Mathematical Models and Analysis of Deformation Processes in Biomaterials with Fractal Structure

Yaroslav Sokolovskyy[1[0000-0003-4866-2575]], Maryana Levkovych[1[0000-0002-0119-3954]]

Olha Mokrytska[1[0000-0002-2887-9585]] Svitlana Yatsyshyn[1[0000-0001-5200-4837]]

Yaroslav Kaspryshyn[1[0000-0003-1118-9059]] and Christine Strauss[2[0000-0003-0276-3610]]

[1] Department of Information Technologies, Ukrainian National Forestry University, UNFU
Lviv, UKRAINE
sokolowskyyyar@yahoo.com, maryana.levkovych@gmail.com,
mokrytska@nltu.edu.ua, svitlana0981@gmail.com
kaspryshyn.ya@gmail.com

[2] Department of Electronic Business, University of Vienna
Vienna, AUSTRIA
christine.strauss@univie.ac.at

Abstract. Mathematical models of heat and mass transfer and deformation processes of biomaterials are investigated, taking into account such properties as memory-effect (eriditarity), self-organization, deterministic chaos, heterogeneity of structure, variability of rheological properties. The obtained results of numerical modelling of non-isothermal moisture transfer and deformation of biomaterials taking into account fractal structure make it possible to estimate – based on the type of material and its thermo-mechanical characteristics – the residual deformation of the material. A mathematical rheological model of two-dimensional visco-elastic deformation of biomaterials with regard to memory-effect and self-organization is constructed, which is described using equilibrium equations with fractional order. The relation between the two-dimensional stress-deformation state of biomaterials for the rheological models of Maxwell, Kelvin and Voigt, which are presented in the integral form, was obtained. The aspects of the algorithm of numerical implementation of two-dimensional mathematical model of visco-elastic deformation in fractured media are presented. The method of splitting fractional-differential parameters of models was adapted, which was used in the problems of identification of non-integer parameters of models. The results of the identification and numerical implementation of the mathematical model of heat and mass transfer processes of biophysical materials are considered, taking into account the fractal structure.

Keywords: eriditarity, biophysical process, non-integer integro-differentiation apparatus.

2019 IDDM Workshops.

1 Introduction

The biophysical process is often characterized by the simultaneous influence on the material of several factors - load, moisture and temperature. Changing at least one of them in the biomaterial, to which belong transplant, implants, joints, medical silicon, leads to formation of deformation and its transition from one type to the other, resulting in the total or partial restoration of the original physical state. This ability of the material characterizes the presence of the memory-effect, which is based on residual deformations. In addition to residual memory, biomaterials are characterized by stochastic heterogeneity of the structure and significant variability of rheological properties. To investigate the above-mentioned properties in biophysical materials, as well as deterministic chaos, the complex nature of spatial correlations and self-organization is possible by formal means of fractional integro-differential operators [1, 10]. This approach provides the basis for the development of mathematical models of non-equilibrium biophysical processes with a fractal structure. At present, a very small number of works [7, 8] is devoted to the development of algorithms and software for studying the processes of deformation and heat-moisture transfer, taking into account the memory-effect-properties and self-organization of materials, which allows us to estimate the residual and elastic stress values.

2 Problem formulation

The mathematical rheological model of two-dimensional visco-elastic deformation of biomaterials, taking into account eriditarity (memory-effect) and self-organization, is described using equilibrium equations with a fractional order γ $(0<\gamma\le 1)$ in spatial coordinates x_1 and x_2:

$$C_{11}\left(\overline{R}_{11}\frac{\partial^{\gamma}\varepsilon_{11}}{\partial x_1^{\gamma}}-\widetilde{R}_{11}\right)+C_{12}\left(\overline{R}_{12}\frac{\partial^{\gamma}\varepsilon_{22}}{\partial x_1^{\gamma}}-\widetilde{R}_{12}\right)+2C_{33}\left(\overline{R}_{33}^{2}\frac{\partial^{\gamma}\varepsilon_{12}}{\partial x_2^{\gamma}}-\widetilde{R}_{33}^{2}\right)=0, \quad (1)$$

$$C_{21}\left(\overline{R}_{21}\frac{\partial^{\gamma}\varepsilon_{11}}{\partial x_2^{\gamma}}-\widetilde{R}_{21}\right)+C_{22}\left(\overline{R}_{22}\frac{\partial^{\gamma}\varepsilon_{22}}{\partial x_2^{\gamma}}-\widetilde{R}_{22}\right)+2C_{33}\left(\overline{R}_{33}^{1}\frac{\partial^{\gamma}\varepsilon_{12}}{\partial x_1^{\gamma}}-\widetilde{R}_{33}^{1}\right)=0. \quad (2)$$

where $\overline{R}_{ij}$, $\widetilde{R}_{ij}$ are the corresponding values of the integrals:

$$\int_0^t R_{ij}(t-z,T,U)dz=\overline{R}_{ij},\ \int_0^t R_{ij}(t-z,T,U)\frac{\partial^{\gamma}\varepsilon_{T1,T2}}{\partial x_k^{\gamma}}dz=\widetilde{R}_{ij},\ (k=1,2)$$

$$\int_0^t R_{ij}(t-z,T,U)\frac{\partial^{\gamma}\varepsilon_{T3}}{\partial x_2^{\gamma}}dz=\widetilde{R}_{33}^{2},\int_0^t R_{ij}(t-z,T,U)\frac{\partial^{\gamma}\varepsilon_{T3}}{\partial x_1^{\gamma}}dz=\widetilde{R}_{33}^{1},$$

