Multicategories of Multiary Lenses

Michael Johnson
CoACT, Departments of
Mathematics and Computing
Macquarie University
michael.johnson@mq.edu.au

Robert Rosebrugh
Department of Mathematics
and Computer Science
Mount Allison University
rrosebrugh@mta.ca

Abstract

Recently lens-like definitions of multidirectional transformations have appeared [2], encompassing both a "propagation" form (in the style of [5]) and a "wide span" form (in the style of [6]). These "multiary lenses" raise a number of new challenges that are addressed in this paper. First, in common with classical symmetric lenses, they need to be studied modulo equivalence relations that factor out differences in hidden data and declare two lenses to be equivalent if their observable behaviours correspond. Then, modulo those equivalence relations the precise relationship between the propagation style lenses and the wide span lenses needs to be established. All of this is carried through here. But more significantly, the compositions of multiary lenses proposed to date have suffered various limitations: well-behaved amendment lenses don't necessarily compose to give well-behaved amendment lenses; technical conditions on model spaces can interfere with composition; junction conditions needed to be introduced because certain kinds of compositions of multiary lenses might not always be defined; and in any case, composition of multiary lenses comes with unusual challenges and can't be expected to form a category structure since when there are more than two data sources there are not even single notions of domain or codomain for multiary lenses. In this paper we introduce a class of asymmetric amendment lenses called spg-lenses (stable putget lenses) which is more general than well-behaved amendment lenses, but is closed under composition, and we show how to use spg-lenses to capture, via wide spans, a wide class of mutidirectional transformations which compose well and form a well-known and long-standing structure, a multicategory — a multicategory of multiary lenses.

1 Introduction

Bidirectional tranformations have always been intended to, in concert, support multiple interacting systems, and a great many example such systems have been in operation for many years. Indeed, it's likely that the majority of systems using bidirectional transformations involve more than two systems, and in some cases they were operating before the term "bidirectional transformation" was coined. Now, finally, with the recent Dagstuhl meeting on Multidirectional Transformations and Synchronisations, and the paper of Diskin et al on Multiple Model Synchronisation with Multiary Delta Lenses [2], the detailed careful analysis of multiary lenses has begun.

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This paper is a contribution to that work. Originally it began as an analysis of some examples of wide spans of d-lenses and how they might interact, but some of those results have been saved for a future paper as the theoretical nature of those wide spans and their "compositions", let us call them linkages for now, took on more importance, and since the introduction by Diskin and his colleagues of amendment lenses promised substantial generalisations, but also some serious complications.

Among the most serious complications, Diskin and his colleagues had needed to introduce so-called junction conditions because apparently reasonable linkages among their multiary delta lenses might not lead to well-defined multiary delta-lenses. But on an even more basic level, the question of what linkages one might expect to have between multiary lenses, and how they might be organised and managed algebraically, needs to be considered. Traditional binary lenses, whether asymmetric or symmetric, form the arrows of categories [7], but multiary lenses don't have an obvious notion of domain or codomain. When a multiary lens has $n > 2$ external facing systems, and a change in one can in principle affect all the others, the lens starts to seem more like a component with n ports each of which can variously at different times deliver both input and output, and as we know from electronics, haphazardly connecting ports is a dangerous activity.

The paper begins by analysing a simplification and generalisation of amendment lenses, called here aa-lenses, and exploring their composability. We identify a new and promising class of aa-lenses which we call spg-lenses because they are in Diskin et al's terminology Stable and satisfy the PutGet condition. These spg-lenses are closed under composition, but are a very general class of amendment lenses, being more general than well-behaved amendment lenses (which is not to imply that they are ill-behaved!), and are thus a promising building block for more complex structures including multiary lenses.

We then study n-wide spans of spg-lenses, beginning with $n = 2$, to see how they might support a "composition" analogous to symmetric lens composition. As Diskin et al discovered, (wide) spans of amendment lenses don't necessarily compose. We organise the linkages among wide spans by requiring that each wide span have an identified leg, herein assumed without loss of generality to be the leftmost leg, which is a closed aa-lens, and show that linkages of the left leg of one wide span with any non-left leg of another wide span do indeed result in a new "composite" wide-span of aa-lenses, again with closed leftmost leg. We should remark that this identified closed leftmost leg is not a loss of generality — one can consider any n-wide span of aa-lenses that might arise in any application, and if it doesn't have any closed leg one can simply adjoin a lens, possibly even an identity lens, to use as the closed leftmost leg. If it helps, the reader may like to think of the identified leftmost leg as a lens which gives access to the head of the span so that compositions can be carried out.

We should also emphasise that the closed leftmost leg does not have any implications for how the n-wide span operates as a multiary lens — any one of the legs, including, but not only, the leftmost leg, might suffer a modification of state, and as in bidirectional transformations, or multidirectional transformations when $n > 2$, the other legs will have their state updated to restore synchronisation.

But what of the algebraic structure that these multiary lenses and their linkages should exhibit? As noted above, it can't be merely a category since there is no consistent notion of domain and codomain. Instead, as the title of the paper suggests, these multiary lenses, consisting of wide spans of spg-lenses with closed left leg, form a well-known and long standing algebraic structure known as a multicategory [9]. The multicategory structure captures succinctly and precisely the wide range of possible compositions of multiple multiary lenses, and all of them are well-defined.

The paper is organised as follows. In Section 2 we set out the basic definitions required based on work of Diskin et al [2], but generalised in an important way (the removal of K as discussed below). Section 3 defines the composition of aa-lenses, studies some of its properties, and identifies spg-lenses as a particularly general class of aa-lenses that is closed under composition and has desirable behaviours (while not being "well-behaved" under the definition presented in [2]). In Section 4 we establish operations for converting between propagation style multiary lenses and wide spans of aa-lenses, and in Section 5 we define the equivalence relations that are appropriate to each style and show that the conversion operations are well-defined for equivalence classes. The operations in fact establish a bijection between the two styles of multiary lenses. Once we have seen that, we can concentrate on either style and results we establish can be carried across to the other. For the remainder of the paper we focus on the wide-span approach to multiary lenses, and in Section 6 we establish composites for a pair of wide spans with closed left legs and prove that these composites can be calculated on arbitrary representatives of equivalence classes. We are then in a position to use those composites to explore the algebraic structure of multiary lenses and in the following section we show how it forms a multicategory. The final sections include a discussion of some of the observations that follow from the work presented here and draw conclusions.

2 Background

We begin with some background. The following definitions are close to those of Diskin et al [2] with the important difference that in this paper model spaces, and the aa-lenses that are the arrows between them, do not have compatibility relations K so that *a model space is simply a category.* The reasons for excluding K, and the implications of so doing, will be dealt with briefly in Section 8. We also include here a brief review of Diskin et al's n-ary symmetric lenses, which involve n-ary multimodel spaces, and in those multimodel spaces we will likewise omit K.

Denote the set of objects of a category $\mathbf{A}$ by $|\mathbf{A}|$. In the context of n-ary symmetric lenses i will always be an integer between 1 and n.

Definition 1 *An* n-ary multimodel space *($n \geq 2$)* $\mathbf{M} = (\mathbf{A}, \mathbf{R}, \delta)$ *consists of the following data:*
(i) An n-tuple of categories $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_n)$
(ii) A set $\mathbf{R}$ *whose elements are called* (consistent) corrs
(iii) A function $\delta : \mathbf{R} \longrightarrow |\mathbf{A}_1| \times \ldots \times |\mathbf{A}_n|$ *called the* boundary function.
A consistent multimodel *is a pair* $(R, (A_1, \ldots, A_n))$ *(with A_i an object of $\mathbf{A}_i$) such that* $\delta(R) = (A_1, \ldots, A_n) = (\delta_1 R, \ldots, \delta_n R)$*. For consistent multimodels* $(R, (A_i)), (R', (A'_i))$*. a* multimodel update *is an n-tuple of morphisms* $(u_1, \ldots, u_n) : (A_1, \ldots, A_n) \longrightarrow (A'_1, \ldots, A'_n)$.

