An Abstract Fixed-point Theorem for Horn Formula Equations
(Abstract)

Stefan Hetzl and Johannes Kloibhofer

Institute of Discrete Mathematics and Geometry, TU Wien, Austria
stefan.hetzl@tuwien.ac.at, j.kloibhofer@gmx.net

We consider the problem of solving a formula equation, i.e., given a formula $\exists \overline{X}\, \varphi$ where φ is first-order and $\overline{X}$ is a tuple of predicate variables, to find a substitution $[\overline{X} \backslash \overline{\psi}]$ s.t. $\varphi[\overline{X} \backslash \overline{\psi}]$ is a valid first-order formula. This problem is also known as Boolean solution problem in the literature [9] and is closely related to second-order quantifier elimination.

More specifically, we focus on the class of Horn formula equations, which is defined by restricting φ to be a Horn clause set w.r.t. the predicate variables. We state and prove a fixed-point theorem for Horn formula equations based on expressing the fixed-point computation of a minimal model (in the sense of logic programming) of a set of Horn clauses on the object level as a formula in first-order logic with a least fixed-point operator. This result is shown by an extension of the fixed-point approach of Nonnengart and Szałas to second-order quantifier elimination [7]. Our fixed-point theorem applies not only to the usual semantics of second-order logic and first-order logic with a least fixed-point operator but also to model abstractions, a semantics for logical formulas that corresponds to abstract interpretation of programs using Galois connections [2].

Our fixed-point theorem allows both new results and simpler proofs of existing results as applications and corollaries.

1. It entails expressibility of the weakest precondition and the strongest postcondition, and thus the partial correctness of an imperative program, in first-order logic with a least fixed-point operator.
2. It allows a generalisation of a result by Ackermann [1] on approximating a second-order formula by first-order formulas in a direction different from the recent generalisation [8].
3. It allows to obtain a result from a recently introduced approach to automated inductive theorem proving with tree grammars [3] as another straightforward corollary.
4. Since it incorporates abstract interpretation, it permits to considerably simplify the proof of the decidability of affine formula equations originally presented in [5].

This work is rooted in the second author's master's thesis [6]. Some of these results have been presented at the 8th Workshop on Horn Clauses for Verification and Synthesis [4].

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