Information Warfare Model with Migration*

Igor Samoilenko[1[0000-0001-6858-8224]], Congzao Dong[2[0000-0001-5877-1877]],

Bogdan Dovgay[3[0000-0003-2840-7884]]

[1]School of Mathematics and Statistics, Xidian University, 710126 Xi'an, P.R. China
Taras Shevchenko National University of Kyiv, 64 Volodyimyrska str., Kyiv, Ukraine
isamoil@i.ua
[2]School of Mathematics and Statistics, Xidian University, 710126 Xi'an, P.R. China
czdong@xidian.edu.cn
[3]Taras Shevchenko National University of Kyiv, 64 Volodyimyrska str., Kyiv, Ukraine
bogdov@gmail.com

Abstract. We construct and study a discrete time model describing the conflict interaction between two complex systems with non-trivial internal structures. The external conflict interaction is based on the model of alternative interaction between a pair of non-annihilating opponents. The internal conflict dynamics is similar to the one of Lotka-Volterra model, namely information warfare model. We show that the typical trajectory of the complex system converges to an asymptotic attractive cycle. We propose an interpretation of our model in terms of migration processes.

Keywords. Lotka-Volterra equations, information warfare model, conflict interaction, dynamical system, cyclic attractor, limiting distributions, migration

1 Introduction

Since the beginning of 20th century the Lotka-Volterra model of prey-predator interaction is one of the main models for simulation of many processes in population theory, social sciences and economics. As a rule, continuous models where Lotka-Volterra equations have ratio-depended parameters are studied (see, for example [3, 5, 6, 11-13, 17, 19, 21]). Application of this approach to information warfare model was proposed in [16].

Authors regard some social community of quantity N_0, potentially exposed some information threat (InfT) of two types, that is, for example, the threat of a negative change in its state by transmitting some information relevant to this group by information two different channels. The values $N_1(t)$, $N_2(t)$ – the numbers of “adherents” depending on time t who accepted the new information, ideas, norms, etc. of

* The work was supported by the Chinese Belt and Road Program DL20180077

the type 1 and 2 respectively. These are the main current characteristics of the degree of prevalence of InfT.

The main model assumptions are:

1. Both InfT are distributed among the community through the two information channels:

– the first one is "external" in relation to the community, for example, advertising media campaign. Its intensity is characterized by the parameters $\alpha_1 > 0$ and $\alpha_2 > 0$ respectively, both are considered to be independent of time;
– the second, "internal" channel is interpersonal communication between members of the social community (its intensity, that is, the number of equivalent informational contacts, characterized by the parameters $\beta_1 > 0$ and $\beta_2 > 0$ respectively, that are also independent of time). As a result, the adherents of the first idea that have been already "recruited" (their number is equal to $N_1(t)$), make their personal contribution to the recruitment process by affecting non-recruited members (their number is equal to the value of $N_0 - N_1(t) - N_2(t)$). The same is for the adherents of the second idea.

2. The rate of change of the number of adherents $N_1(t)$ and $N_2(t)$ (that is, the number recruited into the unit time) consists of:

– external recruitment rate (it is proportional to the product of the intensities α_1 and α_2 and on the number of individuals who are not yet recruited $N_0 - N_1(t) - N_2(t)$), that is, $\alpha_1 \cdot (N_0 - N_1(t) - N_2(t))$ and $\alpha_2 \cdot (N_0 - N_1(t) - N_2(t))$ respectively;
– internal recruitment rate (it is proportional to the product of intensities β_1 and β_2, on the corresponding number of active adherents $N_1(t)$, $N_2(t)$ and on the number non-recruited $N_0 - N_1(t) - N_2(t)$), that is, $\beta_1 \cdot N_1(t)\, (N_0 - N_1(t) - N_2(t))$ and $\beta_2 \cdot N_2(t)\, (N_0 - N_1(t) - N_2(t))$ respectively.

The model is, thus, described by Lotka-Volterra-type equations:

$$\begin{aligned} dN_1(t)/dt &= (\alpha_1 + \beta_1 N_1(t))(N_0 - N_1(t) - N_2(t)), \\ dN_2(t)/dt &= (\alpha_2 + \beta_2 N_2(t))(N_0 - N_1(t) - N_2(t)), \quad t > 0. \end{aligned} \tag{1}$$

The main aim of the work [16] is determination of obvious solution of (1), its stable points, bifurcation points, asymptotic behavior, etc.