Notice that a corr and the boundary function determine a consistent multimodel, so we can just dispense with the (A_i) in the notation, as is implicit below.

Definition 2 *An* n-ary symmetric lens $L = (\mathbf{M}, \mathsf{p})$ *is an n-ary multi model space,* $\mathbf{M}$*, and an n-tuple* $\mathsf{p} = (\mathsf{p}_1, \ldots, \mathsf{p}_n)$ *of operations as follows. For a consistent multimodel* $(R, (A_1, \ldots, A_n))$ *and morphism* $u_i : A_i \longrightarrow A'_i$, p_i *returns:*
(i) A consistent multimodel $(R'', (A''_1, \ldots, A''_n))$ *with* R'' *denoted* $\mathsf{p}_i^R(u_i)$
(ii) An amendment $u_i^a : A'_i \longrightarrow A''_i$ *denoted* $\mathsf{p}_{ii}^R(u_i)$
(iii) A multimodel update $(u'_1, \ldots, u'_n) : (A_1, \ldots, A_n) \longrightarrow (A''_1, \ldots, A''_n)$ *such that* $u'_i = u_i^a u_i$ *and, when* $j \neq i$*, the j-th component* u'_j *is denoted* $\mathsf{p}_{ij}^R(u_i)$.
Call an update u_i *R-*closed *when* $\mathsf{p}_{ii}^R(u_i) = \mathrm{id}_{A'_i}$*. The lens L is called* closed *if all updates are R-closed for all R.*

For consistency with earlier work we might have written the operations $\mathsf{p}_i(R, u_i)$, and $\mathsf{p}_{ii}(R, u_i), \mathsf{p}_{ij}(R, u_i)$ and that would avoid superscripts which complicate the reading and sometimes clash with primes. But for ease of comparison with [2] we have retained their notation.

Definition 3 *An n-ary symmetric lens* $L = (\mathbf{M}, \mathsf{p})$ *is called* well-behaved *if it satisfies the following conditions:*
(Stability): For all j, $\mathsf{p}_{ij}^R(\mathrm{id}_{A_i}) = \mathrm{id}_{A_j}$ *and* $\mathsf{p}_i^R(\mathrm{id}_{A_i}) = R$
(Reflect2): For $j \neq i$, $\mathsf{p}_{ij}^R(u_i^a u_i) = \mathsf{p}_{ij}^R(u_i)$
(Reflect3): $u_i^a u_i$ *is R-closed*

Notice that stability for a well-behaved n-ary symmetric lens means a trivial update propagates trivially. The Reflect conditions mean respectively that an amendment "completes" its update, and that a completed update needs no amendment. There is no (Reflect1) because in [2] it requires that an amendment lie in the relation K with its update, but the relation K is omitted here.

Definition 4 *A well-behaved lens is* (weakly) invertible *if each* p_i *satisfies the following law:*
(Invert): For $j \neq i$, $\mathsf{p}_{ij}^R(\mathsf{p}_{ji}^R(\mathsf{p}_{ij}^R(u_i))) = \mathsf{p}_{ij}^R(u_i)$

The n-ary symmetric lenses, and their basic properties, have been introduced because they correspond well to some workers' intuitions of multiary transformations. Diskin et al [2] study both them, and what are essentially n-wide spans of asymmetric amendment lenses. In this paper we concentrate primarily on these latter structures, and we begin to define them now.

The following is derived from Diskin et al's asymmetric lenses with amendments but we exclude the so-called compatibility relations K, and call the resulting simplification (and generalisation) aa-lenses.

Definition 5 *An* aa-lens $L = (\mathbf{A}, \mathbf{B}, g, p)$*, often abbreviated to just* (g, p) *when there is little risk of confusion, consists of categories* $\mathbf{A}$ *and* $\mathbf{B}$ *sometimes called the* base *and* view *model spaces respectively, with* $g : \mathbf{A} \longrightarrow \mathbf{B}$

a functor called the Get *and* $p = \{p^A \mid A \in \mathbf{A}\}$ *a family of operations as follows: to every view update* $v : g(A) \longrightarrow B'$, p^A *returns a pair* $(p_b^A(v) : A \longrightarrow A'', p_a^A(v) : B' \longrightarrow B'')$ *such that* $B'' = g(A'')$.
A view update $v : g(A) \longrightarrow B'$ *is called* closed *if* $p_a^A(v) = \mathrm{id}_{B'}$. *If all view updates are closed we call* L *a* closed aa-lens.

The requirements for a well-behaved aa-lens based on [2] follow.

Definition 6 *An aa-lens is* $L = (\mathbf{A}, \mathbf{B}, g, p)$ *is called* well-behaved *if the following hold for* $A \in \mathbf{A}$ *and* $v : g(A) \longrightarrow B'$:
(Stability): $p_b^A(\mathrm{id}_{g(A)}) = \mathrm{id}_A$
(Reflect0): $B' = g(X)$ *for some* $X \in \mathbf{A}$ *implies* $p_a^A(v) = \mathrm{id}_{B'}$
(Reflect2): $p_b^A(p_a^A(v)v) = p_b^A(v)$
(PutGet): $p_a^A(v)v = g(p_b^A(v))$

There is no (Reflect1) property since we are following [2] and that condition involves the compatibility relation which we omit — in [2] it says that v and its amendment $p_a^A(v)$ need to be a pair related by the compatibility relation K. In a sense we treat all composable pairs of arrows as compatible.

As noted in [2] (PutGet) and (Reflect2) imply the (weak) invertibility above and (PutGet) and (Reflect0) imply the closure required by (Reflect3) in Definition 3.

An aa-lens defines a well behaved, invertible 2-ary symmetric lens whose base updates (the updates in $\mathbf{A}$) are closed. A closed aa-lens which also satisfies a PutPut law is exactly the same thing as an asymmetric delta lens [1].

A further condition considered by Diskin [personal communication] is
(Reflect0*): If $v = g(w)$ for w an arrow with domain A of $\mathbf{A}$ then $p_a^A(v) = \mathrm{id}_{B'}$.

3 The Category of spg-Lenses

We now consider the composition of aa-lenses.

Definition 7 *Let* $L = (\mathbf{A}, \mathbf{B}, g, p)$ *and* $L' = (\mathbf{B}, \mathbf{C}, h, q)$ *be aa-lenses. Their* composite aa-lens *is* $L'L = (\mathbf{A}, \mathbf{C}, k, r)$ *where* $k = hg$ *and the composite Put* r *is defined as follows. If* $w : hg(A) \longrightarrow C'$, *then* $r_b^A(w) = p_b^A(q_b^{g(A)}(w))$ *and* $r_a^A(w) = h(p_a^A(q_b^{g(A)}(w))) \cdot q_a^{g(A)}(w)$. *See the diagram, where* $v = p_a^A(q_b^{g(A)}(w))$:

$$\begin{tikzcd}
A \arrow[rrr, "p_b^A(q_b^{g(A)}(w))"] \arrow[d, dotted, no head] & & & A'' \arrow[d, dotted, no head] \\
g(A) \arrow[rr, "q_b^{g(A)}(w)"] \arrow[d, dotted, no head] & & B' \arrow[r, "v"] \arrow[d, dotted, no head] & B'' \arrow[d, dotted, no head] \\
hg(A) \arrow[r, "w"'] & C' \arrow[r, "q_a^{g(A)}(w)"'] & C'' \arrow[r, "h(v)"'] & C'''
\end{tikzcd}$$

The lens $L = (\mathbf{A}, \mathbf{A}, \mathrm{id}_{\mathbf{A}}, p_{\mathbf{A}})$ where for $u : A \longrightarrow A'$, we have $(p_{\mathbf{A}})_b^A(u) = u$ and $(p_{\mathbf{A}})_a^A(u) = \mathrm{id}_{A'}$ is the identity for the just defined composition, and the reader can easily verify that the composition is associative. Thus there is a category whose objects are model spaces and whose morphisms are aa-lenses. It generalises the category of (asymmetric) delta lenses.

