On Vertical Grammatical Restrictions that Produce an Infinite Language Hierarchy

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Abstract. This paper introduces *derivation tables* that represent a complete grammatical derivations as whole in a vertical way. These tables are obtained by writing the consecutive sentential forms of grammatical derivations vertically one by one. The present paper places and discusses some restrictions on the columns of these tables. More specifically, these restrictions constrain the order of context-sensitive derivations on boundaries of columns. It is demonstrated, that grammars restricted in this way generate an infinite language hierarchy.

1 Introduction

Standardly, the formal language theory always places context restrictions on the currently rewritten sentential form regardless of the preceding and following sentential forms. In this paper, however, we create derivation tables by writing all the consecutive sentential forms vertically one by one. Then, we place some context restriction on the resulting table's columns. As a result, we actually restrict the grammatical derivations as a whole in a vertical way.

More specifically, we discuss vertical restrictions in terms of the type-0 grammars in Kuroda normal form, where we divide their grammatical tables into columns. Then we restrict the order of context-sensitive derivations in the whole table on boundaries of these columns. More specifically, all context rules applied across the border between columns $i-1$ and i must be applied after all context rules that had been applied across the border between columns i and $i+1$. We demonstrate that under this restriction the grammars satisfying this requirement define an infinite hierarchy of languages that is equal to the hierarchy defined by flip-pushdown automata (see [6] and [7]).

2 Definitions

We assume that the reader is familiar with the language theory (see [1], [2], [3], [4]).

Let V be an alphabet. V^* represents the free monoid generated by V under the operation of concatenation. The unit of V is denoted by ε. Set $V^+ = V^* - \{\varepsilon\}$. For a word, $w \in V^*$, $|w|$ denotes the length of w. For a word of the form xw, where $x, w \in V^*$, $pref(xw) = x$ denotes the prefix of xw. Similarly, for a word of the form wx, where $x, w \in V^*$, then $suf(wx) = x$ denotes the suffix of wx. Let $w = a_1 a_2 \dots a_n$, where $a_i \in V$, $1 \leq i \leq n$. The reversal of w, $rev(w)$, is defined as $rev(w) = a_n \dots a_2 a_1$.

Throughout this paper, a phrase-structure grammar, $G = (V, T, P, S)$, is always specified in Kuroda normal form (see [5], page 741 in [1]), where V is an alphabet, $T \subseteq V$, $S \in V - T$, and P is a finite set of productions, where every production is either of the form $AB \rightarrow CD$, $A \rightarrow x$, or $A \rightarrow \epsilon$, where $A, B, C, D \in V - T$, $x \in T \cup (V - T)^2$. If $x, y \in V^*$, $x = u\alpha v$, $y = u\beta v$, and $\alpha \rightarrow \beta \in P$, where $u, v, \alpha, \beta \in V^*$, then x directly derives y in G by using $\alpha \rightarrow \beta$, symbolically written as $x \Rightarrow y$ $[\alpha \rightarrow \beta]$, or, simply, $x \Rightarrow y$ in G. Furthermore, $\Rightarrow^n$, $\Rightarrow^+$, and $\Rightarrow^*$ denote the n-fold product, transitive closure and transitive-reflexive closure of $\Rightarrow$, respectively. The language generated by G, $L(G)$, is defined as $L(G) = \{y : S \Rightarrow^* y \text{ in } G, y \in T^*\}$. Consider an n-step derivation of the form $y_1 \Rightarrow y_2 \Rightarrow \dots \Rightarrow y_n$ in G for some $n \geq 1$, where $y_1 = S$. We next express this derivation by its *derivation table* as

$$\begin{aligned} S &\Rightarrow x_{11}\ x_{12}\ \dots\ x_{1m} \\ &\Rightarrow x_{21}\ x_{22}\ \dots\ x_{2m} \\ &\ \vdots \\ &\Rightarrow x_{n1}\ x_{n2}\ \dots\ x_{nm} \end{aligned}$$