In this work we construct a model that describes non-studied variant of information warfare model, i.e., a discrete model with migration. Here individuals migrate not randomly, but according to strategies, discussed in section 4.

We construct the model of the conflict interaction between a pair of complex systems A and B. This means that every system consists of some parameters that interact by some non-trivial law. The system is a finite set of positive numbers: $\boldsymbol{P} = (P_1, \ldots, P_K)$ for A and $\boldsymbol{R} = (R_1, \ldots, R_K)$ for B, where K means the quantity of parameters that characterize the system. We study dynamics in the discrete time. So, the evolution of every system is described by the sequence of vectors with non-negative coordinates $\boldsymbol{P}^{(n)} = (P^{(n)}_1, \ldots, P^{(n)}_K)$ for A, and $\boldsymbol{R}^{(n)} = (R^{(n)}_1, \ldots, R^{(n)}_K)$ for B, $n = 1, 2, \ldots$. The vectors ***P*** and ***R*** correspond to the moment $n = 0$. Naturally, each system tries to reach the optimal values of its coordinates. In reality, due to the conflict interaction, every coordinate changes in a complicated way. The evolution of all changes is determined by double dependence: by the conflict interaction between systems (which we shortly describe in section 2), and by the mutual "fight" of coordinates (of the information warfare interaction) inside every system.

The law of evolution inside of each (independent) system is described by a discrete variant of equations (1) (here we use the following notations $P:=N_1$ and $R:=N_2$ to separate the discrete case):

$$\begin{aligned} P^{(n)} &= P^{(n-1)} + (\alpha_1 + \beta_1 P^{(n-1)})(N_0 - P^{(n-1)} - R^{(n-1)}), \\ R^{(n)} &= R^{(n-1)} + (\alpha_2 + \beta_2 R^{(n-1)})(N_0 - P^{(n-1)} - R^{(n-1)}). \end{aligned} \tag{2}$$

Typical behavior of both continuous and discrete information warfare model is shown in Figure 1.

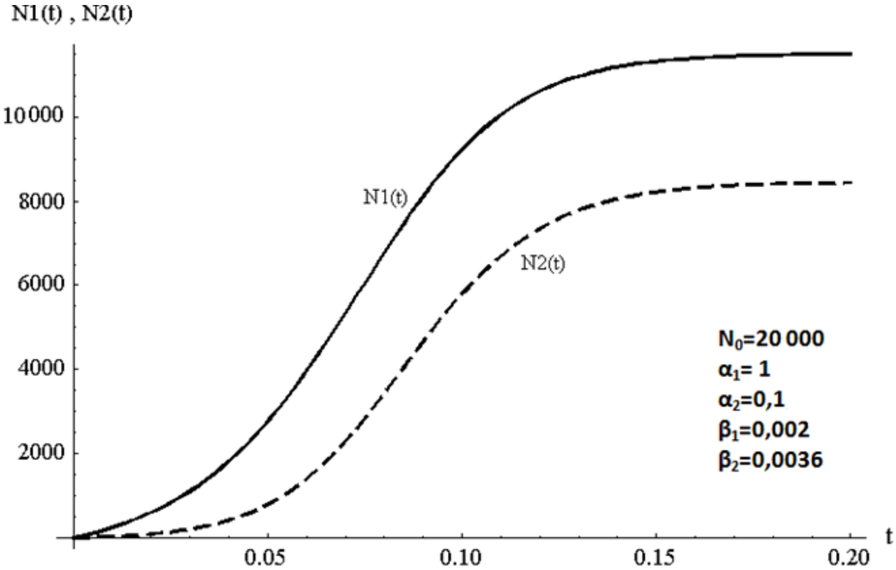

Fig. 1. Typical behavior of continuous information warfare model (Source: [16, p.256])

2 Conflict Interaction Between Non-annihilating Opponents

In this section we shortly remind an alternative approach to describe the redistribution of conflicting positions between two opponents, say A and B, concerning an area of common interests. The main idea of this model is that the influence of every opponent may be redistributed among conflict positions, but no one opponent may be destroyed (that would mean its distribution equals 0 in all the regions). This idea is realized due to the probabilistic character of opponents' distributions (the sum of the each opponent's presence by all regions should be equal to 1).