Proposition 8 *Here are a selection of conditions which are preserved by composition.*

- *If* L *and* L' *satisfy* Stability *then so does* $L'L$.
- *If* L *and* L' *are closed then so is* $L'L$.
- *If* L *and* L' *satisfy* Reflect0 *then so does* $L'L$.
- *If* L *and* L' *satisfy* Reflect0* *then so does* $L'L$.

- *If L and L' have identity amendments for identities, then so does $L'L$.*
- *If L and L' satisfy* PutGet *then so does $L'L$.*

Proof. The proofs all involve routine verifications which we omit. ∎

Note however that if L and L' satisfy Reflect2, it does not apparently follow that $L'L$ does so. Indeed it seems that even if L and L' are well-behaved then the composite need not satisfy Reflect2. This means that we do not have a category of well-behaved aa-lenses. Instead a convenient category of aa-lenses has as arrows those aa-lenses which satisfy Stability and PutGet.

Definition 9 *An aa-lens L is called an* spg-lens *if it satisfies the properties Stability and PutGet. The category of such lenses is denoted* **spg-lens**.

If spg-lenses make up the most convenient category of aa-lenses, we are going to be interested in symmetric spg-lenses — those that arise as equivalence classes of spans of spg-lenses. However, it seems that spans of arbitrary spg-lenses can't be assumed to compose in the usual way, analogous to spans of d-lenses [8]. Instead after a fairly extensive analysis of the kinds of properties of aa-lenses that "pullback" over certain kinds of aa-lenses, we restrict our attention to certain spans, those with closed left legs.

Proposition 10 *There is a composite as shown below for spans in the category **spg-lens** with a closed left leg.*

Proof. Consider two spans of spg-lenses with heads $\mathbf{S}$ and $\mathbf{T}$ shown in the diagram and suppose that the lenses (g, p) and (g', p') are both closed. Let $\mathbf{U}$ be the pullback in **cat** of h and g', with k and m the pullback projections.

$$\begin{tikzcd}
 & & \mathbf{U} \arrow[dl, "(k,r)"'] \arrow[dr, "(m,s)"] & & \\
 & \mathbf{S} \arrow[dl, "(g,p)"'] \arrow[dr, "(h,q)"'] & & \mathbf{T} \arrow[dl, "(g',p')"] \arrow[dr, "(h'q')"] & \\
\mathbf{X} & & \mathbf{Y} & & \mathbf{Z}
\end{tikzcd}$$

We define r and s making the pairs (k, r) and (m, s) spg-lenses. Since $\mathbf{U}$ is a pullback in **cat**, we may suppose without loss of generality that its objects are compatible pairs of objects, one each from $\mathbf{S}$ and $\mathbf{T}$, and that its arrows are compatible pairs of arrows, one from each of $\mathbf{S}$ and $\mathbf{T}$. In each case "compatible" means that applying h to an object or arrow from $\mathbf{S}$, and g' to an object or arrow from $\mathbf{T}$, results in the same object or arrow of $\mathbf{Y}$.

The Put r takes an object (S, T) of $\mathbf{U}$ and an arrow $\alpha : k(S, T) = S \longrightarrow S'$ of $\mathbf{S}$ and returns $r_b^{(S,T)}(\alpha)$ an arrow of $\mathbf{U}$ with domain (S, T) and an amendment $r_a^{(S,T)}(\alpha)$ in $\mathbf{S}$ with domain S'. Let $r_b^{(S,T)}(\alpha) = (\alpha, p_b'^{T}(h(\alpha)))$ and let $r_a^{(S,T)}(\alpha) = \mathrm{id}_{s'}$. To see that (k, r) is an spg-lens, note first that it is stable since h is a functor and (g', p') is stable, and secondly it satisfies PutGet by construction. Furthermore, (k, r) is closed by construction (all amendments are identities).

The definition of s is a little longer because its amendments are not as trivial. The Put s takes an object (S, T) of $\mathbf{U}$ and an arrow $\beta : m(S, T) = T \longrightarrow T'$ and returns $s_b^{(S,T)}(\beta)$ an arrow of $\mathbf{U}$ with domain (S, T) and an amendment $s_a^{(S,T)}(\beta)$ in $\mathbf{T}$ with domain T'. Let $q_a^S(g'(\beta)) = b$ say and define $s_a^{(S,T)}(\beta) = p_b'^{T'}(b) = \tilde{b}$ say. Finally, let $s_b^{(S,T)}(\beta) = (q_b^S(g'(\beta)), \tilde{b}\beta)$. To see that (m, s) is an spg-lens, it is stable by an argument parallel to the argument for (k, r) noting that amendments for identities are identities, and again it satisfies PutGet by construction.

The resulting span with head $\mathbf{U}$ from $\mathbf{X}$ to $\mathbf{Z}$ is again a span of spg-lenses and its left leg is closed since spg-lenses compose and closed spg-lenses compose to give closed spg-lenses. ∎

We remind the reader that compositions like this are analogous to span composition in a category with pullbacks. However, $\mathbf{U}$ need not be the pullback in the category whose arrows are aa-lenses. It is a pullback in **cat**, and we have just given canonical constructions on the projection functors to make them into aa-lenses. To emphasise this distinction, while conserving the analogy, we sometimes refer to $\mathbf{U}$ along with (k, r) and (m, s) as a "pullback" — the inverted commas remind us that it is not in fact a pullback in **spg-lens**.

4 Wide Spans of spg-Lenses and n-ary Symmetric Lenses

In the following sections we will be studying a generalisation of a span of lenses. An *n-wide span of lenses* is a tuple $L = (\mathbf{A}, (\mathbf{B}_i, g_i, p_i)_{i\leq n})$ of lenses with a common domain $\mathbf{A}$. We will mostly be considering aa-lenses.

We next consider two constructions relating n-ary symmetric lenses to n-wide spans of aa-lenses.

For the first construction, we begin with a stable n-ary symmetric lens $M = (\mathbf{M}, \mathsf{p})$ with multimodel space $\mathbf{M} = ((\mathbf{A}_i), \mathbf{R}, \delta)$ and propagation operations p_i. The domain and codomain of an arrow u are denoted $d_0(u)$ and $d_1(u)$ respectively.

We construct an n-wide span L_M of spg-lenses $(L_M)_i : \mathbf{S} \longrightarrow \mathbf{A}_i$ (similar to a construction in [8]). The first step is to define the head $\mathbf{S}$ of the span. The set of objects of $\mathbf{S}$ is defined to be the set $\mathbf{R}$ of corrs of M. The morphisms of $\mathbf{S}$ are defined as follows:
For objects (corrs) R and R', let $\mathbf{S}(R, R') = \{(u_i) \mid d_0u_i = \delta_i(R), d_1u_i = \delta_i(R')\}$ (where we write, as usual, $\mathbf{S}(R, R')$ for the set of arrows of $\mathbf{S}$ from R to R'). Thus an arrow of $\mathbf{S}$ may be thought of as a formal tuple of arrows $(u_i : \delta_i(R) \longrightarrow \delta_i(R'))$ where u_i is an arrow of $\mathbf{A}_i$. Composition is inherited from composition in $\mathbf{A}_i$ component-wise, or more precisely, for $(u_i) \in \mathbf{S}(R, R')$ and $(v_i) \in \mathbf{S}(R', R'')$ we define:

$$(v_i)(u_i) = (v_iu_i)$$

in $\mathbf{S}(R, R'')$. The identities are tuples of identities. It is immediate that $\mathbf{S}$ is a category.