where $m \geq 1$, $x_{ij} \in V^*$, $x_{i1} x_{i2} \dots x_{im} = y_i$, $1 \leq i \leq n$ and $1 \leq j \leq m$. A *segment* is every string x_{ij} in this table, and if $x_{ij} \in T^*$, x_{ij} is a *terminal segment*. A *column* is a vector of the form $\langle x_{1j}, x_{2j}, \dots, x_{nj} \rangle$. Let a column of this form satisfies $x_{1j} = \varepsilon, x_{2j} = \varepsilon, \dots, x_{k-1j} = \varepsilon$, and $|x_{kj}| = 1, |x_{k+1j}| \geq 1, \dots, |x_{nj}| \geq 1$, where $1 \leq k < n$; then, x_{kj} is a *head*. Let $C_1 = \langle x_{1j}, x_{2j}, \dots, x_{nj} \rangle$, $C_2 = \langle x_{1j+1}, x_{2j+1}, \dots, x_{nj+1} \rangle$ be two neighbouring columns, for some $j \geq 1$. Let $suf(x_{ij}) = A$, $pref(x_{ij+1}) = B$, $suf(x_{i+1j}) = C$, $pref(x_{i+1j+1}) = D$, $AB \rightarrow CD \in P$. Then, we say that $AB \rightarrow CD$ is applied *across the boundary* between C_1 and C_2.

A flip-pushdown automaton (see [6], [7]) is a tuple $M = (Q, T, \Gamma, R, q_0, Z_0, \Delta, F)$, where Q is a finite set of states, T is a finite input alphabet, Γ is a finite pushdown alphabet, R is a finite set of rules of the form $Apa \rightarrow vq$, where $A \in \Gamma$, $p, q \in Q$, $a \in T^*$, $v \in \Gamma^*$, $q_0 \in Q$ is the initial state, $Z_0 \in \Gamma$ is the initial symbol on the pushdown, $F \subseteq Q$ is the set of final states and Δ is a mapping from Q to 2^Q.

A *configuration* of M is any string of the form $uApaw$, where $u \in \Gamma^*$, $A \in \Gamma$, $p \in Q$, $a, w \in T^*$. If $uApaw$ and $uvqw$ are two configurations and $Apa \rightarrow vq \in R$, then M makes a *transition* from a configuration $uApaw$ to $uvqw$, symbolically written as $uApaw \vdash uvqw[Apa \rightarrow vq]$, or, simply, $uApaw \vdash uvqw$. If $q \in \Delta(p)$ and $A = Z_0$, then $uApaw \vdash rev(u)Aqaw$. If $A \neq Z_0$, then $uApaw \vdash Arev(u)qaw$. The transition of this form performed by using of the mapping Δ is called the *pushdown reversal*. Whenever, there is a choice between an ordinary pushdown transition or a pushdown reversal, then the automaton nondeterministically chooses the next move. As usual, the n-fold product, the transitive closure and the reflexive-transitive closure of $\vdash$ is denoted by $\vdash^n$, $\vdash^+$ and $\vdash^*$, respectively, where $n \geq 0$. Consider that $c_1 \vdash c_2 \vdash \ldots \vdash c_n$ in M, $n \geq 0$. Then, $c_n \dashv \ldots \dashv c_2 \dashv c_1$ is the same sequence of moves in M written in reversal. We use this notation for maximal lucidity in some cases. Relations $\dashv^+$ and $\dashv^*$ are defined in the usual way.

Let k be a natural number. For a flip-pushdown automaton, M, we define the language accepted by final state and exactly k pushdown reversals to be $L_k(M) = \{w \in T^* | Z_0 q_0 w \vdash^* \gamma f$ with exactly k pushdown reversals, for any $\gamma \in \Gamma^*$ and $f \in F\}$. Denote the family of context-free languages and the family of recursively enumerable languages as **CF** and **RE**, respectively. Next, denote the family of languages accepted by flip-pushdown automata with exactly k pushdown reversals as $\mathbf{FPDA}_k$ and name every language from this family as a *k–flipping language*. From [6] and [7], $\mathbf{CF} = \mathbf{FPDA}_0 \subset \mathbf{FPDA}_1 \subset \ldots \subset \mathbf{FPDA}_\infty = \mathbf{RE}$ holds.