R_{ij} are relaxation kernels of fractional-differential models, which are dependent on time t, temperature T and moisture U; $\varepsilon^T=(\varepsilon_{11},\varepsilon_{22},\varepsilon_{12})$ is a deformation vector, components of which are dependent on time t and spatial variables x_1 and x_2, $(t, x_1, x_2) \in D, D = [0, \tilde{T}] \times [0, l_1] \times [0, l_2]$, $\varepsilon_T = (\varepsilon_{T1}, \varepsilon_{T2}, \varepsilon_{T3})^T$ is deformation vector, components of which are dependent on temperature variations ΔT and moisture content ΔU:

$$\varepsilon_{T1} = \alpha_{11}\Delta T + \beta_{11}\Delta U, \;\; \varepsilon_{T2} = \alpha_{22}\Delta T + \beta_{22}\Delta U, \;\; \varepsilon_{T3} = 0,$$

$\alpha_{11}, \alpha_{22}, \beta_{11}, \beta_{22}$ are coefficients of thermal expansion and moisture-condition shrinkage; C_{ij} are components of the elastic tensor of an orthotropic body:

$$C_{11} = \frac{E_{11}}{(1-\nu_1\nu_2)}, \; C_{12} = \frac{\nu_2 E_{11}}{(1-\nu_1\nu_2)}, \; C_{21} = \frac{\nu_1 E_{22}}{(1-\nu_1\nu_2)}, \; C_{22} = \frac{E_{22}}{(1-\nu_1\nu_2)}, \; C_{33} = \mu,$$

where μ is shear modulus in plane, E_{11}, E_{22}, E_{12} are Young's moduli, ν_1, ν_2 are Poison's ratios.

Let us set the following boundary conditions:

$$\varepsilon_{ij}\Big|_{x_j=0} = 0, \qquad \varepsilon_{ij}\Big|_{x_j=l_j} = 0, \tag{3}$$

and initial conditions, respectively:

$$\varepsilon_{ij}\Big|_{t=0} = 0. \tag{4}$$

The relationship between the components of stress $\sigma^T = (\sigma_{11}, \sigma_{22}, \sigma_{12})$ and strain $\varepsilon^T = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$ for two-dimensional fractional-differential rheological models, respectively, can be written as follows:

Voigt's model

$$\sigma_{11} = \frac{\mathrm{E}_{11}}{(1-\nu_1\nu_2)} D_t^\alpha(\varepsilon_{11} - \varepsilon_{T1}) + \frac{\nu_2 \mathrm{E}_{11}}{(1-\nu_1\nu_2)} D_t^\alpha(\varepsilon_{22} - \varepsilon_{T2}) + \\ + 2\mathrm{E}\tau^\beta \left(D_t^\beta(\varepsilon_{11} - \varepsilon_{T1}) + D_t^\beta(\varepsilon_{22} - \varepsilon_{T2})\right), \tag{5}$$

$$\sigma_{22} = \frac{\nu_1 \mathrm{E}_2}{(1-\nu_1\nu_2)} D_t^\alpha(\varepsilon_{11} - \varepsilon_{T1}) + \frac{\mathrm{E}_{22}}{(1-\nu_1\nu_2)} D_t^\alpha(\varepsilon_{22} - \varepsilon_{T2}) + \\ + 2\mathrm{E}\tau^\beta \left(D_t^\beta(\varepsilon_{11} - \varepsilon_{T1}) + D_t^\beta(\varepsilon_{22} - \varepsilon_{T2})\right), \tag{6}$$

$$\sigma_{12} = \mu D_t^\alpha(\varepsilon_{12} - \varepsilon_{T3}) + \mathrm{E}\tau^\beta D_t^\beta(\varepsilon_{12} - \varepsilon_{T3}), \tag{7}$$

E is elastic modulus of an elastic element of a Voigt's body, $0 \le \alpha < \beta \le 1$;