Next define the spg-lens $(L_M)_i = (g_i, p_i)$ where $g_i : \mathbf{S} \longrightarrow \mathbf{A}_i$ on objects is δ_i, and on arrows is $g_i((u_j)_{j\leq n}) = u_i$. The Put for $(L_M)_i$ has as amendment $p^R_{i,a}(\alpha : g_i(R) \longrightarrow A_i) = \mathsf{p}^R_{ii}(\alpha)$, and has as base Put an arrow of $\mathbf{S}$:

$$\begin{aligned}(p^R_{i,b}(\alpha : g_i(R)) \longrightarrow A_i)_j &= \mathsf{p}^R_{ij}(\alpha) \quad \text{for} \quad j \neq i\\ (p^R_{i,b}(\alpha : g_i(R)) \longrightarrow A_i)_i &= \mathsf{p}^R_{ii}(\alpha)\alpha\end{aligned}$$

which is indeed an arrow of $\mathbf{S}$ from R to $R'' = \mathsf{p}^R_i(\alpha)$.

Lemma 11 *L_M is an n-wide span of spg-lenses.*

Proof. The g_i are evidently functorial. We need to show that p_i satisfy the definition of spg-lens. This follows immediately from the stability of the n-ary symmetric lens M and the construction. ∎

For the construction in the other direction, we begin with an n-wide span L of spg-lenses. Let $L_i = (g_i, p_i)$ where $g_i : \mathbf{S} \longrightarrow \mathbf{A}_i$.

Construct the n-ary symmetric lens $M_L = (\mathbf{M}, \mathsf{p})$ with consistent corrs $\mathbf{R} = |\mathbf{S}|$, with $\delta_i(S) = g_i(S)$ and with p_i defined as follows for $\alpha : g_i(S) \longrightarrow A'_i$:

- $\mathsf{p}^S_i(\alpha) = d_1(p^S_{i,b}(\alpha))$
- $\mathsf{p}^S_{ii}(\alpha) = p^S_{i,a}(\alpha)$
- $\mathsf{p}^S_{ij}(\alpha) = g_j(p^S_{i,b}(\alpha))$ for $j \neq i$.

Lemma 12 *M_L is a stable n-ary symmetric lens.*

Proof. We have defined all of the required parts of the p_i and stability follows from the stability of the component spg-lenses. ∎

5 Equivalence Relations for Lenses and Spans

We are going to need equivalence relations for n-ary symmetric lenses and for n-wide spans of spg-lenses. The idea is that equivalent lenses/spans have the same update behaviour.

Definition 13 *Let $L_1 = (\mathbf{M}, \mathsf{p})$ and $L_2 = (\mathbf{N}, \mathsf{q})$ be n-ary symmetric lenses with $\mathbf{M} = (\mathbf{A}, \mathbf{R}_1, \delta_1)$ and $\mathbf{N} = (\mathbf{A}, \mathbf{R}_2, \delta_2)$ (so they have the same base categories but different corrs).*
We say $L_1 \equiv_n L_2$ if and only if there is a relation σ from $\mathbf{R}_1$ to $\mathbf{R}_2$ with the following properties:
1. σ is compatible with the δ_i's, i.e. $R_1\sigma R_2$ implies $\delta_iR_1 = \delta_iR_2$

2. σ is total in both directions, i.e. for all R_1 in $\mathbf{R}_1$, there is R_2 in $\mathbf{R}_2$ with $R_1\sigma R_2$ and conversely.
3. If $R_1\sigma R_2$ (so $\delta_i R_1 = \delta_i R_2$) and $u_i : A_i \longrightarrow A'_i$ is an arrow of $\mathbf{A}_i$ then $\mathsf{p}_i^{R_1}(u_i)\sigma\mathsf{q}_i^{R_2}(u_i)$ and $\mathsf{p}_{ij}^{R_1}(u_i) = \mathsf{q}_{ij}^{R_2}(u_i)$ for $j = 1, \ldots, n$ (so they have the same Put behaviour and *amendments).*

Notice that σ is an equivalence relation. Condition 3 implies that the codomains for all $\mathsf{p}_{ij}^{R_1}(R_1)$ are the same as the codomains for all $\mathsf{q}_{ij}^{R_2}(R_2)$.

Lemma 14 *The equivalence relation $\equiv_n$ is generated by relations σ, as in the definition, which are surjective functions.*

Proof. To see that $\equiv_n$ is generated by such surjections, consider $L_1 \equiv_n L_2$ by σ and construct the span tabulating σ:

$$\mathbf{R}_1 \longleftarrow \mathbf{Q} \longrightarrow \mathbf{R}_2$$

Notice that each leg of the span is a surjection because of totality of σ in both directions. It is easy to construct a lens with corrs $\mathbf{Q}$ equivalent to both L_1 and L_2 by those surjections. ∎

Definition 15 *Let $L_1 = (\mathbf{S}_1, (\mathbf{A}_i, g_i, p_i)_{i\leq n})$ and $L_2 = (\mathbf{S}_2, (\mathbf{A}_i, g'_i, p'_i)_{i\leq n})$ be n-wide spans of spg-lenses with the same feet. We say L_1 is E-related to L_2 if and only if there is a functor $\Phi : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ such that*
(E1) $g'_i\Phi = g_i, i = 1, \ldots, n$
(E2) Φ is surjective on objects, and
(E3) If $\Phi S_1 = S_2$, then for $u_i : g_i(S_1) = g'_i(S_2) \longrightarrow A'_i$, we have $p'_{i,a}(u_i) = p_{i,a}(u_i)$ and $p'^{S_2}_{i,b}(u_i) = \Phi p^{S_1}_{i,b}(u_i)$.

Condition (E3) says first that the amendments, which are arrows of $\mathbf{A}_i$, agree, and secondly that the Put behaviours correspond by Φ.

Definition 16 *Let $\equiv_{spg}$ be the equivalence relation on n-wide spans of spg-lenses which is generated by E-relatedness.*

Proposition 17 *With the notation of the constructions above, for n-ary symmetric lenses $M = (\mathbf{M}, \mathsf{p})$ and $N = (\mathbf{N}, \mathsf{q})$ with $M \equiv_n N$, we have $L_M \equiv_{spg} L_N$.*

Proof. Suppose $\mathbf{M} = (\mathbf{A}, \mathbf{R}_1, \delta_1)$ and $\mathbf{N} = (\mathbf{A}, \mathbf{R}_2, \delta_2)$ are equivalent by σ relating $\mathbf{R}_1$ and $\mathbf{R}_2$. Without loss of generality, by Lemma 14, it suffices to assume σ is a surjective function. Suppose the wide spans constructed from M and N have heads $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively. We will define a functor $\Phi : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ with σ as its object function. On an arrow of $\mathbf{S}_1$, that is a tuple $u = (u_i)$ of arrows with each u_i an arrow of $\mathbf{A}_i$, define $(\Phi(u))_i = u_i$. Notice that Φ satisfies (E1) and (E2) by construction. For (E3), condition 3 of the definition of $\equiv_n$ guarantees that the required equations hold (both Puts yield the very same tuple of arrows as, respectively, an arrow of $\mathbf{S}_1$ and an arrow of $\mathbf{S}_2$). ∎

Proposition 18 *With the notation of the constructions above, for n-wide spans L and K of spg-lenses with $L \equiv_{spg} K$, we have $M_L \equiv_n M_K$.*