3 Results

Theorem 1. *A language L is $(m-1)$-flipping if and only if there is a constant $k \geq 1$ and a grammar $G = (V, T, P, S)$ in Kuroda normal form, such that $L = L(G)$ and G can generate every $z \in L(G)$ by a derivation of the form*

$$\begin{aligned} S &\Rightarrow x_{11}\ x_{12}\ \ldots\ x_{1m} \\ &\Rightarrow x_{21}\ x_{22}\ \ldots\ x_{2m} \\ &\ \ \vdots \\ &\Rightarrow x_{n1}\ x_{n2}\ \ldots\ x_{nm} \end{aligned}$$

where n, m are two positive integers, and

1. $|x_{ij}| \leq k,\ 1 \leq i \leq n,\ 1 \leq j \leq m$
2. *for all $h = 2, \ldots, m$ there exists $x_{rh} \in V^+$ such that for all $q = 1, \ldots, h$ and $o = h+1, \ldots, m$, $x_{qo} = \varepsilon$*
3. *for all adjacent segments x_{cd} and x_{cd+1}, where $1 \leq c \leq n$ and $d = 1, 2, \ldots, m-1$, if there is a rule that rewrites $x_{cd}x_{cd+1}$ on boundary in a context way, then there is no rule that rewrites any $x_{pq}x_{pq+1}$ on boundary in a context way for all $p = c, c+1, \ldots, n$ and $q = d+1, d+2, \ldots, m$*

Proof. The *only if* portion is trivial. Next, we prove the *if* portion. That is, let G be a grammar satisfying the conditions described in Theorem 1. We construct a flip-pushdown automaton, $M = (Q, T, \Gamma, R, q_0, Z_0, \Delta, F)$ in the following way:

- $Q = \{f\} \cup \{\langle A, \alpha\rangle | \langle A, \alpha\rangle \neq f,\ A \in N,\ \alpha \in V^*,\ |\alpha| \leq k\}$
- $\Gamma = \Gamma_{()} \cup \{Z_0, Z_P\}$, where $\Gamma_{()} = \{(AB \to CD) | AB \to CD \in P\}$
- $s = \langle S, \varepsilon\rangle$
- $R = R_{IN} \cup R_{CS1} \cup R_{CS2} \cup R_I \cup R_{HEAD} \cup R_F$ is constructed by performing following steps:
  - I For every $x \in T^*$ with $|x| \leq k$
    add rules of the form $Z_0\langle X, \varepsilon\rangle x \to Z_P\langle X, x\rangle$ into R_I

II For every $x \in T^*$ with $|x| \leq k$ and for every $p : AB \to CD \in P$ add rules of the form $(p)\langle X, \varepsilon\rangle x \to (p)\langle X, x\rangle$ into R_I

III For every $a \in T \cup \{\varepsilon\}$ and rule $A \to a \in P$ add rules of the form $Z\langle X, \alpha a\beta\rangle \to Z\langle X, \alpha A\beta\rangle$ into R_{IN}, where $Z \in \Gamma$, $Z \neq Z_0$ and $|\alpha a\beta| > 0$

IV For every $AB \to CD$ in P add:
- (a) $Z_P\langle X, \alpha C\rangle \to (AB \to CD)Z_P\langle X, \alpha A\rangle$ into R_{CS1}
- (b) $(AB \to CD)\langle X, D\beta\rangle \to \langle X, B\beta\rangle$ into R_{CS2}
- (c) $Z\langle X, \alpha CD\beta\rangle \to Z\langle X, \alpha AB\beta\rangle$ into R_{IN}, where $Z \in \Gamma$, $Z \neq Z_0$

V For every $A \to BC$ in P add:
- (a) $Z\langle X, \alpha BC\beta\rangle \to Z\langle X, \alpha A\beta\rangle$ into R_{IN}, where $Z \in \Gamma$, $Z \neq Z_0$
- (b) $Z_P\langle A, B\rangle \to Z_P\langle C, \varepsilon\rangle$ into R_{HEAD}