Kelvin's model

$$\sigma_{11}+\frac{E_1\tau^{\alpha}}{(E_1+E_2)}D_t^{\alpha}\sigma_{11}=\frac{E_{11}}{(1-\nu_1\nu_2)}(\varepsilon_{11}-\varepsilon_{T1})+\frac{\nu_2 E_{11}}{(1-\nu_1\nu_2)}(\varepsilon_{22}-\varepsilon_{T2})+$$
$$+\frac{2E_1E_2\tau^{\beta}}{(E_1+E_2)}\left(D_t^{\beta}(\varepsilon_{11}-\varepsilon_{T1})+D_t^{\beta}(\varepsilon_{22}-\varepsilon_{T2})\right), \tag{8}$$

$$\sigma_{22}+\frac{E_1\tau^{\alpha}}{(E_1+E_2)}D_t^{\alpha}\sigma_{22}=\frac{\nu_1 E_{22}}{(1-\nu_1\nu_2)}(\varepsilon_{11}-\varepsilon_{T1})+\frac{E_{22}}{(1-\nu_1\nu_2)}(\varepsilon_{22}-\varepsilon_{T2})+$$
$$+\frac{2E_1E_2\tau^{\beta}}{(E_1+E_2)}\left(D_t^{\beta}(\varepsilon_{11}-\varepsilon_{T1})+D_t^{\beta}(\varepsilon_{22}-\varepsilon_{T2})\right), \tag{9}$$

$$\sigma_{12}+\frac{E_1\tau^{\alpha}}{(E_1+E_2)}D_t^{\alpha}\sigma_{12}=\mu(\varepsilon_{12}-\varepsilon_{T3})+\frac{E_1E_2\tau^{\beta}}{(E_1+E_2)}D_t^{\beta}(\varepsilon_{12}-\varepsilon_{T3}), \tag{10}$$

where E_1 is elastic modulus of an elastic element of a Voigt's body, E_2 is elastic modulus of an elastic element, α, β are fractional derivatives and $0 \le \alpha, \beta \le 1$;

Maxwell's model

$$\sigma_{11}+\tau^{\alpha}D_t^{\alpha}\sigma_{11}=\frac{E_{11}}{(1-\nu_1\nu_2)}(\varepsilon_{11}-\varepsilon_{T1})+\frac{\nu_2 E_{11}}{(1-\nu_1\nu_2)}(\varepsilon_{22}-\varepsilon_{T2})+$$
$$+2E_2\tau^{\beta}\left(D_t^{\beta}(\varepsilon_{11}-\varepsilon_{T1})+D_t^{\beta}(\varepsilon_{22}-\varepsilon_{T2})\right), \tag{11}$$

$$\sigma_{22}+\tau^{\alpha}D_t^{\alpha}\sigma_{22}=\frac{\nu_1 E_2}{(1-\nu_1\nu_2)}(\varepsilon_{11}-\varepsilon_{T1})+\frac{E_{22}}{(1-\nu_1\nu_2)}(\varepsilon_{22}-\varepsilon_{T2})+$$
$$+2E_2\tau^{\beta}\left(D_t^{\beta}(\varepsilon_{11}-\varepsilon_{T1})+D_t^{\beta}(\varepsilon_{22}-\varepsilon_{T2})\right), \tag{12}$$

$$\sigma_{12}+\tau^{\alpha}D_t^{\alpha}\sigma_{12}=\mu(\varepsilon_{12}-\varepsilon_{T3})+E_2\tau^{\beta}D_t^{\beta}(\varepsilon_{12}-\varepsilon_{T3}), \tag{13}$$

where E_2 is elastic modulus of an elastic element for Maxwell's model,, $0 \le \alpha < \beta < 1$.

If we put $\alpha = 0,\ \beta = 1$ in relations (5) - (7), we get the classical two-dimensional Voigt's model in the case of orthotropy. Relations (8) - (13) will describe the classical Maxwell's and Kelvin's models at fractal values $\alpha = 1,\ \beta = 1$.

For the integral representation of relations (5) - (13), we consider the properties of fractional derivatives [5], the definition of fractional derivative $\vartheta,\ (0 \le \vartheta < 1)$:

$$D_t^{\vartheta} f(t)=\frac{1}{\Gamma(1-\vartheta)}D_t\int_0^t (t-\xi)^{-\vartheta} f(\xi)d\xi. \tag{14}$$

as well as the Laplace transform method [9].

Thus, the relations describing the relationship between stress and strain (5) - (13) can be rewritten after the corresponding transformations in the integral form.