Proof. Suppose L and K are given by spg-lenses $L_i = (g_i, p_i)$ where $g_i : \mathbf{S} \longrightarrow \mathbf{A}_i$, and $K_i = (h_i, q_i)$ where $h_i : \mathbf{T} \longrightarrow \mathbf{A}_i$. Without loss of generality, it suffices to assume that $L \equiv_{spg} K$ are equivalent by a functor $\Phi : \mathbf{S} \longrightarrow \mathbf{T}$ satisfying conditions (E1)-(E3). The corrs of M_L are the objects of $\mathbf{S}$ and the corrs of M_K are the objects of $\mathbf{T}$. Let the relation σ be the object function of Φ. We need to show that σ satisfies conditions 1-3 of Definition 13. By (E1) Φ commutes with the Gets so σ is compatible with the δ's (condition 1). Since Φ on objects is a surjective function, σ is total on both sides (condition 2). For condition 3, suppose $S\sigma T$, so

$\Phi(S) = T$, and $\alpha : A_i \longrightarrow A'_i$ is an arrow of $\mathbf{A}_i$. Then for $j \neq i$, in the notation of the construction, and writing q for the propagation constructed using q

$$\begin{aligned}
\mathsf{q}^T_{ij}(\alpha) &= h_j(q^T_{i,b}(\alpha)) \qquad \text{by definition} \\
&= h_j(\Phi p^S_{i,b}(\alpha)) \qquad \text{by hypothesis} \\
&= g_j(p^S_{i,b}(\alpha)) \qquad \text{by } \Phi \text{ commutes with Gets} \\
&= \mathsf{p}^S_{ij}(\alpha) \qquad \text{by definition}
\end{aligned}$$

For the amendments, $\mathsf{q}^T_{ii}(\alpha) = q^T_{i,a}(\alpha) = p^S_{i,a}(\alpha) = \mathsf{p}^S_{i,i}(\alpha)$. And finally, the resulting corrs are related since

$$\begin{aligned}
\mathsf{q}^T_i(\alpha) &= d_1(q^T_{i,b}(\alpha)) \qquad \text{by definition} \\
&= d_1(\Phi p^S_{i,b}(\alpha)) \qquad \text{by hypothesis} \\
&= \Phi(d_1(p^S_{i,b}(\alpha))) = \Phi \mathsf{p}^S_i(\alpha) \qquad \text{by functoriality and definition}
\end{aligned}$$

So condition 3 is satisfied. ∎

The two constructions between n-ary symmetric lenses and n-wide spans of spg-lenses are closely related. Indeed, one composite of the two constructions is actually the identity. The other is, up to $\equiv_{spg}$ equivalence, the identity (up to equivalence because when we start and end with a wide span, the arrows of the head are replaced by the formal sequences described above).

Proposition 19 *With the notation of the constructions above, for any n-ary symmetric lens M,*

$$M = M_{(L_M)}$$

and for any n-wide span L of spg-lenses $L_i = (g_i, p_i)$ where $g_i : \mathbf{S} \longrightarrow \mathbf{A}_i$,

$$L \equiv_{spg} L_{(M_L)}$$

Proof. For the first equation, by inspection, the corrs and δ's of M_{L_M} are those of M. Further, it is easy to see that the propagations of M_{L_M} are identical to those of M.

For the second, the equivalence, notice first that the head of the original wide span, L, is included by an identity on objects functor into the head $\mathbf{S}$ of the wide span $L_{(M_L)}$. Using the notation of the construction, the functor sends an arrow s of $\mathbf{S}$ to the tuple of arrows $(g_i s)$. Call that functor Φ. It certainly commutes with the Gets (by construction) and it is identity on objects, and so certainly surjective on objects, so Φ satisfies conditions (E1) and (E2). To see that it also satisfies condition (E3), and so provides the equivalence required, it suffices to recall that every Put in $L_{(M_L)}$ is obtained from the n different Gets of the corresponding Put in L, and so Φ carries the latter Put to the former Put, as required. ∎

Corollary 20 *The constructions define a bijection between $\equiv_n$ equivalence classes of n-ary symmetric lenses and $\equiv_{spg}$ equivalence classes of n-wide spans of spg-lenses.*

The remainder of this paper will focus on n-wide spans of spg-lenses.

6 The Wide Span Composites

We are now going to use the composite of spans defined in Proposition 10 to define several composites of suitable wide spans of spg-lenses. The composites will take place over a single foot common to both spans. Each wide span will have a specified closed leg, its left-most leg, and the composites will be defined using the closed left-most leg of the second wide span.

We will denote an n-wide span L with the left-most leg, $(g_1, p_1) : \mathbf{S} \longrightarrow \mathbf{A}_1$ assumed to be closed as follows.

$$
\begin{tikzcd}
 & \mathbf{S} \arrow[dl, "(g_1,p_1)"'] \arrow[d, "(g_2,p_2)"'] \arrow[dr, "(g_3,p_3)"] \arrow[drrr, "(g_2,p_n)"] & & & \\
\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots & \mathbf{A}_n
\end{tikzcd}
$$

When we define one of the composites of two such wide spans L and M say, we will compose (as in Proposition 10) across the left-most leg of the right span M and the i-th leg ($i > 1$) of the left span L and call it the i-composite. See the diagram below in which (to avoid clutter) the legs of the left span after position i are not shown.

$$
\begin{tikzcd}
 & & \mathbf{U} \arrow[dl, "(k,r)"'] \arrow[dr, "(m,s)"] & & & \\
 & \mathbf{S} \arrow[dl, "(g_1,p_1)"'] \arrow[d, "(g_2,p_2)"'] \arrow[dr, "(g_i,p_i)"] & \cdots & \mathbf{T} \arrow[dl, "(g'_1,p'_1)"'] \arrow[d, "(g'_2,p'_2)"'] \arrow[dr, "(g'_3,p'_3)"] \arrow[drr, "(g'_{n'},p'_{n'})"] & \cdots & \\
\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_i = \mathbf{A}'_1 & \mathbf{A}'_2 & \mathbf{A}'_3 & \mathbf{A}'_{n'}
\end{tikzcd}
$$

Suppose that all the lenses are spg-lenses and that the two left-most legs, (g_1, p_1) and (g'_1, p'_1), are closed. Proposition 10 shows that (m, s) is an spg-lens and that (k, r) is a closed spg-lens, and so $(g_1, p_1)(k, r)$ is a closed spg-lens by Proposition 8. That last proposition also ensures that all of the other $n + n' - 3$ composites are spg-lenses.

The following definition makes the details of the i-composite precise by specifying all the indices, but the reader may simply focus on the diagram in which the resulting wide span is made evident noting only that the legs ending at $\mathbf{A}_i$ and $\mathbf{A}'_1$ do not appear in the composite.

Definition 21 *The* i-composite *of the two wide spans of spg-lenses shown above and in which* (g_1, p_1) *and* (g'_1, p'_1) *are closed is defined to be the wide span* M_iL *with head* $\mathbf{U}$ *and closed left-most leg* $(g_1, p_1)(k, r) : \mathbf{U} \longrightarrow A_1$*. The feet are* $\mathbf{B}_j$*, for* $j \leq n + n' - 2$ *where* $\mathbf{B}_j = \mathbf{A}_j$*, for* $j < i$*,* $\mathbf{B}_j = \mathbf{A}_{j+1}$*, for* $i < j < n$*, and* $\mathbf{B}_j = \mathbf{A}'_{j-n+2}$*, for* $n \leq j < n+n'-2$*. The lenses are obtained by renumbering similarly and composing:* $(h_j, q_j) = (g_j, p_j)(k, r)$*, for* $j < i$*,* $(h_j, q_j) = (g_{j+1}, p_{j+1})(k, r)$*, for* $i < j < n$*, and* $(h_j, q_j) = (g'_{j-n+2}, p'_{j-n+2})(m, s)$*, for* $n \leq j < n + n' - 2$*.*

Our next goal is to show that we can compose $\equiv_{spg}$ classes of wide spans by composing representatives of each class. For that it suffices to show that the equivalence is a congruence for the i-composites just defined.