We consider the simplest case where the existence space of common interests is a finite set of positions $\Omega = \{\omega_1, \ldots \omega_K,\}$, $K \geq 2$. Each of the opponents A and B tries to occupy a position ω_i, $i = 1, \ldots, K$ with probability $P_A(\omega_i) = p_i \geq 0$ or $P_B(\omega_i) = r_i \geq 0$. The starting distributions of A and B along Ω are arbitrary and normed:

$$\sum_{i=1}^{K} p_i = 1 = \sum_{i=1}^{K} r_i.$$

A and B cannot be present simultaneously in a same position ω_i. The interaction between A and B is considered in discrete time. We introduce the noncommutative conflict composition between real-valued stochastic vectors $\boldsymbol{p}^0 = (p_1, \ldots, p_K)$, $\boldsymbol{r}^0 = (r_1, \ldots, r_K)$:

$$\boldsymbol{p}^1 := \boldsymbol{p}^0 * \boldsymbol{r}^0, \quad \boldsymbol{r}^1 := \boldsymbol{r}^0 * \boldsymbol{p}^0,$$

where the coordinates of $\boldsymbol{p}^1$, $\boldsymbol{r}^1$ are defined as follows:

$$\boldsymbol{p}_i^1 := \frac{\boldsymbol{p}_i^0(1-a\boldsymbol{r}_i^0)}{1-a\sum_{i=1}^{K}\boldsymbol{p}_i^0\boldsymbol{r}_i^0}, \; \boldsymbol{r}_i^1 := \frac{\boldsymbol{r}_i^0(1-a\boldsymbol{p}_i^0)}{1-a\sum_{i=1}^{K}\boldsymbol{p}_i^0\boldsymbol{r}_i^0}, \tag{3}$$

where the coefficient a is from the intervals [-1,0) or (0,1] and stands for the activity interaction. At the nth step of the conflict dynamics we get two vectors

$$\boldsymbol{p}^n := \boldsymbol{p}^{n-1} * \boldsymbol{r}^{n-1} = \boldsymbol{p}^0 *^n \boldsymbol{r}^0, \boldsymbol{r}^n := \boldsymbol{r}^{n-1} * \boldsymbol{p}^{n-1} = \boldsymbol{r}^0 *^n \boldsymbol{p}^0,$$

with coordinates

$$\boldsymbol{p}_i^n := \frac{\boldsymbol{p}_i^{n-1}(1-a\boldsymbol{r}_i^{n-1})}{1-a\sum_{i=1}^{K}\boldsymbol{p}_i^{n-1}\boldsymbol{r}_i^{n-1}}, \; \boldsymbol{r}_i^n := \frac{\boldsymbol{r}_i^{n-1}(1-a\boldsymbol{p}_i^{n-1})}{1-a\sum_{i=1}^{K}\boldsymbol{p}_i^{n-1}\boldsymbol{r}_i^{n-1}}.$$

The behavior of the state $\{\boldsymbol{p}^n, \boldsymbol{r}^n\}$ at time $t = n$ for n tending to infinity has been investigated in [1, 4, 7-10]. We shortly describe the results.

Theorem 1. For any pair of non-orthogonal real-valued stochastic vectors $\boldsymbol{p}$, $\boldsymbol{r}$ such that their inner product $(\boldsymbol{p}, \boldsymbol{r}) > 0$, and any fixed interaction intensity parameter a not equal to $1/(\boldsymbol{p},\boldsymbol{r})$, the sequence of states $\{\boldsymbol{p}^n, \boldsymbol{r}^n\}$ tends to the limit state $\{\boldsymbol{p}^\infty, \boldsymbol{r}^\infty\}$.
This limit state is invariant with respect to the conflict interaction:

$$\boldsymbol{p}^\infty := \boldsymbol{p}^\infty * \boldsymbol{r}^\infty, \boldsymbol{r}^\infty := \boldsymbol{r}^\infty * \boldsymbol{p}^\infty.$$

Moreover,

$$\begin{cases} (\boldsymbol{p}^{\infty}, \boldsymbol{r}^{\infty}) = 0, \text{ if } \boldsymbol{p} \neq \boldsymbol{r} \text{ and } 0 < a \leq 1, \\ \boldsymbol{p}^{\infty} = \boldsymbol{r}^{\infty}, \text{ in all other cases.} \end{cases}$$

We emphasize that in the case of a purely repulsive interaction (parameter a belong to the interval (0,1]), if the starting distributions are different, then the limiting vectors are orthogonal.

Therefore each of the vectors $\boldsymbol{p}^{\infty}$, $\boldsymbol{r}^{\infty}$ contains by necessity some amount of zero coordinates on different positions ω_i. For example the typical limiting picture for $\boldsymbol{p}^n$ is presented in Figure 2.