Two-dimensional fractional-differential Voigt's model:

$$\sigma_{ii}=\frac{C_{ii}}{\Gamma(1-\alpha)}D_t\int_0^t(t-\xi)^{-\alpha}[p_1(\varepsilon_{11}(\xi)-\varepsilon_{T1}(\xi))+p_2(\varepsilon_{22}(\xi)-\varepsilon_{T2}(\xi))]d\xi+$$

$$+\frac{2\mathrm{E}\tau^{\beta}}{\Gamma(1-\beta)}D_t\int_0^t(t-\xi)^{-\beta}[(\varepsilon_{11}(\xi)-\varepsilon_{T1}(\xi))+(\varepsilon_{22}(\xi)-\varepsilon_{T2}(\xi))]d\xi, \qquad (15)$$

$$\sigma_{12}=\frac{\mu}{\Gamma(1-\alpha)}D_t\int_0^t(t-\xi)^{-\alpha}(\varepsilon_{12}(\xi)-\varepsilon_{T3}(\xi))d\xi+\frac{\mathrm{E}\tau^{\beta}}{\Gamma(1-\beta)}D_t\int_0^t(t-\xi)^{-\beta}(\varepsilon_{12}(\xi)-\varepsilon_{T3}(\xi))d\xi. \qquad (16)$$

Two-dimensional fractional-differential Kelvin's and Maxwell's models:

$$\sigma_{ii}=C_iG(t)+A\int_0^t G(t-\xi)[p_1(\varepsilon_{11}(\xi)-\varepsilon_{T1}(\xi))+2BD_t^{\beta}(\varepsilon_{11}(\xi)-\varepsilon_{T1}(\xi))]d\xi+$$

$$+A\int_0^t G(t-\xi)[p_2(\varepsilon_{22}(\xi)-\varepsilon_{T2}(\xi))+2BD_t^{\beta}(\varepsilon_{22}(\xi)-\varepsilon_{T2}(\xi))]d\xi, \qquad (17)$$

$$\sigma_{12}=C_3G(t)+A\int_0^t G(t-\xi)\left[2C_{33}(\varepsilon_{12}(\xi)-\varepsilon_{T3}(\xi))+BD_t^{\beta}(\varepsilon_{12}(\xi)-\varepsilon_{T3}(\xi))\right]d\xi, \qquad (18)$$

where $G(t)=\begin{cases}t^{\alpha-1}E_{\alpha,\alpha}(-At^{\alpha}),\text{Kelvin}\\ t^{\alpha-1}E_{\alpha,\alpha}\left(-\dfrac{t^{\alpha}}{\tau^{\alpha}}\right),\text{Maxwell}\end{cases}$, $B=\begin{cases}\dfrac{\mathrm{E}_1\mathrm{E}_2\tau^{\beta}}{(\mathrm{E}_1+\mathrm{E}_2)},\text{Kelvin}\\ \mathrm{E}_2\tau^{\beta},\text{Maxwell}\end{cases}$, $A=\begin{cases}\dfrac{(\mathrm{E}_1+\mathrm{E}_2)}{\mathrm{E}_1\tau^{\alpha}},\text{Kelvin}\\ \dfrac{1}{\tau^{\alpha}},\text{Maxwell}\end{cases}$ for Maxwell's and Kelvin's models – $\begin{cases}i=1\rightarrow p_1=C_{11},p_2=C_{12}\\ i=2\rightarrow p_1=C_{21},p_2=C_{22}\end{cases}$; for Voigt's model – $\begin{cases}i=1\rightarrow p_1=1,\ p_2=\nu_2\\ i=2\rightarrow p_1=\nu_1,\ p_2=1\end{cases}$.

3 A numerical method for the realization of two-dimensional fractional-differential rheological models

To implement the numerical method, we introduce the space-time grid in the region D:

$$\varpi_{\Delta t,h_1,h_2} = \{(t^k, x_{1(n)}, x_{2(m)}): x_{1(n)} = (n-1)h_1, x_{2(m)} = (m-1)h_2, t^k = k\Delta t, n = 1,...,N;$$
$$h_1 = \frac{l_1}{N-1}; m = 1,...,M; h_2 = \frac{l_2}{M-1}; k = 0,1,...,K; \Delta\tau = \frac{\tilde{T}}{K}\}. \tag{19}$$

Given the Riemann-Liouville formula [10], the difference approximation of a fractional derivative γ $(0 < \gamma \le 1)$ by coordinates x_1, x_2 can be written as follows [4, 5]:

$$\left.\frac{\partial^\gamma u}{\partial x_i^\gamma}\right|_{x_{i(n)}} \approx \frac{u_{n+1} - \gamma u_n}{\Gamma(2-\gamma) h_i^\gamma},\ h_i = x_{i(n+1)} - x_{i(n)},\ i = 1,2, \tag{20}$$

where $\Gamma(\cdot)$ is a Gamma function.