VI For every $\langle X, \varepsilon\rangle \in Q$, where $X \in N$ set $\Delta(\langle X, \varepsilon\rangle) = \{\langle X, \varepsilon\rangle\}$

VII For every $A \in N$, add $Z_P\langle A, A\rangle \to \varepsilon f$ to R_F

The construction is completed. Next, we prove that $L(G) = L(M)$ by demonstrating $L(G) \subseteq L(M)$ and $L(M) \subseteq L(G)$. First, we prove $L(G) \subseteq L(M)$. By induction, we demonstrate Claims A, B and C.

Claim A. $S \Rightarrow^i \lambda\alpha x\beta\mu$ in G implies $\varepsilon f \dashv Z_P\langle X, X\rangle \dashv^* \sigma\langle Z, \alpha x\beta\rangle\omega$ in M, where $X, Z \in N$, $\alpha, \beta, \lambda, \mu \in V^*$, $x \in N^2 \cup N \cup T \cup \{\varepsilon\}$, $\omega \in T^*$, $\sigma \in \Gamma^*$.

Basis

Let $i = 0$. Then $S \Rightarrow^0 S$ in G. By VII, $Z_P\langle X, X\rangle \to \varepsilon f \in R$ for every $X \in N$, so $\varepsilon f \dashv Z_P\langle S, S\rangle \dashv^0 Z_P\langle S, S\rangle$ in M.

Induction Hypothesis. Assume that Claim A holds for every $i \leq n$, where n is a positive integer.

Induction Step. Consider any derivation of the form $S \Rightarrow^{n+1} \lambda\alpha y\beta\mu$ and express this derivation as $S \Rightarrow^n \lambda\alpha x\beta\mu \Rightarrow \lambda\alpha y\beta\mu$, where $x, y \in N^2 \cup N \cup T \cup \{\varepsilon\}$, $\alpha, \beta, \lambda, \mu \in V^*$.

By the induction hypothesis, $f \dashv^* \sigma\langle Y, \alpha x\beta\rangle\omega \dashv \sigma\langle Y, \alpha y\beta\rangle\omega$ in M, where $\sigma \in \Gamma^*$. The next three cases cover all possibilities how G can make the derivation $\lambda\alpha x\beta\mu \Rightarrow \lambda\alpha y\beta\mu$.

a) Let $A \to a \in P$, $x = A$, $y = a$,
where $a \in T \cup \{\varepsilon\}$, $A \in N$.
Then, $\lambda\alpha A\beta\mu \Rightarrow \lambda\alpha a\beta\mu[A \to a]$ in G.
By III, $Z\langle X, \alpha a\beta\rangle \to Z\langle X, \alpha A\beta\rangle \in R$,
so $\delta Z\langle X, \alpha x\beta\rangle\omega \dashv \delta Z\langle X, \alpha y\beta\rangle\omega$ in M, $Z \in \Gamma$, $\delta \in \Gamma^*$.

b) Let $A \to BC \in P$, $x = A$, $y = BC$,
where $A, B, C \in N$.
Then, $\lambda\alpha A\beta\mu \Rightarrow \lambda\alpha BC\beta\mu[A \to BC]$ in G.
By Va, $Z\langle X, \alpha BC\beta\rangle \to Z\langle X, \alpha A\beta\rangle \in R$,
so $\delta Z\langle X, \alpha x\beta\rangle\omega \dashv \delta Z\langle X, \alpha y\beta\rangle\omega$ in M, $Z \in \Gamma$, $\delta \in \Gamma^*$.

c) Let $AB \rightarrow CD \in P$, $x = AB$, $y = CD$,
where $A, B, C, D \in N$.
Then, $\lambda\alpha AB\beta\mu \Rightarrow \lambda\alpha CD\beta\mu[AB \rightarrow CD]$ in G.
By IVc, $Z\langle X, \alpha CD\beta\rangle \rightarrow Z\langle X, \alpha AB\beta\rangle \in R$,
so $\delta Z\langle X, \alpha x\beta\rangle\omega \dashv \delta Z\langle X, \alpha y\beta\rangle\omega$ in M,$Z \in \Gamma$, $\delta \in \Gamma^*$.