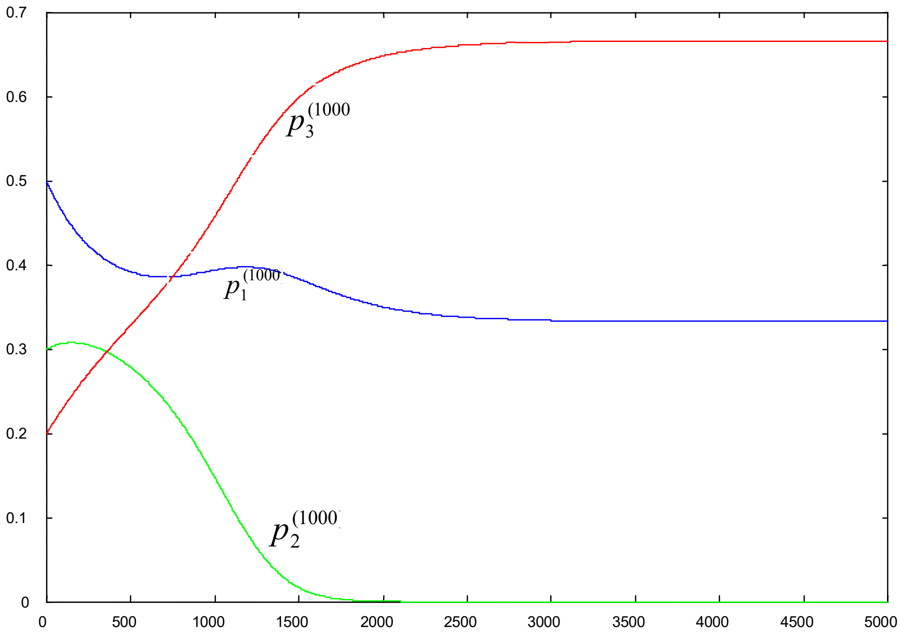

Fig. 2. Typical behavior of a pure conflict model a=1, p^0=(0.5; 0.3; 0.2), r^0=(0.48; 0.34; 0.18), p^{∞}=(0.33; 0; 0.67), r^{∞}=(0; 1; 0)

3 Model of Conflict Interaction Between Complex Systems

In this section we construct a dynamical model of conflict interaction between a pair of complex systems. This again means that every system includes some parameters that interact in non-trivial way described in section 1. But now each of the systems is subject to the inner conflict between their elements. For simplicity, we assume both systems to be similar and described by discrete information warfare models of type (2). We introduce the conflict interaction between these systems using an approach developed in [1, 2, 4, 7-10]. With such a rather complex situation we may obtain a wide spectrum of evolutions. In this work we study qualitative characteristics of the behavior of corresponding dynamical systems for some choice of parameters a, α_1, α_2, β_1, β_2 (see (2), (3)) and values of initial quantities of adherents P_i, R_i.

The coefficient a, that shows intensity of the interaction between systems, has an important effect. The increasing a from zero to unit causes the appearance of a series of bifurcations. For $a = 0$ we have two copies of independent information warfare models. For small values of a both systems behave like pure information warfare systems, coming them to a stable state.

The role of the coefficients α_1, α_2, β_1, β_2 and initial quantities of adherents P_i, R_i in a pure information warfare model is well-known and described (see, [16]).

The state of our dynamical system is fixed by a pair of vectors $P^n = (P^{(n)}_1, \ldots, P^{(n)}_K)$, $R^n = (R^{(n)}_1, \ldots, R^{(n)}_K)$ with non-negative coefficients, where $n = 0, 1, \ldots$ denotes the discrete time, $K \geq 2$ stands for the number of conflict positions. The complex conflict transformation is denoted by the mapping

$$\begin{pmatrix} \boldsymbol{P}^n \\ \boldsymbol{R}^n \end{pmatrix} \xrightarrow{F} \begin{pmatrix} \boldsymbol{P}^{n+1} \\ \boldsymbol{R}^{n+1} \end{pmatrix},$$

where F is the composition of four operations, the specific mathematical transformations: $F = [N^1 * N]U$.

Let us describe them in an explicit form for the first step for the case $K=2$.