Definition 22 *An* n*-wide span of spg-lenses* $L = (\mathbf{S}, (\mathbf{A}_i, g_i, p_i)_{i\leq n})$ *is called a* multiary lens *if* (g_1, p_1) *is closed.*

Proposition 23 *Suppose that* $L = (\mathbf{S}, (\mathbf{A}_i, g_i, p_i)_{i\leq n})$*,* $M = (\mathbf{T}, (\mathbf{B}_i, h_i, q_i)_{i\leq \bar{(n)}})$ *and* $M' = (\mathbf{T}', (\mathbf{B}_i, h'_i, q'_i)_{i\leq \bar{(n)}})$ *are multiary lenses. Further, suppose that* $\Phi : \mathbf{T} \longrightarrow \mathbf{T}'$ *is a functor satisfying conditions E1-E3 so that* $M \equiv_{spg} M'$*, and that* L *and* M *(and so also* L *and* M'*) have a wide span composite over some* $\mathbf{A}_i$ *as in Definition 21. Then the two wide span composites are equivalent, that is* $M_iL \equiv_{spg} M'_iL$*.*

Proof. The diagram below shows some of the lenses in the wide spans L, M and M' chosen to include all those that are required to construct representative parts of the composite wide spans. To avoid clutter the other lenses have been omitted.

The top composite span (M_iL) of spg-lenses has head $\mathbf{U}$, the pullback (in **cat**) of g_i and h_1. Similarly the pullback $\mathbf{U}'$ is the head of the bottom span composite (LM').

$$\begin{tikzcd}
& & \mathbf{U} \arrow[dl, "(k,r)"'] \arrow[dd, dotted, "\Phi'"] \arrow[rr, "(m,s)"] & & \mathbf{T} \arrow[dl, "(h_1,q_1)"'] \arrow[dd, "\Phi"] \arrow[dr, "(h_j,q_j)"] & \\
\mathbf{A}_1 & \mathbf{S} \arrow[l, "(g_1,p_1)"] \arrow[rr, "(g_i,p_i)"'] & & \mathbf{A}_i = \mathbf{B}_1 & & \mathbf{B}_j \\
& & \mathbf{U}' \arrow[ul, "(k',r')"] \arrow[rr, "(m',s')"'] & & \mathbf{T}' \arrow[ul, "(h'_1,q'_1)"'] \arrow[ur, "(h'_j,q'_j)"'] &
\end{tikzcd}$$

In order to show the claimed equivalence, we construct a functor $\Phi' : \mathbf{U} \longrightarrow \mathbf{U}'$. Since $h'_1 \Phi m = g_i k$, the universal property of the pullback $\mathbf{U}'$ can be used to define Φ'.

Since $\mathbf{U}$ and $\mathbf{U}'$ are pullbacks of functors, their objects can be taken to be pairs of objects from $\mathbf{S}$ and $\mathbf{T}$, respectively $\mathbf{S}$ and $\mathbf{T}'$. Similarly, their arrows can be taken to be pairs. Also k and m, respectively k' and m' can be taken to be projections. We can now explicitly describe the action of Φ' on an arrow of $\mathbf{U}$ as $\Phi'(u_0, u_1) = (u_0, \Phi u_1)$.

It remains to show that Φ' satisfies E1-E3.

For E1, $g'_x k' \Phi' = g_x k$ since $k' \Phi' = k$ by the construction of Φ'. Similarly, we have $h'_x m' \Phi' = h'_x \Phi m = h_x m$ by the construction of Φ' and the fact that $h'_x \Phi = h_x$.

For E2, an object of $\mathbf{U}'$ is a pair of objects from $\mathbf{S}$ and $\mathbf{T}'$. Now Φ' is the identity on its first component and Φ, which is surjective on objects, on its second component, hence Φ' is surjective on objects.

Condition E3 considers both parts of a Put and here we will consider the amendment part in detail. The other part is routine and similar to the proof found in [8].

Suppose U is an object of $\mathbf{U}$ and $U' = \Phi'(U)$. Required to show that the amendment for the Put along $(h_j, q_j)(m, s)$ into U is the same as the amendment for the Put along $(h'_j, q'_j)(m', s')$ into U'. In both cases the amendment is obtained, according to Definition 7, by composing an amendment from the lens (h_j, q_j) (respectively (h'_j, q'_j)) which we will call a (respectively a') with h_j (respectively h'_j) applied to an amendment for the lens (m, s) (respectively (m', s')) which we will call b (respectively b'). Explicitly,

$$\begin{aligned}
a &= q_{ja}^{mU}(\beta) \\
a' &= q'^{m'U'}_{ja}(\beta) \\
b &= h_j(s_a^U(q_{jb}^{mU}(\beta))) \\
b' &= h'_j(s'^{U'}_a(q'^{m'U'}_{jb}(\beta)))
\end{aligned}$$

We will show that $a = a'$ and $b = b'$, so that of course the amendments $ba = b'a'$ as required.

$a = a'$:

$$\begin{aligned}
a &= q_{ja}^{mU}(\beta) && \text{by definition} \\
&= q'^{\Phi mU}_{ja}(\beta) && \text{by E3 for } \Phi \\
&= q'^{m'U'}_{ja}(\beta) && \text{by construction of } \Phi' \text{ and } U' = \Phi U
\end{aligned}$$

$b = b'$: First, let $\tau = q_{jb}^{mU}(\beta) : mU \longrightarrow T_1$, say, be the Put of β into mU and correspondingly, let $\tau' =$

$q'^{m'U'}_{jb}(\beta) : m'U' \longrightarrow T'_1$ be the Put of β into $m'U'$ and note that $\Phi\tau = \tau'$ since $m'U' = m'\Phi'U = \Phi m'U'$. Then

$$\begin{aligned}
b &= h_j(s^U_a(q^{mU}_{jb}(\beta))) && \text{by definition} \\
&= h_j(s^U_a(\tau)) \\
&= h_j(q^{T_1}_{1b}(p^{kU}_{ia}(h_1(\tau)))) && \text{by Proposition 10} \\
&= h'_j(\Phi q^{T_1}_{1b}(p^{kU}_{ia}(h_1(\tau)))) && \text{by E1 for } \Phi \\
&= h'_j(q'^{T'_1}_{1b}(p^{kU}_{ia}(h_1(\tau)))) && \text{by E3} \\
&= h'_j(q'^{T'_1}_{1b}(p^{k'U'}_{ia}(h'_1(\Phi\tau)))) && \text{since } h'_1\Phi = h_1 \text{ and } k = k'\Phi' \\
&= h'_j(q'^{T'_1}_{1b}(p^{k'U'}_{ia}(h'_1(\tau')))) \\
&= h'_j(s'^{U'}_a(\tau')) \\
&= h'_j(s'^{U'}_a(q'^{m'U'}_{jb}(\beta))) \\
&= b'
\end{aligned}$$

as required. ∎

Proposition 24 *Suppose that* $L = (\mathbf{S}, (\mathbf{A}_i, g_i, p_i)_{i \le n})$, $L' = (\mathbf{S}', (\mathbf{A}_i, g'_i, p'_i)_{i \le n})$ *and* $M = (\mathbf{T}, (\mathbf{B}_i, h_i, q_i)_{i \le m})$ *are multiary lenses. Further, suppose that* $\Phi : \mathbf{S} \longrightarrow \mathbf{S}'$ *is a functor satisfying conditions E1-E3 so that* $L \equiv_{spg} L'$*, and that* L *and* M *(and so also* L' *and* M*) are composable over some* $\mathbf{A}_i$*. Then the two wide span composites are equivalent, that is* $M_i L \equiv_{spg} M_i L'$*.*

Proof. As above, the diagram below shows some of the lenses in the wide spans L, L' and M chosen to include all those that are required.