Thus Claim A holds. □

Claim B. Let x,$y \in T^*$ are two adjacent terminal segments of any sentence $w \in L(G)$, $|x|$, $|y| \leq k$. If there exists a derivation of the form $\alpha_0 x_0 y_0 \beta_0 \Rightarrow^*$ $\alpha_1 x_1 A_1 B_1 y_1 \beta_1 \Rightarrow \alpha_1 x_1 C_1 D_1 y_1 \beta_1 [p_1] \Rightarrow^* \alpha_2 x_2 A_2 B_2 y_2 \beta_2 \Rightarrow$ $\alpha_2 x_2 C_2 D_2 y_2 \beta_2 [p_2] \Rightarrow^* \ldots \Rightarrow^* \alpha_i x_i A_i B_i y_i \beta_i \Rightarrow \alpha_i x_i C_i D_i y_i \beta_i [p_i] \Rightarrow^* \alpha x y \beta$ in G, where every $p_i = A_i B_i \rightarrow C_i D_i \in P_{CS}$ rewrites A_i and B_i on boundary in a context way ($A_j, B_j, C_j, D_j \in N$, $\alpha, \beta, \alpha_m, \beta_m \in V^*$ for $j = 1, 2, \ldots, i$, $m = 0, 1, \ldots, i$), then

$$\begin{aligned}
&Z_P u\langle X, x\rangle y\omega \vdash^* Z_P\langle X, x_i C_i\rangle y\omega \vdash (p_i) Z_P\langle X, x_i A_i\rangle y\omega \vdash^* \ldots \\
&\ldots \vdash^* (p_i)\ldots(p_3) Z_P\langle X, x_2 C_2\rangle y\omega \vdash (p_i)\ldots(p_3)(p_2) Z_P\langle X, x_2 A_2\rangle y\omega \vdash^* \\
&(p_i)\ldots(p_3)(p_2) Z_P\langle X, x_1 C_1\rangle y\omega \vdash (p_i)\ldots(p_3)(p_2)(p_1) Z_P\langle X, x_1 A_1\rangle y\omega \vdash^* \\
&(p_i)\ldots(p_3)(p_2)(p_1) Z_P\langle X, x_0\rangle y\omega
\end{aligned}$$

and

$$\begin{aligned}
&Z_P(p_1)\ldots(p_{i-1})(p_i)\langle Y, y\rangle\omega \vdash^* Z_P(p_1)\ldots(p_{i-1})(p_i)\langle Y, D_i y_i\rangle\omega \vdash \\
&Z_P(p_1)\ldots(p_{i-1})\langle Y, B_i y_i\rangle\omega \vdash^* \ldots \vdash^* Z_P(p_1)(p_2)\langle Y, D_2 y_2\rangle\omega \vdash \\
&Z_P(p_1)\langle Y, B_2 y_2\rangle\omega \vdash^* Z_P(p_1)\langle Y, D_1 y_1\rangle\omega \vdash Z_P\langle Y, B_1 y_1\rangle\omega \vdash^* Z_P\langle Y, y_0\rangle
\end{aligned}$$

in M, where $x_j, y_j \in V^*$, $u \in \Gamma_{()}^*$, $0 \leq j \leq i$, $\omega \in T^*$, $(p_1), \ldots, (p_i) \in \Gamma_{()}$.

Basis Let $i = 0$. Then $\alpha_0 x_0 y_0 \beta_0 \Rightarrow^* \alpha x y \beta$ in G. In this case, there is no rewriting in a context way between the investigated columns, so productions are applied only inside these columns. Thus, in M, $Z_P\langle X, x\rangle y\omega \vdash^* Z_P\langle X, x_0\rangle y\omega$ and $Z_P\langle Y, y\rangle\omega \vdash^* Z_P\langle Y, y_0\rangle\omega$, where only reductions inside the columns are simulated (see Claim A).