The first operation U describes the interaction between elements inside every system separately according to the information warfare model. Corresponding mathematical transformation of vectors (the interaction composition) $\{\boldsymbol{P}^0, \boldsymbol{R}^0\} \xrightarrow{U} \{\tilde{\boldsymbol{P}}^0, \tilde{\boldsymbol{R}}^0\}$

is described by the system of equations of the form (2):

$$\tilde{P}_1^{(0)} = P_1^{(0)} + (\alpha_1 + \beta_1 P_1^{(0)})(N_0 - P_1^{(0)} - P_2^{(0)}),$$
$$\tilde{P}_2^{(0)} = P_2^{(0)} + (\alpha_2 + \beta_2 P_2^{(0)})(N_0 - P_1^{(0)} - P_2^{(0)}),$$

and

$$\tilde{R}_1^{(0)} = R_1^{(0)} + (\alpha_1 + \beta_1 R_1^{(0)})(N_0 - R_1^{(0)} - R_2^{(0)}),$$
$$\tilde{R}_2^{(0)} = R_2^{(0)} + (\alpha_2 + \beta_2 R_2^{(0)})(N_0 - R_1^{(0)} - R_2^{(0)}),$$

where the passage to new values of coordinates is pointed by tilde, but not by changing of upper index, likely to (2).

The following operation involves the interaction * (see (3)) between previous systems according to the theory of the alternative conflict for non-annihilating opponents (see, e.g. [1, 2, 4, 7-10]). To describe this operation we at first have to normalize the vectors $\tilde{\boldsymbol{P}}^0 = (\tilde{P}_1^{(0)}, \tilde{P}_2^{(0)}), \tilde{\boldsymbol{R}}^0 = (\tilde{R}_1^{(0)}, \tilde{R}_2^{(0)})$, i.e., to work with stochastic vectors. We use the following notation for normalization: $N\{\tilde{\boldsymbol{P}}^0, \tilde{\boldsymbol{R}}^0\} = \{p^0, r^0\}$, where the coordinates of the stochastic vectors $\{p^0, r^0\}$ are determined by formulae

$$p_1^{(0)} = \frac{\tilde{P}_1^{(0)}}{\tilde{z}_P^{(0)}}, p_2^{(0)} = \frac{\tilde{P}_2^{(0)}}{\tilde{z}_P^{(0)}}, r_1^{(0)} = \frac{\tilde{R}_1^{(0)}}{\tilde{z}_R^{(0)}}, r_2^{(0)} = \frac{\tilde{R}_2^{(0)}}{\tilde{z}_R^{(0)}},$$

where $\tilde{z}_P^{(0)} = \tilde{P}_1^{(0)} + \tilde{P}_2^{(0)}, \tilde{z}_R^{(0)} = \tilde{R}_1^{(0)} + \tilde{R}_2^{(0)}$.

The next step exactly corresponds to the conflict interaction between systems. We introduce new stochastic vectors $\{\boldsymbol{p}^1, \boldsymbol{r}^1\}$ with coordinates:

$$p_i^{(1)} := \frac{p_i^{(0)}(1 - a r_i^{(0)})}{1 - a\sum_{i=1}^{2} p_i^{(0)} r_i^{(0)}}, \; r_i^{(1)} := \frac{r_i^{(0)}(1 - a p_i^{(0)})}{1 - a\sum_{i=1}^{2} p_i^{(0)} r_i^{(0)}}, \; i = 1, 2.$$

Finally, we have to come back to the non-normalized vectors, which characterize quantitatively populations in both regions after inner and outer conflicts operations. So, at time $n = 1$ we have the following vectors $N^1\{\boldsymbol{p}^1, \boldsymbol{r}^1\}=\{\boldsymbol{P}^1, \boldsymbol{R}^1\}$, where

$$\boldsymbol{P}^1 = (P_1^{(1)}, P_2^{(1)}), \boldsymbol{R}^1 = (R_1^{(1)}, R_2^{(1)}),$$

and $P_i^{(1)} = p_i^{(1)} \tilde{z}_P^{(0)}, R_i^{(1)} = r_i^{(1)} \tilde{z}_R^{(0)}, i = 1, 2.$

We can repeat this procedure starting from $\{\boldsymbol{P}^1, \boldsymbol{R}^1\}$. So we get $\{\boldsymbol{P}^2, \boldsymbol{R}^2\}$. And so on for any *n*th step.

To find the equilibrium points in the case of the complex conflict interaction described above, we have to solve a very complex system of non-linear equations. Thus, pure mathematical approach seems to be not applicable. We use computer simulation methods to see the behavior of the dynamical system under consideration.