The top composite span $(M_i L)$ of spg-lenses has head $\mathbf{U}$, the pullback (in **cat**) of g_i and h_1. Similarly the pullback $\mathbf{U}'$ is the head of the bottom span composite $(L'M)$.

$$\begin{tikzcd}
 & \mathbf{S} \arrow[dl, "(g_j,p_j)"'] \arrow[dr, "(g_i,p_i)"] \arrow[dd, "\Phi"'] & & \mathbf{U} \arrow[ll, "(m,s)"'] \arrow[dr, "(k,r)"] \arrow[dd, dotted, "\Phi'"] & & \\
\mathbf{A}_j & & \mathbf{A}_i = \mathbf{B}_1 & & \mathbf{T} \arrow[ll, "(h_1,q_1)"'] \arrow[r, "(h_{j'},q_{j'})"] & \mathbf{B}_{j'} \\
 & \mathbf{S}' \arrow[ul, "(g'_j,p'_j)"] \arrow[ur, "(g'_i,p'_i)`"'] & & \mathbf{U}' \arrow[ll, "(m',s')"] \arrow[ur, "(m',s')"'] & &
\end{tikzcd}$$

In order to show the claimed equivalence, we construct a functor $\Phi' : \mathbf{U} \longrightarrow \mathbf{U}'$. Since $G'_R \Phi H = F_L K$, the universal property of the pullback $\mathbf{T}'$ can be used to define Φ'.

The rest of the proof proceeds in parallel to that of the previous proposition and we leave it to the reader. ∎

7 The Multi-Category of Multiary Lenses

We have defined multiary lenses above as certain wide spans of spg-lenses and noted that we may consider composites for them when there is a matching condition on the feet. In this section we show how multiary lenses and their composites may be organised into the structure of a *multicategory*, a concept first introduced by Lambek in the 1960's [9]. We begin with the definition of multicategory.

Definition 25 *[10] A* multicategory $\mathbf{C}$ *consists of the following data and operations:*
(M1) A collection of objects, C_0.
(M2)A collection of multimorphisms, C_1.
(M3) A source *map* $s : C_1 \longrightarrow (C_0)^*$ *to the collection of finite, possibly empty lists of objects (where* $(C_0)^*$ *is the*

free monoid generated by C_0)
(M4)A target *map $t : C_1 \longrightarrow C_0$.*
We write $f : c_1 \dots c_n \longrightarrow c$ to indicate the source and target of a multimorphism f.
(M5) The identity *operation is a map $1_{(-)} : C_0 \longrightarrow C_1$ with $1_c : c \longrightarrow c$.*
(M6) The composition *operation assigns, to each $f : c_1 \dots c_n \longrightarrow c$ and an n-tuple $(f_i : \overrightarrow{c_i} \longrightarrow c_i, i = 1, \dots, n)$, a composite $f(f_1, \dots, f_n) : \overrightarrow{c_1} \dots \overrightarrow{c_n} \longrightarrow c$ where the source is obtained by concatenating lists.*

These operations are subject to the expected associativity and identity axioms.

In general, for a category with pullbacks, there is a multicategory of wide spans constructed much as we do below. Since the category of spg-lenses (like the category of asymmetric delta lenses) does not have pullbacks, we need to describe the multicategory structure in detail, but the idea is the same as for the case with pullbacks, and to preserve the analogy we will write "pullback" and "wide pullback" below to mean the pullback and wide pullback of the relevant Gets in **cat**, endowed with Puts in the canonical way described above for pullbacks (for wide pullbacks the constructions can be done a pullback at a time to generate what we are calling the wide pullback of a given collection of closed aa-lenses).

For our multimorphisms we are going to consider $\equiv_{spg}$ classes multiary lenses. We can now define a multicategory **ML** of multiary lenses.

An object of **ML** is a category.

A multimorphism f of **ML** is an ($\equiv_{spg}$ equivalence class of) multiary lenses.

Suppose $f = [L]$ is a multimorphism which is the equivalence class of the multiary lens $L = (\mathbf{S}, (\mathbf{A}_i, g_i, p_i)_{i \leq n})$. The source of f is $\mathbf{A}_2 \dots \mathbf{A}_n$ and the target of f is $\mathbf{A}_1$.

Identity: For any category $\mathbf{A}_1$, its identity morphism is the equivalence class of the span (so $n = 2$) of identity spg-lenses from $\mathbf{A}_1$ to $\mathbf{A}_1$ The identity spg-lens is closed, so in particular the left leg of that span is closed, as required.

Composition: Suppose that $f = [L]$ is a multimorphism where $L = (\mathbf{S}, (\mathbf{A}_i, g_i, p_i)_{i \leq n})$ is a representative. Suppose also that for $2 \leq i \leq n$, $f_i = [M_i]$ are multimorphisms with representative multiary lenses $M_i = (\mathbf{T}_i, (\mathbf{B}_{ij}, h_{ij}, q_{ij})_{j \leq m_i})$ satisfying $\mathbf{B}_{i1} = \mathbf{A}_i$. The composite multimorphism is denoted

$$f(f_2, \dots, f_n) : \mathbf{B}_{21} \dots \mathbf{B}_{2m_2} \dots \mathbf{B}_{n1} \dots \mathbf{B}_{nm_n} \longrightarrow \mathbf{A}_1$$

For representative multiary lenses as above, we obtain a representative of $f(f_2, \dots, f_n)$ as follows. We denote the composite of L and M_i at $\mathbf{A}_i$ as $L \circ_i M_i$. Denote its head by $\mathbf{U}_i$. Its left leg lens is $(\mathbf{U}_i, \mathbf{A}_1, (k_i, r_i)(g_1, p_1))$. The remaining $m_i - 1$ legs are denoted $(\mathbf{U}_i, (\mathbf{B}_{ij}, k_{ij}, r_{ij})_{2 \leq j \leq m_i})$ where $(k_{ij}, r_{ij}) = (h_{ij}, q_{ij})(m_i, s_i)$. The head of the composite representative is the wide pullback $\mathbf{V}$ of the $(g_1, p_1)(k_i, r_i)$ (which are all closed). Its left leg is the common (closed) spg-lens from $\mathbf{V}$ to $\mathbf{A}_1$. The spg-lens from $\mathbf{V}$ to $\mathbf{B}_{ij}$ is the composite of the projection lens from $\mathbf{V}$ to $\mathbf{U}_i$ followed by (k_{ij}, r_{ij}) to $\mathbf{B}_{ij}$.

Consider the following diagrams showing first how we define $\mathbf{U}_i$:

$$\begin{tikzcd}
 & & \mathbf{U}_i \arrow[dl, "{(k_i,r_i)}"'] \arrow[dr, "{(m_i,s_i)}"] & & & & \\
 & \mathbf{S} \arrow[dl, "{(g_1,p_1)}"'] \arrow[dr, "{(g_i,p_i)}"] & & \mathbf{T_i} \arrow[dl, "{(h_{i1},q_{i1})}"'] \arrow[d, "{(h_{i2},q_{i2})}"'] \arrow[dr, "{(h_{i3},q_{i3})}"] \arrow[drrr, "{(h_{im_i},q_{im_i})}"] & & & \\
\mathbf{A}_1 & & \mathbf{A}_i = \mathbf{B}_{i1} & \mathbf{B}_{i2} & \mathbf{B}_{i3} & \cdots & \mathbf{B}_{im_i}
\end{tikzcd}$$

and next, showing how we define $\mathbf{V}$ and the legs of the composite multiary lens, but with only $\mathbf{U}_1$ and $\mathbf{U}_2$ displayed to minimise the complexity of the diagram:

$$\begin{tikzcd}
 & & & \mathbf{V} \arrow[dll] \arrow[dr] & & & & \\
 & \mathbf{U}_1 \arrow[dl, "(g_1{,}p_1)(k_1{,}r_1)"'] \arrow[d] \arrow[dr] \arrow[drr] & & & \mathbf{U}_2 \arrow[dllll] \arrow[d, "(k_{22}{,}r_{22})"'] \arrow[dr] \arrow[drrr, "(k_{2m_i}{,}r_{2m_2})"] & & \dots & \\
\mathbf{A}_1 & \mathbf{B}_{12} & \mathbf{B}_{13} & \dots \mathbf{B}_{1m_1} & \mathbf{B}_{22} & \mathbf{B}_{23} & \dots & \mathbf{B}_{2m_2}
\end{tikzcd}$$

It remains to show that $f(f_2, \dots, f_n)$ is independent of the representative multiary lenses chosen.