Given (20), the finite-difference approximation of the system of differential equations (1) - (2) will take the form:

$$\begin{aligned} &\frac{C_{11}}{\Gamma(2-\gamma)h_1^\gamma}\bar{R}_{11}\left(\varepsilon^k_{11(n+1,m)} - \gamma\varepsilon^k_{11(n,m)}\right) - C_{11}\tilde{R}_{11} + \\ &+ \frac{C_{12}}{\Gamma(2-\gamma)h_1^\gamma}\bar{R}_{12}\left(\varepsilon^k_{22(n+1,m)} - \gamma\varepsilon^k_{22(n,m)}\right) - C_{12}\tilde{R}_{12} + \\ &+ \frac{2C_{33}}{\Gamma(2-\gamma)h_2^\gamma}\bar{R}^2_{33}\left(\varepsilon^k_{12(n,m+1)} - \gamma\varepsilon^k_{12(n,m)}\right) - 2C_{33}\tilde{R}^2_{33} = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{C_{21}}{\Gamma(2-\gamma)h_2^\gamma}\bar{R}_{21}\left(\varepsilon^k_{11(n,m+1)} - \gamma\varepsilon^k_{11(n,m)}\right) - C_{21}\tilde{R}_{21} + \\ &+ \frac{C_{22}}{\Gamma(2-\gamma)h_2^\gamma}\bar{R}_{22}\left(\varepsilon^k_{22(n,m+1)} - \gamma\varepsilon^k_{22(n,m)}\right) - C_{22}\tilde{R}_{22} + \\ &+ \frac{2C_{33}}{\Gamma(2-\gamma)h_1^\gamma}\bar{R}^1_{33}\left(\varepsilon^k_{12(n+1,m)} - \gamma\varepsilon^k_{12(n,m)}\right) - 2C_{33}\tilde{R}^1_{33} = 0. \end{aligned} \tag{22}$$

The boundary and initial conditions (3) - (4) in the finite-difference form are written, respectively:

$$\varepsilon^{\ k}_{11,22,12(1,m)} = 0,\ \varepsilon^{\ k}_{11,22,12(N,m)} = 0,\ \varepsilon^{\ k}_{11,22,12(n,1)} = 0,\ \varepsilon^{\ k}_{11,22,12(n,M)} = 0, \tag{23}$$

$$\varepsilon^{\ 0}_{11,22,12(n,m)} = 0. \tag{24}$$

4 Splitting two-dimensional fractional-differential kernels

Let a force act on a wooden specimen along the axis OX. Then from the components of the stress tensor $\sigma_{11} \neq 0$, and $\sigma_{22} = 0$.

Since it is known that the average stress is specified by the formula:

$$\tilde{\sigma} = \frac{1}{3}(\sigma_{11} + \sigma_{22}), \tag{25}$$

then in this case $\tilde{\sigma} = \frac{1}{3}\sigma_{11}$.

The stress tensor deviator then takes the form:

$$S_{ij}(t) = \sigma_{ij} - \tilde{\sigma}\delta_{ij} = \begin{vmatrix} \frac{2}{3}\sigma_{11} & 0 \\ 0 & -\frac{1}{3}\sigma_{11} \end{vmatrix}, \tag{26}$$

where $\delta_{ij} = \begin{cases} 1, i = j; \\ 0, i \neq j. \end{cases}$ is the Kronecker symbol.

The strain tensor deviator will take the form:

$$e_{ij}(t) = \varepsilon_{ij} - \tilde{e}\,\delta_{ij} = \begin{vmatrix} \frac{2}{3}(\varepsilon_{11} - \varepsilon_{22}) & 0 \\ 0 & \frac{1}{3}(\varepsilon_{22} - \varepsilon_{11}) \end{vmatrix}, \tag{27}$$

where $\tilde{e} = \frac{1}{3}\theta = \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22})$.

The displacement equation [6] takes the form:

$$e_{ij}(t) = \frac{S_{ij}(t)}{2\mu} + \frac{1}{2\mu}\int_0^t \Pi_{зс}(t-\xi) S_{ij}(\xi)\, d\xi, \tag{28}$$

where $\Pi_{зс}(t-\xi)$ is shear creep kernel, μ is shear modulus.

The equation of volumetric deformation (strain) is written accordingly [11]:

$$\theta(t) = \frac{\tilde{\sigma}(t)}{B} + \frac{1}{B}\int_0^t \Pi_{об}(t-\xi)\tilde{\sigma}(\xi)\, d\xi, \tag{25}$$

where $\Pi_{об}(t-\xi)$ is volumetric creep kernel, B is volumetric modulus of elasticity associated with the longitudinal modulus of elasticity E_{11} and elastic Poisson's ratio ν_0 by the formula $B=\dfrac{\mathrm{E}_{11}}{3(1-2\nu_0)}$.

Based on the creep data of stretched or compressed specimens, using the measured values of longitudinal $\varepsilon_{11}(t)$ and transverse $\varepsilon_{22}(t)$ deformations, we can construct functions of longitudinal $\Pi_{11}(t)$ and transverse $\Pi_{22}(t)$ creep.

We write the equations of the processes of longitudinal and transverse deformation of a wooden specimen that is stretched by stress $\sigma_{11}(t)$ [11]:

$$\varepsilon_{11}(t)=\frac{1}{\mathrm{E}_{11}}\left[\sigma_{11}(t)+\int_0^t \Pi_{11}(t-\xi)\sigma_{11}(\xi)d\xi\right], \tag{30}$$

$$\varepsilon_{22}(t)=-\frac{\nu_0}{\mathrm{E}_{11}}\left[\sigma_{11}(t)+\int_0^t \Pi_{12}(t-\xi)\sigma_{11}(\xi)d\xi\right], \tag{31}$$

where $\nu_0=\nu(0)$ is the value of the elastic Poisson ratio.