Let us consider the case of discrete information warfare model with the conflict interaction between systems. If we take the values of the coefficients N_0=20000, α_1=1, α_2=0,1, β_1=0,002, β_1=0,0036 that corresponds to the case of a pure information warfare model presented at Figure 2 and small value of a=0,00005, then the influence of the inner conflict is minimized and we have, in fact, two separated information warfare models. In this case the equilibrium points have the coordinates P_1 = 14.943507, P_2 = 35.100629. The dynamics is constant with these initial data.

In case of larger a=0,01, when oscillations appear (see Figure 3), the equilibrium point may also be easily found if we put the initial data in both systems to be equal. In this case the behavior is like in the case of a pure information warfare model, and stabilization occurs.

We should also stress that stable points in the models presented at Figures 1 and 3 are extremely different. This effect is caused by the presence of the outer conflict that initiates oscillations and does not allow exponential growth of the quantity of adherents in all the regions. Thus, if we have some information warfare system and want to change the quantity of adherents of some idea inside this system, we may create an analogous "artificial" system, introduce the conflict interaction and obtain the desired

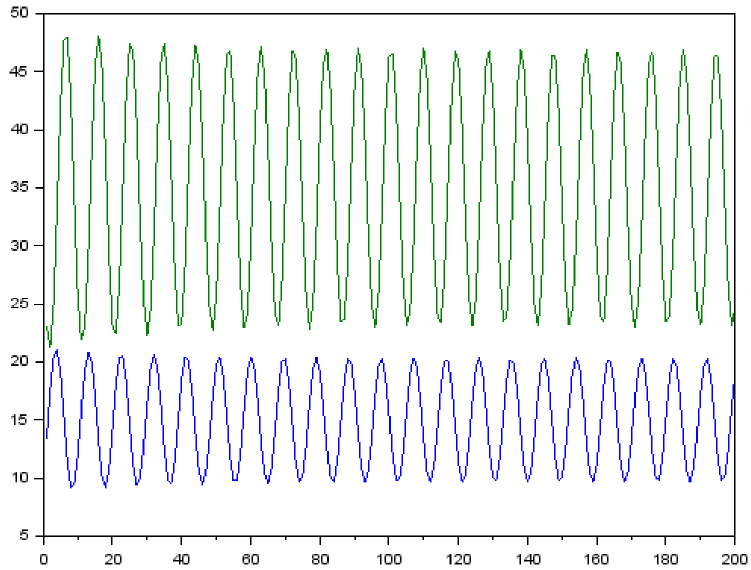

Fig. 3. Oscillations of two adherents' quantities P_1, P_2 inside one region ($0<n<200$) N_0=20000, α_1=1, α_2=0,1, β_1=0,002, β_2=0,0036, a=0.01

shift of the equilibrium point. So, we observed the interesting phenomenon: the equilibrium point of an isolated system is shifted if we come to the case when identical systems are united as an "ensemble".

However, this equilibrium point is unstable, any perturbation of initial data causes the receding of the system from the equilibrium point.

One of more interesting observations concerns the limit cycles. It is known that no such kind of orbits in discrete information warfare model is possible. But under the effect of the outer conflict, as we see at the pictures, the dynamical system reaches the limit cycle starting both from an inside or outside point with respect to the orbit. Partially, in Figure 4 we present the phase-space picture for (P_1, P_2) in the case of the model presented at Figure 3. As it was pointed above, in case of a pure information warfare model, with the stable initial data there is no dynamics. However, in the case of the model with the outer conflict the process tends to a limit cycle.

Thus, the idea of implementation of outer conflict to the standard information warfare model makes it much more complex and allows observing non-classical effects like oscillations, cyclic attractors, shifting of stable points, etc. We hope to study and describe all these effects in our following research.

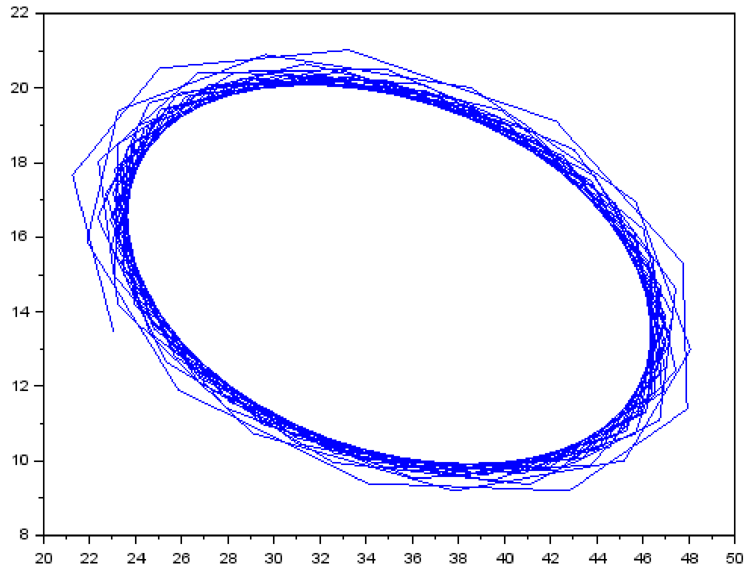

Fig. 4. Phase-space (P_1, P_2) inside one region (corresponding coefficients are presented at Figure 2)