Induction Hypothesis Assume that Claim B holds for every $i \leq t$, where t is a positive integer.

Induction Step By the induction hypothesis,

$$\begin{aligned}
&Z_P u\langle X, x\rangle y\omega \vdash^* Z_P\langle X, x_{t+1} C_{t+1}\rangle y\omega \vdash (p_{t+1}) Z_P\langle X, x_{t+1} A_{t+1}\rangle y\omega \vdash^* \\
&(p_{t+1}) Z_P\langle X, x_t C_t\rangle y\omega \vdash (p_{t+1})(p_t) Z_P\langle X, x_t A_t\rangle y\omega \vdash^* \ldots \\
&\ldots \vdash^* (p_{t+1})(p_t)\ldots(p_3) Z_P\langle X, x_2 C_2\rangle y\omega \vdash \\
&(p_{t+1})(p_t)\ldots(p_3)(p_2) Z_P\langle X, x_2 A_2\rangle y\omega \vdash^* \\
&(p_{t+1})(p_t)\ldots(p_3)(p_2) Z_P\langle X, x_1 C_1\rangle y\omega \vdash \\
&(p_{t+1})(p_t)\ldots(p_3)(p_2)(p_1) Z_P\langle X, x_1 A_1\rangle y\omega \vdash^* \\
&(p_{t+1})(p_t)\ldots(p_3)(p_2)(p_1) Z_P\langle X, x_0\rangle y\omega
\end{aligned}$$

and

$Z_P(p_1)(p_2)\ldots(p_{t-1})(p_t)(p_{t+1})\langle Y, y\rangle\omega \vdash^*$
$Z_P(p_1)(p_2)\ldots(p_{t-1})(p_t)(p_{t+1})\langle Y, D_{t+1}y_{t+1}\rangle\omega \vdash$
$Z_P(p_1)(p_2)\ldots(p_{t-1})(p_t)\langle Y, B_{t+1}y_{t+1}\rangle\omega \vdash^*$
$Z_P(p_1)(p_2)\ldots(p_{t-1})(p_t)\langle Y, D_t y_t\rangle\omega \vdash$
$Z_P(p_1)(p_2)\ldots(p_{t-1})\langle Y, B_t y_t\rangle\omega \vdash^* \ldots \vdash^* Z_P(p_1)(p_2)\langle Y, D_2 y_2\rangle\omega \vdash$
$Z_P(p_1)\langle Y, B_2 y_2\rangle\omega \vdash^* Z_P(p_1)\langle Y, D_1 y_1\rangle\omega \vdash$
$Z_P\langle Y, B_1 y_1\rangle\omega \vdash^* Z_P\langle Y, y_0\rangle\omega$

By IVa and IVb, for every production of the form $AB \to CD \in P$, there are productions of the form $Z_P\langle X, \alpha C\rangle \to (AB \to CD)Z_P\langle X, \alpha A\rangle$ and $(AB \to CD)\langle X, D\beta\rangle \to \langle X, B\beta\rangle$ in R. These productions simulate the transitions of the form $(p_{t+1})\ldots(p_{r-1})Z_P\langle X, x_r C_r\rangle y\omega \vdash (p_{t+1})\ldots(p_{r-1})(p_r)Z_P\langle X, x_r A_r\rangle y\omega$ and $Z_P(p_1)\ldots(p_{s-1})(p_s)\langle Y, D_s y_s\rangle\omega \vdash Z_P(p_1)\ldots(p_{s-1})\langle Y, B_s y_s\rangle\omega$ in M, where $r, s = t+1, t, \ldots, 1$, so Claim B holds. □
All other derivation and transition steps in G and M are investigated in Claims A and C.

Claim C. $\lambda z\mu \Rightarrow^i \lambda' x\mu'$ in G implies $u\langle Z, \alpha z\beta\rangle\omega \vdash^* v\langle A, Y\rangle\omega'$ in M, where $\lambda, \lambda', \mu, \mu' \in V^*$, $z, x \in N^2 \cup N$, $A, Z \in N$, $\omega, \omega' \in T^*$ and $u, v \in \Gamma^*$. For $|x| = 1$, there are $A = x$, $Y \in N$ and for $x = x_1x_2$, $x_1, x_2 \in N$, there are $A = x_2$, $Y = \varepsilon$.