For the component e_{11} from (27) we have:

$$\begin{aligned} e_{11}(t)&=\frac{2}{3}(\varepsilon_{11}-\varepsilon_{22})=\frac{2}{3}\frac{1}{\mathrm{E}_{11}}\left[\sigma_{11}(t)+\int_0^t \Pi_{11}(t-\xi)\sigma_{11}(\xi)d\xi\right]+\\ &+\frac{2}{3}\frac{\nu_0}{\mathrm{E}_{11}}\left[\sigma_{11}(t)+\int_0^t \Pi_{12}(t-\xi)\sigma_{11}(\xi)d\xi\right]=\\ &=\frac{2(1+\nu_0)}{3\mathrm{E}_{11}}\sigma_{11}(t)+\frac{2}{3\mathrm{E}_{11}}\int_0^t (\Pi_{11}(t-\xi)+\nu_0\Pi_{12}(t-\xi))\sigma_{11}(\xi)d\xi. \end{aligned} \tag{32}$$

From (26) it is known that $S_{11}=\frac{2}{3}\sigma_{11}$. Then $\sigma_{11}=\frac{3}{2}S_{11}$ and we get:

$$\begin{aligned} e_{11}(t)&=\frac{1+\nu_0}{\mathrm{E}_{11}}\left[S_{11}(t)+\int_0^t \frac{\Pi_{11}(t-\xi)+\nu_0\Pi_{12}(t-\xi)}{1+\nu_0}S_{11}(\xi)d\xi\right]=\\ &=\frac{1+\nu_0}{\mathrm{E}_{11}}\left[S_{11}(t)+\int_0^t \Pi_{зс}(t-\xi)S_{11}(\xi)d\xi\right], \end{aligned} \tag{33}$$

where $\Pi_{зс}(t-\xi)=\dfrac{\Pi_{11}(t-\xi)+\nu_0\Pi_{12}(t-\xi)}{1+\nu_0}$.

The creep equation in the case of stretching will take the form:

$$\theta = \frac{1}{\mathrm{E}_{11}}\left[\sigma_{11}(t) + \int_0^t \Pi_{11}(t-\xi)\sigma_{11}(\xi)d\xi\right] - \frac{2\nu_0}{\mathrm{E}_{11}}\left[\sigma_{11}(t) + \int_0^t \Pi_{12}(t-\xi)\sigma_{11}(\xi)d\xi\right]. \quad (34)$$

Since $\sigma_{11} = 3\tilde{\sigma}$, then

$$\begin{aligned}\theta &= \frac{1-2\nu_0}{\mathrm{E}_{11}}3\tilde{\sigma}(t) + \int_0^t \left[\frac{1}{\mathrm{E}_{11}}\Pi_{11}(t-\xi) - \frac{2\nu_0}{\mathrm{E}_{11}}\Pi_{12}(t-\xi)\right]3\tilde{\sigma}(t)d\xi = \\ &= \frac{3(1-2\nu_0)}{\mathrm{E}_{11}}\left(\tilde{\sigma}(t) + \int_0^t \frac{\Pi_{11}(t-\xi) - 2\nu_0\Pi_{12}(t-\xi)}{1-2\nu_0}\tilde{\sigma}(t)d\xi\right) = \\ &= \frac{3(1-2\nu_0)}{\mathrm{E}_{11}}\left(\tilde{\sigma}(t) + \int_0^t \Pi_{o\delta}(t-\xi)\tilde{\sigma}(t)d\xi\right),\end{aligned} \quad (35)$$

where $\Pi_{o\delta} = \dfrac{\Pi_{11}(t-\xi) - 2\nu_0\Pi_{12}(t-\xi)}{1-2\nu_0}$.

For fractional-differential models of the shear and volumetric creep kernel, it can be written for each model as follows:

for Voigt's model

$$\begin{aligned}\Pi_{\textit{зс}(F)}(t-\xi) &= \frac{(t-\xi)^{\beta-1}}{2\mathrm{E}\tau^{\beta}(1+\nu_0)}(E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}}{2\mathrm{E}\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta-\alpha}\right) + \\ &+ \nu_0 E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}\nu_2}{2\mathrm{E}\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta-\alpha}\right)),\end{aligned} \quad (36)$$

$$\begin{aligned}\Pi_{o\delta(F)}(t-\xi) &= \frac{(t-\xi)^{\beta-1}}{2\mathrm{E}\tau^{\beta}(1-2\nu_0)}(E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}}{2\mathrm{E}\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta-\alpha}\right) - \\ &- 2\nu_0 E_{\beta-\alpha,\beta}\left(-\frac{\mathrm{E}_{11}\nu_2}{2\mathrm{E}\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta-\alpha}\right)),\end{aligned} \quad (37)$$