4 Interpretation

In many works on mathematical biology and economics [3, 5, 6, 11-13, 17, 19, 21] the modelling of population dynamics or economical processes is based on Lotka-Volterra type equations. As a rule, continuous, not discrete, models are studied. In some works the migration process is considered. It takes place between different regions, inside which an interaction of the Lotka-Volterra type is present. For example, in [5] the migration rate between regions has some fixed probability.

We study discrete information warfare models with an additional interaction between them. That may be interpreted as some kind of correlation between the habitants of different regions. We suppose that discrete models are more natural, partially it is clear that information exchange among individuals happen at some fixed moments of time.

It is well known that in the classical information warfare model [16] a stable point exists. The amount of adherents tends to this point in the phase-space. In this case we observe the following dynamics, after several period of oscillations the populations stabilize (see Figure 1). Thus, we have an attracting point in phase-space. Such a dynamics exists inside every region when "migration" is absent.

When we introduce an additional interaction between the habitants of different regions a redistribution process appears which we interpret as a migration. In some of our complex models there is no stable point, the amount of adherents in both regions oscillates along fixed orbits. Apparently these orbits in a phase-space are attractors.

We note that explicit formulas of conflict interaction between non-annihilating opponents which describe the redistribution of populations are given by (3). The individuals of a certain kind migrate to the region, where their amount more numerous.

Is the "migration strategy" which is described in our model a natural one? We suppose that in many cases individuals may be right behaving in such a way. If we consider an information warfare model, it is clear that every separated individual is unable to estimate all factors that have an influence on the population dynamics like aggressive information influence, real amount of adherents with his own and alternative position, current population dynamics. In other words, the individual "does not know" the parameters of the information warfare equations and their current influence on the population dynamics.

However the individual has the group reflex and will migrate to the region, where, as he supposes, the information background is best (his population should be concentrated there). He suggests, right there are reliable information resources, possibilities for retranslation of his ideas, better conditions to organize large groups. Formula (3) just describes this tendency.

Similar motivations may be proposed in case of the work migration. Here the unemployed may be regarded as playing the role of "neutrals", employees and employed workers as playing the role of "information sources". People, who seek for work and migrate to another country, do not know, as a rule, the real situation in the opposite region. They prefer to migrate to the country where the majority of their friends migrated (group reflex).

So, at the cost of migration accelerates the increasing of one of adherents quantity in one of the regions. But at the same time there is an effect of the inner information warfare "fight" inside every system. As a result, some time later the backward migration starts.

In the Figure 3 we may see the effect of delay, when the amount of adherents of the first idea inside the region decreases, but the adherents of the other idea continue migration to this region, until their amount starts decreasing by following the information warfare model.

We emphasize, that in our model, in comparison with discrete information warfare model, a cyclic oscillations of quantities are observed. Moreover, a cyclic attractor exists in the phase-space, and the adherents trajectory tends to this orbit both from inside or outside point with respect to this cycle. We remark that in our model the normalization was fulfilled by the amount of habitants of the region, so the component of the corresponding vector may be large both at the cost of large population of fixed individuals and at the cost of small whole population of the region. So, a migration to the region with a lot of "free space" is also possible.

We also studied model with the attracting interaction ($a < 0$). In this case we obtained formally a similar dynamics, but now individuals migrate to the region where they are less numerous. Such a migration strategy might be also natural for some species, e.g. for “missioners” who want to spread their ideas among opponents.