Basis Let $i = 0$. Then $\lambda z\mu \Rightarrow^0 \lambda z\mu$ in G. Clearly, $u\langle Z, \alpha z\beta\rangle\omega \vdash^0 u\langle Z, \alpha z\beta\rangle\omega$ in M.

Induction Hypothesis Assume that the implication in Claim C holds for every $i \leq g$, where g is a positive integer.

Induction Step Consider any derivation of the form $\lambda z\mu \Rightarrow^{g+1} \lambda' y\mu'$ and express it as $\lambda z\mu \Rightarrow^g \lambda' x\mu' \Rightarrow \lambda' y\mu'$, where $x, y, z \in N^2 \cup N$, $\lambda, \mu, \lambda', \mu' \in V^*$.

By the induction hypothesis, $u\langle Z, \alpha z\beta\rangle\omega \vdash^g v\langle A, Y\rangle\omega' \vdash v\langle A', Y'\rangle\omega'$ in M, $u, v, v\prime \in \Gamma^*$. Next, we examine the derivation $\lambda' x\mu' \Rightarrow \lambda' y\mu'$ in G.

Let $A \to BC \in P$, $x = A$, $y = y_1y_2 = BC$, where $A, B, C \in N$. Then, $\lambda' A\mu' \Rightarrow \lambda' BC\mu'[A \to BC]$ in G. By Vb, $Z_P\langle A, B\rangle \to Z_P\langle C, \varepsilon\rangle \in R$, so $wZ_P\langle x, y_1\rangle\omega \vdash wZ_P\langle y_2, \varepsilon\rangle\omega$ in M, where $w \in \Gamma^*$. □

Observe that Claim C holds. By Claims A, B, and C, $L(G) \subseteq L(M)$ holds.

The complete proof of the second inclusion, $L(M) \subseteq L(G)$, unfortunately exceeds the page limit of this paper, so we leave it on the reader and we only shortly outline its structure.

The proof of this inclusion is based on the form of computation of flip-pushdown automata, where there can be identified some main parts in the computation, which are closely linked to the corresponding derivation sequences in the grammars restricted in the vertical way described before. By this it is proved that these automata can simulate every derivation in the vertically restricted grammar from the bottom, where the parsed sentence is read and processed column-by-column. There is the pushdown flip performed between every two consecutive columns, so the context-sensitive derivations on the column's boundaries can be efficiently simulated on the pushdown.

By this, $L(M) \subseteq L(G)$ is proved and thus Theorem 1 holds. ■

From the previous result, some corollaries follow. To formalize them, consider a phrase-structure grammar G_i, in Kuroda normal form satisfying the vertical restrictions from Theorem 1, which generates every $w \in L(G_i)$ by using exactly i columns. Name this grammar as an *$i-$column grammar*, where $i \geq 1$. Denote $\mathbf{KNF}_i$ the family of languages generated by $i-$column grammars.

Corollary 1. $\mathbf{FPDA_{i-1}} = \mathbf{KNF}_i$

Corollary 2. $\mathbf{CF} = \mathbf{KNF}_1 \subset \mathbf{KNF}_2 \subset \ldots \subset \mathbf{KNF}_\infty = \mathbf{RE}$

4 Conclusions

The introduced restrictions preserve all forms of derivation rules defined by the Kuroda normal form of phrase-structure grammars. Notice that there were no restrictions on the number of derivation steps in the particular columns introduced, but only the order of context-sensitive derivations on boundaries was restricted. As a result, we found a new infinite language hierarchy generated by grammars restricted in this way, where the lowest family of languages in this hierarchy is equal to the context-free languages and the highest family corresponds to the family of recursively enumerable languages. This result is very interesting in the formal language theory and extends the area of infinite language hierarchies by a new type.

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