	37	ω^2	12	24	25 308	18-GMD	151848	4052620656
30	37	ω^5	12	24	50 616	20-GMD	303696	17698491792
31	37	ω^{20}	3	12	50 616	20-GMD	101232	17623731960
32	37	ω^{25}	12	24	50 616	20-GMD		
33	37	ω^{30}	1	7	25 308	18-GMD		
34	41	ω	6	19	34 440	18-GMD		
35	41	ω^3	3	15	34 440	18-GMD		
36	41	ω^{10}	1	10	34 440	18-GMD		
37	41	ω^{16}	6	12	68 880	20-GMD		
38	41	ω^{25}	4	11	68 880	20-GMD		
39	43	ω^4	14	28	79 464	20-GMD		
40	43	ω^5	3	16	79 464	20-GMD		
41	43	ω^6	2	10	39 732	18-GMD		
42	43	ω^6	12	27	39 732	18-GMD		
43	43	ω^7	8	21	39 732	18-GMD		
44	43	ω^8	3	11	39 732	18-GMD		
45	43	ω^8	5	15	39 732	18-GMD		
46	43	ω^{10}	4	14	39 732	18-GMD		
47	43	ω^{10}	9	23	39 732	18-GMD		
48	43	ω^{10}	12	24	39 732	18-GMD		
49	43	ω^{13}	7	22	39 732	18-GMD		

The table continues on the next page.

Table 2 – continuing from the previous page.

graph	q	g	a	b	order	result	|Aut|	Total distance
50	43	ω^{13}	9	20	39 732	18-GMD		
51	43	ω^{14}	8	19	39 732	18-GMD		
52	43	ω^{17}	4	17	39 732	18-GMD		
53	43	ω^{19}	5	13	79 464	20-GMD		
54	43	ω^{21}	11	23	39 732	18-GMD		
55	43	ω^{23}	2	22	79 464	20-GMD		
56	43	ω^{24}	1	7	39 732	18-GMD		
57	43	ω^{24}	3	6	39 732	19-GMD		
58	43	ω^{24}	3	16	39 732	18-GMD		
59	43	ω^{24}	4	15	39 732	18-GMD		
60	43	ω^{24}	5	10	39 732	18-GMD		
61	43	ω^{25}	11	25	39 732	18-GMD		
62	43	ω^{27}	3	19	39 732	18-GMD		
63	43	ω^{38}	3	6	39 732	18-GMD		
64	47	ω^{5}	7	16	103 776	20-GMD		
65	47	ω^{11}	3	12	51 888	19-GMD		
66	47	ω^{14}	7	14	103 776	20-GMD		
67	47	ω^{16}	13	26	51 888	20-GMD		
68	47	ω^{22}	14	28	51 888	19-GMD		
69	47	ω^{37}	4	18	103 776	20-GMD		
70	49	ω^{2}	16	32	58 800	19-GMD		
71	49	ω^{3}	2	19	58 800	19-GMD		
72	49	ω^{18}	1	14	117 600	22-GMD		
73	49	ω^{20}	6	21	117 600	20-GMD		
74	61	ω^{44}	20	40	226 920	22-GMD		
75	61	ω^{53}	20	40	226 920	22-GMD		
76	64	ω^{1}	21	42	262 080	23-GMD		
77	64	ω^{13}	21	42	262 080	23-GMD		
78	64	ω^{23}	21	42	262 080	23-GMD		
79	64	ω^{31}	21	42	262 080	23-GMD		
80	67	ω^{13}	22	44	150 348	22-GMD		
81	67	ω^{33}	22	44	150 348	22-GMD		
82	67	ω^{59}	22	44	150 348	22-GMD		
83	73	ω^{25}	24	48	388 944	22-GMD		
84	73	ω^{37}	24	48	194 472	22-GMD		
85	73	ω^{65}	24	48	194 472	21-GMD		
86	79	ω^{62}	26	52	246 480	22-GMD		
87	79	ω^{72}	26	52	246 480	22-GMD		
88	97	ω^{5}	32	64	912 576	24-GMD		
89	97	ω^{6}	32	64	912 576	24-GMD		
90	97	ω^{41}	32	64	456 288	23-GMD		
91	97	ω^{47}	32	64	912 576	26-GMD		
92	103	ω^{9}	34	68	546 312	24-GMD		
93	103	ω^{50}	34	68	1 092 624	26-GMD		
94	103	ω^{78}	34	68	546 312	23-GMD		
95	103	ω^{93}	34	68	546 312	24-GMD		