for Kelvin's model

$$\begin{aligned}\Pi_{\textit{зс}(K)}(t-\xi) &= \frac{(\mathrm{E}_1+\mathrm{E}_2)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}(1+\nu_0)}(t-\xi)^{\beta-1}(E_{\beta,\beta}\left(-\frac{\mathrm{E}_{11}(\mathrm{E}_1+\mathrm{E}_2)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta}\right) + \\ &+ \nu_0 E_{\beta,\beta}\left(-\frac{\nu_2\mathrm{E}_{11}(\mathrm{E}_1+\mathrm{E}_2)}{2\mathrm{E}_1\mathrm{E}_2\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta}\right)),\end{aligned} \quad (38)$$

$$\Pi_{об(K)}(t-\xi)=\frac{(E_1+E_2)}{2E_1E_2\tau^{\beta}(1-2\nu_0)}(t-\xi)^{\beta-1}(E_{\beta,\beta}\left(-\frac{E_{11}(E_1+E_2)}{2E_1E_2\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta}\right)-$$
$$-2\nu_0E_{\beta,\beta}\left(-\frac{\nu_2E_{11}(E_1+E_2)}{2E_1E_2\tau^{\beta}(1-\nu_1\nu_2)}(t-\xi)^{\beta}\right)), \quad (39)$$

5 Numerical implementation of a mathematical model

Considering the previous studies [2, 3] regarding the identification of fractional-differential parameters of models, we present the identification results for the rheological Maxwell model (see Fig. 1).

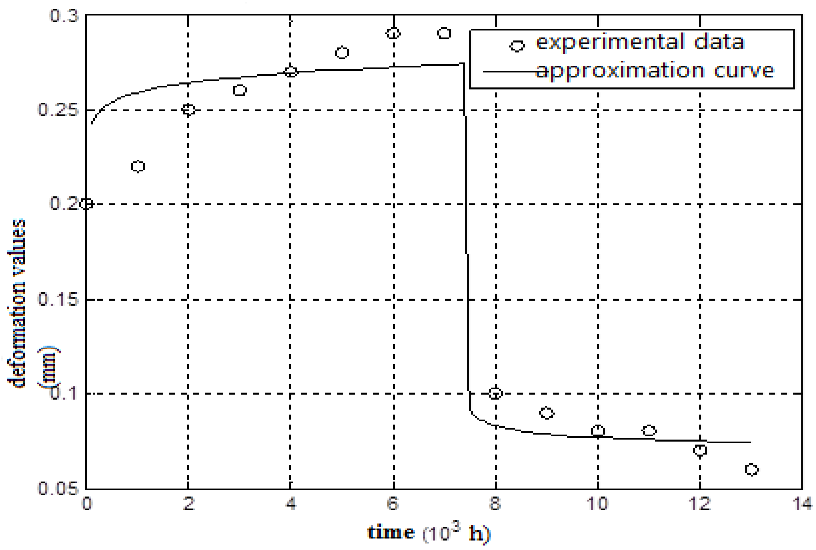

Fig. 1. Identification of fractional-differential parameters of the Maxwell model.

In Fig. 2, the deformation change for a sample of biomaterial (modulus of elasticity $E=16{,}1GPa$) [12] was investigated using a Kelvin rheological model taking into account the fractal structure of the medium without taking it into account. Such studies have shown that by decreasing the fractional-differential parameter β, the deformation functions increase more slowly, and at a value $\beta=0{,}1$, the deformation curve acquires the form in which the deformation of the material is smallest. Thus, it is possible to trace the relationship between the fractal parameters of the model and the process of deformation change.