Proposition 26 *The multicomposition is well-defined independent of choice of representatives.*

Proof. The independence of representatives follows immediately from Propositions 23 and 24 applied to the diagram above. ∎

Corollary 27 *With the operations above* **ML** *is a multicategory whose multiarrows are* $\equiv_{spg}$ *equivalence classes of multiary lenses.*

8 Discussion

We begin the discussion with a few words about why we have excluded from our definitions what Diskin et al [2] call the compatibility relation K. As we understand it, K was introduced to avoid having amendments which might, in the extreme, amend away the intended update. In formal terms, if an intended update $v : g(A) \longrightarrow B'$, happened to be invertible, then the amendment $p_a^A(v)$ might be v^{-1}. More subtly an amendment might amend away some part of the intended update. This could be considered undesirable, so K was introduced as a relation on composable pairs of arrows, and could record those compositions which weren't undesirable. Then, an amendment lens in which an update and its amendment were K-related would be desirable.

We have taken the view that, instead, engineers, immersed in their application environment and knowing what is and isn't desirable in that environment, can choose from among the lenses that the mathematics, without K, offers.

There are further technical reasons for excluding K at this stage too. Having K as part of a model space definition would mean that the correct K needs to be chosen in the domain of an amendment lens, and yet K plays no role at all when it is in the domain of an amendment lens. So, apparently reasonable lenses only become problematic when one goes to compose them with other apparently reasonable amendment lenses and finds that their Ks are incompatible. But in many cases those incompatible Ks were never relevant to the original lenses at all. It is better to be rid of the Ks and instead to let the user choose not to use lenses that they would want to exclude from being legitimate lenses. Having K be part of the model spaces adds complexity to defining composition (one of the principal challenges confronted in this paper) in ways that seem inessential.

Furthermore, when it comes to forming compositions that require "pullbacks" as in Proposition 10, K can have more deleterious effects. In the notation presented there, compatibility relations that occur in the heads of the spans $\mathbf{S}$ and $\mathbf{T}$ have no relevance to the four input lenses, but need to be used to define $\mathbf{U}$, which in turn may be involved in further compositions.

In short, it seems better at this stage to study aa-lenses that don't require any extra relations K, and allow users to choose from among them. If an aa-lens does something that a user might not want to have, and might exclude via a judicious choice of K, that user can just choose those lenses that do not exhibit the undesirable behaviour. This is important because what is undesirable in some applications may not be undesirable in others, and trying to capture it as *part of the lens structure*, or even worse as *part of the model space structure* limits possible compositions and can have severe effects on the existence of "pullback" amendment lenses.

Another way of viewing what we are doing here in excluding K might be to say that the compatibility relation is simply the composability relation – every composable pair is compatible.

Next a few words about our use of closed left legs.

All cospans of d-lenses that satisfy PutGet have "pullbacks", and so can be used to define composites of symmetric d-lenses. A natural plan on moving to aa-lenses might be to calculate similar "pullbacks". But there is a serious problem. As can be seen in the definition of the amendment s_a in Proposition 10, the fact that (h, q) is not closed complicates defining the amendment for its "pullback" (m, s) resulting $s_a^{(S,T)}(\beta) = p_b'^{T'}(q_a^S(g'(\beta))) = \tilde{b}$ say. But if the other leg of the cospan, (g', p'), were also not closed, then $(q_b^S(g'(\beta)), \tilde{b}\beta)$ would not necessarily be an arrow of $\mathbf{U}$ and the $\mathbf{S}$ component would need to adjusted by a similar further amendment, which would in turn require further adjustment of the amendment in $\mathbf{T}$, potentially, *ad infinitum*. In short, composing across a pair of non-closed legs can cause havoc.

Of course, which leg in a cospan is closed has no relevance for the existence of the "pullback". But we do need to be consistent in our choice, and we have chosen the left leg to be the closed leg.

Also of course, but worth remarking, if both legs are closed then the calculations in the proof of Proposition 10 are the same as the calculations for the usual "pullback" of d-lenses [6].

There is an interesting similarity between spans of spg-lenses with closed left leg, and the symmetric lenses studied in the paper by Fong and Johnson [3]. Fong and Johnson study a compositional framework for supervised learning algorithms, and note that the so-called Learners of Fong et al [4] correspond to spans of set-based lenses whose left leg is a constant complement lens. They point out in [3] that such symmetric lenses are in fact left leg closed aa-lenses in a unique way. This was entirely unexpected.

Finally, a remark about our construction of the multicategory of multiary lenses summarised in Corollary 27. We have given in the preceding section a fairly explicit account of the multicategory composition of $n+1$ multiary lenses because that is what's required for a general composition in a multicategory. Meanwhile, it is worth noting that using just the identity and associativity axioms for multicategories gives a coherence result which shows that general multicategory compositions can always be constructed from compositions with at most two non-identity multiarrows, that is, multiarrows of the form $f(1, 1, \ldots, 1, f_i, 1, \ldots 1)$ — these are in a sense the binary multiarrow compositions. An alternative approach to obtaining Corollary 27 is to *define* those binary multiarrow compositions to be the composite given in each case by Definition 21, and then appeal to the coherence result to obtain the general multicategory compositions.

9 Conclusion

This paper has aimed to lay the foundations for work on multiary lenses, whether in the "propagation" form (in the style of the original symmetric lenses [5]) or in "wide span" form (in the style of the spans of d-lenses of [6]). To do that we need to have precise definitions of the equivalences that determine when two propagation style lenses (or indeed two wide span style lenses) have the same behaviour and so should be viewed as representatives of the same lens. Furthermore, those equivalences need to "play well" with composites and with constructions which allow us to convert between the two styles. All that is done here, and the result is a bijection, Corollary 20, that allows us to translate freely between the two forms.

With those foundational matters settled the paper turns to the question of how multiary lenses should be composed, and what algebraic structure they and their composites might form. Aiming for broad generality, we have worked with wide spans of aa-lenses, and shown how to define composites for those wide spans with a closed left leg. Finally we demonstrate that those multiary lenses form a multicategory using the composites calculated across their closed legs.

A final word about the title: Why the plural (Multicategories of Multiary Lenses)? We want to emphasise that while we have isolated general classes of lenses that interact well and through their generality promise wide applicability, we have also determined that adding many kinds of extra requirements to those lenses, for example, that the component lenses are all closed, or that they satisfy (Reflect0) or (Reflect0*), we still obtain a multicategory of such lenses. Furthermore, for workers who might have in mind certain compatibility relations K, something that we have excluded from the analysis presented here, it is possible to look at submulticategories of the multicategory of multiary lenses that contain those lenses that they consider acceptable for their application.

We especially want to emphasise that the work that began this study of multiary lenses, work which was based on wide spans of d-lenses, is just a special case of the results finally presented here.

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