References

1. Albeverio, S., Bodnarchyk, M., Koshmanenko, V.: Dynamics of discrete conflict interactions between non-annihilating opponents. Methods of Functional Analysis and Topology (MFAT). Vol. 11 (4): 309–319 (2005). http://mfat.imath.kiev.ua/article/?id=330
2. Albeverio, S., Koshmanenko, V., Pratsiovytyi, M., Torbin, G.: Spectral properties of image measures under infinite conflict interaction. Positivity. No. 10: 39–49 (2006). doi: 10.1007/s11117-005-0012-3
3. Albeverio S., Koshmanenko, V.D., Samoilenko, I.V.: The conflict interaction between two complex systems: cyclic migration. Journal of interdisciplinary mathematics. V.11, No.2: 163–185 (2008). doi: 10.1007/s11072-006-0054-x
4. Bandyopadhyay, M., Chattopadhayay, J.: Ratio-dependent predator-prey model: effects of environmental fluctuation and stability. Nonlinearity. Vol. 18: 913–936 (2005). doi: 10.1088/0951-7715/18/2/022
5. Colato, A., Mizrahi, S. S.: Effects of random migration in population dynamics. Physical Review E. Vol. 64: 1–14 (2001). doi: 10.1103/PhysRevE.64.011901
6. Cressman, R., Krivan, V., Garay, J.: Ideal free distributions, evolutionary games, and population dynamics in multiple-species environment. The American Naturalist. Vol. 164 (4): 473–489 (2004). doi: 10.1086/423827
7. Koshmanenko, V.: Theorem on conflict for a pair of stochastic vectors. Ukrainian Math. Journal. Vol. 55 (4): 555–560 (2003). doi: 10.1023/B:UKMA.0000010167.63115.37
8. Koshmanenko, V.: Theorem of conflicts for a pair of probability measures. Math. Methods of Operations Research. Vol. 59 (2): 303–313 (2004). doi: 10.1007/s001860300330
9. Koshmanenko, V. D., Kharchenko, N. V.: Invariant points of a dynamical system of conflict in the space of piecewise-uniformly distributed measures. Ukrainian Math. Journal. Vol. 56 (7): 927–938 (2004). doi: 10.1007/s11253-005-0120-z
10. Koshmanenko, V.D., Samoilenko, I.V.: The conflict triad dynamical system. Communications in nonlinear science and numerical simulation. No.16: 2917–2935 (2011). doi: 10.1016/j.cnsns.2010.10.028
11. Kuang, Y., Basic properties of mathematical population models. Journ.Biomath. Vol. 17: 129–142 (2002). https://math.la.asu.edu/~kuang/paper/criteria.pdf
12. Kuang, Y., Beretta, E.: Global qualitative analysis of a ratio-dependent predator-prey system. J. Math. Biology. Vol. 36: 389–406 (1998). doi: 10.1007/s002850050105
13. Lonzonn, Y., Solomon, S., Goldenberg, J., Mazarsky, D.: World size global markets lead to economic instability. Artificial life.: 357–370 (2003). doi: 10.1162/10645460332 2694816
14. Lotka, A. J.: Relation between birth rates and death rates. Science. Vol. 26: 21–22 (1907). doi: 10.1126/science.26.653.21-a
15. Malthus, T. R. : An Essay on the Principle of Population. Reprinted by Macmillan (1894). http://www.esp.org/books/malthus/population/malthus.pdf
16. Mikhailov, A.P., Marevtseva, N.A.: Models of information warfare. Math. Models Comput. Simul. Volume 4, Issue 3.: 251–259 (2012). doi: 10.1134/S2070048212030076
17. Murray, J. D.: Mathematical Biology I: An Introduction. Springer (2002)

18. Takahashi, K. I., Salam, K. Md. M.: Mathematical model of conflict and cooperation with non-annihilating multi-opponent. In: Unifying Themes in Complex Systems. Springer, Berlin, Heidelberg, pp 299-306 (2010). doi: 10.1007/978-3-540-85081-6_38
19. Stone, L., Olinky, R.: Phenomena in ecological systems, Experimental Chaos: 6th Experimental Chaos Conference, pp. 476-487 (2003)
20. Takahashi, K. I., Salam, K. Md. M.: Mathematical model of conflict with non-annihilating multi-opponent. J. Interdisciplinary Math. Vol. 9 (3): 459–473 (2006). doi: 10.1080/09720502.2006.10700457
21. Tufto, J.: Effects of releasing maladapted individuals: a demographic evolutionary model. The American Naturalist. Vol. 158 (4): 331–340 (2001). doi: 10.1086/321987
22. Verhulst, P. P.: Notice sur la loi que la population suit dans son accroissement. Correspondence mathematique et physique publiee par A. Quetelet. Vol. 10: 113–121 (1838)
23. Volterra, V.: Sui tentativi di applicazione della matematiche alle scienze biologiche e sociali. Giornale degli Economisti. Vol. 23: 436–458 (1901). https://www.jstor.org/stable/23221067