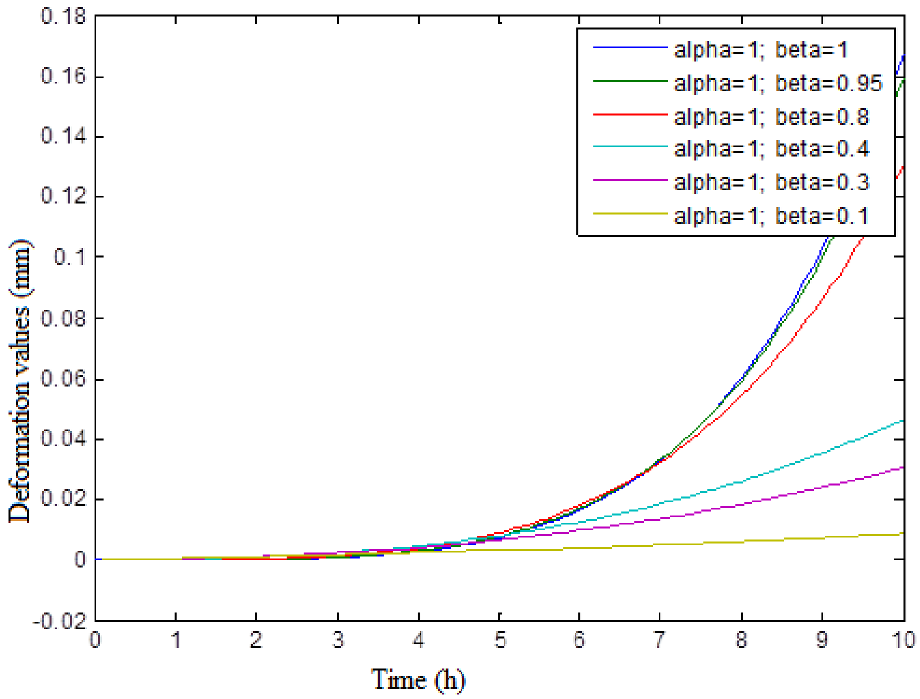

Fig. 2. Change of the deformations of the fractional-differential Kelvin's model.

Our results show that the difference between the stress curves with the fractal structure and without taking into account for more solid types of biomaterials does not exceed 16.7%, whereas the difference between the stress curves for materials with a lower density lies between 19.6 and 24.0%.

6 Conclusions

Two-dimensional mathematical models of deformation processes of biomaterials have been constructed, which make it possible to take into account the fractal struc-ture of a material depending on the initial values of temperature and moisture content, thermo-mechanical characteristics of anisotropy, different types of material. An algorithm for numerical implementation of two-dimensional mathematical models of visco-elastic deformation of biomaterials has been developed, which allows calculating the components of the stress-strain state of a material taking into account the effects of memory and self-organization.

Adaptation has been carried out of the method of splitting fractional-differential creep kernels, which makes it possible to determine the functions of volumetric and shear creep according to the experimental data of one-dimensional models of visco-elastic deformation, to identify fractional-differential parameters of models taking into account the fractal structure of the medium and to estimate the values of elastic and residual stresses of biomaterials. Presented are the results of the numerical implementation of the mathematical model, taking into account the heterogeneity of the structure of biomaterials, self-organization and memory-effect.

References

1. Uchajkin, V.: Method of fractional derivatives. Ulyanovsk: Publishing house “Artishok”, 512 p. (2008).
2. Sokolowskyi, Y., Shymanskyi, V., Levkovych, M., Yarkun, V.: Mathematical and software providing of research of deformation and relaxation processes in environments with fractal structure. Proceedings of the 12th International Scientific and Technical Conference on Computer Sciences and Information Technologies, CSIT 2017, Lviv, Vol. 1, pp. 24-27
3. Sokolovskyy, Y., Levkovych, M., Mokrytska O., Atamanvuk, V.: Mathematical Modeling of Two-Dimensional Deformation-Relaxation Processes in Environments with Fractal Structure. 2nd IEEE International Conference on Data Stream Mining and Processing, DSMP 2018, Lviv, p. 375-380.
4. Beibalaev V.D.: The numerical method of solving regional problems for a two-dimensional weekend. The solution of thermal conductivity with higher order. Vestn. Sam. state tech. un-that. Ser. Phys.-mat. Sciences, 2010, №5 (21), pp, 244-251.
5. Podlubny, I.: Fractional Differential Eguations. Vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA. (1999).
6. Pobedrya B.E.: Models of the linear theory of viscoelasticity. Mechanics of Solids, 2005, Vol. 6, pp. 121-134.
7. Sokolovskyy, Y., Shymanskyi, V., Levkovych, M.: Mathematical Modeling of Non-Isothermal Moisture Transfer and Visco-Elastic Deformation in the Materials with Fractal Structure. 11th International Scientific and Technical Conference on Computer Sciences and Information Technologies, CSIT 2016, Lviv, pp. 91-95.
8. Sokolovskyy, Y., Levkovych, M., Mokrytska O., Kaplunskyy, Y.: Mathematical models of biophysical processes taking into account memory effects and self-similarity. 1st International Workshop on Informatics and Data-Driven Medicine, IDDM 2018, Lviv, p. 215-228.
9. Mainardi F.: The Fundamental Solutions for the Fractional Diffusion-Wave Equation Appl. Math. Lett, Vol. 9, No. 6, 1996, pp. 23-28.
10. Samko S.G., Kilbas A.A., Marichev O.I.: Integrals and Virology and Other Orders and Actions. Minsk. Science and technology, p. 688 (1987).
11. Koltunov M.A., Kravchuk A.S., Mayboroda V.P.: Applied Mechanics of a Deformable solid, M., Higher School, 349 p (1983).
12. Liu Tong: Creep of wood under a large span of loads in constant and varying environments. Experimental observations and analysis. Holz als Roh- und Werkstoff 51, pp. 400-405 (1993).